Abstract

This paper demonstrates how well the \( k-\omega \) turbulence model describes the nonlinear growth of flow instabilities from laminar flow into the turbulent flow regime. Viscous modifications are proposed for the \( k-\omega \) model that yield close agreement with measurements and with Direct Numerical Simulation results for channel and pipe flow. These modifications permit prediction of subtle sublayer details such as maximum dissipation at the surface, \( k \sim y^2 \) as \( y \to 0 \), and the sharp peak value of \( k \) near the surface. With two transition specific closure coefficients, the model equations accurately predict transition for an incompressible flat-plate boundary layer. The analysis also shows why the \( k-\epsilon \) model is so difficult to use for predicting transition.

I. Introduction

There has been renewed interest in development of methods for predicting boundary-layer transition. Current interest in vehicles such as the National Aerospace Plane (NASP), for example, provides the impetus for developing accurate transition prediction tools. Furthermore, because hypersonic boundary layers rarely achieve momentum-thickness Reynolds numbers large enough to sustain fully-developed turbulence, even the post-transition region generally exhibits nontrivial viscous effects. Consequently, accurate low-Reynolds-number turbulence models are also needed.

The standard approach is to view development of a transition model and a low-Reynolds-number turbulence model as two separate issues. The strongest argument in favor of this approach is simply that all spectral effects are lost in the time-averaging process used by turbulence models. Tollmien-Schlichting waves, for example, cannot be distinguished by a turbulence-model. Since a given boundary layer is unstable to perturbations that fall in a specific range of frequencies, conventional turbulence models, which distinguish only magnitude and an average frequency, can never be certain if a given perturbation will actually cause transition. However, if we implement two separate models, one for the transition region and another for the developing turbulent region, achieving a smooth joining of the two models' predictions presents an additional complication.

This complication can be avoided if we view both issues as low-Reynolds-number phenomena that can be addressed in the context of a single model. The strongest argument for this approach is that we can use the same model to describe a smooth transition from laminar to fully turbulent flow, including the transitional region. This approach, plausible provided we restrict our applications to broadband transition-triggering disturbances.

The research of Wilcox, et al\(^6\) provides a great deal of support for the latter approach. Using a \( k-\omega^2 \) turbulence model and transition-specific, low-Reynolds-number modifications, Wilcox simulated boundary-layer transition for a wide range of Mach numbers including pressure gradient, surface roughness, surface heating and cooling, and surface mass transfer. The purpose of this paper is to build upon the work of Wilcox, et al, taking advantage of recent Direct Numerical Simulation (DNS) results in developing appropriate viscous modifications for the Wilcox \( k-\omega \) model. The scope of this paper is confined to incompressible flows.

For incompressible turbulent fluid flow, the complete set of equations that constitute the Wilcox \( k-\omega \) two-equation model are as follows.

\[
\frac{\partial u_i}{\partial x_j} = 0 \quad (1)
\]

\[
\frac{\partial }{\partial t} (\rho u_i) + \frac{\partial }{\partial x_j} (\rho u_i u_j) = \frac{\partial }{\partial x_j} [-p \delta_{ij} + t_{ij} + \tau_{ij}] \quad (2)
\]

\[
\frac{\partial }{\partial t} (\rho k) + \frac{\partial }{\partial x_j} (\rho u_i k) = \tau_{ij} \frac{\partial u_i}{\partial x_j} - \beta^* \rho k^2 + \frac{\partial }{\partial x_j} [(\mu + \sigma \mu_T) \frac{\partial k}{\partial x_j}] \quad (3)
\]

\[
\frac{\partial }{\partial t} (\rho \omega) + \frac{\partial }{\partial x_j} (\rho u_i \omega) = \frac{\omega}{k} \tau_{ij} \frac{\partial u_i}{\partial x_j} - \beta \rho \omega^2 + \frac{\partial }{\partial x_j} [(\mu + \sigma \mu_T) \frac{\partial \omega}{\partial x_j}] \quad (4)
\]

\[
\tau_{ij} = 2 \mu_S \delta_{ij} + \tau_{ij} \quad (5)
\]

\[
\tau_{ij} = 2 \mu_T \delta_{ij} - \frac{2}{3} \rho k \delta_{ij} \quad (6)
\]

\[
\mu_T = \alpha^* \rho k / \omega \quad (7)
\]

\[
S_{ij} = \frac{1}{2} \left[ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right] \quad (8)
\]

