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ON THE THEORY OF OSCILLATING AIRFOILS OF FINITE SPAN IN SUBSONIC COMPRESSIBLE FLOW

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SUMMARY

The problem of the oscillating lifting surface of finite span in subsonic compressible flow is reduced to an integral equation. The kernel of the integral equation is approximated by a simpler expression, on the basis of the assumption of sufficiently large aspect ratio. With this approximation the double integral occurring in the formulation of the problem is reduced to two single integrals, one of which is taken over the chord and the other over the span of the lifting surface. On the basis of this reduction the three-dimensional problem appears separated into two two-dimensional problems, one of them being effectively the problem of two-dimensional flow and the other being the problem of spanwise-circulation distribution. Earlier results concerning the oscillating lifting surface of finite span in incompressible flow are contained in the present more general results.

INTRODUCTION

The present report is concerned with the problem of the oscillating airfoil of finite span, within the frame of the linearized lifting-surface theory. The aim of this study is the development of a theory which incorporates simultaneously the effects of three-dimensionality of the flow and of compressibility of the fluid. As an exact solution of this problem, even within the limitations of the linearized theory, presents very great difficulties, it is worth while to work toward an approximate theory which is valid provided the aspect ratio of the lifting surface is not too small.

The author has previously obtained results of this nature for the case of incompressible flow (references 1 and 2). In this earlier work the known results for the problem of two-dimensional incompressible flow were contained as a special case. The present work generalizes these results so as to take account of compressibility in the subsonic range. Thus the results of this report consist of a system of equations which contain as special cases both the author's results for the wing

of finite span in incompressible flow and the results of Possio's theory of two-dimensional compressible flow (reference 3).

The scope of the present results may briefly be described as follows. The starting point of the work is an integral-equation formulation of the problem of the lifting surface of finite span. The integrals which occur are double integrals and the functions to be determined are functions of two independent variables. The essential step of the present work is to replace the actual kernel of the integral equation by an approximate kernel in such a way that the double integrals are reduced to single integrals over the range of either one of the two independent variables. In this way the problem is reduced to two problems which are to be solved separately. The first of these two problems is of the same nature as the Possio problem of two-dimensional compressible flow. The second of these problems is of the same nature as the problem of the Prandtl lifting-line theory for the wing of finite span in uniform motion.

As in the theory of incompressible flow, this reduction of the double-integral problem to two single-integral problems depends crucially on the assumption of sufficiently large aspect ratio. While "sufficiently large" aspect ratios might be thought to be aspect ratios of about three, definite statements of this nature must be based on experimental evidence, as long as no exact solutions exist for the three-dimensional problem of the oscillating lifting surface in compressible flow.

It is perhaps worth while to state explicitly that the present problem is quite different from the corresponding problem for supersonic flow.

It might also be added that there are reasons to believe that it is not satisfactory, even approximately, to superimpose aspect-ratio corrections for incompressible flow and compressibility corrections for two-dimensional flow in order to obtain corrections for the combined effect. This latter point is one of the reasons for the present study.

In this first report the work is carried to the point where the double-integral equation is reduced to an equation containing single integrals only over the quantities to be determined. Further developments will be given in a subsequent report.

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SYMBOLS

X,Y,Z	Cartesian coordinates
U	main-stream velocity in X-direction
t	time
H	defined by equation of lifting surface $Z = H(X,Y,t)$
u,v,w	components of velocity change caused by presence of lifting surface
R_a	region in X,Y-plane occupied by projection of lifting surface
ρ_o	density of stream flowing with velocity U
ρ,p	density and pressure changes, respectively, associated with velocity changes u, v, and w
a	velocity of sound in main stream $(a^2 = dp/d\rho_o)$
$X_T(Y)$	coordinate of trailing edge of R_a
ϕ	potential of velocity changes u, v, and w
ω	circular frequency of oscillation
M	Mach number of main stream $(M = U/a)$
Re	real part of
b	a length to be identified with the semichord of R_a at midspan
k	reduced-frequency parameter $(k = \omega b/U)$
x,y,z	dimensionless coordinates defined by equation (19)
ψ	function defined by equation (21)
μ	parameter defined as $\mu = \dfrac{kM^2}{1 - M^2}$

κ	parameter defined as $\kappa = \frac{kM}{1 - M^2}$
ν	parameter defined as $\nu = \frac{k}{1 - M^2}$
R_a^*	region in x,y-plane corresponding to region R_a in X,Y-plane
x_T	coordinate of trailing edge of R_a^*
x_L	coordinate of leading edge of R_a^*
R_w^*	region in x,y-plane consisting of the strip to the right of the trailing edge of R_a^*
R_r^*	entire x,y-plane except for regions R_a^* and R_w^*
Λ	function defined by equations (36)
g	function defined by equation (37)
ξ,η	variables of integration in accordance with equation (40)
r	defined by equation (41)
$H_0^{(2)},H_1^{(2)}$	Hankel functions of second kind, and of zeroth and first order, respectively
ζ	auxiliary variable of integration
K	function defined by equation (51)
G	function defined by equation (52)
I_n	functions defined by equations (54), (55), and (56)
O_m	order of magnitude of
λ	function defined by equation (38)
F_M	function defined by equations (73) and (83)

σ	auxiliary variable of integration
τ	auxiliary variable of integration
Ω	function defined by equation (76)
s	ratio of semispan to semichord at midspan
η^*	defined as $\eta^* = \eta/s\sqrt{1 - M^2}$
b^*	local semichord divided by semichord b at midspan
x_m^*	a quantity indicating amount of sweep and defined by equation (86)
x^*	dimensionless coordinate defined as $x^* = (x - x_m^*)/b^*$
κ^*,ν^*,Ω^*	quantities defined by equation (87)
g^*,λ^*	quantities defined as $g^*(x^*,y^*) = g(x,y)$; $\lambda^*(x^*,y^*) = \lambda(x,y)$
N	function defined by equation (89)
S_M	function defined by equation (90)

THE BOUNDARY-VALUE PROBLEM OF THE OSCILLATING LIFTING SURFACE

It is assumed that a nearly plane, impenetrable surface is put into the path of an inviscid flowing fluid which, except for the effect of this surface, possesses a uniform velocity U in the direction of the positive X-axis. The impenetrable surface, henceforth called lifting surface, is taken to lie nearly in the X,Y-plane and its equation is written in the form $Z = H(X,Y,t)$. When $H \equiv 0$ no disturbance is caused. When the lifting surface is not exactly plane and parallel to the direction of U the velocity components $(U,0,0)$ are changed into $(U + u,v,w)$ where u, v, and w depend on the form of the function H and on the shape of the region R_a which is the projection of the lifting surface onto the X,Y-plane.

The disturbances caused by the presence of the lifting surface are assumed to be small, in the sense that the differential equations and boundary conditions of the problem are linearized with respect to

the disturbance velocity components u, v, and w and with respect to the pressure and density changes p and ρ caused by the presence of the lifting surface.

