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THE CALCULATION OF DOWNWASH BEHIND SUPERSONIC WINGS

WITH AN APPLICATION TO TRIANGULAR PLAN FORMS

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THE CALCULATION OF DOWNWASH BEHIND SUPERSONIC WINGS WITH AN APPLICATION TO TRIANGULAR PLAN FORMS

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SUMMARY

A method is developed consistent with the assumptions of small perturbation theory, which provides a means of determining the downwash behind a wing in supersonic flow for a known load distribution. The analysis is based upon the use of supersonic doublets which are distributed over the plan form and wake of the wing in a manner determined from the wing loading.

The first application of the method proves the equivalence in subsonic and supersonic flow of the downwash at infinity corresponding to a given load distribution. The principal application in this report is concerned with the downwash behind a triangular wing with leading edges swept back of the Mach cone from the vertex. A complete solution is given along the center line of the wake and an approximation provided for points in the vicinity of this line.

INTRODUCTION

The linearization of the partial differential equation satisfied by the velocity potential for compressible flow yields, for subsonic flight speeds, an elliptic-type equation which is reducible by means of an elementary transformation to the basic equation in incompressible flow. As a consequence of this result, wing theory in the subsonic realm employs the same concepts and same types of analyses that belong to classical incompressible theory. At supersonic speeds, the differential equation for the velocity potential is hyperbolic in type and for wing theory is equivalent mathematically to the two-dimensional wave equation of physics. In spite of the different character of the basic differential equation in the two flight regimes, certain formal equivalencies can be set up which are intuitively useful in the solution of specific problems. In particular, the velocity potentials of a three-dimensional source and of a doublet each have analogous forms in the two cases. The solution of different boundary-value problems encountered in wing theory has been discussed in reference 1, and it has been shown how suitable distributions of sources and doublets may be used to determine the flow potential associated with a given lifting or nonlifting wing.

The calculation of downwash behind a wing, for incompressible flow, relies almost exclusively on the use of Prandtl's lifting-line theory which is, in turn, developed from the concept of a single horseshoe vortex. The conventional approach to the general downwash problem is to determine, first, the induced field of the simple horseshoe vortex by means of the Biot-Savart law and, then, from a knowledge of the spanwise distribution of loading over the wing, to calculate finally the induced field produced by a vortex sheet composed of superimposed vortices of varying span.

When downwash calculations are to be extended to the case of supersonic wings, it appears at first that the use of vortex sheets is completely inadmissible since no equivalent to the Biot-Savart law exists. It is, in fact, true that the horseshoe vortex no longer plays the outstanding role it has at low speeds. However, when a more detailed investigation is made of the underlying analysis, it becomes apparent that vortex theory and the Biot-Savart law can be developed from the initial use of a constant distribution of doublets over a given surface. (E.g., see references 2 and 3.) These doublets produce a discontinuity in the velocity potential at the surface, and, for incompressible theory, the curve which bounds the surface can be identified with a vortex curve possessing circulation. The proof of the Biot-Savart law and the introduction of vortex sheets are direct consequences of these basic ideas.

Since, as was shown in reference 1, supersonic boundary-value problems involving sources, sinks, and doublets can be solved in a manner analogous to that used in low-speed theory, a method is therefore provided for an attack on the downwash problem for supersonic plan forms through the use of doublet distributions.

The present report has two principal aims: First, to outline in some detail the theoretical approach to the determination of the velocity potential of the flow field associated with a supersonic lifting surface and the subsequent calculation of the downwash; and, second, to apply the theory to the case of a triangular wing swept back of the Mach cone and to present the results of the complete calculations along the center line and in the plane of the wing. Downwash immediately back of the trailing edge and at an infinite distance behind the wing will also be derived and is in agreement with the previously published results of P.A. Lagerstrom (reference 4). The final portion of the analysis will treat the variation of downwash near the center line of the triangular wing.

In the theoretical portion of the report, the boundary-value problem will be introduced and the solutions, obtained from Green's theorem, will be given for low-speed and supersonic flow. In the section of the report devoted to applications, the theory will be used first to evaluate the potential function at an infinite distance downstream from a lifting wing. Since the mathematical problems arising in the physically obvious case of the unswept wing of infinite span correspond closely to those for the more general case, the theory is next applied to this case. From this application, a general procedure is developed for treating wings with supersonic trailing edges. The final application of the report will be devoted to the triangular wing. In all of these applications, it will be seen that the analytic expressions which have been obtained in supersonic theory for the load distributions over certain plan forms afford a means whereby the chordwise distribution of pressure may be introduced into the analysis, and, therefore, such expedients as lifting-line theory are no longer so essential.

No attempt will be made here to discuss the effect of airfoil thickness on the downwash distribution, although this effect is actually simpler to treat mathematically. It suffices to state that the entire theory is postulated on the assumptions of thin-airfoil or small-perturbation theory and that, consequently, thickness effects and lifting-plate solutions are additive. For the results that are given in the plane of the airfoil, the thickness effect, which is necessarily symmetrical with respect to this plane, is zero.

LIST OF IMPORTANT SYMBOLS

a_o	velocity of sound in the free stream
b	wing span
c_o	root chord of wing
E, E_o, E_1, E_2	complete elliptic integrals of the second kind with moduli k, k_o, k_1, k_2, respectively
H	$\dfrac{2\alpha V_o}{E_o\beta}$

k_o	$\sqrt{1-\theta_o{}^2}$
k_1	$\frac{\theta_o}{x_o-1}$
k_2	$\frac{x_o-1}{\theta_o}$
K,K_2	complete elliptic integrals of the first kind with moduli k, k_2, respectively
M	free-stream Mach number $\left(\frac{V_o}{a_o}\right)$
p	static pressure
Δp	p_l-p_u
q	dynamic pressure $\left(\frac{1}{2}\rho_o V_o{}^2\right)$
r	$\sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}$
r_s	$\sqrt{(x-x_1)^2+(y-y_1)^2+z^2}$
r_c	$\sqrt{(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2(z-z_1)^2}$
r_{c_s}	$\sqrt{(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2z^2}$
u,v,w	perturbation velocity components in the direction of the x-, y-, z-axes, respectively
Δu_s	u_u-u_l
V_o	free-stream velocity
w_P	z-component of velocity induced by doublet distribution over plan form
w_W	z-component of velocity induced by doublet distribution over wake
x,y,z	Cartesian coordinates of an arbitrary point
x_1,y_1,z_1	Cartesian coordinates of source or doublet position

x_o x/c_o

y_o $\beta y/c_o$

z_o $\beta z/c_o$

α angle of attack

β $\sqrt{M^2-1}$

δ semivertex angle of triangular wing

η $\dfrac{\beta y_1}{c_o \theta_o}$

θ_o $\beta \tan \delta$

μ Mach angle $\left(\arcsin \dfrac{1}{M}\right)$

ρ_o density in free stream

Φ perturbation velocity potential

$\Delta\Phi_s$ $\Phi_u - \Phi_l$

sign denoting finite part of integral (equations (10) and (11))

Subscripts

u conditions on upper portion of surface

l conditions on lower portion of surface

L.E. conditions at leading edge

T.E. conditions at trailing edge

W refers to wake

P refers to plan form

s conditions on discontinuity surface (at $z_1=0$)

Regions

I,II,III integration regions on plan form (fig. 4)

A,B regions in wake of triangular wing (fig. 5)

THEORY

Boundary Conditions

The proposed problem is one of finding the downwash behind a flat plate which supports a loading consistent with its angle of attack and plan form. It will be assumed throughout the analysis that this load distribution is known. Such values were given for several plan forms in reference 5 and further results can be found in the literature on supersonic wings.

The load distribution over the wing may be obtained from a knowledge of the differences in pressures acting on the lower and upper surfaces. Moreover, in thin-airfoil theory, where boundary conditions are given in the $z = 0$ plane (i.e., the plane of the wing), a simple relation exists between local-pressure coefficient and the streamwise component of the perturbation velocity. Thus, assuming that the free-stream direction coincides with the positive x-axis (fig. 1), and denoting by u the x-component of the perturbation velocity, it follows that

$$\frac{\Delta p}{q} = \frac{p_l - p_u}{q} = \frac{2}{V_o}(u_u - u_l) = \frac{2\Delta u_s}{V_o} \tag{1}$$

where the variables are defined in the table of the symbols. Furthermore, from the definition of the perturbation velocity potential Φ

$$\Phi = \int_a^x u\,dx \tag{2}$$

where a is a point in a region at which the potential is zero. Combining equations (1) and (2), the jump in potential in the plane $z=0$ can be determined by integrating the jump in the u induced velocity or, what amounts to the same thing, the jump in load coefficient. Thus,

$$\Delta\Phi_s = \int_{L.E.}^{x} \Delta u_s dx_1 = \frac{V_o}{2}\int_{L.E.}^{x} \left(\frac{\Delta p}{q}\right) dx_1 \tag{3}$$

where the integration extends from the leading edge to the point x and $\Delta\Phi_s$ represents the jump in Φ in the xy-plane. Since load coefficient $\Delta p/q$ must be zero off the wing and since u is an odd function in z, the value of u must be zero for all points off the wing in the xy-plane. It follows that $\Delta\Phi_s$ remains constant at a given span station for all values of x beyond the trailing-edge position.

Figure 1 indicates an arbitrary lifting surface in the $z=0$ plane together with the distribution of $\Delta\Phi_s$ for given constant values of y and x. In both subsonic and supersonic theory, the wing together with the semi-infinite strip extending downstream of the wing form a discontinuity surface for the velocity potential, while $\Delta\Phi_s$ is equal to 0 throughout the remaining portion of the xy-plane. These conditions, together with the fact that the vertical induced velocity w is a continuous function at $z=0$, are sufficient to determine Φ throughout space. The values of u, v, and w can then be found from the corresponding partial derivatives of Φ with respect to x, y, and z. The attention in the present report is centered on w, the downwash function.

