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EFFECT OF STEADY ROLLING ON LONGITUDINAL AND DIRECTIONAL STABILITY

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SUMMARY

The effects of steady rolling on the longitudinal and directional stability of aircraft have been studied theoretically. Simplifying assumptions have been made with regard to the longitudinal and lateral motions of the airplane in order to obtain a solution which shows the principal effects of the rolling motion. Rolling has been found to cause instability if the directional and longitudinal stabilities are different when the rolling frequency exceeds the lower of the pitching and yawing natural frequencies of the nonrolling airplane. This instability lasts only during the time the airplane is rolling and would not, therefore, affect the normal flight of an airplane. In the case of airplanes of short span and high density, carrying most of their weight in their fuselages, and flying at high altitudes, this instability might cause dangerous attitude changes during rapid rolls. If the directional and longitudinal stabilities are about equal, the instability due to rolling will not occur.

If the rate of roll exceeds both the pitching and yawing natural frequencies of the nonrolling aircraft, the aircraft will be stable. A continuously rolling aircraft will be stable in this case even when the nonrolling aircraft has a certain amount of instability about one axis.

Applications of these conclusions to rolling airplanes and missiles are discussed.

INTRODUCTION

When an airplane rolls about an axis which is not alined with its longitudinal axis, inertia forces are introduced which tend to swing the fuselage out of line with the flight path. These forces are ordinarily neglected when the usual theory of lateral stability of aircraft is used to calculate the motion of an airplane in a roll. This assumption is probably justified for the case of most conventional airplanes because inertia forces involved are small compared with aerodynamic forces on the airplane. Design trends of very high-speed aircraft, however, which include short wing spans, fuselages of high density, and flight at high altitude, all tend to increase the inertia forces due to rolling in comparison with the aerodynamic restoring forces provided by the longitudinal and directional stabilities. It is therefore desirable to investigate the effects of rolling on the longitudinal and directional stabilities of these aircraft. The inertia forces due to rolling velocity are similar to those which are always taken into account in the study of spinning, where they have a predominant effect. The effects of rolling on stability discussed in this report occur only during the period in which an aircraft is rolling, and therefore they do not have any effect on the stability of an aircraft in steady flight.

Some types of research missiles, which were not roll-stabilized and therefore rolled continually in flight, have been employed to investigate longitudinal and lateral stability of airplane configurations. Furthermore, certain types of guided missiles may intentionally roll continually in flight. An analysis would therefore be desirable to determine the effects of the rolling motion on the behavior of these missiles.

The rolling motion introduces coupling between the longitudinal and lateral motion of the aircraft. An exact solution of this problem is very complicated because of the large number of degrees of freedom involved. In the present report, simplifying assumptions have been made with regard to the longitudinal and lateral motions of the aircraft in order to obtain a solution which shows the principal effects of the rolling motion.

SYMBOLS

a, b, c, d, e	coefficients of quartic
A	constant (amplitude ratio)
b	wing span
c	wing chord
C	viscous damping coefficient
C_L	lift coefficient $\left(\frac{L}{\frac{1}{2}\rho V^2 S}\right)$
C_m	pitching-moment coefficient $\left(\frac{M}{\frac{1}{2}\rho V^2 Sc}\right)$
D	differential operator $\left(\frac{d}{dt'}\right)$
e	base of natural logarithms

F	moment-of-inertia parameter $\left(\frac{I_X - I_Y}{I_Z}\right)$
h	shift in aerodynamic center or in stick-fixed maneuver point
I	moment of inertia
I_X	moment of inertia about X-axis
I_Y	moment of inertia about Y-axis
I_Z	moment of inertia about Z-axis
K	spring constant
L	rolling moment; or lift
L_a	aileron rolling moment
M	pitching moment; or Mach number
N	yawing moment
p	rolling velocity about body axis
p_o	steady rolling velocity
q	pitching velocity about body axis
r	yawing velocity about body axis
S	wing area
t	time
t'	nondimensional time $(p_o t)$
$t'_{1/2}$	nondimensional time required to damp to one-half amplitude
V	true airspeed
X, Y, Z	body axes of aircraft
α	angle of attack
β	angle of sideslip
γ	angular displacement of single-degree-of-freedom system
ζ	fraction of critical damping of single-degree-of-freedom system

ζ_θ	fraction of critical damping in pitch of nonrolling aircraft
ζ_ψ	fraction of critical damping in yaw of nonrolling aircraft
θ	angle of pitch relative to flight-path direction
ρ	air density
ϕ	constant (phase angle)
ψ	angle of yaw relative to flight-path direction
ω	actual frequency of single-degree-of-freedom system
ω_1, ω_2	nondimensional frequencies of motion of rolling aircraft with respect to body axes
ω_n	undamped natural frequency of single-degree-of-freedom system
ω_θ	nondimensional undamped natural frequency in pitch of nonrolling aircraft (ratio of pitching frequency to steady rolling frequency)
ω_ψ	nondimensional undamped natural frequency in yaw of nonrolling aircraft

$i = \sqrt{-1}$

$L_\beta = \frac{\partial L}{\partial \beta}, \quad L_r = \frac{\partial L}{\partial r}, \quad . \; . \; .$

Dot over a symbol indicates derivative with respect to time.

ANALYSIS

The motion of the aircraft is studied by means of Euler's equations. These equations are set up in terms of angular velocities and accelerations with respect to axes fixed in the aircraft. The period and damping of any motions obtained as a final result will, therefore, be those which would be measured by instruments, such as accelerometers, mounted in the aircraft during the maneuvers. Euler's equations are as follows:

$$L = I_X\dot{p} - qr(I_Y - I_Z) \tag{1}$$

$$M = I_Y\dot{q} - rp(I_Z - I_X) \tag{2}$$

$$N = I_Z\dot{r} - pq(I_X - I_Y) \tag{3}$$

It is assumed that the mass of aircraft is distributed in a plane, so that $I_Z = I_X + I_Y$. Equation (1), relating to the rolling motion, then becomes

$$\begin{aligned} L &= \beta L_\beta + rL_r + pL_p + L_a \\ &= I_X(\dot{p} + qr) \end{aligned}$$

In the type of motion under consideration, the aileron rolling moment L_a is offset, on the average, mainly by the damping moment pL_p while the quantities β, q, and r in general oscillate about values close to zero. It is assumed in the analysis which follows that the rolling velocity is constant and that the effects of the variations in sideslip, pitching velocity, and yawing velocity in producing rolling moments through aerodynamic or inertia effects may be neglected. Equation (1), therefore, disappears from the analysis and the remaining equations become linear. As a result of this assumption, it is expected that the analysis may not apply very closely in cases where the rolling velocity is small and the dihedral effect is large because a large dihedral effect would result in appreciable variation of rolling velocity during a yawing oscillation.

The equations involving linear accelerations along the X-, Y-, and Z-axes are omitted from the present analysis. The equation involving longitudinal accelerations is omitted because the motion is assumed to occur at constant airspeed. The equations involving lateral and normal accelerations are omitted because, for the purposes of the present analysis, the longitudinal and directional motions of the aircraft which is not rolling are each considered as single-degree-of-freedom motions involving only angular displacements. This assumption does not, however, exclude the possibility of applying the analysis to an aircraft trimmed at an angle of attack different from zero. In this case, as the aircraft rolls, it travels in a helical path. The lift on the aircraft balances the centrifugal force developed by the helical motion. Both the lift and centrifugal force, however, act through the center of gravity and do not influence the moments acting on the aircraft. The stability of angular motions of the aircraft is therefore determined by the moment equations (equations (1) to (3)). The helical motion simply introduces steady angles of pitch and yaw about which the disturbed motions take place.

In the discussion which follows, the terms "oscillation frequency" and "rolling frequency" are often employed. By "oscillation frequency" is meant the circular frequency of a sinusoidal motion, or 2π times the frequency in cycles per second. The term "rolling frequency" is used interchangeably with "rolling velocity" and is the rate of rotation in roll expressed in radians per second. In cases where ratios of these frequencies are used, the frequencies may, of course, be expressed in cycles per second instead of radians per second.

In accordance with the assumption that the longitudinal and directional motions of the aircraft which is not rolling are each considered as single-degree-of-freedom motions, the pitching and yawing equations for the nonrolling aircraft become

$$\theta M_\theta + qM_q = I_Y\dot{q} \tag{4}$$

$$\psi N_\psi + rN_r = I_Z\dot{r} \tag{5}$$

The motions obtained from the solutions of these equations would be damped oscillations in pitch and yaw. The values of natural frequency and damping of these oscillations may differ somewhat from the values of natural frequency and damping obtained from the usual stability theory in which additional degrees of freedom are taken into account. It would be possible and probably desirable, however, to substitute equivalent values for the restoring and damping moment coefficients of equations (4) and (5) such that the same frequency and damping for the single-degree-of-freedom motions would be obtained as from the more complicated stability theory. An alternate method which accomplishes the same result is to set up the equations from the outset in terms of the undamped natural frequency and damping ratios of the motion of the nonrolling aircraft. This procedure, which follows the method and notation of reference 1, may be described briefly by considering a single-degree-of-freedom system consisting of a pivoted beam, such as that shown in figure 1, moving under the influence of a spring restoring force and viscous damping. The equation of motion for the system is

$$I\ddot{\gamma} + C\dot{\gamma} + K\gamma = 0$$

If the following substitutions are made,

$$\omega_n = \sqrt{\frac{K}{I}} \quad \left(\text{or } \omega_n{}^2 = \frac{K}{I}\right) \tag{6}$$

$$\zeta = \frac{C}{2\sqrt{KI}} \quad \left(\text{or } 2\zeta\omega_n = \frac{C}{I}\right) \tag{7}$$

the equation becomes

$$\ddot{\gamma} + 2\zeta\omega_n\dot{\gamma} + \omega_n{}^2\gamma = 0$$

The quantity ω_n is known as the undamped natural frequency and is the frequency of free oscillations of the system when the viscous damping is zero. The quantity ζ is known as the damping ratio and is the ratio of the existing damping of the system to that required for critical damping. The free motion of the system, which is a decreasing oscillation, is given by the expression

$$\gamma = Ae^{-\zeta\omega_n t}\sin(\omega t - \phi)$$

In this formula, A and ϕ are constants depending on the initial conditions. The actual frequency of a free oscillation ω is related to the undamped natural frequency by the formula

$$\omega = \omega_n\sqrt{1 - \zeta^2}$$

If it is desired to calculate from the frequency and damping the restoring-moment and damping-moment coefficients for the single-degree-of-freedom system which simulates the aircraft either in pitch or yaw, the preceding relations for this type of system may be employed.

The substitutions required to express equations (4) and (5) in terms of the natural frequencies and damping ratios of the motions in pitch and yaw may be made in a similar manner to those of equations (6) and (7). In order to simplify the notation of the analysis, the frequencies of the nonrolling aircraft will hereinafter be taken as ratios of the oscillation frequencies to the steady rolling frequency p_o. The undamped natural frequency in pitch is therefore given by the expression

$$\omega_\theta p_o = \sqrt{\frac{-M_\theta}{I_Y}} \quad \left(\text{or} \quad \omega_\theta{}^2 p_o{}^2 = \frac{-M_\theta}{I_Y}\right) \tag{8}$$

The damping ratio in pitch is given by the expression

$$\zeta_\theta = \frac{-M_q}{2\sqrt{-M_\theta I_Y}} \quad \left(\text{or} \quad 2\zeta_\theta\omega_\theta p_o = \frac{-M_q}{I_Y}\right) \tag{9}$$

Analogous expressions are used for the frequency and damping of the yawing motions.