Under these conditions the differential equations of the problem are the following:

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial X} = -\frac{\partial}{\partial X}\left(\frac{p}{\rho_o}\right) \tag{1}$$

$$\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial X} = -\frac{\partial}{\partial Y}\left(\frac{p}{\rho_o}\right) \tag{2}$$

$$\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial X} = -\frac{\partial}{\partial Z}\left(\frac{p}{\rho_o}\right) \tag{3}$$

$$\frac{\partial \rho}{\partial t} + U\frac{\partial \rho}{\partial X} + \rho_o\left(\frac{\partial u}{\partial X} + \frac{\partial v}{\partial Y} + \frac{\partial w}{\partial Z}\right) = 0 \tag{4}$$

$$p = a^2\rho \tag{5}$$

The quantity ρ_o in equations (1) to (4) is the density in the fluid flowing without disturbance and the quantity a in equation (5) is the velocity of sound in the undisturbed fluid, that is, $a^2 = dp/d\rho_o$.

The boundary condition of no relative normal flow at the lifting surface is satisfied, within the frame of the linearized theory, instead of on the lifting surface itself, on the projection of this surface onto the X,Y-plane,

$$X,Y \text{ inside } R_a, \quad w = \frac{\partial H}{\partial t} + U\frac{\partial H}{\partial X} \tag{6}$$

The form of condition (6) (which holds on both sides of the lifting surface) indicates that w is an even function of Z. From equation (3), it follows then that p is an odd function of Z and the condition

that the pressure disturbance p is continuous, except when passing across the lifting surface, means that for $Z = 0$ and

$$X,Y \text{ outside } R_a, \quad p = 0 \tag{7}$$

On the basis of conditions (7) and (6) the problem may be considered to consist in the determination of u, v, w, and p in the half space $Z \geqslant 0$ with the boundary $Z = 0$.

In addition to conditions (6) and (7) the following further conditions are prescribed in order to obtain an unambiguous solution. At the trailing edge of the lifting surface,

$$X = X_T(Y), \quad p \text{ is finite} \tag{8}$$

Finally, it is postulated as a condition "at infinity" that energy is traveling outward without reflection, in a manner to be defined more precisely in what follows for the case of simple harmonic motion.

VELOCITY-POTENTIAL FORMULATION OF THE PROBLEM

It can be shown that the problem may be solved as stated in equations (1) to (8) by means of a velocity potential ϕ, in terms of which

$$\left.\begin{aligned} u &= \frac{\partial\phi}{\partial X} \\ v &= \frac{\partial\phi}{\partial Y} \\ w &= \frac{\partial\phi}{\partial Z} \end{aligned}\right\} \tag{9}$$

Combination of equations (9), (1), (2), and (3) results in the following expression for the pressure change p:

$$p = -\rho_o\left(\frac{\partial\phi}{\partial t} + U\frac{\partial\phi}{\partial X}\right) \tag{10}$$

Combination of equations (10), (5), (9), and (4) results in the following differential equation for ϕ:

$$\frac{\partial^2\phi}{\partial X^2} + \frac{\partial^2\phi}{\partial Y^2} + \frac{\partial^2\phi}{\partial Z^2} - \frac{1}{a^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial X}\right)^2\phi = 0 \tag{11}$$

Equation (11) is to be solved in the half space $Z \geqslant 0$ subject to the following boundary conditions at $Z = 0$:

$$X,Y \text{ inside } R_a, \quad \frac{\partial\phi}{\partial Z} = \frac{\partial H}{\partial t} + U\frac{\partial H}{\partial X} \tag{12}$$

$$X,Y \text{ outside } R_a, \quad \frac{\partial\phi}{\partial t} + U\frac{\partial\phi}{\partial X} = 0 \tag{13}$$

$$X = X_T(Y), \quad \frac{\partial\phi}{\partial t} + U\frac{\partial\phi}{\partial X} \text{ is finite} \tag{14}$$

and subject to the condition of no energy reflection "at infinity."

In what follows attention is restricted to the case of simple harmonic motion and the following equation is written:

$$\phi(X,Y,Z,t) = e^{i\omega t}\bar{\phi}(X,Y,Z) \tag{15}$$

with corresponding expressions for H and p.[1]

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[1]It is perhaps not entirely superfluous to indicate that this is meant in the sense that, corresponding to a surface equation $\mathrm{Re}\left(e^{i\omega t}\bar{H}\right)$, there is a pressure distribution $\mathrm{Re}\left(e^{i\omega t}\bar{p}\right)$.

Equation (11) now assumes the form

$$\frac{\partial^2\overline{\phi}}{\partial X^2} + \frac{\partial^2\overline{\phi}}{\partial Y^2} + \frac{\partial^2\overline{\phi}}{\partial Z^2} - \frac{1}{a^2}\left(i\omega + U\frac{\partial}{\partial X}\right)^2\overline{\phi} = 0 \tag{16}$$

Equation (10) becomes

$$\overline{p} = -\rho_o\left(i\omega\overline{\phi} + U\frac{\partial\overline{\phi}}{\partial X}\right) \tag{17}$$

Now the following dimensionless variables and parameters are introduced:

$$\left.\begin{aligned} M &= \frac{U}{a} \\ k &= \frac{\omega b}{U} \end{aligned}\right\} \tag{18}$$

$$\left.\begin{aligned} x &= \frac{X}{b} \\ y &= \sqrt{1 - M^2}\,\frac{Y}{b} \\ z &= \sqrt{1 - M^2}\,\frac{Z}{b} \end{aligned}\right\} \tag{19}$$

The differential equation (16) then assumes the form

$$\nabla^2\overline{\phi} - i\,\frac{2kM^2}{1 - M^2}\,\frac{\partial\overline{\phi}}{\partial x} + \frac{k^2M^2}{1 - M^2}\,\overline{\phi} = 0 \tag{20}$$

Equation (20) is reduced further, for the purpose of eliminating first-derivative terms, by the following substitution:

$$\bar{\phi} = e^{i\mu x}\psi \tag{21}$$

Choose

$$\mu = \frac{M^2 k}{1 - M^2} \tag{22}$$

and obtain as the equation for ψ,

$$\nabla^2\psi + \kappa^2\psi = 0 \tag{23}$$

where

$$\kappa = \frac{kM}{1 - M^2} \tag{24}$$

Now $\bar{p}$ and the boundary conditions must be expressed in terms of the new independent variables x, y, and z and in terms of the new dependent variable ψ.

From equations (17) and (19), it follows first that

$$\bar{p} = -\frac{\rho_o U}{b}\left(ik\bar{\phi} + \frac{\partial\bar{\phi}}{\partial x}\right) \tag{25}$$

Combination of equations (25) and (21) gives

$$\bar{p} = -\frac{\rho_o U}{b} e^{i\mu x}\left(i\nu\psi + \frac{\partial\psi}{\partial x}\right) \tag{26}$$

where ν is defined as

$$\nu = \frac{k}{1 - M^2} \tag{27}$$

The boundary condition (12) becomes

$$x,y \quad \text{inside} \quad R_a^*, \quad \frac{\partial\psi}{\partial z} = \frac{U}{\sqrt{1 - M^2}} e^{-i\mu x}\left(ik\bar{H} + \frac{\partial\bar{H}}{\partial x}\right) \tag{28}$$

where R_a^* is the region in the x,y-plane corresponding to R_a in the X,Y-plane.

