NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE

No. 1805

REMARKS CONCERNING THE BEHAVIOR OF THE LAMINAR BOUNDARY LAYER IN COMPRESSIBLE FLOWS

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Washington
January 1949

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SUMMARY

The boundary-layer equation of motion and the boundary-layer energy equation for the compressible and steady laminar boundary layer on two-dimensional bodies and bodies of revolution are written in a nondimensional form to provide a clearer indication of the effects of Mach number, Reynolds number, and the properties of the gas.

When the ratio of the local velocity to the free-stream velocity and the ratio of local temperature to the free-stream temperature at all points on the surface of a body and at the outer edge of its boundary layer do not change with Reynolds number and when the Mach number and the physical properties of the gas also do not change with Reynolds number, the boundary-layer thickness at a fixed point on a body is inversely proportional to the square root of the Reynolds number, the surface-friction coefficient at a fixed point is inversely proportional to the square root of the Reynolds number, the friction-drag coefficient of the part of a body covered by a laminar boundary layer is inversely proportional to the square root of the Reynolds number, the separation point is independent of the Reynolds number, and the nondimensional velocity profile is invariable at a fixed fraction of the body length from the stagnation point. By use of the boundary-layer equations, separation of the laminar boundary layer is shown to occur only when the static pressure along the surface rises in the direction of flow.

INTRODUCTION

Some useful results of the Prandtl boundary-layer theory for the incompressible laminar boundary layer are known to be obtainable directly from the form of the boundary-layer equations without having to solve them (reference 1). The boundary-layer thickness and friction drag are found to be inversely proportional to the square root of the Reynolds number, and the nondimensional velocity profile at a fixed point on a body and the separation point are found to be independent of the Reynolds number. These theoretical results are true when the pressure distribution on a body is independent of the Reynolds number.

Because of the increased significance of the laminar boundary-layer for flows in which the effects of compressibility and heating are important, it seemed desirable to determine whether conclusions similar to those for incompressible flow can be drawn from the form of the boundary-layer equations for high-speed flows. An investigation to determine whether separation of the boundary layer can occur when the pressure along the surface does not rise in the direction of the flow also seemed desirable.

Although some of the results obtained are implicit in the work of Von Kármán and Tsien (reference 2) and in the work of others, it was thought worthwhile to develop the results both for two-dimensional flow and for axially symmetric flow over a body of revolution, and to state them explicitly together with the conditions for which they are valid. The results are probably of most interest to experimentalists who require a knowledge of boundary-layer behavior, but who have not had the opportunity to develop these results for themselves.

SYMBOLS

u	velocity inside boundary layer and parallel to surface
v	velocity inside boundary layer and perpendicular to surface
x	distance along surface
y	distance measured from surface in a direction perpendicular to surface
r	radius of body of revolution
T	temperature
μ	coefficient of viscosity
ρ	density
k	coefficient of heat conduction
c_p	specific heat at constant pressure
l	length of body
U	velocity at outer edge of boundary layer and parallel to surface
U_o	free-stream velocity

R_o Reynolds number $\left(\frac{\rho_o l U_o}{\mu_o}\right)$

M_o Mach number $\left(\frac{U_o}{C_o}\right)$

C_o velocity of sound in free stream

p static pressure

ψ stream function for two-dimensional flow

σ_o Prandtl number $\left(\frac{c_{p_o}\mu_o}{k_o}\right)$

τ_S surface shearing stress

γ ratio of specific heats

$\phi = 1 + \gamma M_o^2 p$

R gas constant

F_x component of body force along x-axis

F_y component of body force along y-axis

Ψ stream function for flow over body of revolution

Subscript:

o free-stream conditions

Quantities which contain a bar and which do not refer to free-stream conditions are dimensional.

ANALYSIS

Two-Dimensional Flow

<u>Derivation of boundary-layer equations</u>.- The steady flow of a gas over a wall in a layer having a thickness which is a negligible fraction of the radius of curvature of the wall is described herein by the Navier-Stokes equations of motion in surface coordinates with the terms that involve surface curvature neglected, the equation of continuity, and the energy equation with $\bar{c}_p$ constant.

(1) The $\bar{x}$ component of the equation of motion is

$$\bar{\rho}\bar{u}\frac{\partial\bar{u}}{\partial\bar{x}} + \bar{\rho}\bar{v}\frac{\partial\bar{u}}{\partial\bar{y}} = \bar{\rho}F_x - \frac{\partial\bar{p}}{\partial\bar{x}} + \frac{4}{3}\bar{\mu}\frac{\partial^2\bar{u}}{\partial\bar{x}^2} + \frac{\bar{\mu}}{3}\frac{\partial^2\bar{v}}{\partial\bar{x}\,\partial\bar{y}} + \bar{\mu}\frac{\partial^2\bar{u}}{\partial\bar{y}^2} + \frac{4}{3}\frac{\partial\bar{\mu}}{\partial\bar{x}}\frac{\partial\bar{u}}{\partial\bar{x}} + \frac{\partial\bar{\mu}}{\partial\bar{y}}\frac{\partial\bar{u}}{\partial\bar{y}} + \frac{\partial\bar{v}}{\partial\bar{x}}\frac{\partial\bar{\mu}}{\partial\bar{y}} - \frac{2}{3}\frac{\partial\bar{\mu}}{\partial\bar{x}}\frac{\partial\bar{v}}{\partial\bar{y}} \tag{1}$$