It is now desired to express the equations for the rolling aircraft in terms of these variables. Inasmuch as the rolling does not influence the aerodynamic moments acting due to changes in pitch and yaw, the external moments are the same as those given in equations (4) and (5). The pitching and yawing equations for the rolling aircraft (equations (2) and (3)) then become

$$\begin{aligned} M &= \theta M_\theta + qM_q \\ &= I_Y(\dot{q} - rp) \end{aligned}$$

$$\begin{aligned} N &= \psi N_\psi + rN_r \\ &= I_Z\dot{r} - pq(I_X - I_Y) \end{aligned}$$

Dividing these equations by I_Y and I_Z, respectively, gives the formulas

$$\dot{q} - rp - \theta\frac{M_\theta}{I_Y} - q\frac{M_q}{I_Y} = 0$$

$$\dot{r} - pq\left(\frac{I_X - I_Y}{I_Z}\right) - \psi\frac{N_\psi}{I_Z} - r\frac{N_r}{I_Z} = 0$$

If the expressions for the undamped natural frequencies and damping ratios in pitch and yaw (formulas (8) and (9)) are inserted in these equations, the pitching and yawing equations become

$$\dot{q} - rp_o + 2\zeta_\theta\omega_\theta p_o q + \omega_\theta{}^2 p_o{}^2\theta = 0$$

$$\dot{r} - p_o q\left(\frac{I_X - I_Y}{I_Z}\right) + 2\zeta_\psi\omega_\psi p_o r + \omega_\psi{}^2 p_o{}^2\psi = 0$$

Here the rolling velocity, assumed constant, has been written p_o. For the small angles considered in the present analysis, the angle of pitch θ is taken as the projection on the plane of symmetry of the aircraft (the XZ-plane) of the angle between the flight path and the longitudinal axis of the aircraft. The angle of yaw ψ is taken as the projection on the XY-plane of the angle between the flight path and the longitudinal axis. The axes X, Y, and Z are taken as the body axes of the aircraft.

Since the restoring forces on the aircraft are related to θ and ψ, the angular velocities q and r must be expressed in terms of these angles and their derivatives. It is therefore necessary to resolve the angular velocities q and r, which are measured with respect to the body axes, along the flight-path axes. This procedure is illustrated in figure 2, from which it may be shown that for small angles of pitch and yaw

$$\dot{\theta} = q + p_o\psi$$

$$\dot{\psi} = r - p_o\theta$$

Hence

$$q = \dot{\theta} - p_o\psi$$

$$r = \dot{\psi} + p_o\theta$$

If these substitutions are made, the equations become

$$\ddot{\theta} - p_o\dot{\psi} - p_o\dot{\psi} - p_o^2\theta + 2\zeta_\theta\omega_\theta p_o(\dot{\theta} - p_o\psi) + \omega_\theta^2 p_o^2\theta = 0$$

$$\ddot{\psi} + p_o\dot{\theta} + (p_o^2\psi - p_o\dot{\theta})F + 2\zeta_\psi\omega_\psi p_o(\dot{\psi} + p_o\theta) + \omega_\psi^2 p_o^2\psi = 0$$

where for simplicity $\dfrac{I_X - I_Y}{I_Z}$ has been set equal to F. It is convenient to express time nondimensionally in terms of the frequency of the steady rolling motion. Let the nondimensional time t' equal $p_o t$. Then define

$$D = \frac{d}{dt'}$$

$$= \frac{1}{p_o}\frac{d}{dt}$$

In terms of this operator, the equations become

$$D^2\theta - 2D\psi - \theta + 2\zeta_\theta\omega_\theta D\theta - 2\zeta_\theta\omega_\theta\psi + \omega_\theta^2\theta = 0$$

$$D^2\psi + D\theta + \psi F - D\theta F + 2\zeta_\psi\omega_\psi D\psi + 2\zeta_\psi\omega_\psi\theta + \omega_\psi^2\psi = 0$$

In order to analyze the motion of the rolling aircraft, the determinant of the coefficients of θ and ψ is set equal to zero. This determinant is

$$\begin{vmatrix} D^2 - 1 + 2\zeta_\theta\omega_\theta D + \omega_\theta^2 & -2D - 2\zeta_\theta\omega_\theta \\ D - DF + 2\zeta_\psi\omega_\psi & D^2 + F + 2\zeta_\psi\omega_\psi D + \omega_\psi^2 \end{vmatrix} = 0$$

The determinant may be expanded to give the quartic

$$aD^4 + bD^3 + cD^2 + dD + e = 0 \tag{10}$$

where

$$a = 1$$

$$b = 2\zeta_\psi\omega_\psi + 2\zeta_\theta\omega_\theta$$

$$c = -F + 1 + \omega_\psi^2 + \omega_\theta^2 + 2\zeta_\theta\omega_\theta 2\zeta_\psi\omega_\psi$$

$$d = 2\zeta_\theta\omega_\theta\omega_\psi^2 + 2\zeta_\theta\omega_\theta + 2\zeta_\psi\omega_\psi\omega_\theta^2 + 2\zeta_\psi\omega_\psi$$

$$e = -F + \omega_\theta^2\omega_\psi^2 - \omega_\psi^2 + \omega_\theta^2 F + 2\zeta_\psi\omega_\psi 2\zeta_\theta\omega_\theta$$

From the roots of this quartic, the period and damping of the modes of motion of the rolling aircraft may be determined. Because of the method adopted for expressing time nondimensionally, the frequencies of the motion thus determined are obtained as ratios to the steady rolling frequency. Routh's discriminant for this quartic is given by the formula

$$bcd - d^2 - eb^2$$

Placing this expression equal to zero gives the condition for the boundary between increasing and decreasing oscillations of the system. When the coefficients are substituted in this expression and the operations are carried out, Routh's discriminant becomes

$$\begin{aligned} &12\zeta_\psi\omega_\psi^3\zeta_\theta\omega_\theta - 4\zeta_\psi\omega_\psi^3\zeta_\theta\omega_\theta F + 4\zeta_\psi\omega_\psi^5\zeta_\theta\omega_\theta - 8\zeta_\psi\omega_\psi^3\zeta_\theta\omega_\theta^3 + 16\zeta_\psi^2\omega_\psi^4\zeta_\theta^2\omega_\theta^2 \\ &\quad + 4\omega_\psi^2\zeta_\theta^2\omega_\theta^2 - 4\omega_\psi^2\zeta_\theta^2\omega_\theta^2 F + 16\zeta_\psi\omega_\psi^3\zeta_\theta^3\omega_\theta^3 + 4\zeta_\psi\omega_\psi\zeta_\theta\omega_\theta^3 \\ &\quad + 4\zeta_\theta^2\omega_\theta^4 - 8\zeta_\psi^2\omega_\psi^2\omega_\theta^2 F + 16\zeta_\psi^3\omega_\psi^3\zeta_\theta\omega_\psi^3 - 12\zeta_\psi\omega_\psi\zeta_\theta\omega_\theta^3 F \\ &\quad + 4\zeta_\psi\omega_\psi\zeta_\theta\omega_\theta^5 + 16\zeta_\psi^2\omega_\psi^2\zeta_\theta^2\omega_\theta^4 + 8\zeta_\psi^2\omega_\psi^4 - 4\zeta_\theta^2\omega_\theta^4 F \end{aligned} \tag{11}$$

A condition for the boundary between stability and divergence is obtained by setting the coefficient e of the quartic (formula (10)) equal to zero.

In order to simplify the numerical analysis and at the same time to show the principal effects of the rolling motion, it is helpful to consider the case where the damping ratios ζ_θ and ζ_ψ of the longitudinal and directional oscillations of the nonrolling aircraft are zero. This case of undamped oscillations is of much practical interest because the oscillations of high-density aircraft flying at high altitudes are usually poorly damped.

If the damping ratios ζ_θ and ζ_ψ equal zero, the determinantal equation for the rolling aircraft becomes

$$aD^4 + cD^2 + e = 0$$

where

$$a = 1$$

$$c = -F + 1 + \omega_\psi^2 + \omega_\theta^2$$

$$e = -F + \omega_\theta^2\omega_\psi^2 - \omega_\psi^2 + \omega_\theta^2 F$$

This equation may be solved explicitly as a quadratic in D^2 as follows:

$$D^2 = -\frac{\omega_\psi^2 + \omega_\theta^2 + 1 - F}{2} \pm \sqrt{\left(\frac{\omega_\psi^2 + \omega_\theta^2 + 1 - F}{2}\right)^2 - \omega_\theta^2\omega_\psi^2 + \omega_\psi^2 - \omega_{\psi\theta}^2 F + F} \tag{12}$$

The frequencies of the oscillations of the rolling aircraft are obtained from the numerical values of D, the square root of D^2 (formula (12)), when D is imaginary. There are two frequencies: one designated ω_1, obtained with the minus sign before the radical of formula (12), and the other designated ω_2, obtained with the plus sign before the radical.

DISCUSSION OF RESULTS

Case of zero damping of nonrolling aircraft.– The first case considered is that of an aircraft with frequencies ω_θ and ω_ψ in pitch and yaw when it is not rolling and with zero damping of these oscillations (ζ_θ and $\zeta_\psi = 0$). It will also be assumed that $I_X = 0$, or $F = \frac{I_X - I_Y}{I_Z} = -1$. This case is a reasonable close approximation to many practical aircraft and missiles of short span with slender fuselages in which most of the weight is concentrated.

The characteristics of the motion of a rolling aircraft of this type are shown in figure 3. This figure presents the stable and unstable regions in a plot of ω_θ^2 against ω_ψ^2 and also shows contour lines of the frequencies of the oscillations performed by the rolling aircraft. This figure brings out the symmetry in the effects of ω_θ and ω_ψ which would be expected from physical considerations for the case of $I_X = 0$.

When both the pitching and yawing frequencies of the nonrolling aircraft are greater than the steady rolling frequency under consideration, the motion is stable, in the sense that there is no divergence or increasing oscillation. This condition is shown by the stable region in the upper right-hand part of the diagram where $\omega_\theta > 1$ and $\omega_\psi > 1$. In this region, the rolling aircraft has two modes of oscillation, both of which are undamped and have frequencies different from those of the oscillations of the nonrolling aircraft. If the pitching frequency of the nonrolling aircraft ω_θ equals its yawing frequency ω_ψ, then one mode of oscillation of the rolling aircraft has a frequency equal to this frequency plus the rolling frequency and the other mode of oscillation has a frequency equal to this frequency minus the rolling frequency. In general, for ω_θ not equal to ω_ψ, one frequency of the rolling aircraft is greater than the higher frequency of the nonrolling aircraft; and the other frequency is less than the lower frequency of the nonrolling aircraft.

When one of the frequencies of the nonrolling aircraft equals the frequency of the steady rolling motion (ω_θ or $\omega_\psi = 1$), the aircraft becomes neutrally stable in one mode, as shown by the fact that the frequency of this mode equals zero. This phenomenon may be explained physically on the basis that the restoring forces acting on the nonrolling aircraft which produce a certain oscillation frequency are just offset by the centrifugal forces which attempt to swing the fuselage out of line with the flight path when the aircraft rolls with this frequency. This effect is somewhat analogous to a rotating shaft operating at its critical speed. In fact, if the pitching and yawing frequencies of the aircraft are both equal to the rolling frequency, the conditions are exactly similar to those encountered when a shaft having equal stiffness in all directions rotates at its critical speed. When the frequencies of the aircraft in pitch and yaw are different, and only one of these frequencies equals the rolling frequency, the conditions may be shown to be analogous to those encountered when a shaft of flattened cross section rotates at one of its two critical speeds. It may be of interest to note that the theory for the behavior of such a shaft is identical with the theory developed in this report for the rolling aircraft.

When one frequency of the nonrolling aircraft is less than the steady rolling frequency and the other is greater, the rolling aircraft becomes statically unstable in one mode and performs a straight divergence as measured by instruments fixed in the aircraft. If both frequencies of the nonrolling aircraft are less that the steady rolling frequency, however, the rolling aircraft is stable, as shown by the small stable region in the lower left-hand corner of figure 3 for ω_θ and ω_ψ between 0 and 1. Here again there are two modes of undamped oscillation. In this region, when the values of ω_θ and ω_ψ are equal, the stability is analogous to that of a shaft having equal stiffness in all directions rotating above its critical speed. When ω_θ and ω_ψ both approach zero, which means that the static longitudinal and directional stabilities both approach zero, the two frequencies of the rolling aircraft both approach the rolling frequency. Physically, this condition means that the rolling aircraft can have its axis tilted from the flight path and, because of its lack of static stability, will continue to roll about this tilted axis. This rolling motion will cause periodic changes in the angles of attack and yaw with a frequency equal to the rolling frequency. These periodic changes would be measured as constant-amplitude pitching and yawing oscillations by instruments fixed in the aircraft.