The boundary condition (13) becomes

$$x,y \quad \text{outside} \quad R_a^*, \quad i\nu\psi + \frac{\partial\psi}{\partial x} = 0 \tag{29}$$

The condition of finite pressure at the trailing edge is now,

$$x = x_T(y), \quad i\nu\psi + \frac{\partial\psi}{\partial x} \quad \text{is finite} \tag{30}$$

Finally, the condition of no energy reflection at infinity is written in the form

$$z \rightarrow \infty, \quad \psi \approx f(x,y,z)e^{-i\kappa r} \tag{31}$$

where $r^2 = x^2 + y^2 + z^2$ and where f tends to zero as z tends to infinity.[2]

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[2]This ensures that $\phi \approx fe^{i(\omega t - \kappa r)}$ as z tends to infinity and therewith that waves are traveling away from the source of the disturbance.

The boundary condition (29) is made more specific in the following manner. From condition (29),

$$\psi(x,y,0) = c(y)e^{-i\nu x} \tag{32}$$

For any line y = Constant which does not pass through R_a^*, it can be concluded from the condition of undisturbed flow at $x = -\infty$ that $c(y) = 0$. The same can be said for that portion of any line y = Constant which is situated in front of the leading edge of the airfoil region. The situation is different for portions of lines y = Constant which do pass through R_a^* and which are to the rear of the trailing edge of R_a^*. The region to the rear of the trailing edge and bounded by lines y = Constant which are tangent to R_a^* is called the wake region and is designated by R_w^*. The exterior of the region R_a^* and R_w^* is called the remaining region and is designated by R_r^*. Then $c(y) = 0$ in R_r^* and, in general, $c(y) \neq 0$ in R_w^*.

Equation (29) is thus seen to be equivalent to the following two equations:

$$x,y \text{ inside } R_r^*, \quad \psi = 0 \tag{33a}$$

$$x,y \text{ inside } R_w^*, \quad \psi = c(y)e^{-i\nu x} \tag{33b}$$

There is now introduced a function Λ defined by

$$\Lambda(y) = 2\psi\left[x_T(y),y,0\right] \tag{34}$$

Equation (33b) can then be written in the form

$$x,y \text{ inside } R_w^*, \quad \frac{\partial\psi}{\partial x} = -\frac{i\nu}{2}\Lambda(y)e^{i\nu(x_T-x)} \tag{35}$$

In view of equations (33a), there may also be written

$$\Lambda(y) = 2\int_{x_L}^{x_T} \frac{\partial\psi(x,y,0)}{\partial x}\,dx \tag{36}$$

where $x_L(y)$ is the coordinate of the leading edge of the airfoil region $R_a{}^*$.

SUMMARY OF THE RESULTANT BOUNDARY-VALUE PROBLEM

Before proceeding with the solution of the problem as reduced in the foregoing section of this report its final formulation is recapitulated as follows.

Determine the solution of the differential equation

$$\nabla^2\psi + \kappa^2\psi = 0 \tag{23}$$

in the half space $z \geqslant 0$ subject to the following conditions at $z = 0$:

$$x,y \text{ inside } R_a{}^*, \quad \frac{\partial\psi}{\partial z} = g(x,y) \tag{28a}$$

$$x,y \text{ inside } R_w{}^*, \quad \frac{\partial\psi}{\partial x} = -\frac{i\nu}{2}\Lambda(y)e^{i\nu(x_T-x)} \tag{35}$$

$$x,y \text{ inside } R_r{}^*, \quad \frac{\partial\psi}{\partial x} = 0 \tag{32b}$$

$$x = x_T(y), \quad \frac{\partial\psi}{\partial x} \text{ is finite} \tag{29a}$$

and subject to the following condition at infinity:

$$z \rightarrow \infty, \quad \psi = fe^{-i\kappa r} \tag{31a}$$

where $f = O(z)$ and where $r^2 = x^2 + y^2 + z^2$.

Various quantities occurring in these equations are defined as follows:

$$g(x,y) = \frac{Ue^{-i\mu x}}{\sqrt{1 - M^2}}\left(ik\overline{H} + \frac{\partial\overline{H}}{\partial x}\right) \tag{37}$$

$$\Lambda(y) = \int_{x_L}^{x_T} \lambda(x,y)\, dx \tag{36a}$$

$$\lambda(x,y) = 2\,\frac{\partial\psi(x,y,0)}{\partial x} \tag{38}$$

The parameters κ, μ, and ν are defined by equations (24), (22), and (27), respectively. The region $R_a{}^*$ follows from the airfoil region R_a by multiplication with a scale factor $1/b$ in the x-direction and by multiplication with a scale factor $\sqrt{1 - M^2}/b$ in the y-direction. The region $R_w{}^*$ is the strip extending from the trailing edge $x = x_T$ to $x = \infty$, and the region $R_r{}^*$ is the remainder of the x,y-plane.

The solution of the boundary-value problem is to be used to calculate the pressure-change amplitude $\overline{p}_a$ at the lifting surface in accordance with the relation

$$\overline{p}_a = -\frac{\rho_o U}{b}\, e^{i\mu x}\left(i\nu\psi + \frac{\partial\psi}{\partial x}\right)_{R_a{}^*} \tag{39}$$

which follows from equation (26).

The solution of the problem as summarized will be approached through its reduction to an integral equation for the quantity λ as defined by equation (38).

AN INTEGRAL REPRESENTATION FOR THE VALUES OF $\partial\psi/\partial z$

In this section it is proposed to derive a formula for the values of $\partial\psi/\partial z$, for the purpose of setting up the basic integral equation of the problem under consideration. To begin, results are taken which in essence are known and then are transformed in a way designed to facilitate the subsequent transition from the exact double-integral equation of the problem to the approximate integral equation containing single integrals only.

The first formula is a representation of the values of $\partial\psi/\partial x$ in the interior of the half space $z > 0$ in terms of the values of $\partial\psi/\partial x$ on the boundary $z = 0$ of the half space, as follows:

$$\frac{\partial\psi}{\partial x} = \frac{1}{4\pi}\iint \lambda(\xi,\eta)\frac{\partial}{\partial z}\left(\frac{e^{-i\kappa r}}{r}\right) d\xi\, d\eta \tag{40}$$

In equation (40) and in all that follows,

$$r^2 = (x - \xi)^2 + (y - \eta)^2 + z^2 \tag{41}$$

and the quantity λ is according to equation (38) given by $2\partial\psi(\xi,\eta,0)/\partial\xi$. It is noted that this representation of $\partial\psi/\partial x$ ensures that the conditions at infinity as expressed in equations (31) are satisfied.