Solution to Boundary-Value Problem

In reference 1, the solutions for boundary-value problems of the type under consideration were given for both incompressible and supersonic theory. The basic differential equations satisfied by the perturbation velocity potential are, for the two cases, respectively,

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0 \tag{4}$$

and

$$\beta^2 \frac{\partial^2\Phi}{\partial x^2} - \frac{\partial^2\Phi}{\partial y^2} - \frac{\partial^2\Phi}{\partial z^2} = 0 \tag{5}$$

<u>Incompressible theory</u>.— For boundary conditions prescribed in the $z = 0$ plane; the solution of equation (4) is

$$\Phi\ (x,y,z) = -\frac{1}{4\pi}\int_\tau\int\left[\frac{1}{r_s}\left(\frac{\partial\Phi_u}{\partial z_1} - \frac{\partial\Phi_l}{\partial z_1}\right) - (\Phi_u-\Phi_l)\left(\frac{\partial}{\partial z_1}\frac{1}{r}\right)_s\right]dx_1dy_1 \tag{6}$$

where

$$r = \sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}$$

and τ is the area for which the integrand does not vanish. The terms $\frac{1}{4\pi r_s}$ and $\frac{1}{4\pi}\left(\frac{\partial}{\partial z_1}\frac{1}{r}\right)_s$ are equal to the velocity potential at x,y,z of a unit source and doublet situated at the point $x_1,y_1,0$. The remaining terms in the integrand, which determine the distribution of source and doublet strengths, must be found from known boundary conditions. If a lifting surface fixes the boundary conditions, induced vertical velocities on the upper and lower faces of the surface are equal so that

$$\frac{\partial\Phi_u}{\partial z_1} = \frac{\partial\Phi_l}{\partial z_1}$$

and

$$\Phi\ (x,y,z) = \frac{1}{4\pi}\int_\tau\int\Delta\Phi_s\left(\frac{\partial}{\partial z_1}\frac{1}{r}\right)_s dx_1dy_1 \tag{7}$$

Equations (6) and (7) are well known in potential theory (reference 3, p. 60), but the derivation usually employs the assumption that the value of Φ is zero at all points infinitely distant from the wing. This assumption cannot, of course, be made in aerodynamic applications where the discontinuity surface τ extends to $x = \infty$, as in the case of a lifting wing or lifting line with trailing vortices. These latter problems, with which this report is directly concerned, are of such a nature, however, that the induced effects at an infinite distance are confined to the plane $x = \infty$. An investigation of the derivation of equation (6) reveals that the conditions imposed on Φ, in general, can be relaxed sufficiently to permit a discontinuity in a strip of finite width along the entire extent of the x-axis. The mathematical details of the derivation will not be given here but a statement of the restrictions on Φ at infinity is worthwhile. Thus, denoting by $\frac{\partial\Phi}{\partial n}$ the directional derivative of Φ taken normal to a prescribed surface, the following conditions apply:

1. The functions Φ and $\frac{\partial\Phi}{\partial n}$ are zero at all points having radius vectors which make finite (nonzero) angles with the positive x-axis, the points lying on a spherical surface of infinite radius with center at the wing. (This preserves the usual potential theory assumptions except over the portion of the spherical surface which forms the plane $x = \infty$.)

2. The values Φ and $\frac{\partial\Phi}{\partial x}$ are bounded at all points infinitely distant from the lifting surface and at a noninfinite distance from the positive x-axis. (This condition places restrictions on the values of Φ and $\frac{\partial\Phi}{\partial x}$ in the plane $x = \infty$.)

Conditions (1) and (2) are satisfied for a lifting surface of finite span, and equation (7) is consequently applicable directly to the determination of the velocity potential. As an application of the equation, suppose a sheet of horseshoe vortices is situated as in figure 2 with bound vortices placed on the y-axis, trailing vortices extending parallel to the positive x-axis, and has a spanwise distribution of circulation $\Delta\Phi$ symmetrical to the xz-plane and defined for $-\frac{b}{2} \leq y \leq \frac{b}{2}$. Then the velocity potential corresponding to this vortex sheet is given by the expression

$$\phi(x,y,z) = \frac{z}{4\pi}\int_{-\frac{b}{2}}^{\frac{b}{2}} \Delta\Phi_s dy_1 \int_0^{\infty} \frac{dx_1}{\left[(x-x_1)^2+(y-y_1)^2+z^2\right]^{3/2}} \tag{8}$$

When $\Delta\Phi_s$ = constant, a single horseshoe vortex results.

Supersonic theory.— For supersonic boundary-value problems associated with plan forms as indicated in figure 1(a), where the known conditions are given in the $z = 0$ plane, the general solution of equation (5) is given in reference (1) in the form

$$\Phi(x,y,z) = -\frac{1}{2\pi}\left|\overline{\iint_{\tau}\left[\left(\frac{1}{r_c}\right)_s\left(\frac{\partial\Phi_u}{\partial z_1}-\frac{\partial\Phi_l}{\partial z_1}\right)-\left(\Phi_u-\Phi_l\right)\left(\frac{\partial}{\partial z_1}\frac{1}{r_c}\right)_s\right]dx_1dy_1}\right. \tag{9}$$

where

$$r_c = \sqrt{(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2(z-z_1)^2}$$

and the subscript s on the parentheses indicates that the function is to be evaluated at $z_1 = 0$. The region τ is that portion of the x_1y_1-plane bounded by the leading edge of the wing, the lines parallel to the x-axis and stemming from the lateral tips of the wing, and the trace in the $z_1 = 0$ plane of the Mach forecone with vertex at the point x,y,z. The sign $\left|\overline{\qquad\qquad}\right.$ is to be read "finite part of" and was introduced by Hadamard (reference 6) as a manipulative technique with the property that

$$\left|\overline{\int_a^{x_o}\frac{A(x)dx}{(x_o-x)^{3/2}}}\right. = \int_a^{x_o}\frac{A(x)-A(x_o)}{(x_o-x)^{3/2}}dx - \frac{2A(x_o)}{(x_o-a)^{1/2}} \tag{10}$$

For purposes of calculations, this was modified in reference 1 to

$$\left|\overline{\int_a^{x_o}\frac{A(x)dx}{(x_o-x)^{3/2}}}\right. = \int_a^{*}\frac{A(x)dx}{(x_o-x)^{3/2}} = -F(a) - C \tag{11}$$

the asterisk indicating that no upper limit is to be substituted into the indefinite integral, the latter being determined as

$$F(x) + C$$

where

$$C = \lim_{x \to x_o} \left[\frac{2A(x_o)}{\sqrt{x_o - x}} - F(x) \right]$$

Equation (9) is the direct analogue of equation (6). The terms $\frac{1}{2\pi}\left(\frac{1}{r_c}\right)_s$ and $\frac{1}{2\pi}\left(\frac{\partial}{\partial z_1}\frac{1}{r_c}\right)_s$ are equal to the velocity potential at x,y,z of a unit supersonic source and doublet situated at the point x_1,y_1,o, while the remaining terms in the integrand determine the distribution of source and doublet strength and are determined by the known boundary conditions.

When the potential function associated with a lifting surface is to be evaluated,

$$\frac{\partial \Phi_u}{\partial z_1} = \frac{\partial \Phi_l}{\partial z_1}$$

and equation (9) reduces to the form

$$\Phi\,(x,y,z) = \frac{1}{2\pi}\overline{\int\int_T \Delta\Phi_s \left(\frac{\partial}{\partial z_1}\frac{1}{r_c}\right)_s dx_1 dy_1} \tag{12}$$

In application, the region of integration in equations (7) and (12) can be divided into areas occupied, respectively, by the plan form and the wake region. Thus, for equation (12),

$$\Phi\,(x,y,z) = \frac{-z\beta^2}{2\pi}\overline{\int\int_{\text{plan form}} \frac{\Delta\Phi_s\, dx_1 dy_1}{[(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2 z^2]^{3/2}}}$$

$$- \frac{z\beta^2}{2\pi}\overline{\int\int_{\text{wake}} \frac{\Delta\Phi_s\, dx_1 dy_1}{[(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2 z^2]^{3/2}}} \tag{13}$$

Equation (13) presents a formal solution for the calculation of velocity potential and, subsequently, downwash for a given surface in terms of $\Delta\Phi_s$. Since $\Delta\Phi_s$ was related directly to load distribution in equation (3), it is apparent that the various known solutions to lifting-surface problems are directly applicable. The fact that supersonic theory permits the determination of load distribution in closed analytic form for many simple plan forms provides a distinct advantage that is lacking in subsonic theory wherein virtually all known results are available only in numerical form. Thus, theoretical analysis of problems involving supersonic flight speeds can be carried further before recourse to numerical methods is necessary.

APPLICATIONS

Value of Potential Function at $x = \infty$

It is possible to show, from equations (7) and (12), that the potential functions corresponding to a wing with fixed load distribution are idential at $x = \infty$ for incompressible and supersonic flow. Assuming $\Delta\Phi_s$ known, the values of $\Phi(x,y,z)$ for the two cases are given, respectively, by the equations

$$\Phi(x,y,z) = \frac{z}{4\pi}\iint\limits_{\text{plan form}} \frac{\Delta\Phi_s \, dx_1 dy_1}{[(x-x_1)^2+(y-y_1)^2+z^2]^{3/2}}$$

$$+ \frac{z}{4\pi}\iint\limits_{\text{wake}} \frac{\Delta\Phi_s \, dx_1 dy_1}{[(x-x_1)^2+(y-y_1)^2+z^2]^{3/2}}$$

and

$$\Phi(x,y,z) = \frac{-\beta^2 z}{2\pi}\left|\iint_{\text{plan form}} \frac{\Delta\Phi_s \, dx_1 dy_1}{[(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2 z^2]^{3/2}}\right.$$

$$-\frac{\beta^2 z}{2\pi}\left|\iint_{\text{wake}} \frac{\Delta\Phi_s \, dx_1 dy_1}{[(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2 z^2]^{3/2}}\right.$$

Since, however, $\Delta\Phi_s$ is finite, it follows immediately that for fixed values of y and z the integrals over the plan form in both equations approach zero as x increases indefinitely. Thus, denoting by $x_{T.E.}$ the value of x_1 at the trailing edge of the wing, the potential functions at $x = \infty$ are given by the expressions

$$\Phi(\infty,y,z) = \lim_{x\to\infty} \frac{z}{4\pi}\int_{-\frac{b}{2}}^{\frac{b}{2}} \Delta\Phi_s(x_{T.E.},y_1)dy_1 \int_{x_{T.E.}}^{\infty} \frac{dx_1}{[(x-x_1)^2+(y-y_1)^2+z^2]^{3/2}}$$

and

$$\Phi(\infty,y,z) = \lim_{x\to\infty} \frac{-z\beta^2}{2\pi}\int_{-\frac{b}{2}}^{\frac{b}{2}} \Delta\Phi_s(x_{T.E.},y_1)dy_1 \int_{x_{T.E.}}^{*} \frac{dx_1}{[(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2 z^2]^{3/2}}$$