A small stable region exists where the frequency of one mode of oscillation of the nonrolling aircraft is less than the rolling frequency, and in the other direction the aircraft has a certain degree of static instability. This stabilizing effect of the rolling motion may best be visualized by considering the motion of the aircraft with respect to fixed axes. A fin which provides stability in only one direction (say, yaw) will make the rolling aircraft stable about both axes, provided the rate of roll is fast enough, because the fin rapidly turns from one plane to another. This effect only occurs for a relatively limited range of parameters, however, and is shown in figure 3 as the stable region in the range of negative values of ω_θ^2 and ω_ψ^2. A negative value of ω_θ^2, corresponding to an imaginary value of the frequency, represents an exponential divergence defined by the equation

$$\theta = Ae^{-i\omega_\theta t'}$$

This same equation, of course, represents a sinusoidal oscillation of frequency ω_θ for real values of ω_θ. Figure 3 was plotted in terms of ω_θ^2 and ω_ψ^2 rather than ω_θ and ω_ψ in order to include the imaginary values of these frequencies.

In the lower left-hand corner of figure 3 there is a region of increasing oscillations as measured by instruments fixed in the body. In this region, where the nonrolling aircraft has a large amount of static instability, the longitudinal axis of the rolling aircraft performs a maneuver approximating straight divergence with respect to fixed axes; but because of the rolling, this motion shows up as an increasing oscillation with respect to the body axes.

The effect of distributing weight along the wings as well as along the fuselage on the behavior of the rolling aircraft, again with zero damping in pitch and yaw $(\zeta_\theta = \zeta_\psi = 0)$, is shown in figures 4 and 5. Figure 4 presents the contour lines of the frequencies of the rolling aircraft on a plot of ω_θ^2 against ω_ψ^2 for $F = -0.666$. This value of F corresponds to the case where the moment of inertia about the X-axis equals 0.2 times the moment of inertia about the Y-axis. The results indicated by this figure are similar to those for the case where all the weight is located in the fuselage. A somewhat smaller value of the directional stability is required, however, to avoid divergence in yaw of the rolling aircraft. Figure 5 is a similar plot for $F = 0$. This value of F corresponds to the case where the moment of inertia about the X-axis equals the moment of inertia about the Y-axis. In this case a rolling motion produces no inertia yawing moment on the yawed aircraft. With large stability in pitch, the yawing frequency of the rolling aircraft would therefore be expected to be the same as that of the nonrolling aircraft. The results of figure 5 indicate that the frequency ω_2, which represents mainly a yawing motion with large stability in pitch, approaches asymptotically the yawing frequency ω_ψ as ω_θ becomes large. Furthermore, the divergence boundary in yaw, which occurs at $\omega_\psi = 0$ for the nonrolling aircraft, is unchanged by the rolling motion.

The special case where $\omega_\theta = \omega_\psi$ and $I_X = 0$ may be analyzed more simply by use of the equation of motion of the body with respect to axes fixed in space. This analysis allows a clearer physical interpretation of the motion of the body and serves as a check on the results obtained previously by means of Euler's equations. This special case corresponds to conditions existing along a 45° line through the origin in figure 3. The motion of the system with respect to axes fixed in space is derived in a following section of this paper, but first the results already obtained by means of Euler's equations are stated. It may be seen from figure 3 or derived from formula(12) that the frequencies of the rolling aircraft with respect to body axes for this case are given by the formulas

$$\omega_1 = \omega_\theta + 1$$

$$\omega_2 = |\omega_\theta - 1|$$

Here, ω_1 and ω_2 are nondimensional frequencies expressed as ratios to the steady rolling frequency. The pitching frequency ω_θ of the nonrolling aircraft is equal to the yawing frequency ω_ψ, and either symbol might be used. The member on the right-hand side of the equation for ω_2 indicates the absolute value of the quantity $\omega_\theta - 1$. If these formulas are put in terms of actual frequencies, rather than nondimensional frequencies, they become

$$\omega_1 p_o = \omega_\theta p_o + p_o$$

$$\omega_2 p_o = |\omega_\theta p_o - p_o|$$

Hence, the frequencies of the rolling aircraft are given by the sum and by the absolute value of the difference between the frequency of the nonrolling aircraft and the rolling frequency.

The solution for the motion based on the equations of motion with respect to fixed axes is now considered. The dynamic system is shown in figure 6(a). The restoring forces provided by the fins will usually be the same with respect to fixed axes as with respect to axes rolling with the body. The forces would be exactly the same, for example, if the body had a fin in the form of a circular cylinder. The assumption that the forces are the same would be a close approximation to the conditions existing with a conventional four-fin tail. Because all the weight is located along the X-axis, any rolling motion of the body about the X-axis has no effect whatever on the motion of the X-axis of the body with respect to fixed space. The motion of the X-axis of the body, therefore, is composed of vertical and horizontal oscillations of frequency $\omega_\theta p_o$, exactly as in the case of the nonrolling body. The most general motion of the axis is a combination of these two components with arbitrary amplitudes and phase difference. This combination in general causes the axis to swing so that any point on the axis traces an elliptical path, as shown in figure 6(a). In order to see how this result corresponds to that obtained previously for the motion with respect to body axes, the frequencies measured with respect to fixed axes must be converted to frequencies measured with respect to axes rolling with the body. This conversion is a kinematic transformation, with no dynamics involved. Ordinarily, the motion of the axis would be resolved into vertical and horizontal components as mentioned previously. If the body rolls when the axis is undergoing a vertical or horizontal oscillation, however, the resulting oscillations with respect to body axes will not have constant amplitude. In order to obtain results equivalent to those previously described, it is necessary to break the motion of the axis into components which lead to constant-amplitude oscillations with respect to body axes. Two such motions are possible: one a clockwise and the other a counterclockwise rotation of a point on the rear of the body. This point moves in a circular path with frequency $\omega_\theta p_o$. These motions are shown in figure 6(b). These circular motions of the body with frequency $\omega_\theta p_o$ are possible motions because they may be obtained by combining vertical and horizontal oscillations of equal amplitude with a phase difference of 90°. Any possible motion of the aircraft may be produced by combining these two circular motions with the correct phase difference and amplitude. Examples of possible combinations are given in figures 6(c) and 6(d). Figure 6(d) shows that the elliptical path, which is the most general type of motion, may be produced by this combination.

The frequencies of the rolling aircraft as seen from body axes may be derived by considering the angle-of-attack changes as the body rolls when its axis is performing one of the two circular motions. The case where the axis revolves in a counterclockwise direction with frequency $\omega_\theta p_o$ while the body rolls clockwise with a frequency p_o corresponds to the formula

$$\omega_1 p_o = \omega_\theta p_o + p_o$$

The case where the axis revolves in a clockwise direction with frequency $\omega_\theta p_o$ while the body rolls clockwise with a frequency p_o corresponds to the formula

$$\omega_2 p_o = \left|\omega_\theta p_o - p_o\right|$$

The results obtained by the analysis based on fixed axes may therefore be converted to body axes to give the same result as that obtained directly from the analysis based on body axes.

A case in which the two solutions might be considered to disagree is one in which the rolling frequency equals the pitching (and yawing) frequencies. The results plotted in figure 3 show a condition of neutral stability to exist at this point, whereas the stability of the axis of the body in the analysis based on fixed axes was stated to be independent of the rate of roll. It should be noted, however, that any slight out-of-trim pitching moment applied to the rolling aircraft at this point would produce a vertical and horizontal moment varying sinusoidally with time at the natural frequency of the axis. A condition of resonance would therefore exist and the vertical and horizontal amplitudes of the undamped system would increase indefinitely. In the analysis based on body axes, this same out-of-trim pitching moment applied to the neutrally stable system would cause the angle of pitch to increase indefinitely. The two methods of analysis, therefore, lead to the same result.

If under the conditions where the rate of roll equals the pitching (and yawing) frequencies, the axis of the body is displaced in pitch, then a yawing velocity will be introduced with respect to body axes. Any damping forces proportional to yawing velocity would extract energy from the system and prevent the amplitude from building up. The damping is therefore expected to increase the stability of the system, at least under conditions where the pitching and yawing frequencies are close to the rolling frequency. The effects of damping are now considered on the basis of the theory.

Case of damped oscillations of nonrolling aircraft.- The rate of decrease of amplitude of the oscillations of the nonrolling aircraft is determined by the damping ratio ζ. The fraction of the original amplitude to which the oscillation decays in one cycle is shown as a function of ζ in figure 7. For $\zeta = 0.2$, the oscillation damps to 0.28 of its original amplitude in one cycle. This amount of damping is greater than that usually found for either the pitching or yawing oscillation of an aircraft of high density and is used to give an extreme example of the effect of damping on the stability of the rolling aircraft.

The divergence boundary for the rolling aircraft is determined by setting the coefficient e of the quartic (equation (10)) equal to zero. The divergence boundary for the case $\zeta_\theta = \zeta_\psi = 0.2$ and $I_X = 0$ is given on a plot of ω_θ^2 against ω_ψ^2 in figure 8. This figure also corresponds to any values of ζ_θ and ζ_ψ satisfying the relation $\zeta_\theta\zeta_\psi = 0.04$ because these quantities enter into the coefficient e only as a product. By comparing the boundaries of figure 8 with those of figure 3, it may be seen that the addition of damping has broadened the stable region in the neighborhood of the point $\omega_\theta = 1$, $\omega_\psi = 1$, that is, where the frequencies in pitch and yaw are close to the rolling frequency. In other parts of the figure, the boundaries are but little changed. The boundary between increasing and decreasing oscillations is not shown in figure 8.

In practice, when the frequency of the nonrolling aircraft is changed, the damping ratio also changes. For example, if the frequency in pitch is changed by varying the center-of-gravity location, the damping ratio increases as the aircraft approaches neutral stability because the damping moment provided by the tail remains nearly constant while the restoring moment decreases to zero. The condition encountered in practice is more nearly represented by the condition that $\zeta_\theta\omega_\theta$ equals a constant. This condition, for a single-degree-of-freedom system, is fulfilled when the viscous damping device remains the same as the spring restoring force is varied. The divergence boundaries for the case $\zeta_\theta\omega_\theta = 0.2$ and $\zeta_\psi\omega_\psi = 0.2$ are plotted in figure 9. The results are similar to those of figure 8 although, of course, the damping coefficient is less at large values of the nondimensional frequencies and greater at frequencies approaching zero. In figure 8, the actual damping moment decreases to zero when the corresponding frequency equals zero, and for this reason the boundaries cross the same point as those of figure 3 when $\omega_\theta = 0$ or $\omega_\psi = 0$.

The boundary between decreasing and increasing oscillations for the case of damped motion is obtained by setting Routh's discriminant equal to zero. This boundary is also plotted in figure 9 for the case $\zeta_\theta\omega_\theta = 0.2$ and $\zeta_\psi\omega_\psi = 0.2$. This boundary is almost coincident with the boundary between constant-amplitude oscillations and increasing oscillations given in figure 3. Thus, the boundary between constant-amplitude and increasing oscillations, which cannot be strictly termed a stability boundary, goes over into the Routh boundary as soon as any damping is present.

The effect of damping on the characteristics of the motion for representative combinations of frequency and damping has been studied by determining the roots of the stability quartics obtained from formula (10). The results are presented in figure 10 which shows the roots on enlarged plots of ω_θ^2 against ω_ψ^2 similar to those previously given in figures 3 and 9. Three conditions have been investigated, namely, zero damping $(\zeta_\theta\omega_\theta = \zeta_\psi\omega_\psi = 0)$, equal damping about each axis $(\zeta_\theta\omega_\theta = \zeta_\psi\omega_\psi = 0.2)$, and zero damping about one axis combined with damping about the other axis $(\zeta_\theta\omega_\theta = 0$ and $\zeta_\psi\omega_\psi = 0.2$ or vice versa). The real roots represent convergences and divergences whereas conjugate complex roots represent oscillations. Real roots or real parts of complex roots determine the nondimensional time to decrease to one-half amplitude, if they are negative, or to double amplitude, if they are positive, in accordance with the formula

$$t'_{1/2} = \frac{0.693}{\text{Real part of complex root}}$$

The imaginary parts of complex roots give the nondimensional frequencies directly. Figure 10(a), which shows the results with zero damping, is simply a repetition of what has previously been presented in figure 3; but the roots are given to facilitate comparison with the cases of damped motion. This figure shows that constant amplitude oscillations exist within the stability boundary and that a divergence is present outside these boundaries in the region shown.