From equation (40), it follows that

$$\frac{\partial^2\psi}{\partial x\partial z} = \frac{1}{4\pi}\iint \lambda(\xi,\eta)\frac{\partial^2}{\partial z^2}\left(\frac{e^{-i\kappa r}}{r}\right) d\xi\, d\eta \tag{42}$$

If it is now observed that the quantity $r^{-1}\, e^{-i\kappa r}$ is a solution of the differential equation (23) for ψ, then equation (42) may be written in the alternate form

$$\frac{\partial^2\psi}{\partial x\partial z} = \frac{-1}{4\pi}\iint \lambda(\xi,\eta)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \kappa^2\right)\left(\frac{e^{-i\kappa r}}{r}\right) d\xi\, d\eta \tag{43}$$

In equation (43), it is noted that $\partial^2/\partial y^2 = \partial^2/\partial \eta^2$ and the term in question is integrated by parts with respect to η. In addition to this, by making use of the obvious identity

$$\lambda(\xi,\eta) = \lambda(\xi,y) + \left[\lambda(\xi,\eta) - \lambda(\xi,y)\right]$$

equation (43) is written as follows:

$$\frac{\partial^2\psi}{\partial x\partial z} = \frac{-1}{4\pi}\iint \lambda(\xi,y)\left(\frac{\partial^2}{\partial x^2} + \kappa^2\right)\left(\frac{e^{-i\kappa r}}{r}\right) d\xi\, d\eta -$$

$$\frac{1}{4\pi}\iint \left[\lambda(\xi,\eta) - \lambda(\xi,y)\right]\left(\frac{\partial^2}{\partial x^2} + \kappa^2\right)\left(\frac{e^{-i\kappa r}}{r}\right) d\xi\, d\eta -$$

$$\frac{1}{4\pi}\iint \frac{\partial\lambda}{\partial\eta}\frac{\partial}{\partial y}\left(\frac{e^{-i\kappa r}}{r}\right) d\xi\, d\eta \tag{44}$$

In the first integral on the right of equation (44) the integration with respect to η may be carried out explicitly. When the remaining two integrals are absent, there is thereby obtained the appropriate form of the integral relation (44) which would follow if two-dimensional flow had been assumed from the beginning.

The following formula expresses the integral in question in terms of a Hankel function

$$\int_{-\infty}^{\infty} \frac{e^{-i\kappa\sqrt{(x-\xi)^2+(y-\eta)^2+z^2}}}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + z^2}}\, d\eta = i\pi H_0^{(2)}\left[\kappa\sqrt{(x-\xi)^2 + z^2}\right] \tag{45}$$

The next step is to reduce the second integral on the right of equation (44) to an integral involving $\partial\lambda/\partial\eta$. To this end, there is written

$$\int_{-\infty}^{\infty} \left[\lambda(\xi,\eta) - \lambda(\xi,y)\right]\frac{e^{-i\kappa r}}{r}\, d\eta = \int_{-\infty}^{y} + \int_{y}^{\infty} \tag{46}$$

The two integrals on the right of equation (46) are integrated by parts and the constants of integration are chosen in such a way that the integrated portions vanish. After some elementary transformations, this leads to the following formula:

$$\int_{-\infty}^{\infty} \left[\lambda(\xi,\eta) - \lambda(\xi,y)\right] \frac{e^{-i\kappa\sqrt{(x-\xi)^2+(y-\eta)^2+z^2}}}{\sqrt{(x-\xi)^2+(y-\eta)^2+z^2}}\, d\eta =$$

$$-\int_{-\infty}^{\infty} \frac{\partial\lambda}{\partial\eta} \frac{|y-\eta|}{y-\eta} \left[\int_{-\infty}^{-|y-\eta|} \frac{e^{-i\kappa\sqrt{(x-\xi)^2+z^2+\zeta^2}}}{\sqrt{(x-\xi)^2+z^2+\zeta^2}}\, d\zeta\right] d\eta \quad (47)$$

Equations (45) and (47) are introduced into equation (44) and the following relation is obtained:

$$\frac{\partial^2\psi}{\partial x\partial z} = \frac{-1}{2\pi}\int \lambda(\xi,y)\left(\frac{\partial^2}{\partial x^2} + \kappa^2\right)\left[\frac{i\pi}{2} H_0^{(2)}\left(\kappa\sqrt{(x-\xi)^2+z^2}\right)\right] d\xi +$$

$$\frac{1}{4\pi}\iint \frac{\partial\lambda}{\partial\eta}\left\{\frac{|y-\eta|}{y-\eta}\left(\frac{\partial^2}{\partial x^2} + \kappa^2\right)\int_{-\infty}^{-|y-\eta|} \frac{e^{-i\kappa\sqrt{(x-\xi)^2+z^2+\zeta^2}}}{\sqrt{(x-\xi)^2+z^2+\zeta^2}}\, d\zeta +\right.$$

$$\left.\frac{\partial}{\partial\eta}\left[\frac{e^{-i\kappa\sqrt{(x-\xi)^2+z^2+(y-\eta)^2}}}{\sqrt{(x-\xi)^2+z^2+(y-\eta)^2}}\right]\right\} d\xi\, d\eta \quad (48)$$

The final step now consists in integrating equation (48) with respect to x between the limits $-\infty$ and x. The condition of

undisturbed flow far in front of the lifting surface makes $(\partial\psi/\partial z)_{x=-\infty} = 0$. There is then obtained the relation

$$\frac{\partial\psi}{\partial z} = \frac{-1}{2\pi}\int \lambda(\xi,y)\left(\frac{\partial}{\partial x}\left\{\frac{i\pi}{2} H_0^{(2)}\left[\kappa\sqrt{(x-\xi)^2+z^2}\right]\right\} + \right.$$

$$\left.\kappa^2\int_{-\infty}^{x} \frac{i\pi}{2} H_0^{(2)}\left[\kappa\sqrt{(x'-\xi)^2+z^2}\right] dx'\right) d\xi +$$

$$\frac{1}{4\pi}\iint \frac{\partial\lambda}{\partial\eta}\left\{\frac{|y-\eta|}{y-\eta}\frac{\partial}{\partial x}\int_{-\infty}^{-|y-\eta|} \frac{e^{-i\kappa\sqrt{(x-\xi)^2+z^2+\zeta^2}}}{\sqrt{(x-\xi)^2+z^2+\zeta^2}} d\zeta + \right.$$

$$\frac{|y-\eta|}{y-\eta}\kappa^2\int_{-\infty}^{x}\left[\int_{-\infty}^{-|y-\eta|} \frac{e^{-i\kappa\sqrt{(x'-\xi)^2+z^2+\zeta^2}}}{\sqrt{(x'-\xi)^2+z^2+\zeta^2}} d\zeta\right] dx' +$$

$$\left.\int_{-\infty}^{x}\frac{\partial}{\partial\eta}\left[\frac{e^{-i\kappa\sqrt{(x'-\xi)^2+z^2+(y-\eta)^2}}}{\sqrt{(x'-\xi)^2+z^2+(y-\eta)^2}}\right] dx'\right\} d\xi\, d\eta \tag{49}$$

Equation (49) expresses the values of $\partial\psi/\partial z$ for $z > 0$ in terms of the values of $\lambda(\xi,\eta) = 2\partial\psi(\xi,\eta,0)/\partial\xi$ and is thus the result, the attainment of which was the aim in the present section.

THE INTEGRAL EQUATION OF THE OSCILLATING LIFTING SURFACE

The integral equation of the problem is obtained by substituting the information contained in equations (28), (32), and (35) in equation (49) and by then letting z tend to zero. In so doing the integrals which contain infinities in the integrand are to be interpreted as Cauchy principal values. This latter step has been discussed in detail for the case of incompressible flow in reference 1. No additional difficulties in this regard are encountered for the present problem of compressible subsonic flow and the explicit justification for this step is therefore omitted in the present report.