(2) The $\bar{y}$ component of the equation of motion is

$$\bar{\rho}\bar{u}\frac{\partial\bar{v}}{\partial\bar{x}} + \bar{\rho}\bar{v}\frac{\partial\bar{v}}{\partial\bar{y}} = \bar{\rho}F_y - \frac{\partial\bar{p}}{\partial\bar{y}} + \bar{\mu}\frac{\partial^2\bar{v}}{\partial\bar{x}^2} + \frac{\bar{\mu}}{3}\frac{\partial^2\bar{u}}{\partial\bar{y}\,\partial\bar{x}} + \frac{4}{3}\bar{\mu}\frac{\partial^2\bar{v}}{\partial\bar{y}^2} + \frac{\partial\bar{\mu}}{\partial\bar{x}}\frac{\partial\bar{v}}{\partial\bar{x}} + \frac{4}{3}\frac{\partial\bar{\mu}}{\partial\bar{y}}\frac{\partial\bar{v}}{\partial\bar{y}} + \frac{\partial\bar{\mu}}{\partial\bar{x}}\frac{\partial\bar{u}}{\partial\bar{y}} - \frac{2}{3}\frac{\partial\bar{\mu}}{\partial\bar{y}}\frac{\partial\bar{u}}{\partial\bar{x}} \tag{2}$$

(3) The equation of continuity is

$$\frac{\partial\bar{\rho}\bar{u}}{\partial\bar{x}} + \frac{\partial\bar{\rho}\bar{v}}{\partial\bar{y}} = 0 \tag{3}$$

(4) The energy equation for $\bar{c}_p$ constant is

$$\bar{\rho}\bar{u}\bar{c}_p\frac{\partial\bar{T}}{\partial\bar{x}} + \bar{\rho}\bar{v}\bar{c}_p\frac{\partial\bar{T}}{\partial\bar{y}} - \bar{u}\frac{\partial\bar{p}}{\partial\bar{x}} - \bar{v}\frac{\partial\bar{p}}{\partial\bar{y}} = -\frac{2}{3}\bar{\mu}\left(\frac{\partial\bar{u}}{\partial\bar{x}} + \frac{\partial\bar{v}}{\partial\bar{y}}\right)^2 + 2\bar{\mu}\left[\left(\frac{\partial\bar{u}}{\partial\bar{x}}\right)^2 + \left(\frac{\partial\bar{v}}{\partial\bar{y}}\right)^2\right] + \bar{\mu}\left(\frac{\partial\bar{u}}{\partial\bar{y}} + \frac{\partial\bar{v}}{\partial\bar{x}}\right)^2 + \bar{k}\left(\frac{\partial^2\bar{T}}{\partial\bar{x}^2} + \frac{\partial^2\bar{T}}{\partial\bar{y}^2}\right) + \frac{\partial\bar{k}}{\partial\bar{x}}\frac{\partial\bar{T}}{\partial\bar{x}} + \frac{\partial\bar{k}}{\partial\bar{y}}\frac{\partial\bar{T}}{\partial\bar{y}} \tag{4}$$

The usual assumption of the boundary-layer theory (reference 1) that the flow takes place in a thin layer in which the velocity is almost parallel to the wall and in which the largest viscous terms are of the same order of magnitude as the inertia terms is now made by the following substitutions:

$$\left.\begin{aligned}
\bar{x} &= lx & \bar{\mu} &= \mu\mu_o \\
\bar{y} &= \left(\frac{l}{R_o^{1/2}}\right)y & \bar{\rho} &= \rho\rho_o \\
\bar{u} &= U_o u & \bar{p} - p_o &= \rho_o U_o^2 p \\
\bar{v} &= \left(\frac{U_o}{R_o^{1/2}}\right)v & \bar{c}_p &= c_p c_{p_o} \\
\bar{U} &= U_o U & \bar{k} &= kk_o \\
R_o &= \frac{\rho_o l U_o}{\mu_o} & \bar{T} &= TT_o
\end{aligned}\right\} \tag{5}$$

When substitutions (5) are used in equation (1), one group of terms has the factor $1/R_o$. For large Reynolds numbers this group is neglected and equation (1) becomes

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{6}$$

where the body forces $\bar{\rho}F_x$ are also neglected.

When substitutions (5) are used in equation (2) and all terms containing the factor $1/R_o$ or $1/R_o^2$ are neglected and the body forces $\bar{\rho}F_y$ are neglected, the result is

$$0 = \frac{\partial p}{\partial y} \tag{7}$$

The use of substitutions (5) in equation (3) results in

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \tag{8}$$

When equation (7) and substitutions (5) are used in equation (4) and terms containing the factors $\frac{(\gamma - 1)M_o^2}{R_o}$, $\frac{1}{\sigma_o R_o}$, and $\frac{(\gamma - 1)M_o^2}{R_o^2}$ are neglected, equation (4) becomes

$$\rho u \frac{\partial T}{\partial x} + \rho v \frac{\partial T}{\partial y} = (\gamma - 1)M_o^2 \left[u \frac{\partial p}{\partial x} + \mu\left(\frac{\partial u}{\partial y}\right)^2\right] + \frac{1}{\sigma_o}\frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) \tag{9}$$

Equations (6), (7), (8), and (9), together with the equation of state for a gas and relations between μ, k, and T, describe the flow in the boundary layer. Because of equation (7) the static pressure in the boundary layer is a function only of x; therefore, $\partial p/\partial x$ in equations (6) and (9) can be replaced by dp/dx.