In Equations (1-8), \( t \) is time, \( x_i \) is position vector, \( u_i \) is velocity, \( \rho \) is density, \( p \) is pressure, \( \mu \) is molecular viscosity, and \( \tau_{ij} \) is the sum of molecular and Reynolds stress tensors. Also, \( \delta_{ij} \) is the Kronecker delta, \( k \) is the turbulence kinetic energy, \( \omega \) is specific dissipation rate, \( \tau_{ij} \) is Reynolds stress tensor, and \( \mu_T \) is eddy viscosity. The six parameters \( \alpha, \alpha^*, \beta, \beta^*, \sigma \) and \( \sigma^* \) are closure coefficients whose values are given below.

\[
\alpha^* = \frac{\alpha_0^* + Re_T/R_k}{1 + Re_T/R_k} \quad (9)
\]

\[
\alpha = \frac{5}{9} \frac{\alpha_0 + Re_T/R_k}{1 + Re_T/R_k} \quad (\alpha^*) \quad (10)
\]
\[
\begin{align*}
\beta^* &= \frac{9}{100} \cdot \frac{5/18 + (Re_\tau/R_a)^4}{1 + (Re_\tau/R_a)^4} \\
\delta &= 3/40, \quad \sigma^* = \sigma = 1/2 \\
\alpha^*_o &= \delta/3, \quad \alpha_o = 1/10 \\
R_\delta &= 8, \quad R_s = 6, \quad R_\omega = 2.7
\end{align*}
\]

Section II explains in detail how the turbulence model simulates transition, and justifies the form of the viscous modifications. Section III explains why the most popular low-Reynolds-number k-\epsilon models are unsuitable for predicting transition. Section IV demonstrates how well the model performs for low-Reynolds-number channel flow and for pipe flow. Section V includes transition predictions for an incompressible flat-plate boundary layer. Section VI presents a summary of and conclusions drawn from the study.

II. Simulating Transition with Turbulence Model Equations

Turbulence model equations can be used to predict transition from laminar to turbulent flow, although most models predict transition to turbulence at Reynolds numbers that are at least an order of magnitude too low. To understand why and how the k-\omega model predicts transition, consider the flat-plate boundary layer. For the k-\omega model, the incompressible, two-dimensional boundary-layer form of the equations for \( k \) and \( \omega \) is as follows.

\[
\begin{align*}
\frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= \frac{\partial}{\partial y} \left[ (\nu + \nu_\tau) \frac{\partial u}{\partial y} \right] \\
\frac{\partial k}{\partial x} + v \frac{\partial k}{\partial y} &= \nu_\tau \left( \frac{\partial u}{\partial y} \right)^2 - \frac{\beta^* \omega \delta}{\partial y} \left[ (\nu + \sigma^* \nu_\tau) \frac{\partial k}{\partial y} \right] \\
\frac{\partial \omega}{\partial x} + v \frac{\partial \omega}{\partial y} &= \nu_\tau \left( \frac{\partial u}{\partial y} \right)^2 + \frac{\sigma \delta^2}{\partial y} \left[ (\nu + \sigma^* \nu_\tau) \frac{\partial \omega}{\partial y} \right] + \nu \frac{\partial \nu}{\partial y} \\
\nu_\tau &= \alpha^* k / \omega
\end{align*}
\]

where \( u \) and \( v \) are velocity components in the streamwise \((x)\) and normal \((y)\) directions, respectively, \( \nu \) is kinematic molecular viscosity, and \( \nu_\tau \) is kinematic eddy viscosity. We can most clearly illustrate how the model equations predict transition by rearranging terms in Equations (17) and (18) as follows.