The integral equation of the oscillating lifting surface in subsonic compressible flow is thus of the following form:

$$g(x,y) = -\frac{1}{2\pi} \oint_{x_L(y)}^{x_T(y)} \lambda(\xi,y) K\left[(x - \xi);\ \kappa\right] d\xi +$$

$$\frac{i\nu}{2\pi} \Lambda(y) e^{i\nu x_T(y)} \int_{x_T(y)}^{\infty} e^{-i\nu\xi}\ K\left[(x - \xi);\ \kappa\right] d\xi +$$

$$\frac{1}{4\pi} \oint\oint_{R_a^*} \frac{\partial\lambda}{\partial\eta} G\left[(x - \xi),\ (y - \eta);\ \kappa\right] d\xi\ d\eta -$$

$$\frac{i\nu}{4\pi} \iint_{R_w^*} \frac{d}{d\eta}\left(\Lambda e^{i\nu x_T}\right) e^{-i\nu\xi}\ G\left[(x - \xi),\ (y - \eta);\ \kappa\right] d\xi\ d\eta \tag{50}$$

The kernels K and G are of the following form:

$$K = \frac{\partial}{\partial x}\left[\frac{i\pi}{2} H_0^{(2)}\left(|x - \xi|\right)\right] +$$

$$\kappa^2 \int_{-\infty}^{x} \frac{i\pi}{2} H_0^{(2)}\left(|x' - \xi|\right) dx' \tag{51}$$

$$G = \frac{|y-\eta|}{y-\eta}\left\{\frac{\partial}{\partial x}\int_{-\infty}^{-|y-\eta|}\frac{e^{-i\kappa\sqrt{(x-\xi)^2+\zeta^2}}}{\sqrt{(x-\xi)^2+\zeta^2}}\,d\zeta + \right.$$

$$\left.\kappa^2\int_{-\infty}^{x}\left[\int_{-\infty}^{-|y-\eta|}\frac{e^{-i\kappa\sqrt{(x'-\xi)^2+\zeta^2}}}{\sqrt{(x'-\xi)^2+\zeta^2}}\,d\zeta\right]dx'\right\} +$$

$$-\int_{-\infty}^{x}\frac{\partial}{\partial\eta}\left[\frac{e^{-i\kappa\sqrt{(x'-\xi)^2+(y-\eta)^2}}}{\sqrt{(x'-\xi)^2+(y-\eta)^2}}\right]dx' \tag{52}$$

Equation (50) holds when x and y are inside $R_a{}^*$ and is to be solved for λ, in terms of Λ and g, where Λ and g are defined by equations (36) and (37), respectively.

For the case of two-dimensional flow, $\partial\lambda/\partial\eta = 0$ and the last two integrals in equation (50) are absent. The remainder of the present work has as its object the derivation of a procedure to take account of these last two integrals in an approximate manner which permits the calculation of the effect of three-dimensionality of the flow in a simpler way than by actually solving the complete equation (50). In the derivation of this procedure the integral over the airfoil region $R_a{}^*$ and the integral over the wake region $R_w{}^*$ are treated separately.

REDUCTION OF THE DOUBLE INTEGRAL OVER THE AIRFOIL REGION $R_a{}^*$

Write

$$\iint_{R_a{}^*}\frac{\partial\lambda}{\partial\eta}\,G\,d\xi\,d\eta = \iint_{R_a{}^*}\frac{\partial\lambda}{\partial\eta}\left[\frac{|y-\eta|}{y-\eta}(I_1+I_2)+I_3\right]d\xi\,d\eta \tag{53}$$

where

$$I_1 = \frac{\partial}{\partial x} \int_{-\infty}^{-|y-\eta|} \frac{e^{-i\kappa\sqrt{(x-\xi)^2+\zeta^2}}}{\sqrt{(x-\xi)^2+\zeta^2}} d\zeta \tag{54}$$

$$I_2 = \kappa^2 \int_{-\infty}^{x} \left[\int_{-\infty}^{-|y-\eta|} \frac{e^{-i\kappa\sqrt{(x'-\xi)^2+\zeta^2}}}{\sqrt{(x'-\xi)^2+\zeta^2}} d\zeta \right] dx' \tag{55}$$

$$I_3 = \int_{-\infty}^{x} \frac{\partial}{\partial \eta} \left[\frac{e^{-i\kappa\sqrt{(x'-\xi)^2+(y-\eta)^2}}}{\sqrt{(x'-\xi)^2+(y-\eta)^2}} \right] dx' \tag{56}$$

When dealing with lifting surfaces of sufficiently elongated form, that is, with surfaces of sufficiently high aspect ratio, $|y-\eta| \gg |x-\xi|$ over the major portion of the surface. When this latter inequality holds, the terms I_n can, as will be seen, be written in such a way that a relatively simple dominant term can be separated in each of them. These dominant terms will be used for the approximation to be developed. There remains then the question concerning the validity of the approximation over that portion of $R_a{}^*$ where the inequality $|x-\xi| \ll |y-\eta|$ does not hold. This question is answered as follows. It is assumed that $\partial\lambda/\partial\eta$ varies sufficiently slowly with η so that in this portion of the region $R_a{}^*$ it is effectively constant. If this is the case, all that is necessary is to take account of the fact that both the kernel G and the approximation to G to be obtained are odd functions of $y-\eta$, so that in both cases the contribution to the value of the integral coming from this portion of $R_a{}^*$ can be neglected.

The aforementioned argument is also implicit in the earlier derivations for the corresponding problem for incompressible flow (references 1 and 2). There appears to be no reason to believe that this particular argument should be less applicable to the problem of subsonic compressible flow than to the problem of incompressible flow.

Equation (54) is written in the form

$$I_1 = -\int_{-\infty}^{-|y-\eta|} \frac{e^{-i\kappa\sqrt{(x-\xi)^2+\zeta^2}}}{(x-\xi)^2+\zeta^2}\left[\frac{x-\xi}{\sqrt{(x-\xi)^2+\zeta^2}} + i\kappa(x-\xi)\right] d\zeta$$

and this implies the following order-of-magnitude relations:

$$I_1 \approx -(x-\xi)\int_{-\infty}^{-|y-\eta|} \frac{e^{-i\kappa|\zeta|}}{\zeta^2}\left(\frac{1}{|\zeta|} + i\kappa\right) d\zeta$$

$$I_1 \approx -(x-\xi)\left[O_m\left(\frac{e^{-i\kappa|y-\eta|}}{|y-\eta|^3}\right) + O_m\left(\frac{\kappa e^{-i\kappa|y-\eta|}}{|y-\eta|^2}\right)\right] \tag{57}$$

Equation (55) is written in the form

$$I_2 = \kappa^2 \int_{-\infty}^{-|y-\eta|}\left[\left(\int_{-\infty}^{0} + \int_{0}^{x-\xi}\right)\frac{e^{-i\kappa\sqrt{(x'')^2+\zeta^2}}}{\sqrt{(x'')^2+\zeta^2}}\, dx''\right] d\zeta$$

$$I_2 = \kappa^2 \int_{-\infty}^{-|y-\eta|}\left[\frac{i\pi}{2} H_0^{(2)}\left(\kappa|\zeta|\right) + \int_{0}^{x-\xi} \frac{e^{-i\kappa\sqrt{(x'')^2+\zeta^2}}}{\sqrt{(x'')^2+\zeta^2}}\, dx''\right] d\zeta \tag{58}$$

In equation (58), note that, when κ is not too small and when $|y-\eta| \gg |x-\xi|$,

$$H_0^{(2)}\left(\kappa|\zeta|\right) = O_m\left(\frac{e^{-i\kappa|\zeta|}}{\sqrt{\kappa\zeta}}\right) \tag{59}$$

$$\int_0^{x-\xi} \frac{e^{-i\kappa\sqrt{(x'')^2+\zeta^2}}}{\sqrt{(x'')^2+\zeta^2}}\,dx'' = O_m\left(\frac{e^{-i\kappa|\zeta|}}{|\zeta|}\right)(x-\xi) \tag{60}$$

Neglect the term in equation (60) compared with the term in equation (59); then,

$$I_2 \approx \kappa^2 \int_{-\infty}^{-|y-\eta|} \frac{i\pi}{2} H_0^{(2)}\left(\kappa|\zeta|\right) d\zeta \tag{61}$$

Note that this approximation ceases to be correct as κ becomes smaller and smaller. However, when this is the case the contribution to the total coming from I_2 becomes negligible because of the factor κ^2 in front of the integral in I_2.