A nondimensional stream function ψ is then introduced, where

$$\left.\begin{aligned} u &= \frac{1}{\rho}\frac{\partial \psi}{\partial y} \\ v &= -\frac{1}{\rho}\frac{\partial \psi}{\partial x} \end{aligned}\right\} \tag{10}$$

The equation of continuity (equation (8)) is automatically satisfied. Equation (6) becomes

$$\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial \psi}{\partial y}\right) - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\left(\frac{1}{\rho}\frac{\partial \psi}{\partial y}\right) = -\frac{dp}{dx} + \frac{\partial}{\partial y}\left[\mu \frac{\partial}{\partial y}\left(\frac{1}{\rho}\frac{\partial \psi}{\partial y}\right)\right] \tag{11}$$

and equation (9) becomes

$$\frac{\partial \psi}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial T}{\partial y} = (\gamma - 1)M_o^2 \left\{\frac{1}{\rho}\frac{\partial \psi}{\partial y}\frac{dp}{dx} + \mu\left[\frac{\partial}{\partial y}\left(\frac{1}{\rho}\frac{\partial \psi}{\partial y}\right)\right]^2\right\} + \frac{1}{\sigma_o}\frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) \tag{12}$$

The relative density ρ can be replaced by a function of T, M_o, γ, and p by the following development. The static pressure in the boundary layer is a function only of x; therefore, the density at a point in the boundary layer depends only on the temperature at the point and on the static pressure at the edge of the boundary layer. Then, from the perfect gas law

$$\rho = \frac{\bar{p}}{p_o}\frac{1}{T} \tag{13}$$

where $\overline{p}$ is the static pressure at the boundary-layer edge. From the definition

$$\overline{p} - p_o = \rho_o U_o^2 p$$

the perfect gas law

$$p_o = \rho_o R T_o$$

and the expression

$$c_{p_o}(\gamma - 1)T_o = C_o^2$$

the following relation can be obtained:

$$\frac{\overline{p}}{p_o} = 1 + \gamma M_o^2 p$$

Equation (13) then becomes

$$\rho = \frac{\phi}{T} \tag{14}$$

where

$$\phi = 1 + \gamma M_o^2 p$$

When equation (14) is substituted in equations (11) and (12), equation (11) becomes

$$\left(\frac{\partial\psi}{\partial y}\right) \frac{\partial}{\partial x}\left(T \frac{\partial\psi}{\partial y}\right) - \frac{\partial\psi}{\partial x} \frac{\partial}{\partial y}\left(T \frac{\partial\psi}{\partial y}\right) = \frac{dp}{dx}\left[\frac{\gamma M_o^2}{\phi} T\left(\frac{\partial\psi}{\partial y}\right)^2 - \phi\right] + \frac{\partial}{\partial y}\left[\mu \frac{\partial}{\partial y}\left(T \frac{\partial\psi}{\partial y}\right)\right] \tag{15}$$

and equation (12) becomes

$$\frac{\partial\psi}{\partial y} \frac{\partial T}{\partial x} - \frac{\partial\psi}{\partial x} \frac{\partial T}{\partial y} = \frac{(\gamma - 1)M_o^2}{\phi}\left\{T \frac{\partial\psi}{\partial y} \frac{dp}{dx} + \frac{\mu}{\phi}\left[\frac{\partial}{\partial y}\left(T \frac{\partial\psi}{\partial y}\right)\right]^2\right\} + \frac{1}{\sigma_o} \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) \tag{16}$$

In equations (15) and (16) μ and k are assumed to be functions only of T and σ_o is assumed to depend only on the gas. Equations (15) and (16) describe the behavior of the laminar boundary layer in a compressible flow. The solution in a specific case requires the determination of ψ (x,y) and of T (x,y) subject to the following boundary conditions:

$$\left.\begin{array}{ll} \text{When } y = 0, & \\ & u = 0 \\ & T = T(x)_{y=0} \\ & v = v(x)_{y=0} \\ \text{when } y \to \infty, & \\ & U = U(x) \\ & T = T(x) \end{array}\right\} \qquad (17)$$

When ϕ and the boundary conditions are independent of the Reynolds number, it is seen from equations (15) and (16) that although the Mach number and the physical properties of the gas appear, the Reynolds number does not. The conclusion, therefore, is that ψ (x,y) and T (x,y) are independent of the Reynolds number but are dependent on the physical properties of the gas and on the Mach number.

<u>Boundary-layer thickness, skin friction, separation point, and velocity profile</u>.- The value of y at the edge of the boundary layer is determined by the requirement that $\frac{1}{\rho}\frac{\partial\psi}{\partial y} \cong U$, where ρ (x,y) and ψ (x,y) are independent of Reynolds number. Substitutions (5) state that $\overline{y} = \left(\frac{l}{R_o^{1/2}}\right)y$. Therefore, for a given relative pressure and temperature distribution along the body as well as for a given Mach number and gas, the boundary-layer thickness at a fixed point on the body is inversely proportional to the square root of the Reynolds number.

The surface shearing-stress coefficient is

$$\frac{\tau_S}{\rho_o U_o^{\,2}} = \frac{\left(\bar{\mu}\,\frac{\partial \bar{u}}{\partial \bar{y}}\right)_{\bar{y}=0}}{\rho_o U_o^{\,2}} = \frac{1}{R_o^{\,1/2}}\left(\mu\,\frac{\partial u}{\partial y}\right)_{y=0}$$

or with $u = 0$ at $y = 0$ and $\rho = \frac{\phi}{T}$

$$\frac{\tau_S}{\rho_o U_o^{\,2}} = \frac{1}{\phi R_o^{\,1/2}}\left(\mu T\,\frac{\partial^2 \psi}{\partial y^2}\right)_{y=0} \tag{18}$$

Because $\psi\,(x,y)$, $T\,(x,y)$ and $\phi\,(x)$ are independent of R_o, the surface-friction coefficient $\frac{\tau_S}{\rho_o U_o^{\,2}}$ at a fixed point on a body is inversely proportional to the square root of the Reynolds number R_o when the boundary conditions (equations (17)), the Mach number M_o, and the gas are fixed. The friction drag coefficient of the part of a body covered by a laminar boundary layer also varies as $1/\sqrt{R_o}$ when the boundary conditions (equations (17)), the Mach number M_o, and the gas are fixed.

The separation point is the point at which $\tau_S = 0$; that is, where $\left(\frac{\partial^2 \psi}{\partial y^2}\right)_{y=0} = 0$. The value of x at which $\left(\frac{\partial^2 \psi}{\partial y^2}\right)_{y=0} = 0$ is independent of R_o when $\psi\,(x,y)$, $T\,(x,y)$, and $\phi\,(x)$ are independent of R_o. The ratio $\frac{\bar{x}}{l} = x$ is also independent of the Reynolds number. Therefore, the separation point on a body, when the Mach number, the gas, and the boundary conditions (equations (17)) are fixed, is independent of the Reynolds number.