Figure 10(b) shows the results when both modes of oscillation of the nonrolling aircraft have equal damping and corresponds to the case for which divergence boundaries are shown in figure 9. It may be seen that in most of the stable region two modes of oscillation occur, both of which damp to one-half amplitude in the same time as the damped oscillations of the nonrolling aircraft. The periods of the oscillations are very nearly equal to those existing with zero damping. As the divergence boundary is approached very closely, however, one mode of oscillation changes into a pair of convergences. One of these convergences becomes weaker upon closer approach to the boundary until at the boundary it is transformed into a divergence. For the one point investigated, the rate of divergence is slower than that for the case with zero damping. The damping changes the real root, which determines rate of divergence, by about the same amount as it changes the real part of the complex root, which determines the damping of the oscillation. It may be concluded that, within the region of constant-amplitude oscillations of the undamped motion, damping is very effective in providing stability and causes the motion to disappear in the same time as in the case of the nonrolling aircraft. Outside the divergence boundary for the damped motion damping reduces the rate of divergence, but for practical values of damping this reduction would not be important.

Figure 10(c) presents the results for the case when one mode of oscillations of the nonrolling aircraft is well damped and the other mode has zero damping. Although this condition is not likely to exist in practice, it represents an extreme example of this inequality. This example is intended to bring out the differences between this case and the case of equal damping of the two modes. The divergence boundaries in this case are the same as those for zero damping (fig. 10(a)). Physically, this fact means that when the aircraft has a mode of oscillation of the same frequency as the rolling motion, it may be oriented in such a way that no angular velocity occurs about the axis around which damping forces exist. Thus, the damping can have no effect on this mode. In the region where oscillations exist, the damping is one-half as great as in the case of equal damping. This result means that when the airplane is rolling the damping is effective about half the time. The rate of divergence in the unstable region is intermediate between that for the case with zero damping and that with equal damping about the two axes.

APPLICATION OF RESULTS

Full-scale airplanes.- The previous analysis indicates that instability may be caused by very rapid rates of roll in small heavily loaded airplanes carrying a large proportion of their weight in their fuselages and flying at high altitudes. This instability lasts only as long as the airplane rolls and would not, therefore, cause difficulty in normal flight. The instability in a roll might, however, cause an airplane to reach dangerous attitudes if the divergence were sufficiently rapid.

The rate of roll of an airplane with a given aileron deflection and true airspeed remains approximately constant as the altitude is increased, but the periods of the longitudinal and directional oscillations increase because of the reduced indicated airspeed. The rolling frequency may possibly exceed one or the other oscillation frequencies, with the result that instability of the type discussed would be encountered. For example, the rolling frequency and the frequencies of the pitching and yawing oscillations of an existing transonic research airplane are plotted as a function of altitude in figure 11. This airplane has a large amount of directional stability, so that the yawing oscillation has a higher frequency than the pitching oscillation. With the assumed value of $pb/2V$ of 0.05 and Mach number of 0.8, the rolling frequency exceeds the pitching frequency at an altitude of about 28,000 feet when the static margin $\left(\frac{dC_m}{dC_L}\right)_M$ is 0.05, or at 46,000 feet when the static margin is 0.10. The airplane would perform a longitudinal divergence in rolls of this rate at higher altitudes. Higher rates of roll would, of course, cause instability at lower altitudes.

The instability would not be present if the periods of the pitching and yawing oscillations were equal. It would appear advisable to provide approximately equal values of longitudinal and directional stability on airplanes that are intended to roll rapidly. Because the longitudinal stability inevitably varies with changes in center-of-gravity position, however, this condition may not be easy to realize in practice. It is, therefore, desirable to provide fairly large values of both longitudinal and directional stability on airplanes with high rates of roll in order to avoid the instability due to rolling.

The rates of divergence for the unstable cases investigated are generally not large enough to cause unduly large changes in attitude of the airplane in rolling to angles of bank up to 90°, but they may cause serious attitude changes in a complete 360° roll. Large yawing moments due to rolling, and pitching moments due to sideslip, are usually present which cause displacements in pitch and yaw during the early stages of a roll. These displacements would increase rapidly if instability were present. Consideration of these disturbing moments alone leads to the conclusion that relatively large values of directional and longitudinal stability are desirable on airplanes intended to roll rapidly.

The effect of rolling on the longitudinal stability in the case when the directional stability is very large may be considered as a forward shift in aerodynamic center or in the stick-fixed maneuver point. This shift is given as a fraction of the chord by the expression

$$h = -\frac{8I_Y\left(\frac{pb}{2V}\right)^2}{C_{L_\alpha}\rho Scb^2}$$

On a given airplane the shift in maneuver point is thus proportional to $(pb/2V)^2$ and varies inversely as the air density. This same formula applies approximately for practical values of directional stability, provided that the longitudinal stability is small compared with the directional stability. For the airplane used as an example in figure 11, the shift in maneuver point with a value of $pb/2V$ of 0.05 at sea level is 4.6 percent chord and at 50,000 feet altitude is 31 percent chord.

Missiles.- Some missiles differ from full-scale airplanes in having much smaller wing span, higher density, and a greater proportion of weight in the fuselage. The rolling frequency of these missiles may therefore be larger in comparison with the frequencies of their longitudinal and directional oscillations.

Some research missiles, which were not roll-stabilized, have been used to investigate the longitudinal and directional stability of airplane configurations. The preceding analysis shows that the frequencies of oscillations recorded by instruments in the missile cannot be used directly to compute the longitudinal and directional stability unless the rolling frequency is very small in comparison with these frequencies. If the rate of roll were recorded and a sufficiently long record of the motion were obtained under steady conditions to enable determination of the frequencies of both modes of oscillation, it would be theoretically possible by use of charts such as figures 3 to 5 to compute the frequencies of the nonrolling aircraft. These steady conditions are rarely obtained in practice, however. Devices to limit the rate of roll or to roll-stabilize such research missiles therefore should be used unless their inherent rates of roll are very small.

The preceding analysis may be used to indicate the design features required for stability of missiles that are intended to roll continually in flight. If such missiles roll at a smaller frequency than the frequencies of their longitudinal and directional oscillations, then equal stability in both planes is desirable, as it was in the case of the full-scale airplane. If the rolling frequency is greater than that of the more rapid oscillation, as is usually the case with such missiles, then a fin providing stability in only one plane is adequate to stabilize the missile. The instability in the other direction should not be so great as to place the system in the unstable region of figure 3, however.

The results of figure 3 indicate that a body which is unstable in both planes cannot be stabilized by spinning, a result which appears to disagree with the normal practice of stabilizing projectiles by spinning. The stabilization of projectiles by spinning cannot be studied from the numerical results presented in figure 3 because these results apply to the case where the moment of inertia about the longitudinal axis I_X is zero. The stability of spinning projectiles depends on the value of I_X. The preceding analysis may readily be extended to include the case of an axially symmetrical body with a finite value of I_X. In this case, it may be shown that the axis of the rapidly spinning body will perform constant-amplitude oscillations in the absence of damping forces whether or not it is stable in pitch and yaw. If the body is unstable, however, damping forces in pitch and yaw will produce an increasing oscillation; whereas, if the body is stable, damping forces will produce a decreasing oscillation. Inasmuch as most artillery projectiles are unstable in pitch and yaw, they cannot be called truly stable in flight. The rate of divergence of the oscillation is small enough, however, to avoid appreciable increase in amplitude during the time of flight.

CONCLUSIONS

An analysis has been made to show the effects of rolling on the stability of aircraft. In this analysis, it was assumed that the longitudinal and directional motions involved only pitching and yawing, respectively, and that the rolling velocity was constant. The neglect of the additional degrees of freedom and of the possible effect of sideslip on the rolling velocity may lead to some inaccuracy, particularly in cases where the rolling velocity is small and the dihedral effect, large. The analysis is expected to apply closely, however, in the cases of greatest interest where the aircraft has high density and is rolling rapidly. From the results presented, the combinations of directional and longitudinal stability that produce stable motion with different rates of roll may be calculated and the effect of rolling on the characteristics of the motion in pitch and yaw may be found. The analysis leads to the following conclusions:

1. Rolling of an airplane may introduce inertia forces that tend to swing the fuselage out of line with the flight path. These forces tend to produce (1) longitudinal instability if the longitudinal stability of the nonrolling airplane is small compared with its directional stability and (2) directional instability if the directional stability of the nonrolling airplane is small compared with its longitudinal stability. This tendency toward instability lasts only as long as the airplane rolls and, therefore, would not affect normal flying of an airplane. The destabilizing effect may be appreciable on airplanes of short span and high density, carrying most of their weight in their fuselages, and flying at high altitudes. On such airplanes, dangerous attitudes might be reached in rapid rolls, particularly if the rolling continued through 360°. Instability occurs when the rolling

frequency exceeds the lower of the pitching and yawing natural frequencies. This type of instability does not occur if the stabilities about the two axes are about equal though rolling reduces the stability in this case.

2. The pitching and yawing oscillation frequencies as recorded by instruments in an aircraft are changed when the aircraft is rolling. These frequencies measured in a rolling aircraft cannot, therefore, be used directly to calculate the longitudinal and directional stability of the nonrolling aircraft.

3. Missiles rolling rapidly may be stabilized by a fin in only one plane, provided that the frequency of the rolling motion is greater than the natural frequency of the oscillation of the nonrolling missile in the plane in which the fin produces stabilizing moments and provided that the instability in the other plane is not too great.

Langley Memorial Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Field, Va., March 25, 1948

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1. Draper, C. S., and Schliestett, G. V.: General Principles of Instrument Analysis. Instruments, vol. 12, no. 5, May 1939, pp. 137-142.

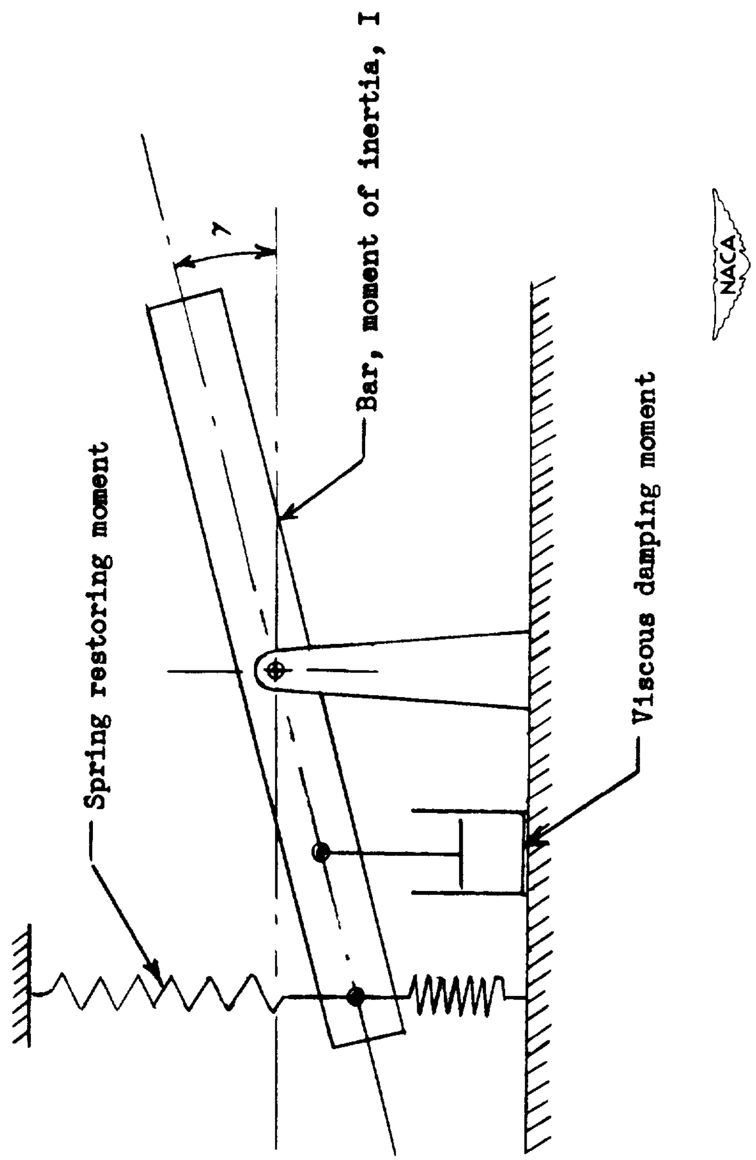


Figure 1.- Sketch of single-degree-of-freedom system.

