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BUCKLING OF A LONG SQUARE TUBE IN TORSION AND COMPRESSION

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SUMMARY

The buckling of an infinitely long square tube under combined torsion and compression is investigated by means of an exact energy method utilizing Lagrangian multipliers. An interaction curve is obtained from which it is possible to determine the amount of one loading required to produce buckling when a given amount of the other loading is present.

INTRODUCTION

The local buckling of a long, thin-wall, square tube subjected to a combination of torsion and longitudinal compression is investigated theoretically in the present paper. The walls of the tube are considered to be infinitely long flat plates continuous over nondeflecting line supports at the corners. An exact theoretical analysis of the problem by means of the Lagrangian multiplier method, presented in detail in the appendix, is used to derive an interaction curve which gives the combinations of torsion and compression required to produce local buckling.

SYMBOLS

b	width of tube wall
t	wall thickness
E	Young's modulus
μ	Poisson's ratio
D	plate stiffness in bending $\left(\frac{Et^3}{12(1 - \mu^2)}\right)$
σ	compressive stress

τ	shear stress
T	torque
k_c	compressive-stress coefficient $\left(\sigma \dfrac{b^2t}{\pi^2D}\right)$
k_s	shear-stress coefficient $\left(\tau \dfrac{b^2t}{\pi^2D}\right)$
R_s	ratio of shear stress present to critical shear stress for pure torsion
R_c	ratio of compressive stress present to critical compressive stress for pure compression
x	plate coordinate parallel to length
y	plate coordinate parallel to width
w	deflection normal to plane of plate
a_m, b_m, c_m, d_m	Fourier coefficients
α', η'	Lagrangian multipliers
λ	half wave length
$\beta = \dfrac{\lambda}{b}$	
V	internal bending energy
T_c	external work of compressive stress
T_s	external work of shear stress
m, j	integers

RESULTS AND DISCUSSION

The critical combination of compressive and shear stress of a long square tube (see fig. 1) is given by the formulas

$$\sigma = k_c \frac{\pi^2 D}{b^2 t}$$

and

$$\tau = k_s \frac{\pi^2 D}{b^2 t}$$

where τ is related to the total torque T by the formula

$$\tau = \frac{T}{2b^2t}$$

An interaction curve for the critical combinations of the stress coefficients k_c and k_s is presented in figure 2; the theoretical analysis from which this curve is obtained is given in the appendix.

Essentially the interaction curve is a curvilinear portion joined to a straight portion; it is of interest to compare this curve with the interaction curve for an isolated simply supported plate also shown in figure 2 (see reference 1). The disparity in behavior of the two curves may be explained in terms of nodal patterns. In the case of the isolated plate, the inclination as well as the spacing of the transverse nodal lines adjust themselves so as to cause the buckling stress to be a minimum. Thus, as relatively more compression is applied, the nodal lines become less inclined, until, for the case of pure compression, they are straight and perpendicular to the edges of the plate. However, in the case of the square tube, the inclination and spacing of the nodal lines are always constrained to be such that a nodal line be continuous all the way around the tube. But, as relatively more compression is applied, at a certain ratio of compressive to shear stress, the nodal lines suddenly cease to be inclined and become straight and perpendicular to the tube corners (and thus still remain continuous around the tube). Buckling with this type of nodal pattern corresponds to the straight portion of the interaction curve. The closeness of the two curves in the shear-predominating range is due to the fact that the nodal pattern for the isolated simply supported plate in pure shear happens to be one that would very nearly be continuous around a square tube made up of four such plates.

The interaction curve for the square tube is shown in stress-ratio form in figure 3 and is compared with a parabolic interaction curve that is shown in reference 1 to hold very closely for an isolated plate having equal elastic restraint of any magnitude along the edges. The comparison shows clearly that, at least in the compression-predominating range, it would be unduly conservative to consider the walls of the tube to behave as isolated elastically restrained plates.

CONCLUDING REMARKS

A theoretically computed interaction curve for the buckling of an infinitely long square tube in torsion and compression is presented.

From this curve it can be concluded that an appreciable amount of torsion may be present without in any way reducing the compression required for buckling.