These relations can be integrated once to give for the subsonic case

$$\Phi(\infty,y,z) = \lim_{x\to\infty} \frac{-z}{4\pi}\int_{-\frac{b}{2}}^{\frac{b}{2}} \Delta\Phi_s(x_{T.E.},y_1)dy_1 \left\{\frac{(x-x_1)}{[(y-y_1)^2+z^2]\sqrt{(x-x_1)^2+(y-y_1)^2+z^2}}\right\}_{x_{T.E.}}^{\infty}$$

$$= \frac{z}{2\pi}\int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\Delta\Phi_s \ (x_{T.E.},y_1)dy_1}{(y-y_1)^2+z^2} \qquad (14a)$$

and for the supersonic case

$$\Phi(\infty,y,z) = \lim_{x\to\infty} \frac{-z}{2\pi} \int_{-\frac{b}{2}}^{\frac{b}{2}} \Delta\Phi_s \left(x_{T.E.},y_1\right) dy_1 \left\{\frac{(x-x_1)}{\left[(y-y_1)^2+z^2\right]\sqrt{(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2z^2}}\right\}^{*}_{x_{T.E.}}$$

$$= \frac{z}{2\pi} \int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\Delta\Phi_s \left(x_{T.E.},y_1\right)dy_1}{(y-y_1)^2+z^2} \tag{14b}$$

From these equations, it follows that the sidewash and downwash at $x = \infty$ are invariant with Mach number provided the load distribution is fixed. In fact, their values depend solely on the spanwise load distribution, since the terms corresponding to the chordwise distribution disappeared in the analysis. This has been pointed out elsewhere in the literature. It should be stressed, however, that the result which has been obtained here states that equal span load distributions in the two cases yield equal values of the potential function at $x = \infty$. This does not imply that a wing at low and supersonic speeds maintains the same potential function at infinity. When the wing is kept fixed, the distribution of w on the wing is fixed, but the load distribution is a function of speed.

Downwash On and Off the Wing

As a further application, the unswept wing of infinite span will be treated for supersonic speeds. In this case, as is well known, the induced velocities are zero at all points downstream of the upper and lower Mach waves stemming from the trailing edge. An abrupt jump in vertical velocity therefore occurs at the trailing edge of the wing. Consideration of this jump for the unswept wing furnishes considerable insight into the nature of the mathematical difficulties inherent in the calculation of downwash on and off wings of arbitrary plan form. The calculation for the particular case will therefore be followed by a more general discussion which will be of value in connection with the later treatment of the triangular wing.

Unswept wing of infinite span.– The pressure distribution for the wing of infinite aspect ratio is constant. For this so-called Ackeret-type loading, $\frac{\Delta p}{q}$ is equal to $\frac{4\alpha}{\beta}$, so that, when the leading edge lies along the y-axis,

$$\Delta\Phi_s = \frac{V_o}{2}\int_0^{x_1}\frac{\Delta p}{q}\,dx = \frac{2\alpha V_o}{\beta}\,x_1 \tag{15a}$$

where α is angle of attack. In the wake

$$\Delta\Phi_s = \frac{V_o}{2}\int_0^{c_o}\frac{\Delta p}{q}\,dx = \frac{2\alpha V_o}{\beta}\,c_o \tag{15b}$$

The downwash, or vertical velocity, will first be found when the point is between the Mach waves from the leading and trailing edges and then when the point is downstream of the trailing-edge wave.

Since the wing is of infinite width and experiences no variation with y, it is possible to consider the problem at $y = 0$. Thus, from equation (13), for the case when the point under consideration is between the Mach waves from the leading and trailing edges with its forecone cutting the wing as shown in figure 3(a),

$$\Phi = \lim_{\epsilon \to 0}\frac{-z\beta^2 V_o\alpha}{\pi\beta}\int_0^{x-\beta\sqrt{\epsilon^2+z^2}} x_1\,dx_1 \left|\overline{\,2\int_\epsilon^{\frac{1}{\beta}\sqrt{(x-x_1)^2-\beta^2z^2}}\frac{dy_1}{\left[(x-x_1)^2-\beta^2y_1^2-\beta^2z^2\right]^{3/2}}}\right.$$

where the symmetry of the problem with respect to the y_1-axis has been used. Computing the finite part of the integral

$$\Phi = \lim_{\epsilon \to 0} \frac{2z\beta V_0\alpha}{\pi} \epsilon \int_0^{x-\beta\sqrt{\epsilon^2+z^2}} \frac{x_1\, dx_1}{[(x-x_1)^2-\beta^2 z^2]\sqrt{(x-x_1)^2-\beta^2\epsilon^2-\beta^2 z^2}} \tag{16}$$

and

$$w = \frac{\partial \Phi}{\partial z} = \frac{\partial}{\partial z} \lim_{\epsilon \to 0} \frac{2z\beta V_0\alpha\epsilon}{\pi} \int_0^{x-\beta\sqrt{\epsilon^2+z^2}} \frac{x_1\, dx_1}{[(x-x_1)^2-\beta^2 z^2]\sqrt{(x-x_1)^2-\beta^2\epsilon^2-\beta^2 z^2}} \tag{17}$$

Equations (16) and (17) can be evaluated directly to give the results

$$\left.\begin{aligned} \Phi &= -\frac{V_0\alpha}{\beta}\frac{z}{|z|}(x \pm \beta z) \\ w &= -V_0\alpha \end{aligned}\right\} \tag{18}$$

For a point behind the trailing-edge wave (fig. 3(b)), the two quantities can be determined in a similar manner. Thus,

$$\Phi = \lim_{\epsilon \to 0} \frac{-2z\beta V_0\alpha\epsilon}{\pi} \int_0^{c_0} \frac{x_1\, dx_1}{[(x-x_1)^2-\beta^2z^2]\sqrt{(x-x_1)^2-\beta^2z^2-\beta^2\epsilon^2}}$$

$$+ \frac{2z\beta V_0\alpha c}{\pi} \int_0^{\frac{1}{\beta}\sqrt{(x-c)^2-\beta^2z^2}} dy_1 \int_{c_0}^{x-\beta\sqrt{y_1^2+z^2}} \frac{dx_1}{[(x-x_1)^2-\beta^2y_1^2-\beta^2z^2]^{3/2}} \qquad (19)$$

and w is given by the partial derivative of Φ with respect to z. The term containing the single integral is zero, since the integral itself is bounded for all values of ϵ, while the term containing the double integral is readily calculable. Thus, the values of the velocity potential and the downwash for a point behind the trailing-edge wave are given by the relations

$$\Phi = -\frac{V_0\alpha c_0}{\beta} \frac{z}{|z|} \qquad (20)$$

$$w = 0$$

These results are the familiar equations associated with two-dimensional supersonic flat-plate theory.

The point of principal interest in this development is the jump in the induced vertical velocity w in passing from a point just ahead of the trailing-edge wave to a point immediately behind the wave. A study of equations (17) and (19) shows that this jump is the result of the discontinuity in the contribution to the downwash of the term containing the single integral. Ahead of the trailing-edge wave, this term yielded the result that

$$w = -V_o\alpha$$

whereas behind the wave, the contribution of the term to w was zero.

The method of attack used in the study of the unswept wing can be generalized to apply to arbitrary plan forms. A discussion of this case follows.

Arbitrary plan forms.- As will be shown later, for any plan form with supersonic trailing edge, the jump in the value of w in the plane of the wing at the trailing edge can be calculated directly by means of simple momentum methods. At this point, however, it is of more interest to consider in a genral manner the nature of the integrations involved when the point x,y,z is either ahead of or behind the trailing-edge wave. Figures 4(a) and 4(b) show a plan form with a straight trailing edge with areas of integration indicated for the point P in each of the two positions. (The straight trailing edge is not a necessary restriction and is only introduced for convenience of notation.) The regions of integration are divided under the assumption that the first integration on the plan form in equation (13) will be made with respect to y_1. When the point P is ahead of the trailing-edge wave, therefore, the contribution of the wake is zero and the integration over the plan form is made to conform with regions I and II. When the point is behind the trailing-edge wave, three integrals are evaluated corresponding to regions I, II, and III. In the case of the infinite aspect ratio, unswept wing region I was, of course, nonexistent and, in general, no essential difficulty in regard to the limits of integration is introduced by this region regardless of where P is situated. In region II, however, the problem must be treated in more detail.

Consider first the case when P is ahead of the wave and denote by Φ_{IIa} the contribution of one side of region II to the total potential. Then

$$\Phi_{IIa} = \lim_{\epsilon \to 0} \frac{-\beta^2 z}{2\pi} \int_{x}^{x-\beta\sqrt{\epsilon^2+z^2}} dx_1 \left| \int_{y+\epsilon}^{Y_1} \frac{\Delta\Phi_1\ (x_1,y_1)dy_1}{[(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2 z^2]^{3/2}} \right. \tag{21a}$$

where

$$Y_1 = y + \frac{1}{\beta}\sqrt{(x-x_1)^2-\beta^2 z^2}$$

Similarly, when P is behind the wave and the same subscript notation is used to refer to one side of region II, the value of Φ_{IIa} is

$$\Phi_{IIa} = \lim_{\epsilon \to 0} \frac{-\beta^2 z}{2\pi} \int_{x}^{\infty} dx_1 \left| \int_{y+\epsilon}^{Y_1} \frac{\Delta\Phi_1\ (x_1,y_1)dy_1}{[(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2 z^2]^{3/2}} \right. \tag{21b}$$

where Y_1 is as defined above.

The contributions of the other side of region II to the potential will not be considered separately as the behavior is identical. When P lies ahead of the wave, ϵ appears in the limits for integration with respect to both y_1 and x_1. This corresponds to the situation in equations (16) and (17) and, as for that problem, the limiting process is carried out after the integration is completed. When P is behind the wave, it is not necessary to defer the limiting process, since

$$\Phi_{IIa} = \frac{-\beta^2 z}{2\pi} \int_{x}^{\infty} dx_1 \lim_{\epsilon \to 0} \left| \int_{y+\epsilon}^{Y_1} \frac{\Delta\Phi_1\ (x_1,y_1)dy_1}{[(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2 z^2]^{3/2}} \right.$$

and if $\{\ \}$ represents the integrand, then

$$\left| \int_{y+\epsilon}^{Y_1} \{\ \}\, dy_1 \right. = \left| \int_{y}^{Y_1} \{\ \}\, dy_1 \right. - \int_{y}^{y+\epsilon} \{\ \}\, dy_1$$

But since

$$\frac{\Delta\Phi_1\ (x_1,y_1)}{[(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2z^2]^{3/2}} \leqq M$$

for $c_o \leqq x_1 \leqq x_o$ (i.e., $\{\ \}$ is bounded for all values of x_1 in the interval of the first integration); and further, since

$$\lim_{\epsilon \to 0} M \int_y^{y+\epsilon} dy_1 = \lim_{\epsilon \to 0} M\epsilon = 0$$

therefore, for P situated behind the trailing-edge wave, the contribution of region IIa is given by

$$\Phi_{IIa} = \frac{-\beta^2 z}{2\pi} \int_x^{c_o} dx_1 \left| \int_y^{Y_1} \frac{\Delta\Phi_1\ (x_1,y_1)dy_1}{[(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2z^2]^{3/2}} \right. \quad . \ . \ . \qquad (22)$$

Equation (22) will be applied directly in the determination of downwash behind the triangular wing. The significance of the result is that, when the point P at x,y,z is behind the Mach wave from a supersonic trailing edge, the limiting process associated with region II need not be considered. When P is ahead of the Mach wave, the term ϵ must be retained in the analysis and the limiting process used. As was previously noted, the general analysis developed in this report places no restriction on the orientation of the trailing edge; however, it should be pointed out that region II exists only for the case in which the trailing edge is supersonic. Therefore, the jump in downwash, obtained from the integration over region II, is associated only with supersonic trailing edges; whereas both the downwash and loading are continuous across a subsonic trailing edge.