Finally, write for I_3

$$I_3 = \frac{\partial}{\partial\eta}\left(\int_{-\infty}^{0} + \int_0^{x-\xi}\right)\frac{e^{-i\kappa\sqrt{(x'')^2+(y-\eta)^2}}}{\sqrt{(x'')^2+(y-\eta)^2}}\,dx''$$

$$I_3 = \frac{\partial}{\partial\eta}\left[\frac{i\pi}{2} H_0^{(2)}\left(\kappa|y-\eta|\right) + \int_0^{x-\xi} \frac{e^{-i\kappa\sqrt{(x'')^2+(y-\eta)^2}}}{\sqrt{(x'')^2+(y-\eta)^2}}\,dx''\right] \tag{62}$$

Now, when $\kappa|y-\eta|$ is sufficiently large,

$$\frac{\partial}{\partial\eta}\left[H_0^{(2)}\left(\kappa|y-\eta|\right)\right] = O_m\left(\sqrt{\frac{\kappa}{|y-\eta|}}\,e^{-i\kappa|y-\eta|}\right) \tag{63}$$

and

$$\frac{\partial}{\partial\eta}\int_0^{x-\xi}\frac{e^{-i\kappa\sqrt{(x'')^2+(y-\eta)^2}}}{\sqrt{(x'')^2+(y-\eta)^2}}\,dx'' = O_m\left[\frac{(x-\xi)e^{-i\kappa|y-\eta|}}{|y-\eta|^2}\right] + O_m\left[\frac{(x-\xi)\kappa}{|y-\eta|}\,e^{-i\kappa|y-\eta|}\right] \tag{64}$$

Neglecting the terms in equation (64) compared with the terms in equation (63),

$$I_3 \approx \frac{\partial}{\partial\eta}\left[\frac{i\pi}{2}\,H_0^{(2)}\left(\kappa|y-\eta|\right)\right] = \frac{i\pi}{2}\,\kappa\frac{|y-\eta|}{y-\eta}\,H_1^{(2)}\left(\kappa|y-\eta|\right) \tag{65}$$

Comparison of expressions (57), (61), and (65) shows that the contribution of I_1 to the total may also be neglected. Introducing then expressions (61) and (65) into equation (53), the following approximation is obtained

$$\oint_{R_a^*}\oint \frac{\partial\lambda}{\partial\eta}\,G\,d\xi\,d\eta \approx \oint_{R_a^*}\oint \frac{\partial\lambda}{\partial\eta}\,\frac{|y-\eta|}{y-\eta}\left[\kappa^2\int_{-\infty}^{-|y-\eta|}\frac{i\pi}{2}\,H_0^{(2)}\left(\kappa|\zeta|\right)d\zeta + \kappa\,\frac{i\pi}{2}\,H_1^{(2)}\left(\kappa|y-\eta|\right)\right]d\xi\,d\eta \tag{66}$$

Equation (66) contains the fundamental simplification of the kernel G in the region R_a^*.

Noting that the factor of $\partial\lambda/\partial\eta$ in equation (66) does not depend on ξ, there is written further

$$\int_{x_L(\eta)}^{x_T(\eta)} \frac{\partial\lambda}{\partial\eta}\, d\xi = \frac{\partial}{\partial\eta} \int_{x_L(\eta)}^{x_T(\eta)} \lambda\, d\xi - (\lambda)_{x_T} \frac{dx_T}{d\eta} + (\lambda)_{x_L} \frac{dx_L}{d\eta} \tag{67}$$

Take for $(\lambda)_{x_L}$ its value zero immediately in front of the leading edge and for $(\lambda)_{x_T}$ its value $-i\nu\Lambda$ which follows from equations (35) and (38). Therewith, and with equations (36), equation (67) becomes

$$\int_{x_L(\eta)}^{x_T(\eta)} \frac{\partial\lambda}{\partial\eta}\, d\xi = e^{-i\nu x_T(\eta)} \frac{d}{d\eta}\left(\Lambda e^{-i\nu x_T}\right) \tag{68}$$

By introducing equation (68) into equation (66), the following equation is obtained:

$$\iint_{R_a^*} \frac{\partial\lambda}{\partial\eta}\, G\, d\xi\, d\eta = \int e^{-i\nu x_T} \frac{d}{d\eta}\left(\Lambda e^{i\nu x_T}\right) \frac{|y-\eta|}{y-\eta} \left[\kappa^2 \int_{-\infty}^{-|y-\eta|} \frac{i\pi}{2} H_0^{(2)}\left(\kappa|\zeta|\right) d\zeta + \frac{i\pi\kappa}{2} H_1^{(2)}\left(\kappa|y-\eta|\right)\right] d\eta \tag{69}$$

Equation (69) represents the final result of the present section.

REDUCTION OF THE DOUBLE INTEGRAL OVER THE WAKE REGION R_w^*

In reducing the double integral over the airfoil region R_a^* to a simpler approximate form, use has been made of the fact that the span

of R_a^* is appreciably larger than the chord of R_a^*. Evidently this is not the situation for the wake region R_w^* and thus additional considerations are necessary for the integral extended over R_w.

Proceed as follows. Write the last integral on the right of equation (50) in the form

$$\iint_{R_w^*} d\xi\, d\eta = \int \left[\int_x^{\infty} d\xi - \int_x^{x_T(\eta)} d\xi \right] d\eta \tag{70}$$

The second of the two inner integrals on the right of equation (70) may be treated exactly like the integral over R_a^*. The first of the integrals has the property, as will be seen, that it is dependent on x in a simple explicit form.

Taking the second integral first, there results, in analogy to equation (56),

$$\int \frac{d}{d\eta}\left(\Lambda e^{i\nu x_T}\right) \left(\int_x^{x_T(\eta)} e^{-i\nu\xi} G\, d\xi \right) d\eta \approx$$

$$\int \frac{d}{d\eta}\left(\Lambda e^{i\nu x_T}\right) \frac{|y-\eta|}{y-\eta} \left[\kappa^2 \int_{-\infty}^{-|y-\eta|} \frac{i\pi}{2} H_0^{(2)}(\kappa|\zeta|)\, d\zeta + \right.$$

$$\left. \kappa \frac{i\pi}{2} H_1^{(2)}(\kappa|y-\eta|) \right] \int_x^{x_T(\eta)} e^{-i\nu\xi}\, d\xi\, d\eta \tag{71}$$

The remaining integral with respect to ξ in expression (71) will be introduced in evaluated form in the final collected form of the results.