For fixed boundary conditions (17) and for a fixed gas and a fixed Mach number, the curve of u against y is independent of Reynolds number; thus the curve of $\bar{u}/\bar{U}$ against $\frac{\bar{y}}{l}\sqrt{R_o}$ is invariable at a station $\bar{x}/l$. This criterion can be used to test whether a velocity profile is laminar.

Necessity of a positive pressure gradient for separation.- The separation point is defined as the point at which $\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)_{\bar{y}=0} = 0$ and $\bar{u} > 0$ for $\bar{y} > 0$.

Consider the possibility of having $\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)_{\bar{y}=0} = 0$ when $\left(\frac{\partial \bar{p}}{\partial \bar{x}}\right)_{\bar{y}=0} < 0$

From the equation of continuity, (equation (3)) and the equation of motion (equation (1)) with $\bar{u} = \bar{v} = 0$ at $\bar{y} = 0$ it follows that at $\bar{y} = 0$

$$0 = -\frac{\partial \bar{p}}{\partial \bar{x}} + \bar{\mu}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + \frac{\partial \bar{\mu}}{\partial \bar{y}}\frac{\partial \bar{u}}{\partial \bar{y}}$$

Then for $\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right) = 0$

$$\bar{\mu}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} = \frac{\partial \bar{p}}{\partial \bar{x}}$$

It is now assumed that the velocity can be expanded in a Taylor's series. Thus,

$$\bar{u} = \left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)_{\bar{y}=0}\bar{y} + \left(\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right)_{\bar{y}=0}\frac{\bar{y}^2}{2!} + \left(\frac{\partial^3 \bar{u}}{\partial \bar{y}^3}\right)_{\bar{y}=0}\frac{\bar{y}^3}{3!} + \cdot \cdot \cdot$$

For $\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)_{\bar{y}=0} = 0$

$$\bar{u} = \frac{\bar{y}^2}{2!}\left[\left(\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right)_{\bar{y}=0} + \left(\frac{\partial^3 \bar{u}}{\partial \bar{y}^3}\right)_{\bar{y}=0}\frac{\bar{y}}{3} + \cdot \cdot \cdot\right]$$

or for small $\bar{y}$

$$\bar{u} = \left(\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right)_{\bar{y}=0}\frac{\bar{y}^2}{2!}$$

For $\left(\frac{\partial \bar{p}}{\partial \bar{x}}\right)_{\bar{y}=0} < 0$ it follows that $\left(\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right)_{\bar{y}=0} < 0$. Therefore $\bar{u} < 0$ for small $\bar{y}$. This result, however, disagrees with the requirement that $\bar{u} > 0$ for $\bar{y} > 0$. Therefore $\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)_{\bar{y}=0}$ cannot be zero when $\left(\frac{\partial \bar{p}}{\partial \bar{x}}\right)_{\bar{y}=0} < 0$. Thus separation cannot occur when $\left(\frac{\partial \bar{p}}{\partial \bar{x}}\right)_{\bar{y}=0} < 0$. If the boundary-layer assumptions are used then $\left(\frac{\partial \bar{p}}{\partial \bar{x}}\right)_{\bar{y}=0} = \frac{d\bar{p}}{d\bar{x}}$ and it follows that separation cannot occur when $\frac{d\bar{p}}{d\bar{x}} < 0$.

Consider the possibility of having $\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)_{\bar{y}=0} = 0$ when $\frac{d\bar{p}}{d\bar{x}} = 0$. From the full equation of motion with $\left(\frac{\partial \bar{p}}{\partial \bar{x}}\right)_{\bar{y}=0} = 0$ it follows that $\left(\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right)_{\bar{y}=0} = 0$ when $\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)_{\bar{y}=0} = 0$. Thus, at $\bar{y} = 0$

$$\bar{u} = \left(\frac{\partial \bar{u}}{\partial \bar{y}}\right) = \left(\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right) = 0$$

By using the boundary-layer equation of motion (equation (6)) with $\frac{dp}{dx} = \bar{u}_{y=0} = \left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)_{\bar{y}=0} = \left(\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right)_{\bar{y}=0} = 0$ it can be shown that

$$\left(\frac{\partial^3 \bar{u}}{\partial \bar{y}^3}\right)_{\bar{y}=0} = \left(\frac{\partial^4 \bar{u}}{\partial \bar{y}^4}\right)_{\bar{y}=0} = \cdot \cdot \cdot = \left(\frac{\partial^n \bar{u}}{\partial \bar{y}^n}\right)_{\bar{y}=0} = 0$$

Therefore, if $\frac{d\bar{p}}{d\bar{x}} = 0$ and $\left(\frac{\partial\bar{u}}{\partial\bar{y}}\right)_{\bar{y}=0} = 0$, it follows that $\bar{u} = 0$ for all $\bar{y}$ when it is assumed that $\bar{u}$ can be expanded in a Taylor's Series in $\bar{y}$. The conclusion that $\bar{u} = 0$ for all $\bar{y}$, however, contradicts the requirement that $\bar{u} > 0$ for $\bar{y} > 0$. Therefore, $\left(\frac{\partial\bar{u}}{\partial\bar{y}}\right)_{\bar{y}=0}$ cannot be zero when $\frac{d\bar{p}}{d\bar{x}} = 0$. Thus, separation cannot occur when $\frac{d\bar{p}}{d\bar{x}} = 0$.