Triangular Wing

Consider a triangular wing (fig. 5) with leading edges swept back of the Mach cone from the vertex. The loading over the wing is known to be (references 7 and 5)

$$\frac{\Delta p}{q} = \frac{4\theta_o{}^2\alpha x_1}{E_o\beta\sqrt{\theta_o{}^2x_1{}^2-\beta^2y_1{}^2}} \tag{23}$$

where E_o is the complete elliptic integral of the second kind with modulus $k_o = \sqrt{1-\theta_o{}^2}$ and $\theta_o = \beta \tan \delta$, δ being the semivertex angle of the triangle. From equation (3)

$$\Delta\Phi_s = H\sqrt{\theta_o{}^2x_1{}^2-\beta^2y_1{}^2} \tag{24}$$

where

$$H = \frac{2\alpha V_o}{E_o\beta} \tag{25}$$

Sufficient information is now at hand to permit the use of equation (13). Setting, for convenience, $\Phi = \Phi_P + \Phi_W$ the velocity potential at x,y,z is given by the sum of the two expressions

$$\Phi_P = -\frac{zH\beta^2}{2\pi}\sqrt{\iint_{\text{plan form}} \frac{\sqrt{\theta_o{}^2x_1{}^2-\beta^2y_1{}^2}\,dx_1\,dy_1}{[(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2z^2]^{3/2}}} \tag{26}$$

$$\Phi_W = -\frac{zH\beta^2}{2\pi}\sqrt{\iint_{\text{wake}} \frac{\sqrt{\theta_o{}^2c_o{}^2-\beta^2y_1{}^2}\,dx_1\,dy_1}{[(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2z^2]^{3/2}}} \tag{27}$$

Equation (26) represents the contribution to the velocity potential furnished by the doublets distributed over the plan form, while equation (27) represents the contribution furnished by the doublets within the wake. The latter equation is the mathematical equivalent in supersonic flow of the subsonic velocity potential of a sheet of horseshoe vortices corresponding to an elliptic span load distribution. Equations (14a) and (14b) showed that the expression for Φ_W at $x = \infty$ is identical to the velocity potential of the subsonic vortex sheet. However, in the vicinity of the $x = c_o$ line, the behavior is entirely different.

In the present report, equations (26) and (27) will be applied only to the determination of downwash along the center line in the wake of the airfoil. Further extensions are limited only by difficulty of integration. It will appear in the development that recourse to numerical methods is probably necessary in general.

Downwash induced by doublets in the wake.— Setting $y = 0$ in equation (27) and integrating with respect to x_1, it follows that

$$\Phi_W = \frac{zH(x-c_o)}{2\pi}\int_{-Y_1}^{Y_1} \frac{\sqrt{\theta_o^2 c_o^2-\beta^2 y_1^2}\, dy_1}{(y_1^2+z^2)\sqrt{(x-c_o)^2-\beta^2 y_1^2-\beta^2 z^2}} \tag{28}$$

where the limits on the integral are not yet specified, since they differ in the regions A and B indicated in figure 5. In either case, however, the limits are seen to be the roots of one of the two radicals in the integrand.

In order to derive an expression for downwash in the plane of the airfoil, it is convenient to express equation (28) in a different form. Integrating by parts

$$\frac{\Phi_W}{H} = \frac{\sqrt{\theta_o^2 c_o^2-\beta^2 y_1^2}}{2\pi} \arctan \frac{y_1(x-c_o)}{z\sqrt{(x-c_o)^2-\beta^2 y_1^2-\beta^2 z^2}}\Bigg]_{-Y_1}^{Y_1}$$

$$+ \frac{\beta^2}{2\pi}\int_{-Y_1}^{Y_1} \frac{y_1}{\sqrt{\theta_o^2 c_o^2-\beta^2 y_1^2}} \arctan \frac{y_1(x-c_o)}{z\sqrt{(x-c_o)^2-\beta^2 y_1^2-\beta^2 z^2}}\, dy_1$$

This form of the equation is easier to treat when $\lim\limits_{z\to o}\dfrac{\partial\Phi_W}{\partial z}$ is to be considered. In both regions A and B, it can be shown that w_W, the contribution to the downwash made by the doublets in the wake, is given by the expression

$$w_W = -\frac{H\beta^2}{2\pi(x-c_o)} \int_{-Y_1}^{Y_1} \sqrt{\frac{(x-c_o)^2-\beta^2 y_1^2}{\theta_o^2 c_o^2-\beta^2 y_1^2}}\, dy_1 \tag{29}$$

In region B, $Y_1 = \frac{\theta_o c_o}{\beta}$. Using the fact that the integrand is an even function and introducing the notation

$$x_o = \frac{x}{c_o} \qquad \eta = \frac{\beta y_1}{c_o \theta_o} \tag{30}$$

it follows that

$$\frac{w_W \pi}{H\beta} = -\int_0^1 \sqrt{\frac{1-\left(\frac{\theta_o}{x_o-1}\right)^2 \eta^2}{1-\eta^2}}\, d\eta$$

Setting $k_1 = \frac{\theta_o}{x_o-1}$ and noting that $k_1 < 1$ in the region, the expression for w_W in region B is

$$w_W = -\frac{H\beta}{\pi} E_1 \tag{31}$$

where E_1 is the complete elliptic integral of the second kind and the subscript denotes that the modulus is k_1.

In region A

$$Y_1 = \frac{x-c_o}{\beta}$$

By means of the transformations

$$x = \frac{x}{c_o} \qquad \eta = \frac{\beta y_1}{x-c_o}$$

the integral is transformed to the form

$$\frac{w_W\pi}{H\beta} = -\int_0^1 \sqrt{\frac{1-\eta^2}{\left(\frac{\theta_o}{x_o-1}\right)^2-\eta^2}}\,d\eta$$

Setting $k_2 = \frac{x_o-1}{\theta_o} = \frac{1}{k_1}$, noting that $k_2 < 1$ in region A, and introducing Jacobian elliptic functions in the transformation $\eta = \operatorname{sn} u$, we have

$$\frac{w_W\pi}{H\beta} = -k_2\int_0^{K_2} \operatorname{cn}^2 u\,du = \frac{(1-k_2^2)K_2-E_2}{k_2} \tag{32}$$

where K_2 and E_2 are complete elliptic integrals of the first and second kind with modulus k_2.

The values of the downwash given by equations (31) and (32) can be presented in terms of w_o, the downwash on the wing, where from equation (25) and from the boundary condition, $\alpha = \frac{-w_o}{V_o}$,

$$\frac{w_o\pi}{H\beta} = -\frac{\pi}{2}E_o \tag{33}$$

Thus,

$$\frac{w_W}{w_o} = 2\,\frac{E_2-(1-k_2^2)K_2}{\pi E_o k_2} \qquad \text{(region A)}$$

$$\frac{w_W}{w_o} = \frac{2E_1}{\pi E_o} \qquad \text{(region B)}$$

Figure 6 shows this variation in $\frac{w_W}{w_o}$ plotted as a function of x_o for values of θ_o equal to 0.2, 0.4, 0.6, and 0.8.

Downwash induced by doublets on the plan form.- Along the center line, the contribution to the potential function due to the distribution of doublets on the plan form is given by equation (26), where the value of y is zero. Just as in the case of the wake distribution, it is seen that the nature of the solution changes in passing from region A to region B so that the derivations for these regions will be given independently.

In region B, no singularities occur in equation (26) so the finite part sign can be disregarded. Since, moreover, the integrand is an even function of y_1, the expression for velocity potential becomes

$$\Phi_P = -\frac{zH\beta^2}{\pi}\int_0^{c_o} dx_1 \int_0^{\frac{\theta_o x_1}{\beta}} \frac{\sqrt{\theta_o{}^2 x_1{}^2-\beta^2 y_1{}^2}}{[(x-x_1)^2-\beta^2 y_1{}^2-\beta^2 z^2]^{3/2}}\, dy_1 \tag{34}$$

After changing variables by the transformations

$$\xi = \frac{\beta y_1}{\theta_o x_1} \qquad\qquad k = \frac{\theta_o x_1}{\sqrt{(x-x_1)^2-\beta^2 z^2}}$$

equation (34) becomes

$$\Phi_P = -\frac{zH\beta}{\pi}\int_0^{c_o} \frac{k^3}{\theta_o x_1}\, dx_1 \int_0^1 \frac{\sqrt{1-\xi^2}}{(1-k^2\xi^2)^{3/2}}\, d\xi$$

Substituting $\xi = \text{sn } u$ and noting that $\text{sn } K=1$, $\text{sn } 0=0$ where K is the complete elliptic integral with modulus k, we have

$$\Phi_P = -\frac{zH\beta}{\pi}\int_0^{c_o} \frac{k^3}{\theta_o x_1}\, dx_1 \int_0^K \text{cd}^2 u\, du = -\frac{zH\beta}{\pi}\int_0^{c_o} \frac{(K-E)\,dx_1}{\sqrt{(x-x_1)^2-\beta^2 z^2}}$$

The downwash w_P in the $z=0$ plane can now be found by considering $\lim\limits_{z\to 0} \dfrac{\partial\Phi_P}{\partial z}$. This leads to the result

$$w_P = -\frac{H\beta}{\pi}\int_0^{c_o} \frac{(K-E)}{x-x_1}\, dx_1$$

where the modulus reduces to

$$k = \frac{\theta_o x_1}{x-x_1}$$

Rewriting the equation so that k is the variable of integration and $k_1 = \frac{\theta_o}{x_o-1}$

$$\frac{w_p \pi}{H\beta} = -\int_o^{k_1} \frac{K-E}{k+\theta_o}\, dk \tag{35}$$

As is to be expected, the value of w_p is seen to approach zero as x_o becomes infinitely large. The upstream boundary of region B lies at $x_o=1+\theta_o$ and, as will be seen, w_p is continuous at this point.