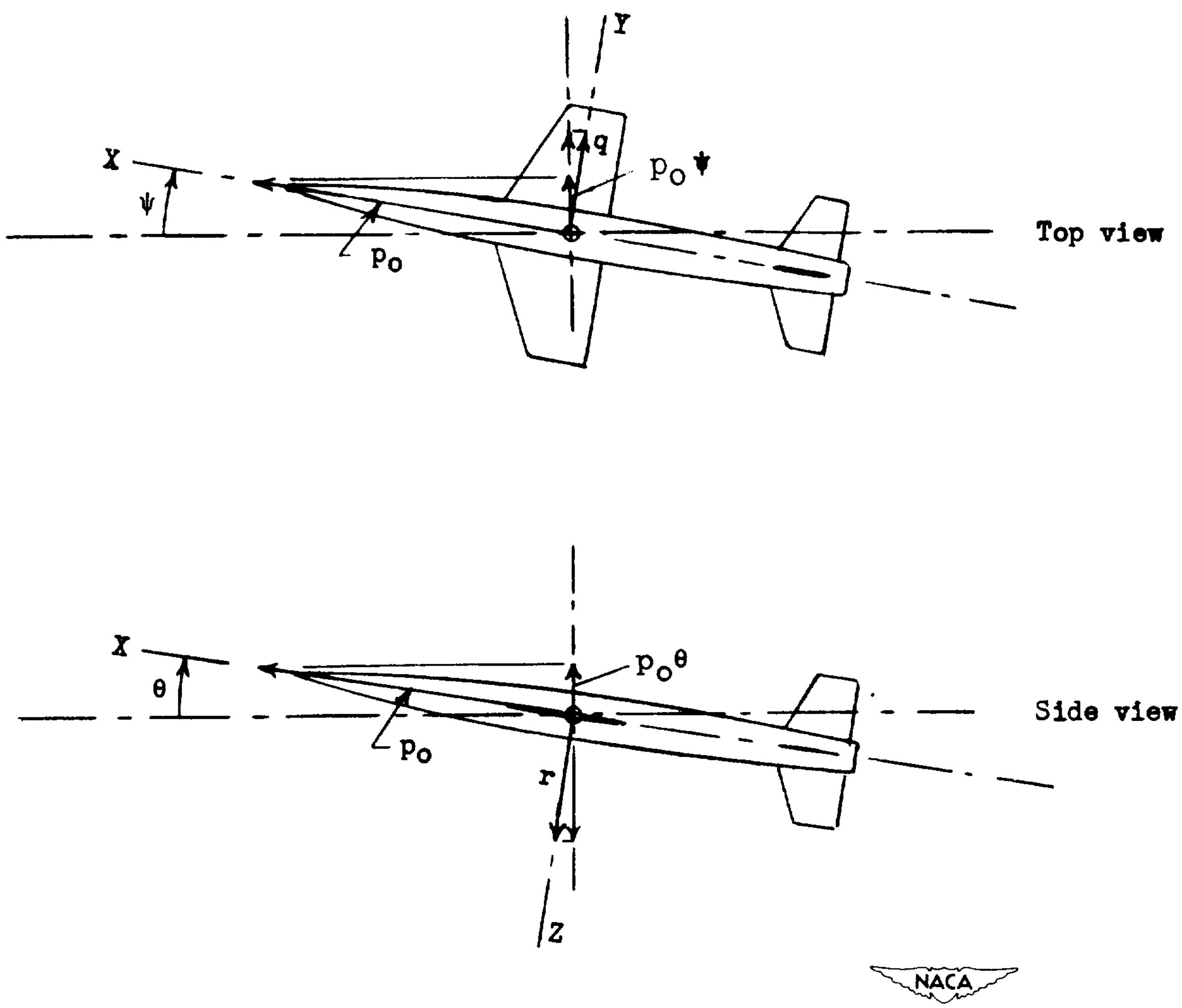


Figure 2.- Relations between angles of pitch and yaw and angular velocities about body axes and flight-path axes. (Views perpendicular to flight path.)

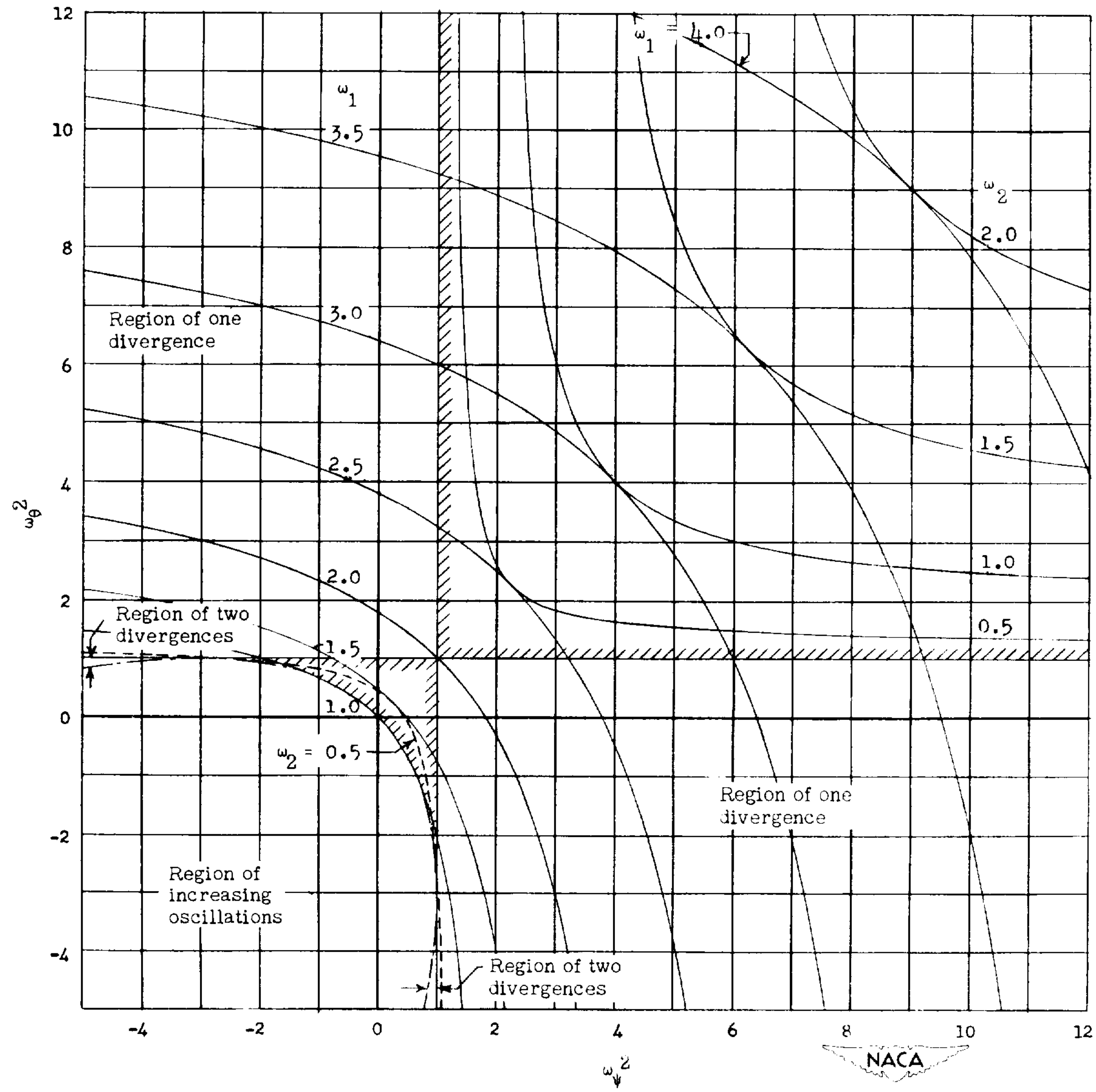


Figure 3.- Contour lines of nondimensional oscillation frequencies of rolling aircraft on a plot of ω_θ^2 against ω_ψ^2 for the case $I_X = 0$, $\zeta_\theta = \zeta_\psi = 0$. Regions of diagram free from divergence or increasing oscillations indicated by cross-hatching.

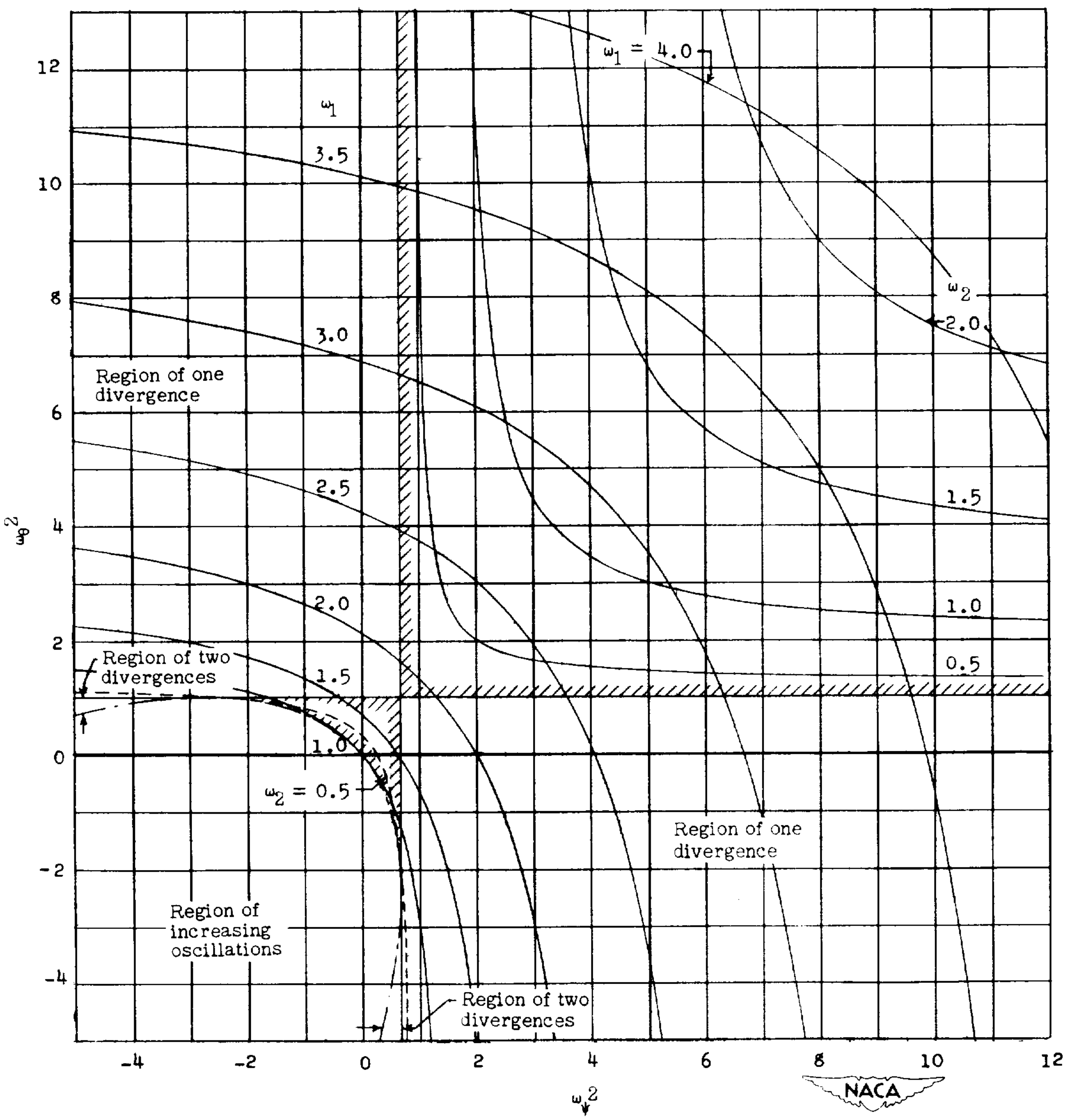


Figure 4.- Contour lines of nondimensional oscillation frequencies of rolling aircraft on a plot of ω_θ^2 against ω_ψ^2 for the case $I_X = 0.2 I_Y$, $\zeta_\theta = \zeta_\psi = 0$. Regions of diagram free from divergence or increasing oscillations indicated by cross-hatching.