Next the remaining integral in equation (70) is transformed as follows

$$\int \frac{d}{d\eta}\left(\Lambda e^{i\nu x_T}\right)\left[\int_x^\infty e^{-i\nu\xi}G(x-\xi,\ y-\eta)\ d\xi\right] d\eta$$

$$= \int \frac{d}{d\eta}\left(\Lambda e^{i\nu x_T}\right)\left[\int_0^\infty e^{-i\nu(x+\sigma)}G(-\sigma,\ y-\eta)\ d\sigma\right] d\eta$$

$$= e^{-i\nu x}\int \frac{d}{d\eta}\left(\Lambda e^{i\nu x_T}\right)F_M(y-\eta)\ d\eta \tag{72}$$

It may be noted that the function F_M reduces to the function F first introduced by Cicala when $M = 0$, that is, when the fluid is incompressible (references 4 and 5). Combination of equations (72) and (52) results in the following form of F_M,

$$F_M = \int_0^\infty e^{-i\nu\sigma}G(-\sigma,\ y-\eta;\ \kappa)\ d\sigma$$

$$= \frac{|y-\eta|}{y-\eta}\int_0^\infty e^{-i\nu\sigma}\left\{\int_{-\infty}^{-|y-\eta|}\frac{\sigma e^{-i\kappa\sqrt{\sigma^2+\zeta^2}}}{\sigma^2+\zeta^2}\left(\frac{1}{\sqrt{\sigma^2+\zeta^2}}+i\kappa\right)d\zeta\ +\right.$$

$$\kappa^2\int_{-\infty}^{-\sigma}\left(\int_{-\infty}^{-|y-\eta|}\frac{e^{-i\kappa\sqrt{\tau^2+\zeta^2}}}{\sqrt{\tau^2+\zeta^2}}\ d\zeta\right)d\tau\ +$$

$$\left.\int_{-\infty}^{-\sigma}\frac{|y-\eta|e^{-i\kappa\sqrt{\tau^2+(y-\eta)^2}}}{\tau^2+(y-\eta)^2}\left[\frac{1}{\sqrt{\tau^2+(y-\eta)^2}}+i\kappa\right]d\tau\right\}d\sigma \tag{73}$$

Now equations (71) and (72) are combined in accordance with equation (70) in order to obtain the following approximation to the double integral over $R_w{}^*$:

$$\iint_{R_w{}^*} \frac{d}{d\eta}\left(\Lambda e^{i\nu x_T}\right) e^{-i\nu\xi} G \, d\xi \, d\eta \approx$$

$$e^{-i\nu x} \int \frac{d}{d\eta}\left(\Lambda e^{i\nu x_T}\right) F_M(y - \eta) \, d\eta +$$

$$\int \frac{e^{-i\nu x_T} - e^{-i\nu x}}{i\nu} \frac{d}{d\eta}\left(\Lambda e^{i\nu x_T}\right) \frac{|y - \eta|}{y - \eta} \times$$

$$\left[\kappa^2 \int_{-\infty}^{-|y-\eta|} \frac{i\pi}{2} H_0{}^{(2)}\left(\kappa|\zeta|\right) d\zeta + \right.$$

$$\left. \kappa \frac{i\pi}{2} H_1{}^{(2)}\left(\kappa|y - \eta|\right)\right] d\eta \tag{74}$$

THE APPROXIMATE INTEGRAL EQUATION OF THE PROBLEM

Equations (74) and (69) are substituted into equation (50) and the following approximate integral equation is obtained:

$$g(x,y) = -\frac{1}{2\pi} \oint_{x_L}^{x_T} \lambda(\xi,y)K(x - \xi;\ \kappa)\ d\xi +$$

$$\frac{i\nu}{2\pi} \Lambda(y)e^{i\nu x_T} \int_{x_T}^{\infty} e^{-i\nu\xi}K(x - \xi;\ \kappa)\ d\xi +$$

$$\frac{e^{-i\nu x}}{4\pi} \oint \frac{d}{d\eta}\left(\Lambda e^{i\nu x_T}\right) \frac{|y - \eta|}{y - \eta} \left[\kappa^2 \int_{-\infty}^{-|y-\eta|} \frac{i\pi}{2} H_0^{(2)}\left(\kappa|\zeta|\right) d\zeta +\right.$$

$$\left.\kappa \frac{i\pi}{2} H_1^{(2)}\left(\kappa|y - \eta|\right)\right] d\eta - \frac{i\nu}{4\pi} e^{-i\nu x} \int \frac{d}{d\eta}\left(\Lambda e^{i\nu x_T}\right) F_M\ d\eta \qquad (75)$$

Equation (75) represents in preliminary form the result to be obtained in this report. This result is reduced to a somewhat simpler form as follows.

Set

$$\Lambda e^{i\nu x_T} = \Omega \qquad (76)$$

and introduce a new dimensionless spanwise coordinate y^* defined by

$$y^* = \frac{y}{s\sqrt{1 - M^2}} \qquad (77)$$

If b represents the semichord at midspan and sb the semispan, then it follows from the definitions of x and y in equation (19) that the coordinate y^* assumes values in the interval $(-1,1)$ only.

Further, set

$$x_L(y) = x_L^*(y^*), \quad x_T(y) = x_T^*(y^*) \tag{78}$$

and note that $x_L^*(0) = -1$, $x_T(0) = 1$. For a rectangular lifting surface, $-x_L^* = x_T^* = 1$ throughout.

From equations (51) and (65), there follows for the first integral on the right of equation (75),

$$\int_{x_L}^{x_T} \lambda(\xi,y)K\,d\xi = \int_{x_L^*}^{x_T^*} \lambda(\xi,y)\left[-\frac{i\pi\kappa}{2}\frac{|x-\xi|}{x-\xi}H_1^{(2)}\left(\kappa|x-\xi|\right) + \frac{i\pi\kappa}{2}\int_{-\infty}^{\kappa(x-\xi)} H_0^{(2)}\left(|\zeta|\right)d\zeta\right]d\xi \tag{79}$$

The second integral on the right of equation (75) becomes

$$\int_{x_T}^{\infty} e^{-i\nu\xi}K\,d\xi = \int_{x_T^*}^{\infty} e^{-i\nu\xi}\left[-\frac{i\pi\kappa}{2}\frac{|x-\xi|}{x-\xi}H_1^{(2)}\left(\kappa|x-\xi|\right) + \frac{i\pi\kappa}{2}\int_{-\infty}^{\kappa(x-\xi)} H_0^{(2)}\left(|\zeta|\right)d\zeta\right]d\xi \tag{80}$$

The third integral on the right of equation (75) becomes, with κ from equation (24),

$$\int \frac{d}{d\eta}\left(\Lambda e^{i\nu x_T}\right)\frac{|y-\eta|}{y-\eta}\left[\kappa^2 \int_{-\infty}^{-|y-\eta|} \frac{i\pi}{2} H_0^{(2)}\left(\kappa|\zeta|\right) d\zeta + \kappa \frac{i\pi}{2} H_1^{(2)}\left(\kappa|y-\eta|\right)\right] d\eta =$$

$$\int_{-1}^{1} \frac{d\Omega}{d\eta^*} \frac{|y^*-\eta^*|}{y^*-\eta^*}\left[\frac{kM}{1-M^2} \int_{-\infty}^{-\frac{ksM}{\sqrt{1-M^2}}|y^*-\eta^*|} \frac{i\pi}{2} H_0^{(2)}\left(|\zeta|\right) d\zeta + \frac{kM}{1-M^2} \frac{i\pi}{2} H_1^{(2)}\left(\frac{ksM}{\sqrt{1-M^2}}|y^*-\eta^*|\right)\right] d\eta^* \tag{81}$$