Consider now region A. In this case, the traces of the Mach forecone of the point (x,o,z) cut across the plan form. For ease of calculation, the area within the wing leading and trailing edges and the traces of the forecone will be divided into two parts. Thus, making use again of the symmetry with respect to the x_1-axis, and also the results discussed in connection with equation (22), it follows from equation (26) that

$$\Phi_p = -\frac{zH\beta^2}{\pi}\int_o^{X_1} dx_1 \int_o^{\frac{\theta_o x_1}{\beta}} \frac{\sqrt{\theta_o{}^2 x_1{}^2-\beta^2 y_1{}^2}}{[(x-x_1)^2-\beta^2 y_1{}^2-\beta^2 z^2]^{3/2}}\, dy_1$$

$$-\frac{zH\beta^2}{\pi}\int_{X_1}^{c_o} dx_1 \int_o^{\frac{1}{\beta}\sqrt{(x-x_1)^2-\beta^2 z^2}} \frac{\sqrt{\theta_o{}^2 x_1{}^2-\beta^2 y_1{}^2}}{[(x-x_1)^2-\beta^2 y_1{}^2-\beta^2 z^2]^{3/2}}\, dy_1 \tag{36}$$

where

$$X_1 = \frac{x-\sqrt{\theta_o{}^2 x^2+\beta^2 z^2(1-\theta_o{}^2)}}{1-\theta_o{}^2}$$

The reduction of the first double integral in equation (36) can be carried out by methods exactly equivalent to those used in reducing equation (34) to (35). As a result of this calculation, the contribution made to w_p in the plane $z=0$ by the first double integral is

$$-\frac{H\beta}{\pi}\int_{0}^{1}\frac{K-E}{k+\theta_o}\,dk \tag{37}$$

In the second double integral of equation (36) the following substitution is made

$$\xi = \frac{\beta y_1}{\sqrt{(x-x_1)^2-\beta^2 z^2}} \qquad k = \frac{\sqrt{(x-x_1)^2-\beta^2 z^2}}{\theta_o x_1}$$

Then, if

$$I = \int_{0}^{\frac{1}{\beta}\sqrt{(x-x_1)^2-\beta^2 z^2}} \frac{\sqrt{\theta_o{}^2 x_1{}^2-\beta^2 y_1{}^2}}{[(x-x_1)^2-\beta^2 y_1{}^2-\beta^2 z^2]^{3/2}}\,dy_1$$

I can be written

$$I = \frac{1}{\beta\theta_o x_1 k^2}\int_{0}^{1}\frac{\sqrt{1-k^2\xi^2}}{(1-\xi^2)^{3/2}}\,d\xi$$

The integration can best be completed here through the use of equation (14) where

$$A(\xi) = \frac{\sqrt{1-k^2\xi^2}}{(1+\xi)^{3/2}}$$

and the auxiliary transformation $\xi = \mathrm{sn}\, u$ is used. As a result of this calculation

$$I = \frac{1}{\beta\theta_o x_1 k^2}(K-E)$$

so that the second double integral of equation (36) is equal to

$$-\frac{zH\beta}{\pi}\int_{X_1}^{c_o}\frac{K-E}{\theta_o x_1 k^2}\,dx_1$$

Using methods similar to those employed in the derivation of equation (35), the contribution to w_p is found to be

$$\frac{H\beta}{\pi}\int_1^{k_2}\frac{K-E}{k^2(1+\theta_o k)}\,dk \tag{38}$$

where $k_2 = \frac{x_o-1}{\theta_o}$. Equations (37) and (38) may now be combined to give the total downwash in region A induced by the doublets on the plan form. Thus,

$$\frac{w_p\pi}{H\beta} = \int_1^{k_2}\frac{K-E}{k^2(1+\theta_o k)}\,dk - \int_o^1 \frac{K-E}{k+\theta_o}\,dk \tag{39}$$

The integration of equation (39) requires numerical methods. A slight simplification can, however, be introduced by using the known value of the downwash w_3 just behind the trailing edge. As will be shown in the next section,

$$\frac{w_3\pi}{H\beta} = -\frac{\pi}{2}\left(E_o-\theta_o\right)$$

Equation (39) must, of course, yield this result for x_o on the trailing edge, that is, for $x_o=1$ and $k_2=0$. Using this relation together with equation (39) it follows that

$$\frac{w_p\pi}{H\beta} = -\frac{\pi}{2}\left(E_o-\theta_o\right) + \int_o^{k_2}\frac{K-E}{k^2(1+\theta_o k)}\,dk \tag{40}$$

Values given by equations (35) and (40) are consistent at the point $k_1=k_2=1$, that is, at the point where the Mach cones from the two trailing-edge tips intersect on the x-axis. By means of these equations, the contribution to the downwash of the doublets distributed over the plan form is determined.

In summary, these relations are

$$\frac{w_P}{w_o} = \frac{E_o-\theta_o}{E_o} - \frac{2}{\pi E_o}\int_0^{k_2} \frac{K-E}{k^2(1+\theta_o k)}\,dk \qquad \text{(region A)}$$

$$\frac{w_P}{w_o} = \frac{2}{\pi E_o}\int_0^{k_1} \frac{K-E}{k+\theta_o}\,dk \qquad \text{(region B)}$$

Figure 7 shows the results of the integrations, the function $\frac{w_P}{w_o}$ being plotted as a function of x_o for various values of θ_o.

Conditions at the trailing edge.– The value of the vertical induced velocity immediately ahead of and behind the trailing-edge wave must, of course, be determinable directly from equations (21) and (22), respectively, by setting $x=c_o \pm z\beta$. If, however, the discussion is restricted to the $z=0$ plane, a much simpler method exists for finding w at these points. The approach taken here follows essentially that given by Lagerstrom in reference 4.

Let conditions just ahead of the trailing-edge wave be denoted by the subscript 2 and conditions just behind the wave by the subscript 3. Figure 8 shows a section of a given wing in the plane y=constant. The Mach waves at the leading and trailing edge make the angle $\mu=\arcsin\frac{1}{M}$ with the $z=0$ plane, and the wing is presumed to be at angle of attack α. Assuming the trailing edge to be normal to the free-stream direction, the variation in the x-component in velocity when passing through the trailing-edge wave can be treated as a two-dimensional problem with the condition imposed that $u_3=0$ in the $z=0$ plane. It is known that continuity of flow together with balance of tangential momentum across the wave lead to the result that the component of velocity tangential to the wave is continuous. The tangential components of velocity immediately ahead of and behind the wave are given, respectively, by the expressions

$$(V_t)_2 = (V_o+u_2)\cos\mu + w_2\sin\mu$$

$$(V_t)_3 = V_o\cos\mu + w_3\sin\mu$$

Equating these relations, it follows that

$$\begin{aligned} w_3 &= w_2 + u_2 \cot \mu \\ &= -V_o\alpha + \frac{V_o\beta}{4}\left(\frac{\Delta p}{q}\right)_2 \end{aligned} \qquad (41)$$

For the two plan forms which have been considered, equation (41) gives the following results

(a) Unswept wing of infinite span

In this case

$$\left(\frac{\Delta p}{q}\right)_2 = \frac{4\alpha}{\beta}$$

and

$$w_3 = -V_o\alpha + \frac{V_o\beta}{4} \frac{4\alpha}{\beta} = 0 \qquad (42)$$

Equation (20) showed that this result actually applies to all points behind the Mach wave.

(b) Triangular wing

From equation (23)

$$\left(\frac{\Delta p}{q}\right)_2 = \frac{4\theta_o^2\alpha c_o}{E_o\beta\sqrt{\theta_o^2 c_o^2 - \beta^2 y_1^2}}$$

from which it follows that

$$w_3 = w_o\left(1 - \frac{\theta_o^2 c_o}{E_o\sqrt{\theta_o^2 c_o^2 - \beta^2 y_1^2}}\right) \qquad (43)$$

Setting $y_1=0$ and introducing H from equation (25)

$$\frac{w_s \pi}{H\beta} = -\frac{\pi}{2}\left(E_o - \theta_o\right) \tag{44}$$

which is the expression used in equation (40). These results are equivalent to those given by Lagerstrom in reference 4.

Values of downwash near center line of wake.— The values of downwash, which were obtained on the center line of the wake, were worked out in exact detail and subjected to no restrictions other than those originally imposed by the use of the linearized equations of flow. It is possible, moreover, to get an indication of the variation in the downwash function in a portion of the wake for points near the center line through the use of a generalized Taylor's expansion in the vicinity of the line $y=z=0$. The next higher terms in the expansion can be found without too much difficulty for the region bounded as follows

(a) $-\frac{1}{2}b < y < \frac{1}{2}b$

(b) Both y and z lying within the Mach cones from the trailing-edge tips

The problem resolves itself into one of finding the first nonvanishing coefficients of y_o and z_o in the series

$$\frac{w}{w_o} = A_o + A_1 z_o + B_1 y_o + A_2 z_o^2 + C_2 z_o y_o + B_2 y_o^2 + \ldots \tag{45}$$

The value of A_o has, of course, already been computed and is known from equations (31) and (35) to be

$$A_o = \frac{2}{\pi E_o}\left(E_1 + \int_o^{k_1} \frac{K-E}{k+\theta_o}\,dk\right) \tag{46}$$

where $k_1 = \frac{\theta_o}{x_o - 1}$.

The coefficient B_1 in the expansion is known to be equal to $\left(\frac{\partial w}{\partial y_o}\right)_{y_o=0}$ The evaluation of this term follows a devlopment similar to that used for equations (29) and (34) except that y is retained in the analysis. The derivation yields the relation

$$\frac{w\pi}{H\beta} = \frac{1}{2(x_o-1)}\int_{-\theta_o}^{\theta_o} \frac{\eta}{y_o-\eta}\sqrt{\frac{(x_o-1)^2-(y_o-\eta)^2}{\theta_o{}^2-\eta^2}}\,d\eta$$

$$-\frac{1}{2}\int_0^1 d\xi\int_{-\xi\theta_o}^{\xi\theta_o} \frac{\sqrt{\theta_o{}^2\xi^2-\eta^2}}{[(x_o-\xi)^2-(y_o-\eta)^2]^{3/2}}\,d\eta$$

Carrying out the differentiation, with proper regard for the singularity in the first integral, it follows that $B_1 = 0$. Similarly, it can be shown that $C_2 = 0$, while the coefficient $B_2 = \frac{1}{2}\left(\frac{\partial^2 w}{\partial y_o{}^2}\right)_{y_o=0}$ is given by the expression

$$B_2 = \frac{1}{\pi E_o\theta_o{}^2}\left(2K_1 - \frac{2-k_1{}^2}{1-k_1{}^2}E_1\right)$$
$$-\frac{1}{\pi E_o\theta_o{}^2 x_o{}^2}\int_0^{k_1}(k+\theta_o)\left[E\,\frac{1+k^2}{(1-k^2)^2} - K\,\frac{1}{1-k^2}\right]dk \qquad (47)$$

where again $k_1 = \frac{\theta_o}{x_o-1}$.