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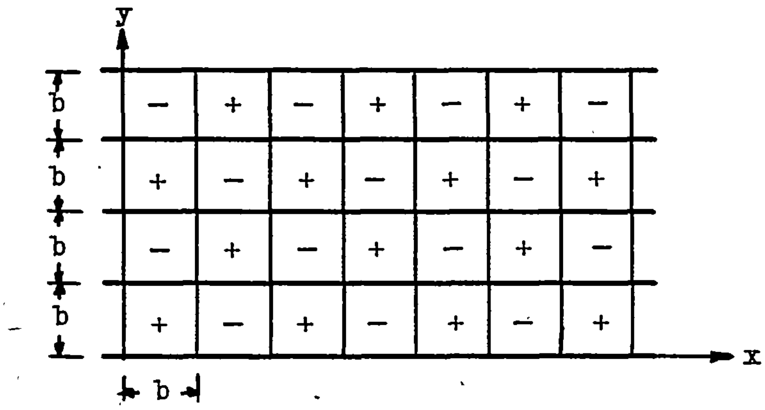
APPENDIX

THEORETICAL ANALYSIS

For the purpose of the present analysis, the square tube is idealized into four infinitely long flat plates continuous over non-deflecting line supports. Evidently two distinct types of buckle pattern must be considered. The first type (see sketch) given by

$$w = \sin \frac{\pi x}{b} \sin \frac{\pi y}{b}$$

represents the case in which the nodal lines are straight and do not advance longitudinally as they proceed around the tube:

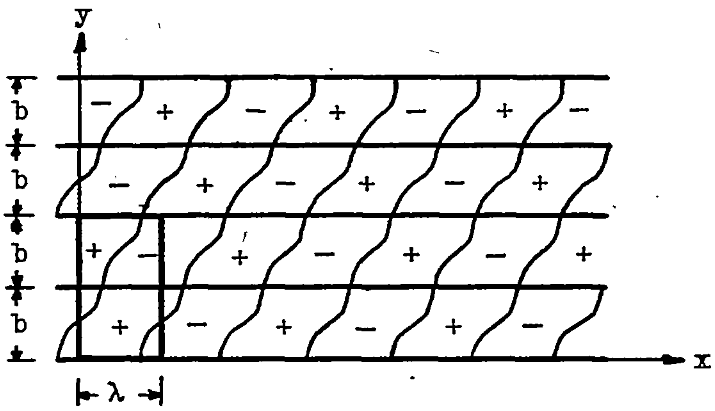


For this buckling mode, the tube will buckle only if

$$\sigma = 4 \frac{\pi^2 D}{b^2 t}$$

This type of buckling is represented by the straight portion of the interaction curve (see figs. 2 and 3); for this type of buckling, the shear stresses do no external work.

In the second type of buckling deformation, the nodal lines are inclined and advance longitudinally exactly one full wave length as they traverse the four walls of the tube as shown in the following sketch:



From the general remarks concerning Fourier series in appendix B of reference 2, it is seen that in the region $(0,\lambda)$, $(0,2b)$ the deflection may be given by

$$w = \sin\frac{\pi x}{\lambda}\left(\sum_{m=1,3,5,\ldots}^{\infty} a_m \sin\frac{m\pi y}{2b} + \sum_{m=1,3,5,\ldots}^{\infty} b_m \cos\frac{m\pi y}{2b}\right)$$

$$+ \cos\frac{\pi x}{\lambda}\left(\sum_{m=1,3,5,\ldots}^{\infty} c_m \sin\frac{m\pi y}{2b} + \sum_{m=1,3,5,\ldots}^{\infty} d_m \cos\frac{m\pi y}{2b}\right) \tag{1}$$

(Single, rather than double, Fourier series are required because the deflection is sinusoidal along any line in the infinite direction.)

This function satisfies the requirement that

$$w(x,y) = -w(x,y+2b)$$

Additional conditions that must be satisfied are

$$w(x,y) = w(\lambda-x,b-y) \tag{2}$$

and

$$w(x,y) = -w(\lambda-x,3b-y) \tag{3}$$

Equations (2) and (3) are, respectively, conditions of symmetry in the region $(0,\lambda)$, $(0,b)$ and antisymmetry in the region $(0,\lambda)$, $(b,2b)$. These conditions are fulfilled by making $b_m = a_m(-1)^{\frac{m-1}{2}}$ and $d_m = -c_m(-1)^{\frac{m-1}{2}}$. Thus the deflection function becomes

$$w = \sin\frac{\pi x}{\lambda}\sum_{m=1,3,5,\ldots}^{\infty} a_m\left[\sin\frac{m\pi y}{2b} + (-1)^{\frac{m-1}{2}}\cos\frac{m\pi y}{2b}\right]$$

$$+ \cos\frac{\pi x}{\lambda}\sum_{m=1,3,5,\ldots}^{\infty} c_m\left[\sin\frac{m\pi y}{2b} - (-1)^{\frac{m-1}{2}}\cos\frac{m\pi y}{2b}\right] \tag{4}$$

This function will be used in an exact stability analysis by the Lagrangian multiplier method as described in reference 2.