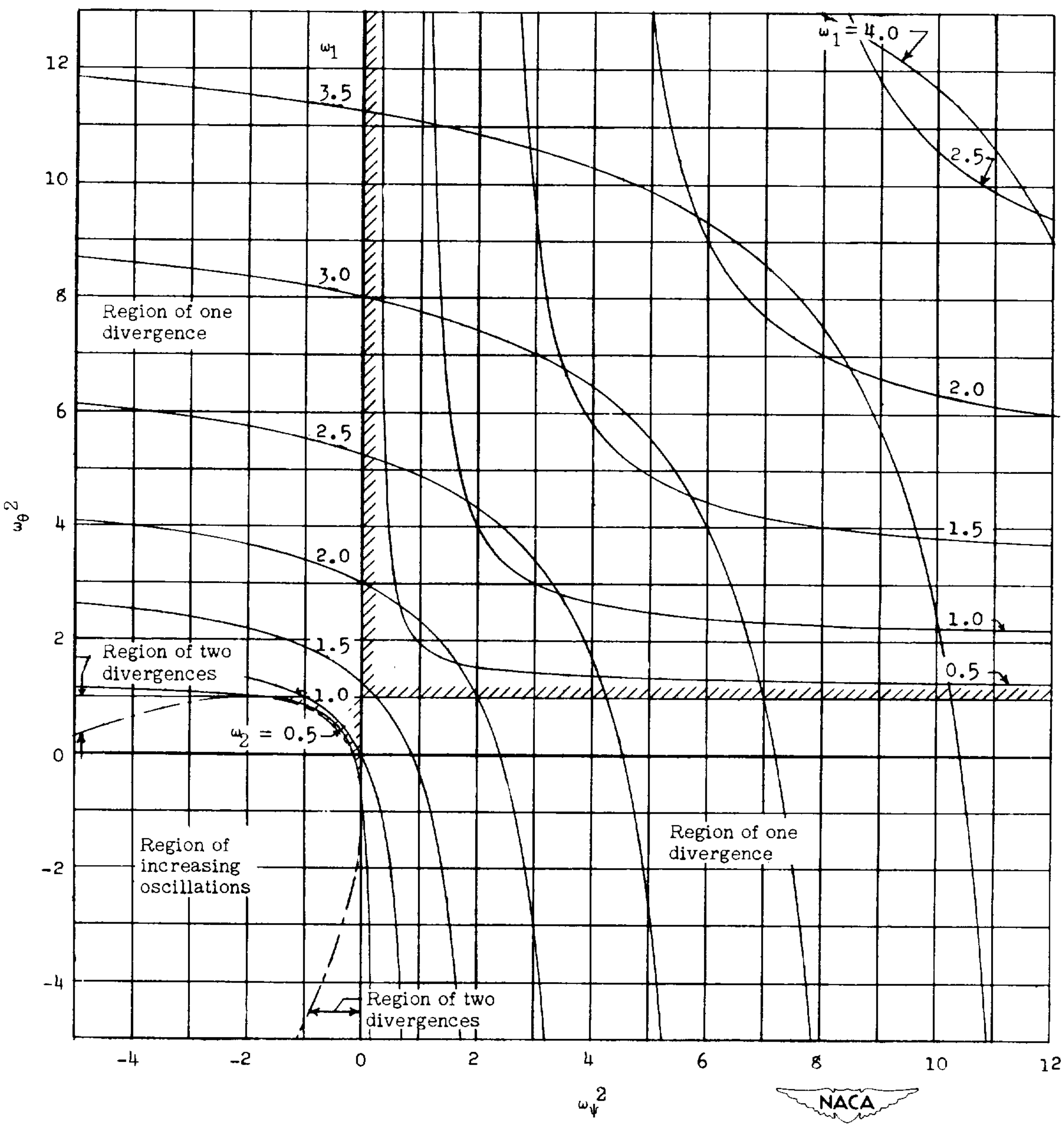
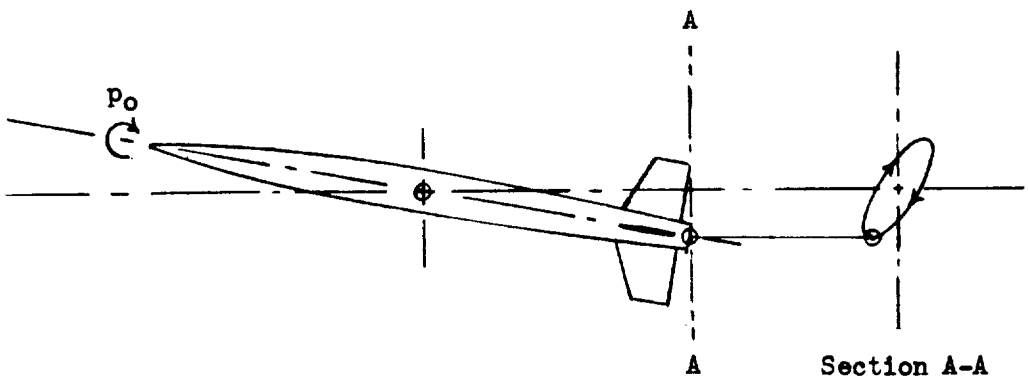
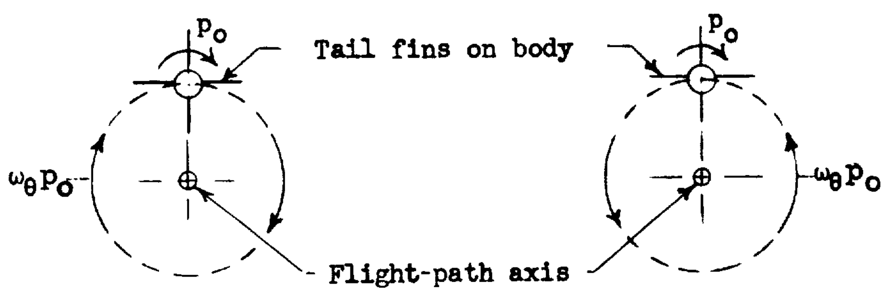


Figure 5.- Contour lines of nondimensional oscillation frequencies of rolling aircraft on a plot of ω_θ^2 against ω_ψ^2 for the case $I_X = I_Y$, $\zeta_\theta = \zeta_\psi = 0$. Regions of diagram free from divergence or increasing oscillations indicated by cross-hatching.

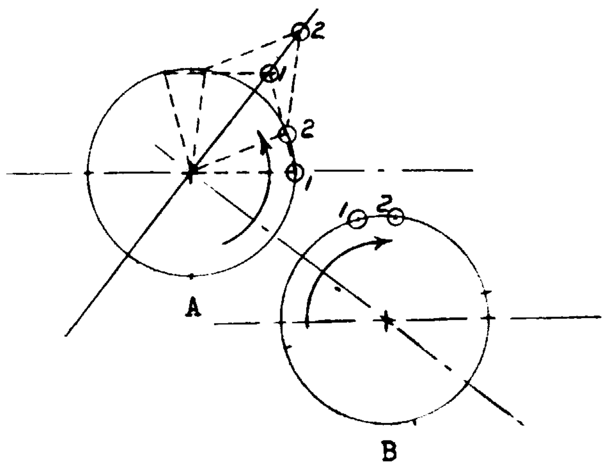


(a) Symmetrical rolling body stabilized by fins, with weight distributed along the longitudinal axis. Sectional view illustrates general motion of a point on the axis.

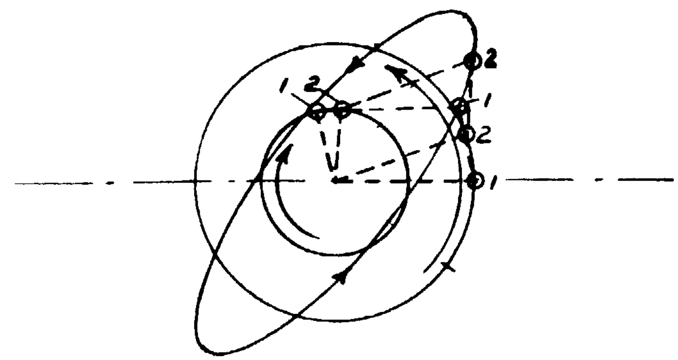


(b) Sectional views through tail fins of symmetrical rolling body, showing the two types of motion which lead to constant amplitude yawing and pitching oscillations with respect to body axes.

Figure 6.- Motion of a symmetrical rolling body stabilized by fins, with weight distributed along the longitudinal axis.



(c) Generation of motion on a tilted straight line by combination of two circular motions with the same frequency and amplitude, opposite direction and arbitrary phase difference. Circle B is shown displaced from circle A for clarity.



(d) Generation of motion on an ellipse with tilted axis by combination of two circular motions with the same frequency, unequal amplitude, opposite direction, and arbitrary phase difference.

Figure 6.- Concluded.

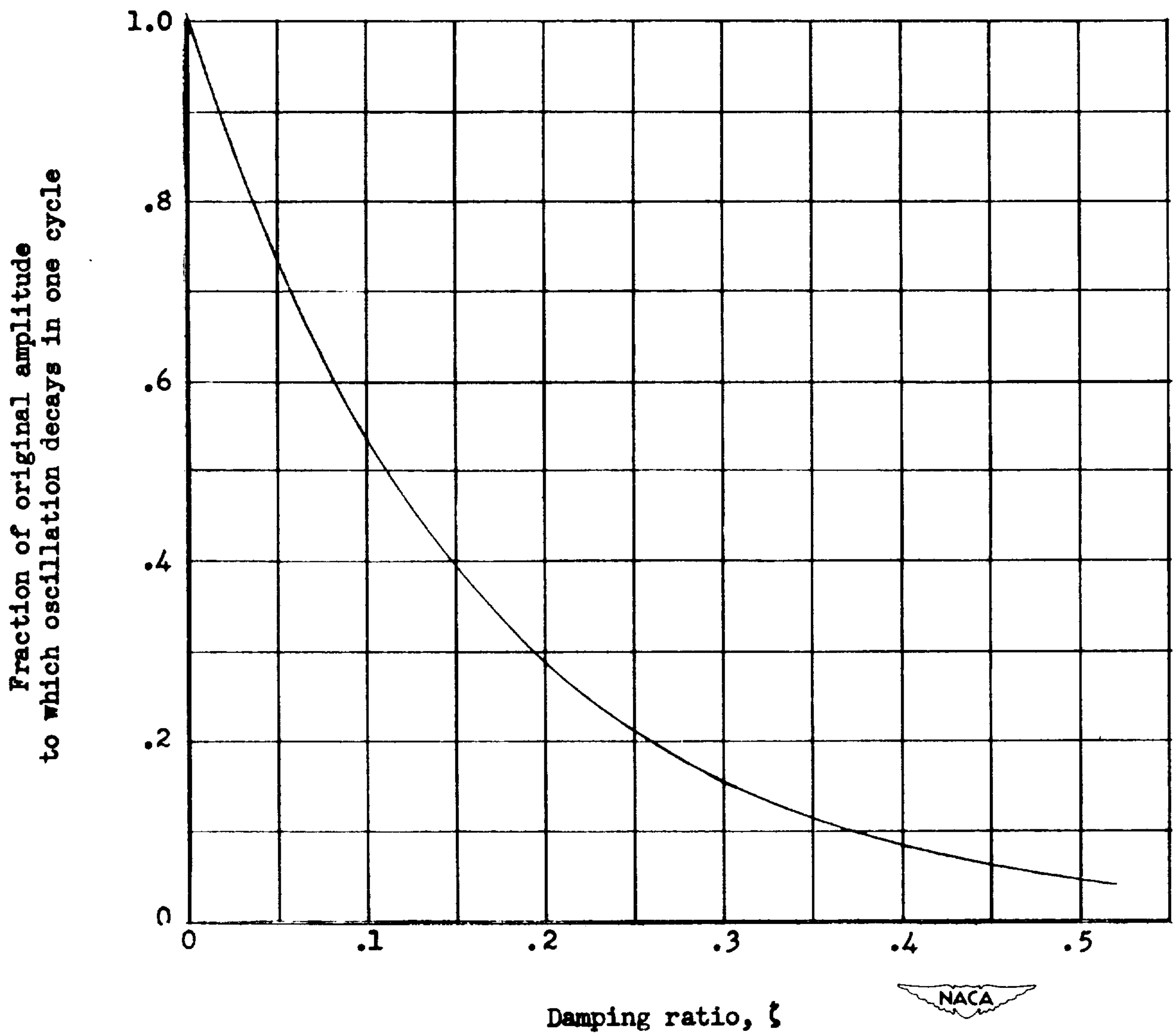


Figure 7.- Fraction of original amplitude to which oscillation decays in one cycle as a function of the damping ratio ζ.