\[
\begin{align*}
\frac{\partial k}{\partial x} + v \frac{\partial k}{\partial y} &= P_k \delta^* \omega \delta k + \frac{\partial}{\partial y} \left[ (\nu + \sigma^* \nu_\tau) \frac{\partial k}{\partial y} \right] \\
\frac{\partial \omega}{\partial x} + v \frac{\partial \omega}{\partial y} &= P_\omega \omega \delta^2 + \frac{\sigma \delta^2}{\partial y} \left[ (\nu + \sigma^* \nu_\tau) \frac{\partial \omega}{\partial y} \right]
\end{align*}
\]

The net production per unit dissipation for the two equations, \( P_k \) and \( P_\omega \) are defined by:

\[
\begin{align*}
P_k &= \frac{\alpha^* \beta}{\beta^*} \frac{\partial u}{\partial y} \left( \frac{\partial u}{\partial y} \right)^2 \frac{\partial u}{\partial y} \\
P_\omega &= \frac{\alpha \alpha^* \beta}{\beta} \frac{\partial u}{\partial y} \left( \frac{\partial u}{\partial y} \right)^2 \frac{\partial u}{\partial y}
\end{align*}
\]

There are two important observations worthy of mention at this point. First, if the turbulence energy is zero, Equation (21) has a well-behaved solution. That is, when \( k = 0 \), the eddy viscosity vanishes and the \( \omega \) equation uncouples from the \( k \) equation. Consequently, the \( k-\omega \) model has a nontrivial laminar-flow solution for \( \omega \). Second, the signs of \( P_k \) and \( P_\omega \) determine whether \( k \) and \( \omega \) are amplified or reduced in magnitude. However, it is not obvious by inspection of Equations (22) and (23) how the signs of these terms vary with Reynolds number as we move from the plate leading edge to points downstream. We can make the variation obvious by rewriting Equations (22) and (23) in terms of the Blasius transformation.

Before we introduce the Blasius transformation, we must determine the appropriate scaling for \( \omega \). To do this, we note that close to the surface of a flat plate boundary layer, the specific dissipation rate behaves according to

\[
\omega \to \frac{6 \nu}{\beta y^2} \text{ as } y \to 0
\]

In terms of the Blasius similarity variable, \( \eta \), defined by

\[
\eta = y / (\nu x / U_\infty)^{1/2}
\]

where \( U_\infty \) is freestream velocity, the asymptotic behavior of \( \omega \) approaching the surface is

\[
\omega \to \frac{6 U_\infty}{\beta x \eta^2} \text{ as } \eta \to 0
\]

Consequently, we conclude that the appropriate scaling for \( \omega \) in the Blasius boundary layer is given by

\[
\omega = \frac{U_\infty}{x} W(x, \eta)
\]

where \( W(x, \eta) \) is a dimensionless function to be determined as part of the solution. Hence, if we write the velocity in terms of dimensionless velocity, \( F(x, \eta) \), i.e.,

\[
u = U_\infty F(x, \eta)
\]

the net production per unit dissipation terms become

\[
\begin{align*}
P_k &= \frac{\alpha^* \beta}{\beta^*} \frac{\partial F}{\partial \eta} \left( \frac{\partial F}{\partial \eta} \right)^2 \\
P_\omega &= \frac{\alpha \alpha^* \beta}{\beta} \frac{\partial F}{\partial \eta} \left( \frac{\partial F}{\partial \eta} \right)^2
\end{align*}
\]