The boundary conditions of zero deflection along the idealized supports (corners)

$$w(x,0) = w(x,b) = 0$$

lead to the constraining relationships

$$\sum_{m=1,3,5,\ldots}^{\infty} a_m(-1)^{\frac{m-1}{2}} = 0 \tag{5}$$

and

$$\sum_{m=1,3,5,\ldots}^{\infty} c_m(-1)^{\frac{m-1}{2}} = 0 \tag{6}$$

The internal energy V and the external work of the stresses T_c and T_s are given by the expressions

$$V = \frac{D}{2}\int_0^{\lambda}\int_0^{2b}\left\{\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\mu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\right]\right\} dx\,dy$$

$$T_c = \frac{\sigma t}{2}\int_0^{\lambda}\int_0^{2b}\left(\frac{\partial w}{\partial x}\right)^2 dx\,dy$$

$$T_s = -\tau t\int_0^{\lambda}\int_0^{2b}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\,dx\,dy$$

Substituting the Fourier expansion of w (equation (4)) into the energy expressions yields

$$V = \frac{\pi^4 D}{32b^2\beta^3}\sum_{m=1,3,5,\ldots}^{\infty}\left(4 + m^2\beta^2\right)^2\left(a_m^2 + c_m^2\right) \tag{7}$$

$$T_c = \frac{k_c\pi^4 D}{2b^2\beta}\sum_{m=1,3,5,\ldots}^{\infty}\left(a_m^2 + c_m^2\right) \tag{8}$$

and

$$T_s = -\frac{k_s\pi^4 D}{b^2}\sum_{m=1,3,5,\ldots}^{\infty} m a_m c_m (-1)^{\frac{m-1}{2}} \tag{9}$$

where

$$\beta = \frac{\lambda}{b}$$

$$k_c = \sigma \frac{b^2 t}{\pi^2 D}$$

and

$$k_s = \tau \frac{b^2 t}{\pi^2 D}$$

The function to be minimized is

$$F = V - T_c - T_s - \alpha' \sum_{m=1,3,5,\ldots}^{\infty} (-1)^{\frac{m-1}{2}} a_m - \eta' \sum_{m=1,3,5,\ldots}^{\infty} (-1)^{\frac{m-1}{2}} c_m \tag{10}$$

Minimizing equation (10) with respect to the coefficients a_j and c_j yields

$$\frac{\partial F}{\partial a_j} = (4 + j^2\beta^2)^2 a_j - 16k_c\beta^2 a_j + 16k_s\beta^3 j(-1)^{\frac{j-1}{2}} c_j - \alpha(-1)^{\frac{j-1}{2}} = 0 \tag{11}$$

and

$$\frac{\partial F}{\partial c_j} = (4 + j^2\beta^2)^2 c_j - 16k_c\beta^2 c_j + 16k_s\beta^3 j(-1)^{\frac{j-1}{2}} a_j - \eta(-1)^{\frac{j-1}{2}} = 0 \tag{12}$$

where

$$j = 1,3,5,\ldots$$

$$\alpha = \alpha' \frac{16b^2\beta^3}{\pi^4 D}$$

and

$$\eta = \eta' \frac{16b^2\beta^3}{\pi^4 D}$$

After simplification, equations (11) and (12) become

$$A_j a_j + B_j c_j = \alpha(-1)^{\frac{j-1}{2}}$$

and

$$B_j a_j + A_j c_j = \eta(-1)^{\frac{j-1}{2}}$$

where

$$A_j = (4 + j^2\beta^2)^2 - 16k_c\beta^2$$

and

$$B_j = 16k_s\beta^3 j(-1)^{\frac{j-1}{2}}$$

Then

$$a_j = \frac{A_j\alpha(-1)^{\frac{j-1}{2}} - B_j\eta(-1)^{\frac{j-1}{2}}}{A_j^2 - B_j^2} \tag{13}$$

and

$$c_j = \frac{A_j\eta(-1)^{\frac{j-1}{2}} - B_j\alpha(-1)^{\frac{j-1}{2}}}{A_j^2 - B_j^2} \tag{14}$$