It has been shown from the complete equation of motion (equation (1)) that separation cannot occur when $\left(\frac{\partial\bar{p}}{\partial\bar{x}}\right)_{\bar{y}=0} < 0$. If the boundary-layer assumptions are valid, then it follows that separation cannot occur when $\frac{d\bar{p}}{d\bar{x}} < 0$. It has also been shown by using the boundary-layer equation of motion (equation (6)) that separation cannot occur when $\frac{d\bar{p}}{d\bar{x}} = 0$. Therefore, separation can occur only when $\frac{d\bar{p}}{d\bar{x}} > 0$.

Axially Symmetrical Flow over a Body of Revolution

Derivation of boundary-layer equations.- The steady flow of a gas over the surface of a body of revolution in a layer having a thickness which is a negligible fraction of the radius of curvature of the surface in a meridian plane is described herein by the Navier-Stokes equations of motion in surface coordinates with the terms that involve surface curvature neglected, the equation of continuity, and the energy equation with $\bar{c}_p$ constant.

(1) The $\bar{x}$ component of the equation of motion is

$$\bar{\rho}\bar{u}\frac{\partial\bar{u}}{\partial\bar{x}} + \bar{\rho}\bar{v}\frac{\partial\bar{u}}{\partial\bar{y}} = \bar{\rho}\bar{F}_x - \frac{\partial\bar{p}}{\partial\bar{x}} + \frac{\bar{\mu}}{3}\frac{\partial^2\bar{v}}{\partial\bar{x}\,\partial\bar{y}} + \bar{\mu}\frac{\partial^2\bar{u}}{\partial\bar{y}^2} - \frac{\bar{\mu}}{\bar{r}}\frac{\partial\bar{r}}{\partial\bar{y}}\frac{\partial\bar{v}}{\partial\bar{x}} + \frac{\bar{\mu}}{\bar{r}}\frac{\partial\bar{r}}{\partial\bar{y}}\frac{\partial\bar{u}}{\partial\bar{y}} + \frac{4}{3}\bar{\mu}\frac{\partial^2\bar{u}}{\partial\bar{x}^2}$$
$$+ \frac{4}{3}\bar{\mu}\frac{\partial}{\partial\bar{x}}\left(\frac{\bar{u}}{\bar{r}}\frac{\partial\bar{r}}{\partial\bar{x}} + \frac{\bar{v}}{\bar{r}}\frac{\partial\bar{r}}{\partial\bar{y}}\right) + \frac{\partial\bar{u}}{\partial\bar{y}}\frac{\partial\bar{\mu}}{\partial\bar{y}} + \frac{4}{3}\frac{\partial\bar{u}}{\partial\bar{x}}\frac{\partial\bar{\mu}}{\partial\bar{x}} - \frac{2}{3}\frac{\partial\bar{v}}{\partial\bar{y}}\frac{\partial\bar{\mu}}{\partial\bar{x}}$$
$$- \frac{2}{3}\frac{\bar{v}}{\bar{r}}\frac{\partial\bar{r}}{\partial\bar{y}}\frac{\partial\bar{\mu}}{\partial\bar{x}} + \frac{\partial\bar{v}}{\partial\bar{x}}\frac{\partial\bar{\mu}}{\partial\bar{y}} - \frac{2}{3}\frac{\bar{u}}{\bar{r}}\frac{\partial\bar{r}}{\partial\bar{x}}\frac{\partial\bar{\mu}}{\partial\bar{x}} \qquad (19)$$

(2) The $\bar{y}$ component of the equation of motion is

$$\bar{\rho}\bar{u}\frac{\partial\bar{v}}{\partial\bar{x}} + \bar{\rho}\bar{v}\frac{\partial\bar{v}}{\partial\bar{y}} = \bar{\rho}F_y - \frac{\partial\bar{p}}{\partial\bar{y}} + \bar{\mu}\frac{\partial^2\bar{v}}{\partial\bar{x}^2} + \frac{1}{3}\bar{\mu}\frac{\partial^2\bar{u}}{\partial\bar{y}\,\partial\bar{x}} + \frac{\bar{\mu}}{\bar{r}}\frac{\partial\bar{r}}{\partial\bar{x}}\frac{\partial\bar{v}}{\partial\bar{x}} - \frac{\bar{\mu}}{\bar{r}}\frac{\partial\bar{r}}{\partial\bar{x}}\frac{\partial\bar{u}}{\partial\bar{y}} + \frac{4}{3}\bar{\mu}\frac{\partial^2\bar{v}}{\partial\bar{y}^2}$$
$$+ \frac{4}{3}\bar{\mu}\frac{\partial}{\partial\bar{y}}\left(\frac{\bar{u}}{\bar{r}}\frac{\partial\bar{r}}{\partial\bar{x}} + \frac{\bar{v}}{\bar{r}}\frac{\partial\bar{r}}{\partial\bar{y}}\right) - \frac{2}{3}\frac{\partial\bar{u}}{\partial\bar{x}}\frac{\partial\bar{\mu}}{\partial\bar{y}} + \frac{\partial\bar{v}}{\partial\bar{x}}\frac{\partial\bar{\mu}}{\partial\bar{x}} - \frac{2}{3}\frac{\bar{u}}{\bar{r}}\frac{\partial\bar{\mu}}{\partial\bar{y}}\frac{\partial\bar{r}}{\partial\bar{x}}$$
$$+ \frac{\partial\bar{u}}{\partial\bar{y}}\frac{\partial\bar{\mu}}{\partial\bar{x}} + \frac{4}{3}\frac{\partial\bar{v}}{\partial\bar{y}}\frac{\partial\bar{\mu}}{\partial\bar{y}} - \frac{2}{3}\frac{\bar{v}}{\bar{r}}\frac{\partial\bar{r}}{\partial\bar{y}}\frac{\partial\bar{\mu}}{\partial\bar{y}} \tag{20}$$

(3) The equation of continuity is

$$\frac{\partial\bar{r}\bar{\rho}\bar{u}}{\partial\bar{x}} + \frac{\partial\bar{r}\bar{\rho}\bar{v}}{\partial\bar{y}} = 0 \tag{21}$$