Thus, both \( P_k \) and \( P_\omega \) increase linearly with Reynolds number, \( Re_\infty \). From the exact laminar solution for \( F(\eta) \) and \( W(\eta) \) [the \( x \) dependence vanishes for the Blasius boundary layer], the maximum value of the ratio of \( \partial F / \partial \eta \) to \( W \) is given by

\[
\frac{\partial F / \partial \eta}{W} \text{ max } = \frac{1}{300}
\]
Hence, as long as the eddy viscosity remains small compared to the molecular viscosity, we can specify the precise points where $P_k$ and $P_\omega$ change sign. In general, using Equation (31), we conclude that the sign changes occur at the following Reynolds numbers:

$$(Re)_k = 9 \times 10^4 \frac{\beta^*}{\alpha^*}$$  (32)

$$(Re)_\omega = 9 \times 10^4 \frac{\beta}{\alpha^*}$$  (33)

With no viscous modifications, the closure coefficients $\alpha^*$, $\beta^*$ and $\beta^*$ are $5/9$, 1, 3/40 and 9/100, respectively. These values correspond to the limiting form of Equations (9-14) as $Re_T \to \infty$. Using these fully turbulent values, we find $(Re)_k = 8,100$ and $(Re)_\omega = 12,150$. Thus, starting from laminar flow at the leading edge of a flat plate, the following sequence of events occurs.

1. The computation starts in a laminar region with $k = 0$ in the boundary layer and a small freestream value of $k$.
2. Initially, because $P_k < 0$ and $P_\omega < 0$, dissipation of both $k$ and $\omega$ occurs in the production term and turbulence energy is amplified. At some point in this process, the eddy viscosity grows rapidly and this corresponds to the transition point.
3. At the critical Reynolds number, $Re_T = 8,100$, production catches dissipation in the $k$ equation. Downstream of $x*$, $k$ production exceeds $k$ dissipation and turbulence energy is amplified. At some point in this process, the eddy viscosity grows rapidly and this corresponds to the transition point.
4. $k$ continues to be amplified and, beyond $Re_T = 12,150$ production catches dissipation in the $\omega$ equation. $\omega$ is now amplified and continues growing until a balance between production and dissipation is achieved in the $k$ equation. When this balance is achieved, transition from laminar to turbulent flow is complete.

Consistent with experimental measurements, the entire process is very sensitive to the freestream value of $k$. There is also a sensitivity to the freestream value of $\omega$, although the sensitivity is more difficult to quantify.

Three key points are immediately obvious. First, $k$ begins growing at a Reynolds number of 8,100. By contrast, linear-stability theory tells us that Tollmien-Schlichting waves begin forming in the Blasius boundary layer at a Reynolds number of 90,000. This is known as the minimum critical Reynolds number. Correspondingly, we find that the model predicts transition at much too low a Reynolds number. Second, inspection of Equations (32) and (33) shows that the width of the transition region is controlled by the ratio of $\beta$ to $\alpha^*$. Third, transition will never occur if $P_\omega$ reaches zero earlier than $P_k$. Thus, occurrence of transition requires

$a^* < \frac{\beta^*}{\beta}$  (34)

This fact must be preserved in any viscous modification to the model. The viscous modifications in Equations (9-14), i.e., the dependence of $\alpha$, $a^*$ and $\beta^*$ upon $Re_T$, are designed to accomplish two objectives. The most important objective is to match the minimum critical Reynolds number. Reference to Equation (32) shows that we must require

$\beta^*/\alpha^* \to 1$ as $Re_T \to 0$  (35)

A secondary objective is to achieve asymptotic consistency with the exact behavior of $k$ and dissipation. $\epsilon = \beta^*k\omega_a$ approaching a solid boundary. That is, we would like to have

$k/y^2 \to \text{constant}$, $\epsilon/k \to 2v/y^2$ as $y \to 0$  (36)

Close to a solid boundary, Wilcox\textsuperscript{7} shows that the dissipation and molecular diffusion terms balance in both the $k$ and $\omega$ equations. The very-near-wall solution for $\omega$ is given by Equation (24). The solution for $k$ is of the form