Finally, there is written for the fourth integral on the right of equation (75),

$$\int \frac{d}{d\eta}\left(\Lambda e^{i\nu x_T}\right) F_M(y-\eta)\, d\eta = \int_{-1}^{1} \frac{d\Omega}{d\eta^*} F_M\left[\frac{sk}{\sqrt{1-M^2}}(y^*-\eta^*)\right] d\eta^* \tag{82}$$

where the function F_M, as defined by equation (73), may be written in the following alternate form:

$$F_M(x) = \frac{|x|}{x} \int_0^{\infty} e^{-i\sigma} \left[\int_{-\infty}^{-|x|} \frac{\sigma e^{-iM\sqrt{\sigma^2+\zeta^2}}}{\sigma^2 + \zeta^2} \left(\frac{1}{\sqrt{\sigma^2 + \zeta^2}} + iM \right) d\zeta + \right.$$

$$M^2 \int_{-\infty}^{-\sigma} \left(\int_{-\infty}^{-|x|} \frac{e^{-iM\sqrt{\tau^2+\zeta^2}}}{\sqrt{\tau^2 + \zeta^2}} d\zeta \right) d\tau +$$

$$\left. \int_{-\infty}^{-\sigma} \frac{|x| e^{-iM\sqrt{\tau^2+x^2}}}{\tau^2 + x^2} \left(\frac{1}{\sqrt{\tau^2 + x^2}} + iM \right) d\tau \right] d\sigma \tag{83}$$

Now collect equations (82), (81), (80), and (79) and substitute the result in equation (75). This gives

$$g(x,y^*) = -\frac{1}{2\pi} \oint_{x_L^*}^{x_T^*} \lambda(\xi,y)\frac{i\pi\kappa}{2}\left[-\frac{|x-\xi|}{x-\xi} H_1^{(2)}\left(\kappa|x-\xi|\right) + \right.$$

$$\left. \int_{-\infty}^{-\kappa(x-\xi)} H_0^{(2)}\left(|\zeta|\right) d\zeta\right] d\xi +$$

$$\frac{i\nu}{2\pi}\,\Omega(y) \int_{x_T^*}^{\infty} e^{-i\nu\xi}\,\frac{i\pi\kappa}{2}\left[-\frac{|x-\xi|}{x-\xi} H_1^{(2)}\left(\kappa|x-\xi|\right) + \right.$$

$$\left. \int_{-\infty}^{-\kappa(x-\xi)} H_0^{(2)}\left(|\zeta|\right) d\zeta\right] d\xi +$$

$$\frac{e^{-i\nu x}}{4\pi s\sqrt{1-M^2}} \oint_{-1}^{1} \frac{d\Omega}{d\eta^*}\left\{\frac{ksM}{\sqrt{1-M^2}}\,\frac{i\pi}{2}\left[H_1^{(2)}\left(\frac{ksM}{\sqrt{1-M^2}}\,|y^*-\eta^*|\right) + \right.\right.$$

$$\left. \int_{-\infty}^{-\frac{ksM}{\sqrt{1-M^2}}|y^*-\eta^*|} H_0^{(2)}\left(|\zeta|\right) d\zeta\right] -$$

$$\left.\frac{iks}{\sqrt{1-M^2}}\,F_M\left[\frac{ks}{\sqrt{1-M^2}}(y^*-\eta^*)\right]\right\} d\eta^* \tag{84}$$

Finally, there are introduced dimensionless coordinates x^* and ξ^* as follows:

$$x^* = \frac{2x - (x_T + x_L)}{x_T - x_L} \tag{85}$$

so that the interval $x_L \leqslant x \leqslant x_T$ goes over into the interval $-1 \leqslant x^* \leqslant 1$. Further,

$$\left.\begin{aligned} x_T - x_L &= 2b^* \\ \frac{1}{2}(x_T + x_L) &= x_m^* \end{aligned}\right\} \tag{86}$$

where b^* is the ratio of local semichord to semichord b at midspan. Finally, the following quantities are introduced:

$$\left.\begin{aligned} \kappa^* &= b^*\kappa \\ \nu^* &= b^*\nu \\ \Omega^* &= \Omega/b^* \end{aligned}\right\} \tag{87}$$

Note that, for the rectangular plan form, $b^* = 1$, $x_m^* = 0$ and, for the elliptical plan form, $b^* = \sqrt{1 - y^{*2}}$, $x_m^* = 0$.

Equation (84) then assumes the following form:

$$g^*(x^*,y^*) = -\frac{1}{2\pi} \oint_{-1}^{1} \lambda^*(\xi^*,y^*)\kappa^* N\left[\kappa^*(x^* - \xi^*)\right] d\xi^* +$$

$$\frac{i\nu^*}{2\pi} \Omega^*(y^*)e^{-i\nu^* x_m^*} \int_{1}^{\infty} e^{-i\nu^*\xi^*} \kappa^* N\left[\kappa^*(x^* - \xi^*)\right] d\xi^* +$$

$$\frac{e^{-i\nu^*(x^*+x_m^*)}}{4\pi s\sqrt{1 - M^2}} \oint_{-1}^{1} \frac{d\Omega^*}{d\eta^*} S_M\left[\frac{ks}{\sqrt{1 - M^2}}(y^* - \eta^*)\right] d\eta^* \tag{88}$$

Comparison of equations (88) and (84) gives for the kernels N and S_M the following expressions:

$$N(x) = -\frac{i\pi}{2}\left[\frac{|x|}{x} H_1^{(2)}(|x|) - \int_{-\infty}^{x} H_0^{(2)}(|\zeta|)\, d\zeta\right] \tag{89}$$

$$S_M(x) = \frac{i\pi}{2} \frac{ksM}{\sqrt{1 - M^2}} \frac{|x|}{x}\left[H_1^{(2)}(M|x|) + \int_{-\infty}^{-M|x|} H_0^{(2)}(|\zeta|)\, d\zeta\right] -$$

$$i \frac{ks}{\sqrt{1 - M^2}} F_M(x) \tag{90}$$

The task from here on is the following. Equation (88) must be solved for λ^*, in terms of g^* and Ω^*. This part of the problem is exactly as in the two-dimensional theory. The function Ω^* is then to be determined by an integrodifferential equation which is obtained by expressing Ω^* in terms of λ^* in accordance with the definition of Ω^*. This part of the problem is similar to the earlier

work on incompressible flow of references 1, 2, and 6. Finally the solution of the integral equation for Ω^* must be used to obtain expressions for the pressure distribution at the airfoil, as affected by the three-dimensionality of the flow about a wing of finite span.

The results, as expressed by equations (88), (89), (90), and (83), include the special case of two-dimensional flow for which $d\Omega^*/d\eta^* = 0$, and the special case of incompressible flow for which $M = 0$. They also include essentially known results on compressible steady flow for which $k = 0$.

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