(4) The energy equations for constant $\bar{c}_p$ is

$$\bar{\rho}\bar{c}_p\bar{u}\frac{\partial\bar{T}}{\partial\bar{x}} + \bar{\rho}\bar{c}_p\bar{v}\frac{\partial\bar{T}}{\partial\bar{y}} - \bar{u}\frac{\partial\bar{p}}{\partial\bar{x}} - \bar{v}\frac{\partial\bar{p}}{\partial\bar{y}} = \bar{\mu}\left[\frac{4}{3}\left(\frac{\partial\bar{u}}{\partial\bar{x}}\right)^2 + 2\frac{\partial\bar{v}}{\partial\bar{x}}\frac{\partial\bar{u}}{\partial\bar{y}} + \frac{4}{3}\left(\frac{\partial\bar{v}}{\partial\bar{y}}\right)^2 + \left(\frac{\partial\bar{v}}{\partial\bar{x}}\right)^2\right.$$
$$+ \left(\frac{\partial\bar{u}}{\partial\bar{y}}\right)^2 - \frac{2\bar{u}^2}{\bar{r}}\frac{\partial^2\bar{r}}{\partial\bar{x}^2} - \frac{4\bar{u}\bar{v}}{\bar{r}}\frac{\partial^2\bar{r}}{\partial\bar{x}\,\partial\bar{y}} + \frac{4}{3}\frac{\bar{u}^2}{\bar{r}^2}\left(\frac{\partial\bar{r}}{\partial\bar{x}}\right)^2 + \frac{8}{3}\frac{\bar{u}\bar{v}}{\bar{r}^2}\left(\frac{\partial\bar{r}}{\partial\bar{x}}\right)\left(\frac{\partial\bar{r}}{\partial\bar{y}}\right) + \frac{4}{3}\frac{\bar{v}^2}{\bar{r}^2}\left(\frac{\partial\bar{r}}{\partial\bar{y}}\right)^2$$
$$\left. - \frac{4}{3}\frac{\bar{u}}{\bar{r}}\frac{\partial\bar{u}}{\partial\bar{x}}\frac{\partial\bar{r}}{\partial\bar{x}} - \frac{4}{3}\frac{\partial\bar{u}}{\partial\bar{x}}\frac{\partial\bar{v}}{\partial\bar{y}} - \frac{4}{3}\frac{\bar{v}}{\bar{r}}\frac{\partial\bar{u}}{\partial\bar{x}}\frac{\partial\bar{r}}{\partial\bar{y}} - \frac{4}{3}\frac{\bar{u}}{\bar{r}}\frac{\partial\bar{r}}{\partial\bar{x}}\frac{\partial\bar{v}}{\partial\bar{y}} - \frac{4}{3}\frac{\bar{v}}{\bar{r}}\frac{\partial\bar{v}}{\partial\bar{y}}\frac{\partial\bar{r}}{\partial\bar{y}}\right] + \frac{\partial\bar{k}\frac{\partial\bar{T}}{\partial\bar{x}}}{\partial\bar{x}}$$
$$+ \frac{\partial\bar{k}\frac{\partial\bar{T}}{\partial\bar{y}}}{\partial\bar{y}} + \frac{\bar{k}}{\bar{r}}\left(\frac{\partial\bar{r}}{\partial\bar{x}}\frac{\partial\bar{T}}{\partial\bar{x}} + \frac{\partial\bar{r}}{\partial\bar{y}}\frac{\partial\bar{T}}{\partial\bar{y}}\right) \tag{22}$$

The usual assumptions of boundary-layer theory are now made and expressed by equations (5). To these substitutions is added $\bar{r} = rr_o$. Use of these substitutions, the fact that $\frac{\partial\bar{r}}{\partial\bar{y}}$ and $\frac{\partial\bar{r}}{\partial\bar{x}}$ are of the order of magnitude of unity, the assumption that $\frac{l}{r_o} << R_o^{1/2}$, neglecting terms containing the factor $\frac{1}{R_o^{1/2}}$, $\frac{1}{R_o}$, or $\frac{1}{R_o^{3/2}}$, and neglecting the term ρF_x reduces equation (19) to

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = - \frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \tag{23}$$

When the development used to obtain equation (23) from equation (19) is applied to equation (20) and terms containing the factor $\frac{1}{R_o}$, $\frac{1}{R_o^{3/2}}$, or $\frac{1}{R_o^2}$ and the term $\rho \bar{F}_y$ are neglected, the result is

$$0 = \frac{\partial p}{\partial y} \tag{24}$$

The use of substitutions (5) in equation (21) results in

$$\frac{\partial r \rho u}{\partial x} + \frac{\partial r \rho v}{\partial y} = 0 \tag{25}$$

When the development used to obtain equation (23) from equation (19) is applied to equation (22) and terms containing the factor $\frac{(\gamma - 1)M_o^2}{R_o}$, $\frac{(\gamma - 1)M_o^2}{R_o^2}$, $\frac{(\gamma - 1)M_o^2}{R_o^{3/2}}$, $\frac{1}{\sigma_o R_o}$, or $\frac{1}{\sigma_o R_o^{1/2}}$ are neglected, the result is

$$\rho u \frac{\partial T}{\partial x} + \rho v \frac{\partial T}{\partial y} = (\gamma - 1)M_o^2 \left[u \frac{\partial p}{\partial x} + \mu\left(\frac{\partial u}{\partial y}\right)^2\right] + \frac{1}{\sigma_o} \frac{\partial}{\partial y} k\left(\frac{\partial T}{\partial y}\right) \tag{26}$$