$k/y^n \to \text{constant}$ as $y \to 0$  (37)

where $n$ is given by

$n = \sqrt[3]{1 + (1 + 24\beta^*/\beta)^{1/2}}$  (38)

Noting that dissipation is related to $k$ and $\omega$ by

$\epsilon = \beta^*k\omega$  (39)

we can achieve the desired asymptotic behavior of Equation (36) by requiring

$\beta^*/\beta \to 1/3$ as $Re_T \to 0$  (40)

Requiring this limiting behavior as $Re_T \to 0$ is sufficient to achieve the desired asymptotic behavior as $y \to 0$ since the eddy viscosity, and hence, $Re_T$ vanish at a solid boundary.

If we choose to have $\beta$ constant for all values of $Re_T$, Equations (34), (35) and (40) are sufficient to determine the limiting values of $\alpha^*$ and $\beta^*$ and an upper bound for $\alpha^*$ as turbulence Reynolds number becomes vanishingly small. Specifically, we find

$$a^* < \frac{\beta^*}{\beta}$$

$$\begin{align*}
\alpha^* &\to \beta/3 \quad \text{as } Re_T \to 0 \\
\beta^* &\to \beta/3
\end{align*}$$

Wilcox, et al.,\textsuperscript{14} make the equivalent of $\alpha a^*$ in their $k-\omega$ models approach the same limiting value and obtain excellent agreement with measured transition width for incompressible boundary layers. Numerical experimentation with the $k-\omega$ model indicates the optimum choice for incompressible boundary layers is $\alpha a^* \to 0.748$, or

$\alpha a^* \to 1/18$ as $Re_T \to 0$  (42)

Equations (9-14) postulate functional dependencies upon $Re_T$ that guarantee the limiting values in Equations (41) and (42), as well as the original fully turbulent values for $Re_T \to \infty$.

The three coefficients $R_k$, $R_\omega$ and $R_\omega$ control the rate at which the closure coefficients approach their fully turbulent values. As in previous analyses based on the $k-\omega$ model,\textsuperscript{3,5} we assume in the Blasius boundary layer that

$$B = \lim_{r \to \infty} \left[ \frac{u^+}{r^*} \right]$$  (43)
where \( u^* = u/u_* \) and \( y^* = u_*/v \) are standard sublayer sealed coordinates. Also, \( \kappa = 0.41 \) is Kármán's constant. For a given value of \( R_e \), there is a unique value of \( R_\infty \) that yields a constant in the law of the wall of 5.0. For small values of \( R_\infty \), the peak value of \( k \) near the surface is close to the value achieved without viscous corrections, viz. \( u_*^2/(2\nu) \). As \( R_\infty \) increases, the maximum value of \( k \) near the surface increases. Comparison of computed sublayer structure with Direct Numerical Simulation (DNS) results of Mansour, et al\(^{10}\) indicates the optimum choice for these three coefficients is as indicated in Equation (14). Section IV presents a complete comparison of computed channel flow properties with the Mansour, et al DNS results.

The only flaw in the model's asymptotic consistency occurs in the Reynolds shear stress, \( \tau_{xy} \). While the exact asymptotic behavior is \( \tau_{xy} \sim y^3 \), the model as formulated predicts \( \tau_{xy} \sim y^2 \). This discrepancy could easily be removed with another viscous modification. However, results obtained to date indicate this is of no significant consequence. It has no obvious bearing on either the model's ability to predict transition or properties of interest in turbulent boundary layers. The additional complexity and uncertainty involved in achieving this subtle feature of the very-near-wall behavior of \( \tau_{xy} \) does not appear to be justified.