Substituting for a_j and b_j in the constraining relationships (5) and (6) yields

$$\sum_{m=1,3,5,\ldots}^{\infty} \frac{A_m\alpha - B_m\eta}{A_m^2 - B_m^2} = 0 \tag{15}$$

and

$$\sum_{m=1,3,5,\ldots}^{\infty} \frac{A_m\eta - B_m\alpha}{A_m^2 - B_m^2} = 0 \tag{16}$$

For the Lagrangian multipliers, α and η, to have values other than zero, the condition that must be satisfied is

$$\left(\sum_{m=1,3,5,\ldots}^{\infty} \frac{A_m}{A_m^2 - B_m^2}\right)^2 - \left(\sum_{m=1,3,5,\ldots}^{\infty} \frac{B_m}{A_m^2 - B_m^2}\right)^2 = 0 \tag{17}$$

Equation (17) can be factored into

$$\sum_{m=1,3,5,\ldots}^{\infty} \frac{1}{A_m + B_m} = 0 \tag{18}$$

and

$$\sum_{m=1,3,5,\ldots}^{\infty} \frac{1}{A_m - B_m} = 0 \tag{19}$$

Both equations (18) and (19) will give identical results. The positive or negative sign merely indicates the direction of shear.

Equation (18) may be written

$$0 = \sum_{m=1,3,5,\ldots}^{\infty} \frac{1}{(4 + m^2\beta^2)^2 - 16k_c\beta^2 + 16k_s m\beta^3(-1)^{\frac{m-1}{2}}} \tag{20}$$

The critical combination of compressive-stress coefficient k_c and shear-stress coefficient k_s for an infinitely long square tube can be calculated from equation (20). For a specified value of one of the stress coefficients and wave-length ratio β, the corresponding value of the other stress coefficient that will satisfy equation (20) can be obtained. This procedure is used for several different values of β until a minimum value of the corresponding stress coefficient is obtained. The curvilinear portion of interaction curve was drawn using these minimum values. (See table 1.)

REFERENCES

1. Stowell, Elbridge Z., and Schwartz, Edward B.: Critical Stress for an Infinitely Long Flat Plate with Elastically Restrained Edges under Combined Shear and Direct Stress. NACA ARR No. 3K13, 1943.

2. Budiansky, Bernard, Hu, Pai C., and Connor, Robert W.: Notes on the Lagrangian Multiplier Method in Elastic-Stability Analysis. NACA TN No. 1558, 1948.

TABLE 1

CRITICAL COMBINATIONS OF STRESS COEFFICIENTS AND RATIOS

Compression		Shear	
k_c	R_c	k_s	R_s
0	0	5.343	1
2	.50	3.92	.735
3	.75	3.03	.568
3.8	.95	2.25	.422
4	1	2.00	.374

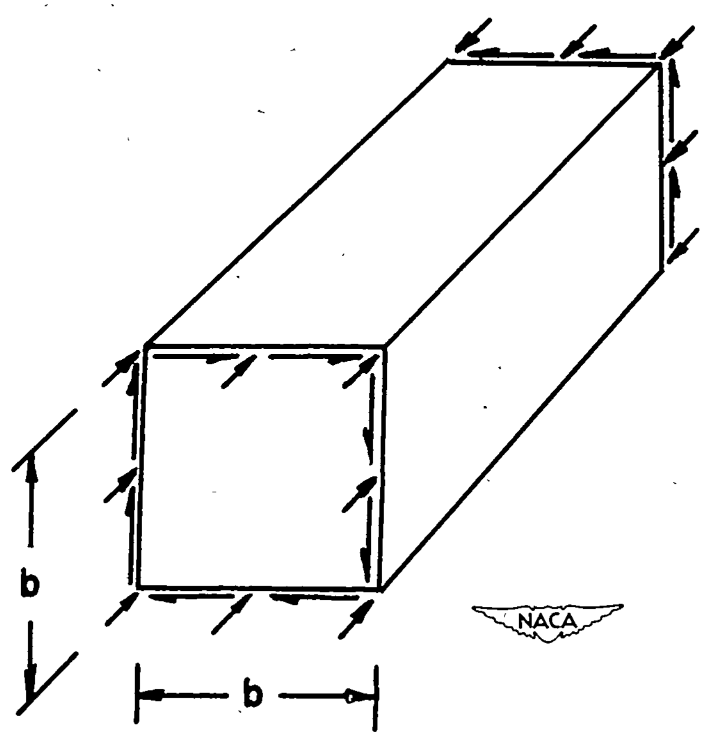


Figure 1.- Long square tube in torsion and compression.