In obtaining equations (23), (24), and (26) from equations (19), (20), and (22) it has been assumed that all terms containing the factor $1/r$ are finite. Equations (23), (24), (25), and (26) describe the flow in the boundary layer. Because of equation (24), the static pressure in the boundary layer is a function only of x; therefore $\partial p/\partial x$ in equations (23) and (26) can be replaced by dp/dx

A nondimensional stream function Ψ is then introduced, where

$$\left.\begin{aligned} u &= \frac{1}{r\rho} \frac{\partial \Psi}{\partial y} \\ v &= - \frac{1}{r\rho} \frac{\partial \Psi}{\partial x} \end{aligned}\right\} \tag{27}$$

The equation of continuity, equation (25), is automatically satisfied. Equation (23) becomes

$$\frac{1}{r}\frac{\partial\Psi}{\partial y}\frac{\partial}{\partial x}\left(\frac{1}{r\rho}\frac{\partial\Psi}{\partial y}\right)-\frac{1}{r}\frac{\partial\Psi}{\partial x}\frac{\partial}{\partial y}\left(\frac{1}{r\rho}\frac{\partial\Psi}{\partial y}\right)=-\frac{dp}{dx}+\frac{\partial}{\partial y}\left[\mu\frac{\partial}{\partial y}\left(\frac{1}{r\rho}\frac{\partial\Psi}{\partial y}\right)\right] \tag{28}$$

and equation (26) becomes

$$\frac{1}{r}\frac{\partial\Psi}{\partial y}\frac{\partial T}{\partial x}-\frac{1}{r}\frac{\partial\Psi}{\partial x}\frac{\partial T}{\partial y}=(\gamma-1)M_o^2\left\{\frac{1}{r\rho}\frac{\partial\Psi}{\partial y}\frac{dp}{dx}+\mu\left[\frac{\partial}{\partial y}\left(\frac{1}{r\rho}\frac{\partial\Psi}{\partial y}\right)\right]^2\right\}$$
$$+\frac{1}{\sigma_o}\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) \tag{29}$$

The relative density ρ is now replaced by ϕ/T, just as for two-dimensional flow. Equation (28) then becomes

$$\frac{1}{r}\frac{\partial\Psi}{\partial y}\frac{\partial}{\partial x}\left(\frac{T}{r}\frac{\partial\Psi}{\partial y}\right)-\frac{1}{r}\frac{\partial\Psi}{\partial x}\frac{\partial}{\partial y}\left(\frac{T}{r}\frac{\partial\Psi}{\partial y}\right)=\frac{dp}{dx}\left[\frac{T}{r^2}\left(\frac{\partial\Psi}{\partial y}\right)^2\frac{\gamma M_o^2}{\phi}-\phi\right]$$
$$+\frac{\partial}{\partial y}\left[\mu\frac{\partial}{\partial y}\left(\frac{T}{r}\frac{\partial\Psi}{\partial y}\right)\right] \tag{30}$$

and equation (29) becomes

$$\frac{1}{r}\frac{\partial\Psi}{\partial y}\frac{\partial T}{\partial x}-\frac{1}{r}\frac{\partial\Psi}{\partial x}\frac{\partial T}{\partial y}=\frac{(\gamma-1)M_o^2}{\phi}\left\{\frac{T}{r}\frac{\partial\Psi}{\partial y}\frac{dp}{dx}+\frac{\mu}{\phi}\left[\frac{\partial}{\partial y}\left(\frac{T}{r}\frac{\partial\Psi}{\partial y}\right)\right]^2\right\}$$
$$+\frac{1}{\sigma_o}\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) \tag{31}$$

In equations (30) and (31) μ and k are assumed to be functions only of T, σ_o is assumed to depend only on the gas, and r is a function of x and y. Equations (30) and (31) describe the laminar boundary layer on a body of revolution in a compressible flow. The solution in a specific case requires the determination of Ψ (x,y) and of T (x,y) subject to the boundary conditions, equations (17).

When $y = 0$,

$$\left.\begin{aligned} u &= 0 \\ T &= T(x)_{y=0} \\ v &= v(x)_{y=0} \\ \text{when } y &\to \infty, \\ U &= U(x) \\ T &= T(x) \end{aligned}\right\} \tag{17}$$

When ϕ and the boundary conditions are independent of the Reynolds number, it is seen from equations (30) and (31) that although the Mach number M_o and the physical properties of the gas appear, the Reynolds number does not. The conclusion, therefore, is that $\bar{\Psi}$ (x,y) and T (x,y) are independent of the Reynolds number but dependent on the physical properties of the gas and on the Mach number.

Boundary-layer thickness, skin friction, separation point, and velocity profile.- The conclusions concerning the boundary-layer thickness, skin friction, separation point, and velocity profile are the same as those obtained for two-dimensional motion. The conclusions are obtained in the same way as those for two-dimensional motion, except that for the body of revolution equation (18) is replaced by

$$\frac{\tau_S}{\rho_o U_o^2} = \frac{1}{\phi R_o^{1/2}} \left(\frac{\mu T}{r} \frac{\partial^2 \bar{\Psi}}{\partial y^2}\right)_{y=0} \tag{32}$$

Necessity of a positive pressure gradient for separation.- The separation point is defined as the point at which $\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)_{\bar{y}=0} = 0$ and $\bar{u} > 0$ for $\bar{y} > 0$.

Consider the possibility of having $\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)_{\bar{y}=0} = 0$ when $\left(\frac{\partial \bar{p}}{\partial \bar{x}}\right)_{\bar{y}=0} < 0$.