### III. Difficulties Attending Use of the k-\( \varepsilon \) Model

Given the information developed in Section II, it is a simple matter to explain why little progress has been made in predicting transition with the k-\( \varepsilon \) model.\(^{11}\) The primary difficulties can be easily demonstrated by focusing upon incompressible boundary layers. If we use the standard form of the k-\( \varepsilon \) model, Equations (17-19) are replaced by

\[
\frac{\partial k}{\partial t} + \frac{\partial k}{\partial x} = \nu \left( \frac{\partial}{\partial y} \left( \frac{\partial k}{\partial y} \right) - \epsilon \frac{\partial \tau_{xx}}{\partial y} \right) \tag{44}
\]

\[
\frac{\partial \varepsilon}{\partial t} + \frac{\partial \varepsilon}{\partial x} = C_{tl} \left( \frac{\partial^2 \varepsilon}{\partial y^2} - \frac{\partial k}{\partial y} \right) - C_{m} \nu \left( \frac{\partial \tau_{xx}}{\partial y} \right) \tag{45}
\]

\[
\nu = C_{fl} f_{\nu} k^2 / \epsilon \tag{46}
\]

Values of the closure coefficients/functions \( C_{fl}, C_m, C_e, f_{\nu}, C_{tl}, \) and \( C_m \) differ amongst the various versions of this model. In the absence of viscous modifications, the standard values used are \( C_{fl} = 1.44, C_{tl} = 1.92, C_m = 0.09, f_{\nu} = 1, C_\epsilon = 1, \) and \( \sigma = 0.769 \).

One critical difference from the k-\( \omega \) model is obvious by inspection of Equations (44-46). Specifically, if the turbulence energy is zero, \( \epsilon \) must also be zero. We cannot simply drop the eddy viscosity in the \( \epsilon \) equation because of the presence of \( k \) in the denominator of the \( \epsilon \) equation's dissipation term. The model does possess a laminar-flow solution for the ratio of \( \epsilon \) to \( k \). That is, if we make the formal change of variables

\[
\epsilon = C_{\mu} k \omega \tag{47}
\]

and assume \( \nu \tau \ll \nu \), the following laminar-flow equation for \( \omega \) results.

\[
u \frac{\partial \omega}{\partial x} + \frac{\partial \omega}{\partial y} = \left( C_{tl} - 1 \right) f_{\omega} \frac{\partial \omega}{\partial y} - \left( C_{m} - 1 \right) C_{\omega} \omega^2 + \nu \frac{\partial^2 \omega}{\partial y^2} + \frac{2 \nu \partial k \partial \omega}{k \partial y \partial y} \tag{48}
\]

Equation (48) is nearly identical to the limiting form of Equation (18) for \( \nu / \nu \rightarrow 0 \). The only significant difference is the last term on the right-hand side of Equation (48). Except close to the surface where \( k \) is exactly zero, this term is unlikely to have a significant effect on the solution for small nonzero values of \( k \). However, in a numerical solution, products of dependent-variable gradients are generally destabilizing, and the problem can only be aggravated by having a coefficient inversely proportional to \( k \). This is not an insurmountable problem. However, establishing starting conditions is clearly more difficult with the k-\( \epsilon \) model than with the k-\( \omega \) model.

Given the diverse nature of viscous modifications that have been proposed for the k-\( \epsilon \) model,\(^{12}\) it is impossible to make any universal statements about why a specific model fails to predict realistic transition Reynolds numbers. Perhaps the strongest statement that can be made is, no one has approached the problem from the transition point of view. Most researchers have sought only to achieve asymptotic consistency and attempted transition predictions only as an afterthought. We can gain some insight by examining the net production per unit dissipation terms for the \( k \) and \( \epsilon \) equations that are analogous to Equations (29) and (30), viz,

\[
P_k = f_{\nu} \frac{\partial F}{\partial \eta} \frac{\partial \eta}{\partial w} \tag{49}
\]

\[
P_\epsilon = \frac{C_{fl} f_{\nu}}{C_{m}} \frac{\partial F}{\partial \eta} \frac{\partial \eta}{\partial w} \tag{50}
\]

On the one hand, without viscous damping, if we assume Equation (31) is valid, we find \( (Re)_k = 8,100 \) and \( (Re)_\epsilon = 10,800 \). Consequently, like the high-Reynolds-number version of the k-\( \omega \) model, transition will occur at too low a Reynolds number.