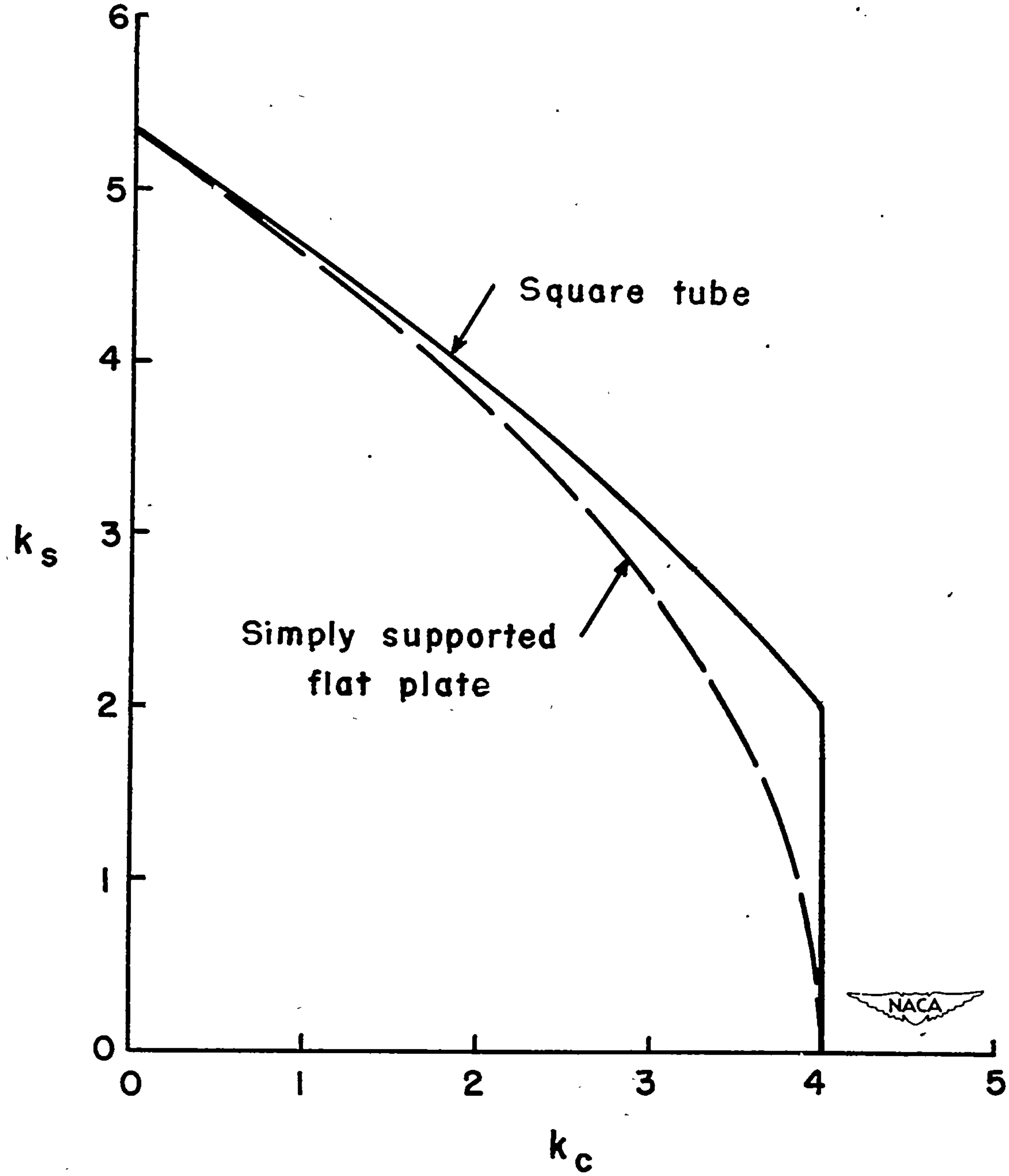


Figure 2.- Critical combinations of shear-stress and compressive-stress coefficients for buckling of a long square tube in torsion and compression.