In order to calculate the variation with z, it is necessary to evaluate $A_1 = \left(\frac{\partial^2\Phi}{\partial z_o{}^2}\right)_{z_o=0}$

where

$$\frac{\Phi\pi}{H\beta} = \beta \int_0^{\frac{\theta_o c_o}{\beta}} \frac{y_1}{\sqrt{\theta_o^2 c_o^2 - \beta^2 y_1^2}} \text{ arc tan } \frac{y_1(x-c_o)}{z\sqrt{(x-c_o)^2 - \beta^2 y_1^2 - \beta^2 z^2}} \, dy_1$$

$$- z\beta \int_0^{c_o} dx_1 \int_0^{\frac{\theta_o x_1}{\beta}} \frac{\sqrt{\theta_o^2 x_1^2 - \beta^2 y_1^2}}{[(x-x_1)^2 - \beta^2 y_1^2 - \beta^2 z^2]^{3/2}} \, dy_1$$

The double integral contributes nothing to the coefficient, and the remaining portion of the expression can be evaluated without integrating by differentiating twice and using Cauchy's integral theorem

$$\oint \frac{f(y_1)dy_1}{(y_1 - iz)^2} = 2\pi i \left(\frac{\partial f}{\partial y_1}\right)_{y_1 = iz}$$

Thus,

$$A_1 = -\frac{1}{E_o\theta_o}\frac{z}{|z|} \tag{48}$$

The coefficient A_2 will not be evaluated, since the first higher order term in z has been found. Thus, to the first order in y_o and z_o, the downwash function $\frac{w}{w_o}$ is

$$\frac{w}{w_o} = \frac{2}{\pi E_o}\left(E_1 + \int_0^{k_1} \frac{K-E}{k+\theta_o} \, dk\right) - \frac{z_o}{E_o\theta_o}\frac{z}{|z|} \tag{49}$$

DISCUSSION

The variation of $\frac{w_W}{w_o}$, the downwash due to the doublets in the wake in terms of the downwash on the triangular wing, was given in figure 6 as a function of x_o for various values of θ_o. This distribution of downwash along the center line of the wake can be presented as a single curve if $\frac{w_W}{w_o} E_o$ is expressed in terms of $\frac{x_o-1}{\theta_o}$. A graph of this function is shown in figure 9.

Figure 10 presents the resultant downwash on the center line induced by the doublets distributed over the wake and over the plan form. This is the complete value of downwash on the x_o-axis and shows that, following the discontinuity in downwash at the trailing edge, the magnitude of $\frac{w}{w_o}$ builds up to its asymptotic value and achieves an almost constant value within approximately one chord length aft of the trailing edge for values of θ_o less than 0.8. The table given on the figure relates θ_o to free-stream Mach number M and to β for the particular case of a triangular wing with leading edges swept back 45°.

It has been shown that at $x = \infty$ the downwash has the same form for both incompressible and supersonic flow, provided the span load distributions are equal. An exact solution for the downwash on the center line of the triangular wing has been developed and extended to include a region near the center line of the wake by means of an approximation (equation (45)). A measure of the exactness of the approximation may be obtained by comparing these results with the exact results for the incompressible case for an elliptic span load distribution, as obtained from reference 8, page 151. Figure 11 shows the exact values of $\frac{w}{w_o}$ for incompressible theory for $0 < \frac{y_o}{\theta_o} < 0.3$ and $0 < \frac{z_o}{\theta_o} < 0.3$. Also included are the linear approximations in supersonic theory to $\frac{w}{w_o} E_o$ for the same range of the geometric variables. The span load distributions for both wings are elliptical, but, in order that the same value of lift is maintained for the low- and high-speed cases, the angle of attack must be modified. This accounts for the use of the factor E_o in one set of ordinates. It is apparent from the comparison that, at a large distance behind the wing, the approximation of equation (49) yields results within 4 percent of the exact value for the region considered.

In application, it is desirable to know not only the order of the variables retained in the analysis, but also the magnitudes of the neglected terms. In this respect, equation (49) does not supply as much information as might be desired, since no estimation is furnished of further coefficients in the series. It appears, however, from the nature of the agreement obtained in figure 11 for conditions at a large distance from the wing together with observations as to the asymptotic behavior apparent in figure 10, that equation (49) furnishes a reasonable estimate of downwash in region B, provided that the point of calculation is not in the immediate vicinity of the tip Mach cones. For values in region A, no conclusions can be drawn from this investigation about the variation of downwash with respect to z. The spanwise variation indicated by equation (49) agrees with the value given by equation (43) at the trailing edge; that is, to the first order in y, the downwash is a constant. Thus, in the xy-plane, equation (49) can be expected to give a reasonably close estimate to the downwash near the center line and away from the tip Mach cones for all points behind the trailing edge. For variation in the z-direction, however, the validity of equation (49) is restricted to points in region B.

The behavior of the downwash in the vicinity of the trailing edge indicates that in this region an important difference exists between lifting-line theory in supersonic and in subsonic flow. Figure 6 shows the variation of downwash along the center line of a doublet sheet (lifting-line theory in incompressible flow) with an elliptic span load distribution. The error introduced by not using the chordwise distribution of load is given in figure 7 and is sizeable for about one chord length back of the wing.

The methods of analysis presented here were shown in the first application to lead immediately to the value of the potential function at $x = \infty$, and thus provide a ready method for the determination of vortex drag of a supersonic wing.

A more detailed study of downwash behind the wing will necessarily involve considerable labor. The methods given in the report are, however, general and are directly applicable.

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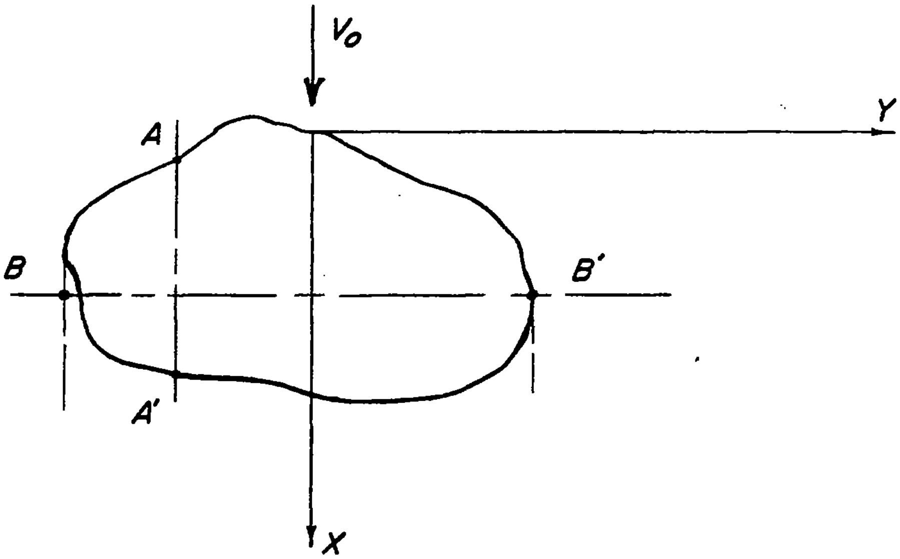
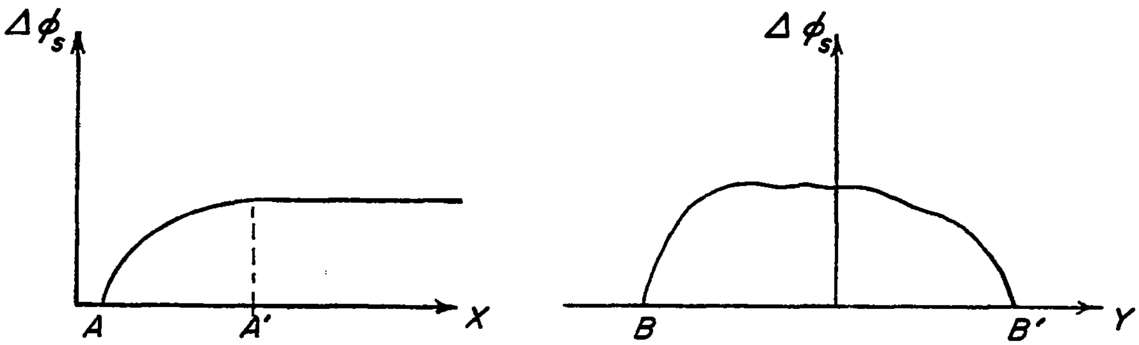


(a) Plan form



(b) Sections showing distribution of $\Delta\phi_s$

Figure 1.- Sketch showing arbitrary lifting surface together with distribution of $\Delta\phi_s$, the jump in perturbation velocity potential in the plane of the surface.