On the other hand, because \( f_{\nu}, C_{\mu}, \) and sometimes \( C_{fl} \) are often permitted to be functions of distance from the surface and/or functions of \( Re_T \), we cannot simply use Equation (31). Furthermore, some modelers add terms to the \( k \) and \( \epsilon \) equations in addition to damping the closure coefficients. Each set of values for the closure coefficients and additional terms must be used in solving Equation (48) to determine the laminar-flow solution for \( \epsilon / k \). While it is clearly impossible to make a quantitative evaluation of all variants of the k-\( \epsilon \) model, we can nevertheless make some general observations.

From the analysis of the k-\( \omega \) model, it is obvious that having \( f_{\nu} \ll 1 \) will tend to delay transition. Virtually all modelers implement an \( f_{\nu} \) that will accomplish this end. However, the modifications of Jones and Launder,\(^{11}\) Chien,\(^{13}\) and Lam and Brethorst,\(^{14}\) for example, damp \( C_{fl} \) to the extent that \( (Re)_\epsilon \) is smaller than \( (Re)_k \). This is the opposite of what is needed and will have an undesirable effect on both the onset of and the extent of the transition region.

This discussion is not intended as an exhaustive survey of the numerous low-Reynolds-number versions of the k-\( \epsilon \) model. Rather, it is intended to illustrate how difficult it is to apply the model to the transition problem. Given enough additional closure coefficients and damping functions, the k-\( \epsilon \) model can probably be modified to permit satisfactory transition predictions. However, even if this is done, establishing starting conditions will ultimately require a solution to Equation (48). That is, to initialize the computation, we must effectively transform to the k-\( \omega \) model. Since this is the natural starting point, it seems illogical to perform subsequent computations in terms of \( k \) and \( \epsilon \).
IV. Turbulent Flow Applications

To achieve a complete description of the transition from laminar to turbulent flow, we must be able to accurately describe the flow in the turbulent regime. This is, after all, the primary advantage of using turbulence model equations to describe transition. In this section, we examine channel and pipe flow to demonstrate how well the low-Reynolds-number form of the k-\(\omega\) model predicts properties of turbulent flows.

Figures 1 and 2 compare computed channel-flow skin friction, \(\tau_v\), with the Halleen and Johnston\(^{16}\) correlation for Reynolds number based on channel height, \(H\), and average velocity ranging from \(10^3\) to \(10^5\). The correlation is

\[
\tau_v = 0.0706 \sqrt{Re_H}
\]

As shown, computed \(\tau_v\) differs from the correlation by less than 3% except at the lowest Reynolds number shown where the correlation probably is inaccurate. Figure 3 compares computed pipe flow \(\tau_v\) with Prandtl's universal law of friction, viz.

\[
\tau_v^{1/2} = 4 \log_{10}(2 Re_D \tau_v^{1/2}) - 1.6
\]

Reynolds number based on pipe diameter, \(D\), and average velocity varies from \(10^3\) to \(10^6\). As with channel flow, computed \(\tau_v\) falls within 5% of the correlation except at the lowest Reynolds number indicated where the correlation is likely to be in error.

For more detailed comparisons, we consider two low-Reynolds-number channel flow cases corresponding to the DNS results of Mansour, et al\(^{10}\) and one high-Reynolds-number pipe flow case corresponding to measurements of Laufer\(^{16}\).

For purposes of identification, the three cases are referred to in terms of the parameter

\[
Re_c = u_r R / v
\]

where \(u_r\) is friction velocity and \(R\) is either channel half height or pipe radius. Figures 1 and 2 compare various computed profiles with the Mansour, et al DNS results for \(Re_c = 180\) and 395, respectively.

