ROBUST CONTROL FOR UNCERTAIN STRUCTURES

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APPROACH

• Assume full-state feedback

• Try to guarantee stability and performance robustness of classical LQR design
  - Guaranteed stability
  - Reasonable guaranteed robustness (gain and phase margin properties)

• Apply to benchmark problem to see interesting properties

\[
\begin{align*}
  & \overset{\cdot}{x}_1 = k x_1 + u, \\
  & m_1 = 1, \\
  & m_2 = 1, \\
  & x_1 \rightarrow x_2 = y, \\
  & \frac{1}{2} \leq k \leq 2
\end{align*}
\]
ROBUST LQR FORMULAS

- Standard LQR design when there is no uncertainty

\[ J = \int_0^\infty (x^T(t)Q_0x(t) + \rho u^T(t)u(t))dt \]

\[ PA_0 + A_0^TP + Q_0 - \frac{1}{\rho} PBB^TP = 0 \]

- Apply Petersen-Hollot bounds to derive robust Riccati Equation

\[ A = A_0 + \sum_{i=1}^{p} q_i E_i \quad |q_i| \leq 1 \]

\[ E_i = l_i n_i^T \quad L = [l_1 \ l_2 \ l_3 \ldots]; \quad N = [n_1 \ n_2 \ n_3 \ldots] \]

\[ PA_0 + A_0^TP + (Q_0 + \gamma NN^T) - P \left( \frac{1}{\rho} B B^T - \frac{1}{\gamma} L L^T \right) P = 0 \]

- Control

\[ G = \frac{1}{\rho} B^TP \quad u = -Gx \]
MISMATCHED LQR DESIGN

\begin{align*}
\text{Time} & \quad \text{Time} \\
0 & \quad 0 \\
5 & \quad 5 \\
10 & \quad 10 \\
15 & \quad 15 \\
20 & \quad 20
\end{align*}

\begin{align*}
\text{Time} & \quad \text{Time} \\
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5 & \quad 5 \\
10 & \quad 10 \\
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\end{align*}
RLQR DESIGN
INTERPRETATIONS OF RLQR DESIGN

- Equivalent to an optimal design where we minimize the cost functional

\[ J = \int_{0}^{\infty} (x^T(t)Q_0x(t) + x^T(t)\gamma NN^T x(t) + x^T(t)\frac{1}{\gamma} PLL^T P x(t) + \rho u^T(t)u(t))dt - \beta d^T(t)d(t) \]

- \(x^T(t)Q_0x(t)\) is the state weighting
- \(x^T(t)NN^T x(t)\) has been shown to be uncertain potential energy of an uncertain spring (or rate of dissipation for a damper)
- \(x^T(t)PLL^T P x(t)\) is an equivalent \(\mathcal{H}_\infty\) term.

- Parameter \(\gamma\) is therefore a tradeoff between minimizing unknown uncertain energy and worst case disturbance arising from forces due to parameter errors.
\[ k = 1.625 \]

\[ 0.5 \leq k \leq 2 \]
$K = 1.625 \ (RLQR)$
DISTURBANCE REJECTION

- Does the RLQR controller reject disturbances?
- Add a white noise disturbance at the output
- Apply both mismatched LQR and RLQR designs

\[ .5 \leq k \leq 2 \]
MISMATCHED LQR DESIGN

- $k = 0.5$
- $k = 0.875$
- $k = 1.25$
- $k = 1.625$
- $k = 2$

Time vs. $y$ and Disturbance vs. Time graphs for different values of $k$. The plots show the response of the system over time with varying gain factors. The disturbance graph on the bottom right illustrates the oscillatory nature of the disturbance signal.
RLQR DESIGN

\[ k = 0.5 \]
\[ k = 1.25 \]
\[ k = 1.625 \]
\[ k = 2 \]

Disturbance

Time

Disturbance
THREE-MASSES, TWO UNCERTAIN SPRINGS

\[ m_1 = 1 \quad u_1 \quad k_1 \quad m_2 = 1 \quad k_2 \quad m_3 = 1 \]

\[ x_1 \quad x_2 \quad x_3 = y \quad 0.5 \leq k_1, k_2 \leq 2 \]
PERFORMANCE COMPARISONS:
RLQR (LEFT) VS MISMATCHED LQR (RIGHT)
RLQR TRANSIENTS: 2-SPRING SYSTEM

\[ k_1 = .5, \quad k_2 = 1.2 \]
MISMATCHED LQR TRANSIENTS: 2-SPRING SYSTEM

\[ K_1 = 0.5, \quad K_2 = 1.2 \]
CONCLUSIONS

- RLQR design is a full state method
- Guarantees stability as well as some robustness
- Interesting energy interpretations
- Understanding underlying fundamentals will help us when we extend to output feedback