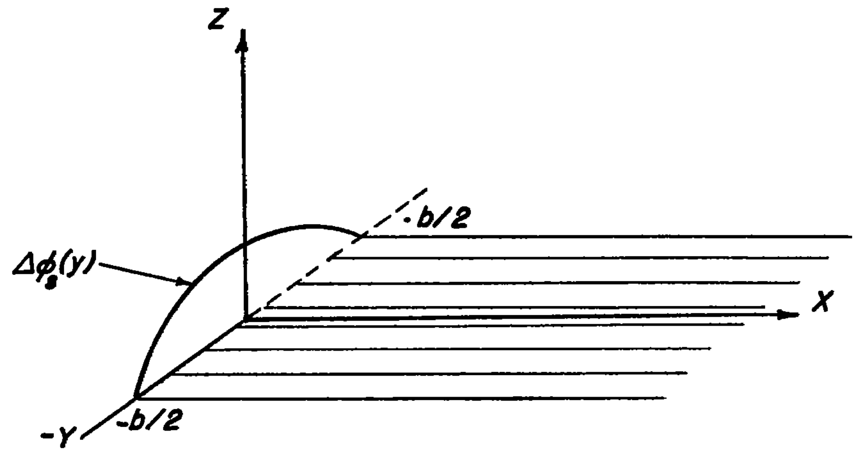
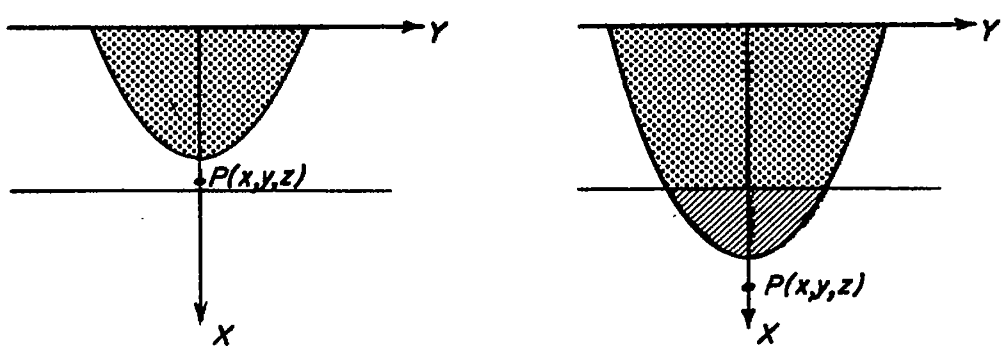
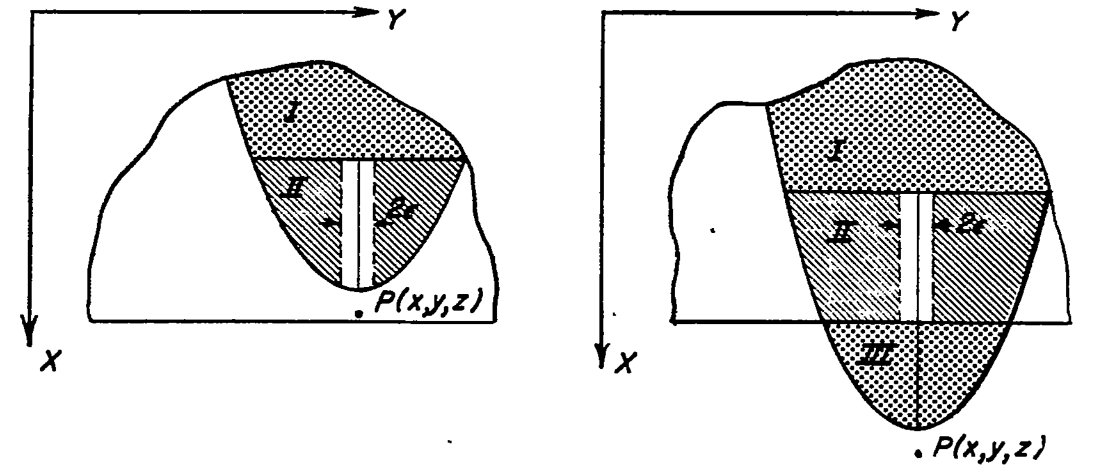


Figure 2.- Vortex sheet with bound vortices on y-axis and distribution of circulation equal to $\Delta\phi_s$.



(a) Point P ahead of trailing edge *(b) Point P behind trailing edge*

Figure 3.- Areas of integration for infinite span wing.



(a) Point P ahead of trailing edge (b) Point P behind trailing edge

Figure 4.- Areas of integration for arbitrary plan form with supersonic trailing edge.

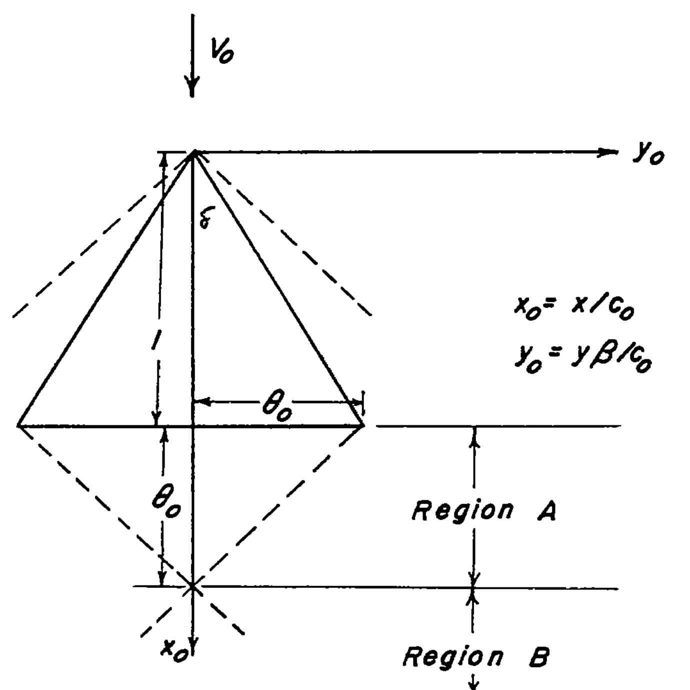


Figure 5.- Triangular wing swept behind Mach cone showing location of regions A and B.

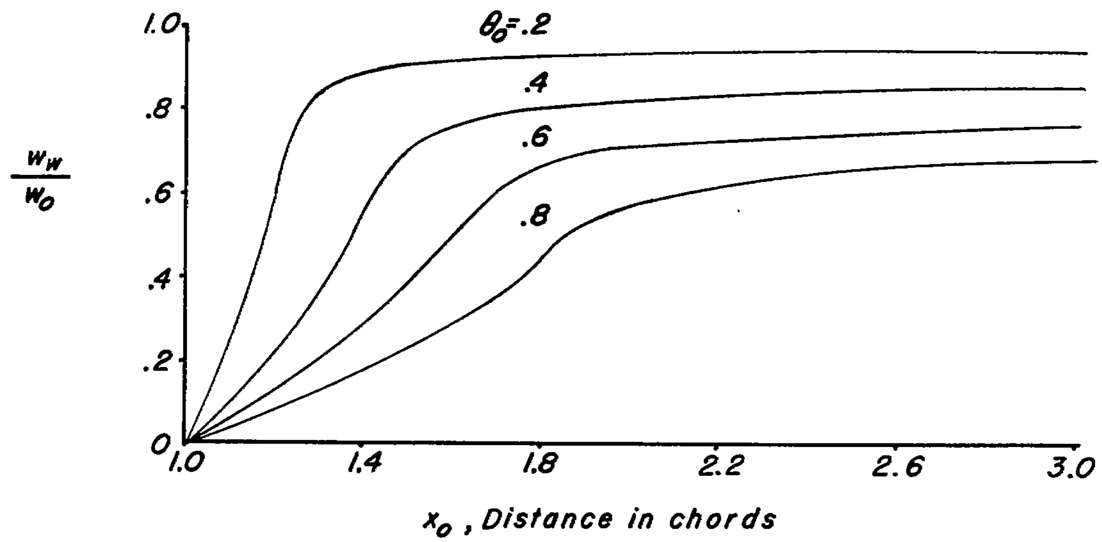


Figure 6.- Variation of the part of downwash on x-axis induced by doublets in wake with distance downstream in chord lengths, $x_0 = x/c_0$. Triangular wing.

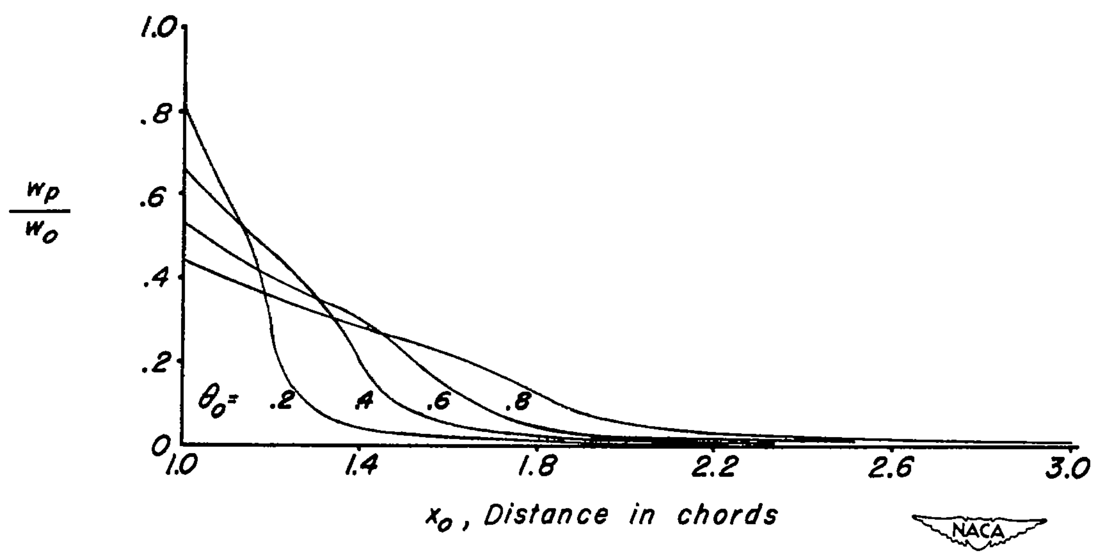


Figure 7.- Variation of the part of downwash on x-axis induced by doublets on the plan form with distance downstream in chord lengths, $x_0 = x/c_0$. Triangular wing.

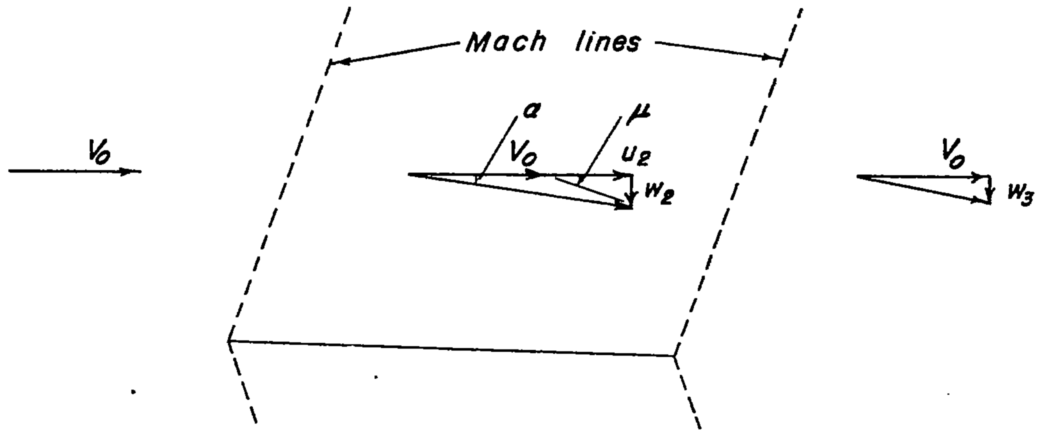


Figure 8.- Sketch of velocity vectors of the air before reaching, on, and after leaving supersonic airfoil.