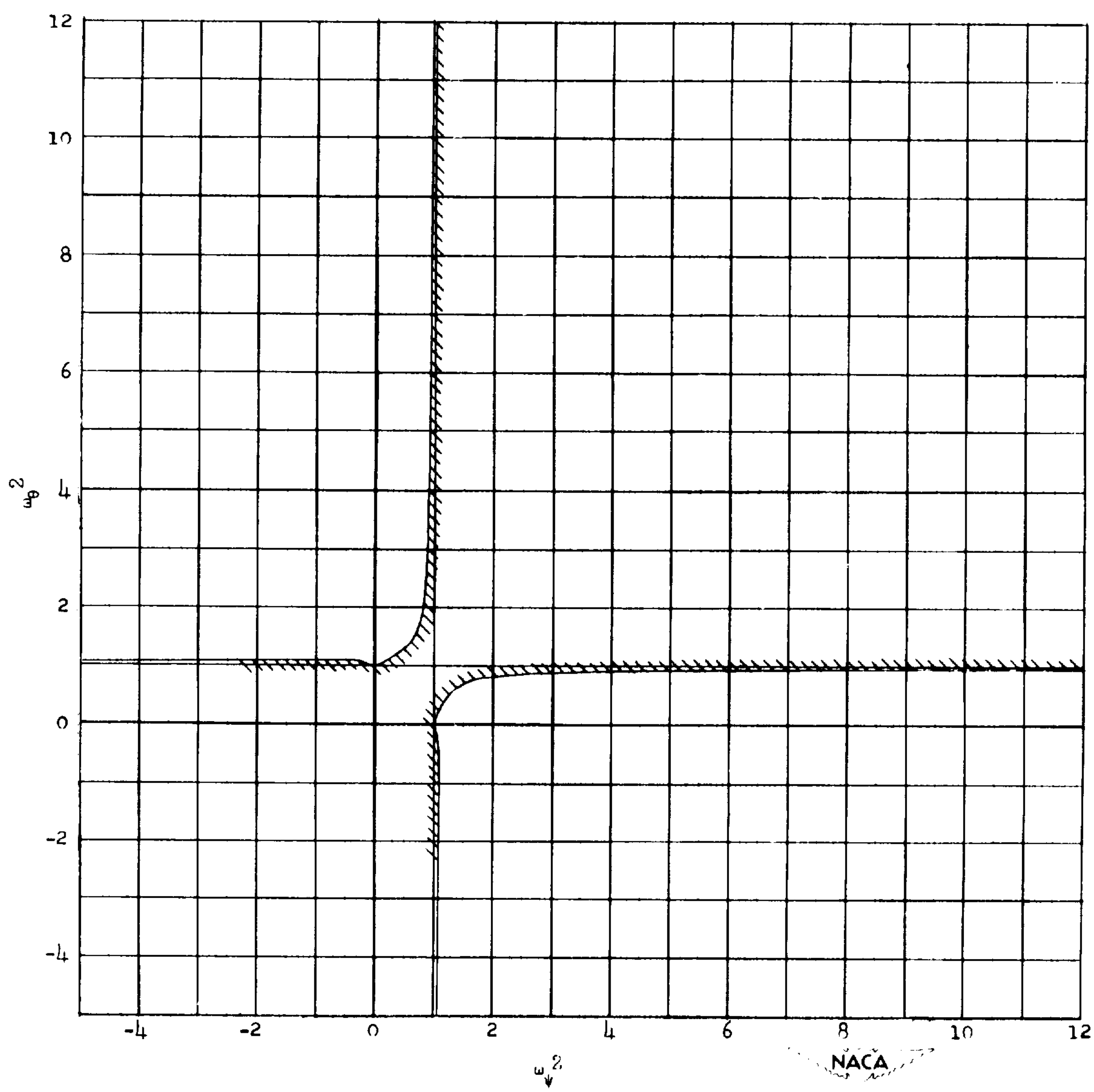


Figure 8.- Divergence boundaries on a plot of ω_θ^2 against ω_ψ^2 for the case $I_X = 0$, $\zeta_\theta = \zeta_\psi = 0.2$. Stable side of boundaries indicated by cross-hatching. Comparison with figure 3 shows effect of increased damping.

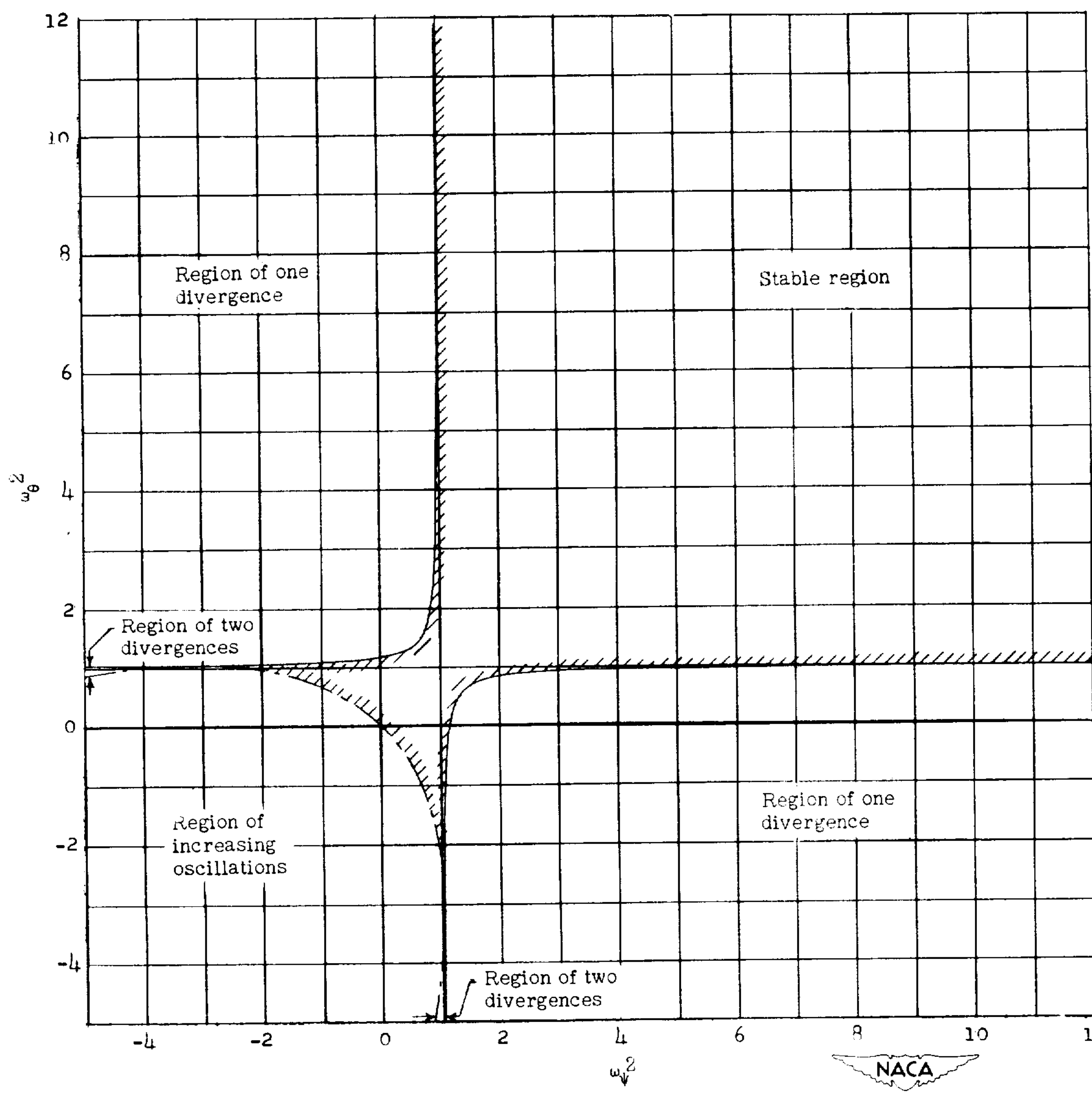
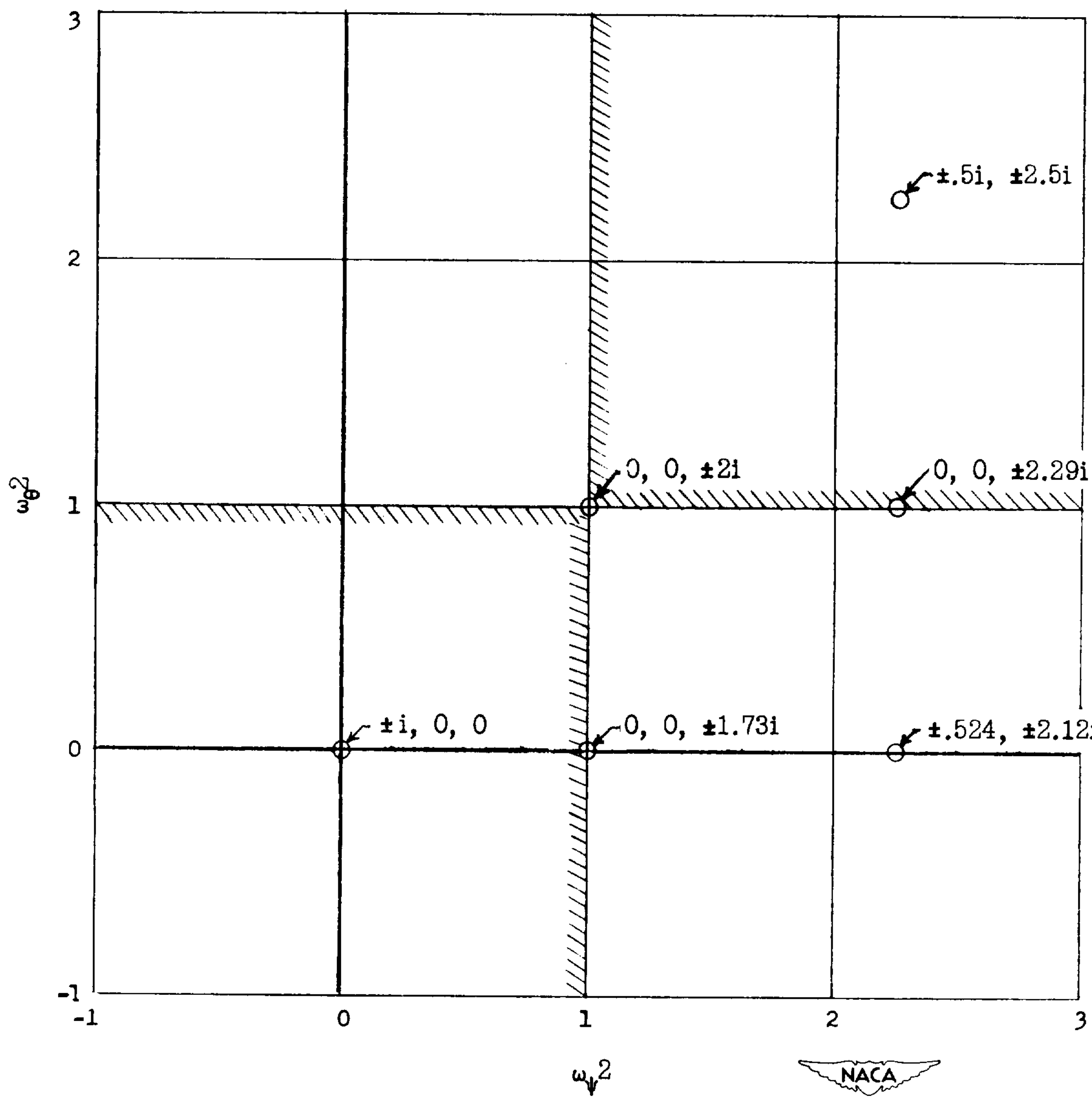
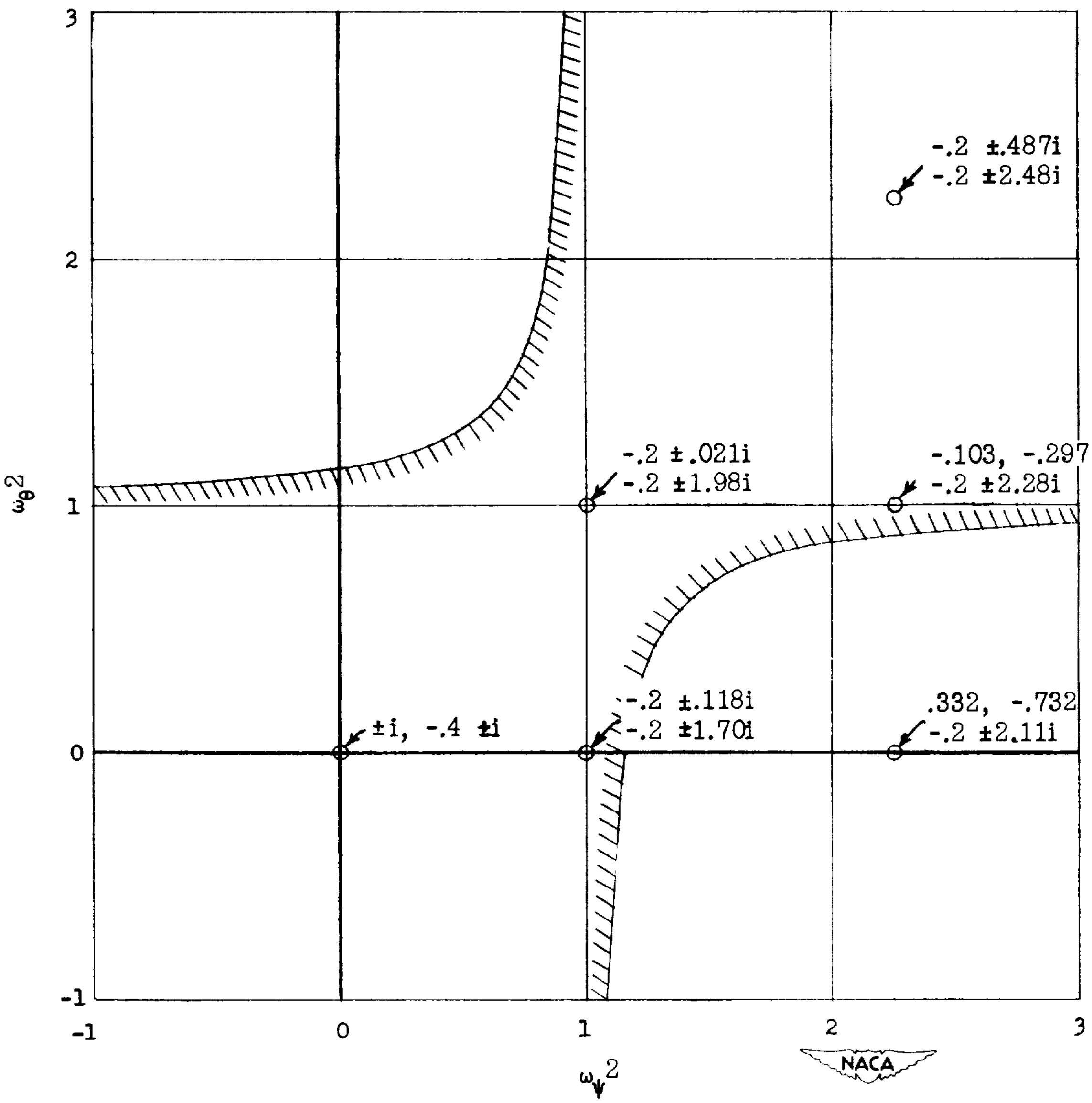