From the equation of continuity (equation (21)) and the equation of motion (equation (19)) with $\bar{u} = \bar{v} = 0$ at $\bar{y} = 0$ and with $\frac{1}{\bar{r}}\frac{\partial \bar{r}}{\partial \bar{y}}$, $\frac{1}{\bar{r}}\frac{\partial \bar{r}}{\partial \bar{x}}$, $\frac{\partial}{\partial \bar{x}}\left(\frac{1}{\bar{r}}\frac{\partial \bar{r}}{\partial \bar{x}}\right)$, and $\frac{\partial}{\partial \bar{x}}\left(\frac{1}{\bar{r}}\frac{\partial \bar{r}}{\partial \bar{y}}\right)$ finite

$$0 = -\frac{\partial \bar{p}}{\partial \bar{x}} + \frac{\bar{\mu}}{\bar{r}}\frac{\partial \bar{r}}{\partial \bar{y}}\frac{\partial \bar{u}}{\partial \bar{y}} + \frac{\partial \bar{u}}{\partial \bar{y}}\frac{\partial \bar{\mu}}{\partial \bar{y}} + \bar{\mu}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}$$

or for $\frac{\partial \bar{u}}{\partial \bar{y}} = 0$

$$\frac{\partial \bar{p}}{\partial \bar{x}} = \bar{\mu}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}.$$

When $\left(\frac{\partial \bar{p}}{\partial \bar{x}}\right)_{\bar{y}=0} < 0$,

$$\left(\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right)_{\bar{y}=0} < 0$$

Then, by the same reasoning as in the two-dimensional case, separation cannot occur when $\left(\frac{\partial \bar{p}}{\partial \bar{x}}\right)_{\bar{y}=0} < 0$. If the boundary-layer assumptions are used then $\left(\frac{\partial \bar{p}}{\partial \bar{x}}\right)_{\bar{y}=0} = \frac{d\bar{p}}{d\bar{x}}$ and separation cannot occur when $\frac{d\bar{p}}{d\bar{x}} < 0$.

Consider the possibility of having $\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)_{\bar{y}=0} = 0$ when $\frac{d\bar{p}}{d\bar{x}} = 0$.

The development is the same as that for two-dimensional flow with the exception that equation (23) is used instead of equation (6) and it is assumed that $\bar{r} \neq 0$ and that all its derivatives with respect to $\bar{y}$ are finite. The conclusion is that separation cannot occur when $\frac{d\bar{p}}{d\bar{x}} = 0$.

The complete equation of motion (equation (19) thus indicates that separation cannot occur when $\left(\frac{\partial \bar{p}}{\partial \bar{x}}\right)_{\bar{y}=0} < 0$. If the boundary-layer assumptions are valid, it follows that separation cannot occur when $\frac{d\bar{p}}{d\bar{x}} < 0$. It also follows from the boundary-layer equation of motion (equation (23)) that separation cannot occur when $\frac{d\bar{p}}{d\bar{x}} = 0$. Therefore, separation can occur only when $\frac{d\bar{p}}{d\bar{x}} > 0$.

DISCUSSION

The conclusions of the present work concerning the behavior of the laminar boundary layer were reached by the following assumptions:

(1) The boundary-layer thickness is a negligible fraction of the radius of curvature of the wall in the plane of the velocity.

(2) The flow in the boundary layer is almost parallel to the surface.

(3) The boundary-layer thickness is a small fraction of the distance to the stagnation point.

(4) The inertia and largest viscous forces are of equal order of magnitude.

(5) The body forces are negligible.

(6) The coefficients of specific heat are constant.

(7) The coefficients μ and k are functions only of T.

(8) The Prandtl number depends only on the gas.

(9) The perfect gas law is applicable.

(10) The Reynolds number is large.

For the body of revolution it is also assumed that terms which contain r or its derivatives are finite and that the body is not very slender. The conclusions concerning the effects of Reynolds number contain the additional requirements that the conditions at the surface and at the outer edge of the boundary layer, when expressed nondimensionally, equation (17), are independent of Reynolds number.

The assumptions of the boundary-layer theory may be invalid downstream of the separation point and perhaps even at the separation point. No evidence is available, however, to indicate that the boundary-layer approximations become so poor at the separation point that the conclusions concerning the separation point are invalid. The region near the base of a shock wave is another at which the boundary-layer assumptions may be invalid, but here again a definite statement cannot yet be made.

The conclusions concerning the effect of Reynolds number are noted to be the same for compressible flow as for incompressible flow. For compressible flows, however, the boundary conditions involve the temperature distribution as well as the pressure distribution. The Mach number and the ratio of the specific heats appear as parameters.

CONCLUSIONS

The boundary-layer equation of motion and the boundary-layer energy equation for the compressible and steady laminar boundary layer on two-dimensional bodies and bodies of revolution are written in a nondimensional form to provide a clearer indication of the effects of Mach number, Reynolds number, and the properties of the gas.

When the ratio of the local velocity to the free-stream velocity and the ratio of the local temperature to the free-stream temperature at all points on the surface of a body and at the outer edge of its boundary layer do not change with Reynolds number and when the Mach number and the physical properties of the gas also do not change with Reynolds number, then it follows that:

1. The boundary-layer thickness at a fixed point on a body is inversely proportional to the square root of the Reynolds number R_o.

2. The surface-friction coefficient at a fixed point is inversely proportional to the square root of the Reynolds number R_o.

3. The friction drag coefficient of the part of a body covered by a laminar boundary layer is inversely proportional to the square root of the Reynolds number R_o.

4. The separation point is independent of the Reynolds number R_o.

5. The nondimensional velocity profile is invariable when the velocity ratio $\bar{u}/\bar{U}$ (where $\bar{u}$ is the velocity inside the boundary layer and parallel to the surface and $\bar{U}$ is the velocity at the outer edge of the boundary layer and parallel to the surface) is plotted against $\frac{\bar{y}}{l}\sqrt{R_o}$

(where $\bar{y}$ is the distance measured from the surface in a direction perpendicular to the surface, l is the length of the body, and R_0 is the Reynolds number) at a fixed fraction of the body length from the stagnation point. By use of the boundary-layer equations, separation of the laminar boundary layer is shown to occur only when the static pressure along the surface rises in the direction of flow.

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Langley Air Force Base, Va., November 22, 1948

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