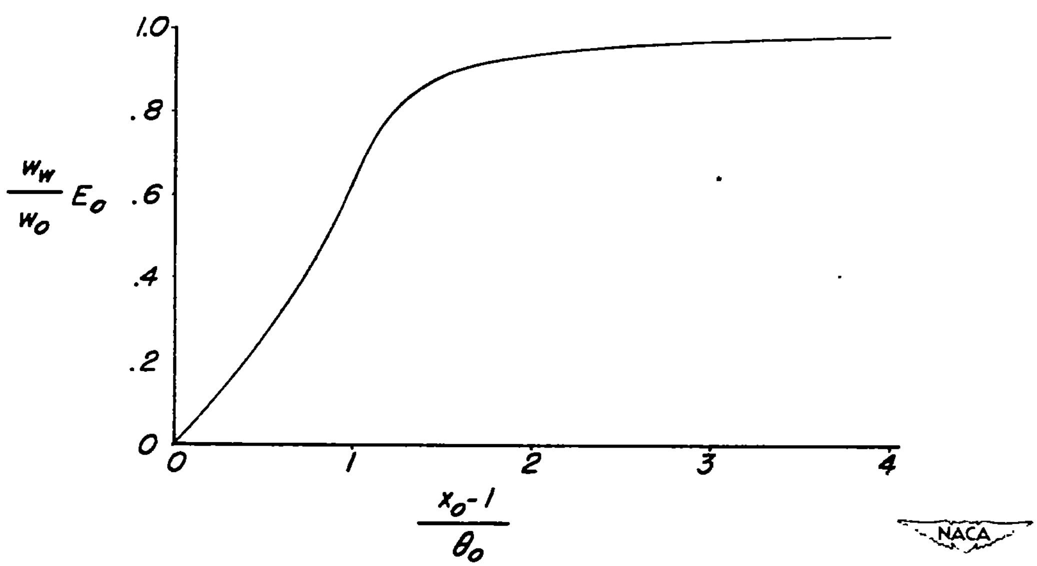


Figure 9.- Downwash factor induced by doublets in wake, $w_W E_o / w_o$, plotted against factor representing distance downstream from trailing edge, $(x_o - 1)/\theta_o$. Triangular wing.

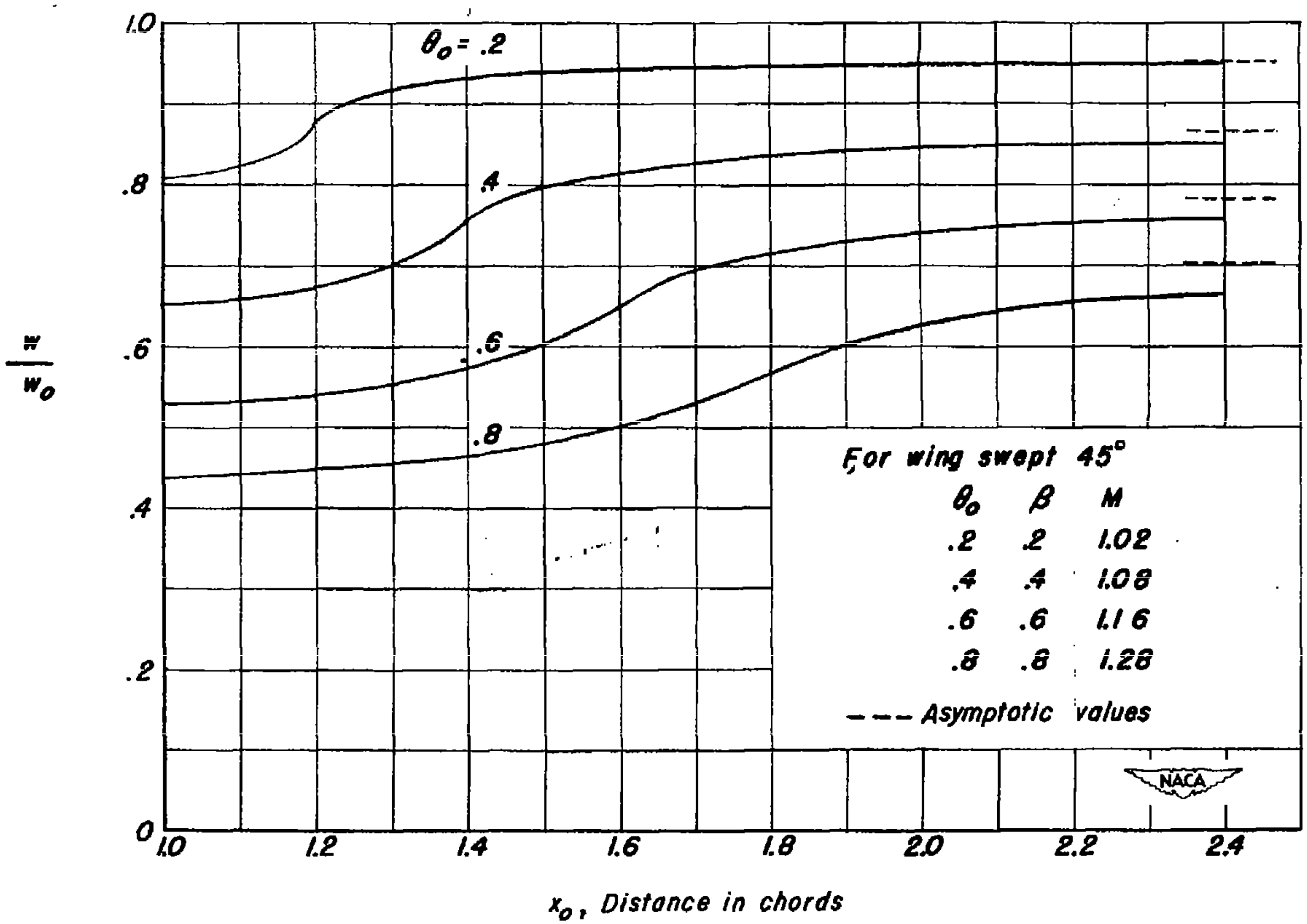


Figure 10.- Variation of the total downwash on x-axis behind a triangular wing swept behind Mach cone with distance downstream in chord lengths, $x_o = x/c_o$.

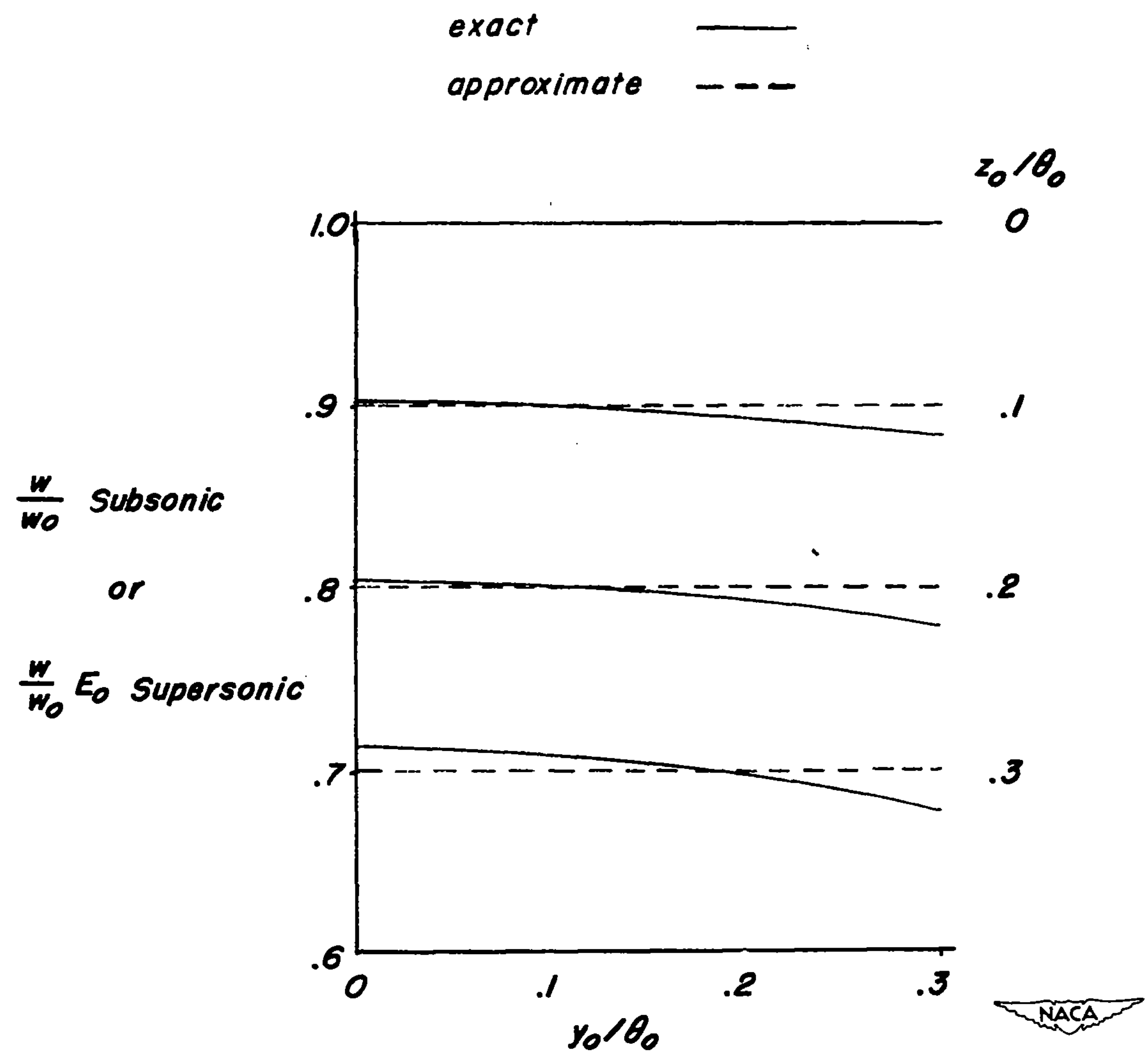


Figure 11.-Downwash at a large distance behind a wing with elliptical span loading in either subsonic or supersonic flight compared with first order expansion determined from equation (44).