Figure 9.- Stability boundaries on a plot of ω_θ^2 against ω_ψ^2 for the case $I_X = 0$, $\zeta_\theta\omega_\theta = 0.2$, and $\zeta_\psi\omega_\psi = 0.2$. Stable side of boundaries indicated by cross-hatching.



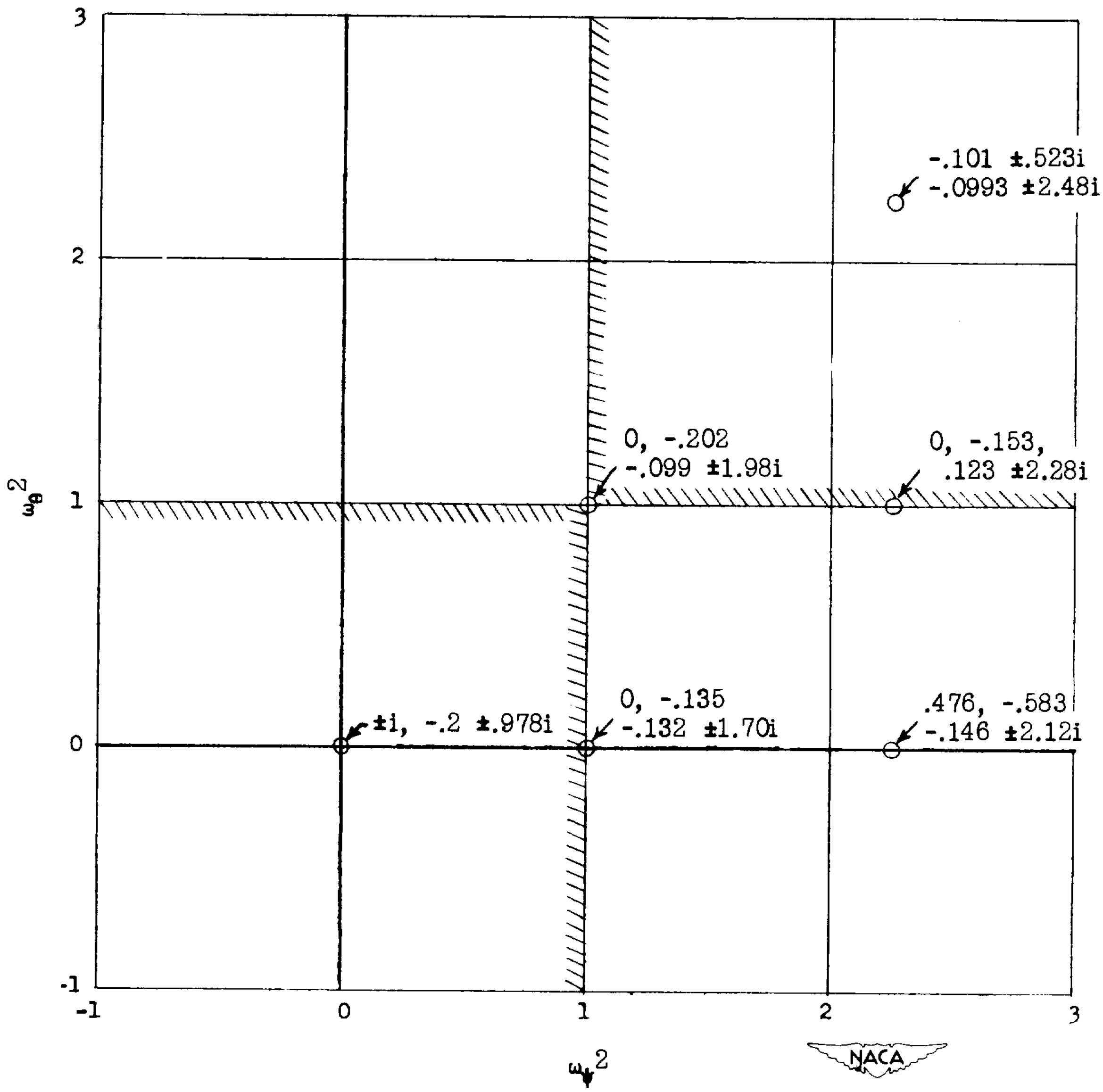
(a) $\zeta_\theta\omega_\theta = \zeta_\psi\omega_\psi = 0.$

Figure 10.- Roots of the stability equation for various combinations of frequency and damping. $I_X = 0.$



(b) $\zeta_\theta\omega_\theta = \zeta_\psi\omega_\psi = 0.2.$

Figure 10.- Continued.



(c) $\zeta_\theta \omega_\theta = 0$, $\zeta_\psi \omega_\psi = 0.2$ or vice versa.

Figure 10.- Concluded.

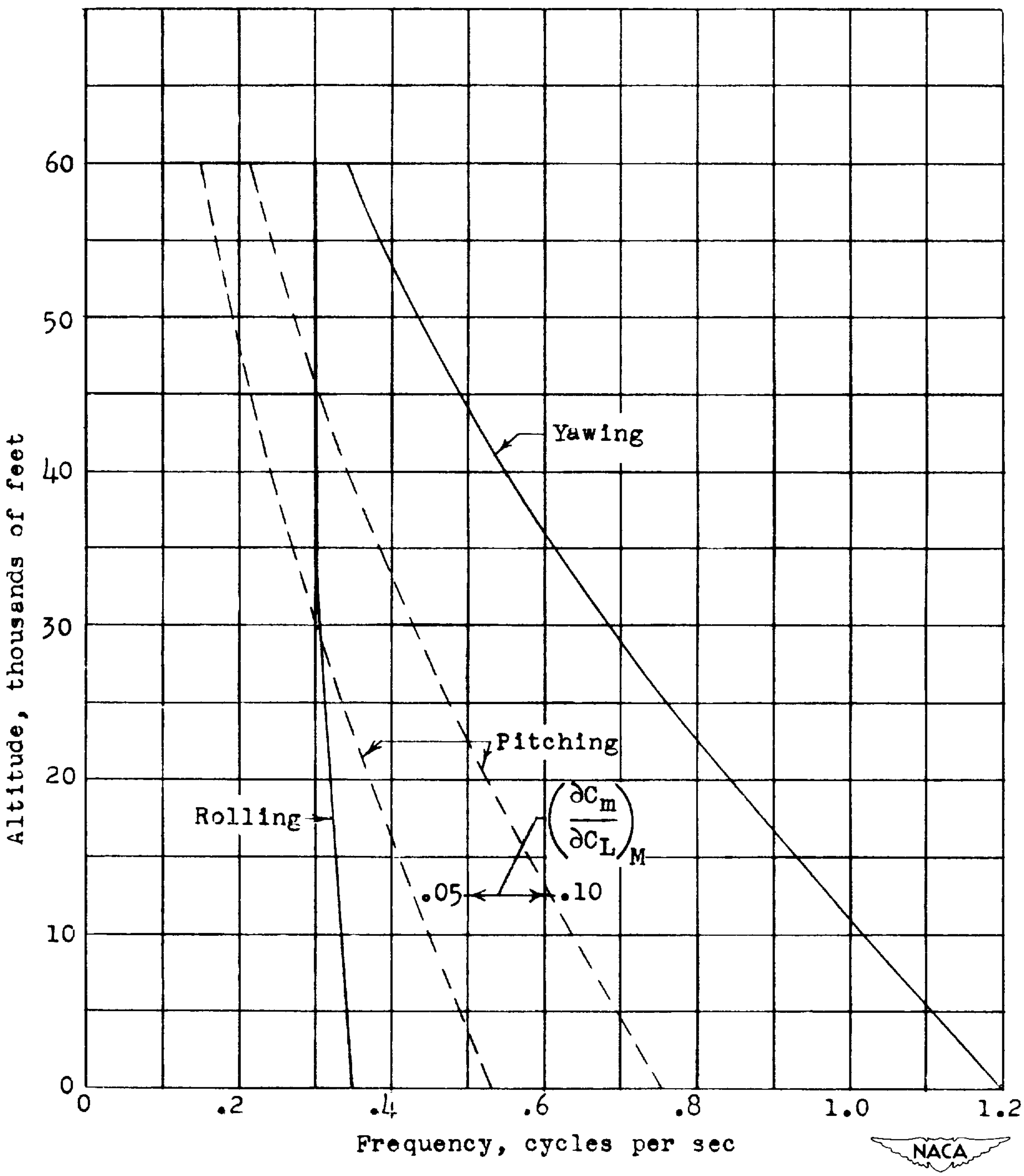


Figure 11.- Yawing, pitching, and rolling frequencies as a function of altitude for an existing transonic research airplane. $\frac{pb}{2V} = 0.05$; $M = 0.8$.