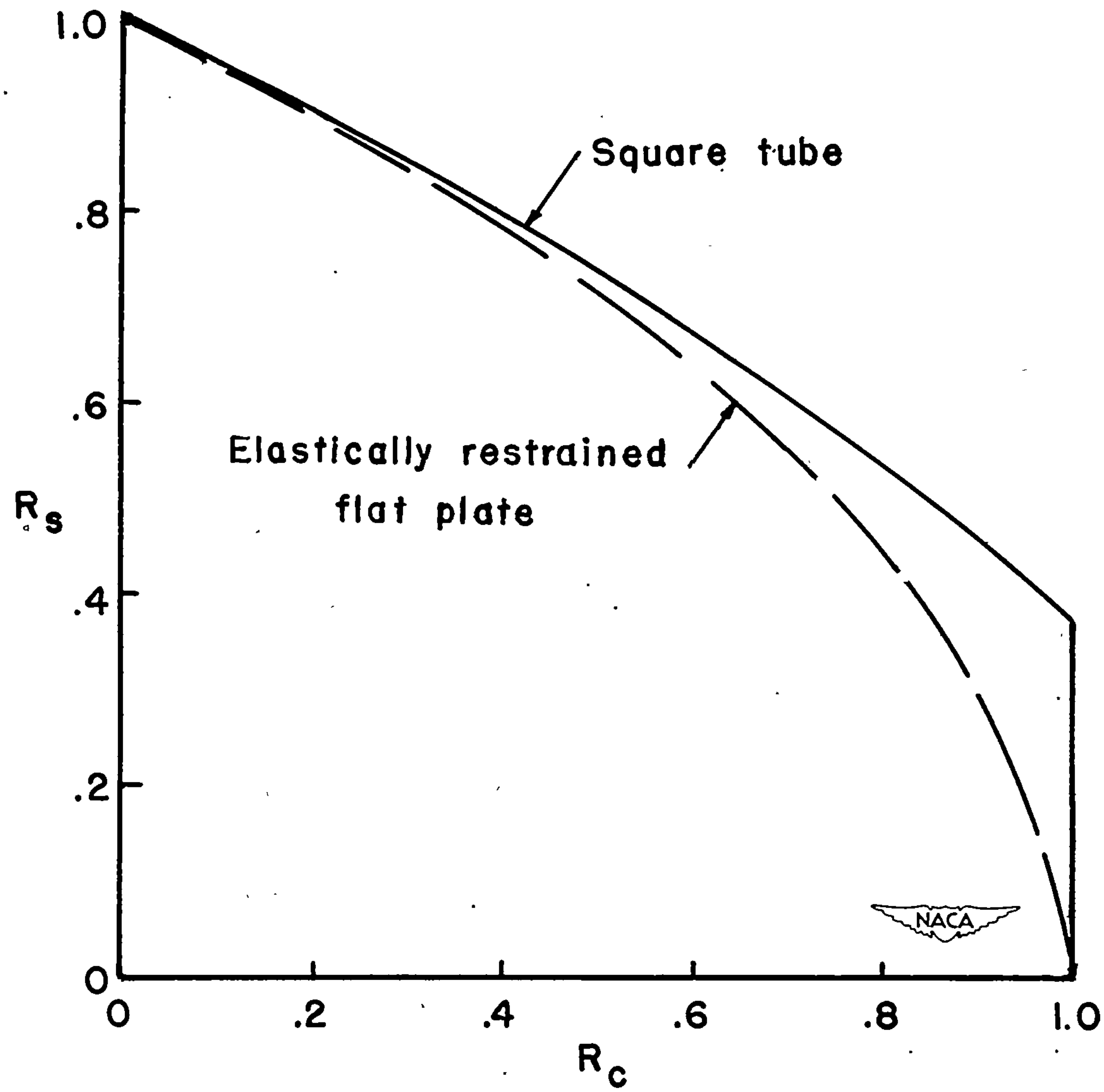


Figure 3.- Interaction curve in stress-ratio form for buckling of a long square tube in torsion and compression.