Six different comparisons are shown in each figure, including mean velocity, skin friction, \(\tau_v\), Reynolds shear stress, turbulence kinetic energy, turbulence energy production and dissipation rate. For both cases, velocity, \(\tau_v\), Reynolds shear stress, and turbulence kinetic energy profiles differ by less than 7%. Most notably, for both Reynolds numbers, the model predicts the peak value of \(k\) near the channel wall to within 4% of the DNS value. Additionally, approaching the surface, the turbulence-energy production, \(r_{\tau} \partial U / \partial y\), and dissipation, \(\epsilon\), are within 10% of the DNS results except very close to the surface.

Figure 3 compares \(k-\omega\) model pipe flow results with Laufer's\(^{16}\) measurements at a Reynolds number based on pipe diameter and average velocity of 40,000. As shown, computed and measured velocity and Reynolds shear stress profiles differ by less than 8%. As with channel flow, computed and measured turbulence kinetic energy differ by about 5% including close to the surface where the sharp peak occurs. Note that, at this high a Reynolds number, the \(k\) profile has a sharp spike near \(y = 0\) and this feature is captured in the computations. Except very close to the surface, computed turbulence energy production and dissipation differ from measured values by less than 10%. This may actually be a desirable result. That is, some controversy exists about the accuracy of Laufer's dissipation measurements close to the surface.

![Figure 1. Incompressible channel flow with \(Re_c = 180\), \(Re_H = 5,590\); o Mansour, et al; \(\square\) Johnston Formula.](image1)

![Figure 2. Incompressible channel flow with \(Re_c = 395\), \(Re_H = 13,750\); o Mansour, et al; \(\square\) Johnston Formula.](image2)
V. Transition Applications

Figure 4 compares computed and measured transition Reynolds number, $Re_{tr}$, for an incompressible flat-plate boundary layer. We define the transition Reynolds number as the point where the skin friction achieves its minimum value. Results are displayed as a function of freestream turbulence intensity, $T'$, defined by

$$T' = 100 \left( \frac{2}{3 \ U_e^*} \right)^{1/2}$$

(54)

where subscript $e$ denotes the value at the boundary-layer edge. As shown, consistent with the data compiled by Dryden,17 $Re_{tr}$ increases as the freestream intensity decreases. Because $\omega$ can be thought of as an averaged frequency of the freestream turbulence, it is reasonable to expect the predictions to be sensitive to the freestream value of $\omega$. To assess the effect, the freestream value of the turbulence length scale defined by $l = k^{1/2}/\omega$ has been varied from .0016 to 1.06 where $\delta$ is boundary layer thickness. As shown, computed $Re_{tr}$ values bracket virtually all of the data. These predictions are markedly superior to the preliminary efforts of Wilcox18 in developing low-Reynolds-number modifications for the $k-\omega$ model.

Figure 5 compares computed width of the transition region with measurements of Dhawan and Narasimha.19

We define transition width, $\Delta x$, as the distance between minimum and maximum skin-friction points. The computed width falls within experimental data scatter for $Re_{tr} < 10^4$, and lies a bit above the data for larger values. $\Delta x$, is unaffected by the freestream value of $\omega$.

Figure 5. Comparison of computed and measured width of the transition region for an incompressible flat-plate boundary layer.

VI. Summary and Conclusions

The primary objective of this paper has been to illustrate how two-equation turbulence models, most notably the $k-\omega$ model, predict transition. While the long-term goal of this research is to develop a transition model for all Mach numbers, this paper has focused on the case about which we know the most, viz, incompressible flow. The low-Reynolds-number modifications proposed not only facilitate accurate transition predictions, but also yield reasonably close agreement with DNS results for low-Reynolds-number channel flow.

The degeneracy of the $\epsilon$ equation in the $k-\epsilon$ model is a major stumbling block that impedes successful application
to the transition problem. By transforming the model to an equivalent k-ε model, it would be possible to remove some of the difficulties. After making such a transformation however, there is little reason to transform back.

The applications presented in Sections IV and V indicate we have formed a solid foundation for future low-Reynolds-number and transition research.

**VII. Acknowledgements**

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