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ON THE STABILITY OF THE FREE LAMINAR BOUNDARY LAYER BETWEEN PARALLEL STREAMS

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SUMMARY

An analysis and calculations on the stability of the free laminar boundary layer between parallel streams were made for an incompressible fluid using the Tollmien-Schlichting theory of small disturbances. Because the boundary conditions are at infinity, two solutions of the Orr-Sommerfeld stability equation need not be considered, and the remaining two solutions are exponential in character at the infinite boundaries.

The solution of the stability equation is obtained in powers of $(-i/\alpha R)$, where R is Reynolds number and α is disturbance wave number. With an asymptotic solution as a start, the stability equation is numerically integrated. The eigenvalue problem between α, R, and the disturbance phase velocity c was thus explored by trial-and-error process.

The calculations show that the flow is unstable except for very low Reynolds numbers. The regions of stability and instability in the α, R plane were checked by obtaining damped and amplified solutions on opposite sides of the neutrally stable solution.

INTRODUCTION

Some of the classical problems that have been of interest to many investigators in the field of the stability of parallel flows are the stability of Couette-type motion, Poiseuille-type motion, and boundary-layer flows. Couette flow between rotating cylinders was successfully treated by G. I. Taylor (reference 1). The problem of plane Couette flow, however, has not been decisively settled. Plane Poiseuille flow was treated by Heisenberg (reference 2), Pekeris (references 3 and 4), Goldstein, and Lin (references 5 and 6) and highly controversial results were obtained. Poiseuille motion in a circular pipe was treated by Sexl (references 7 and 8) who, after an incomplete investigation, concluded that the flow was

stable. The problem of boundary-layer stability was treated by Tollmien (references 9 to 11), Schlichting (references 12 to 15) Lin, and others, and an experimental verification of the results was obtained by Schubauer and Skramstad (reference 16).

In all of the previous investigations, the problem of the stability of parallel flows was treated for a case in which there was at least one solid boundary. In the investigation discussed herein, a flow that has no solid boundaries is treated, the case of the stability of the laminar, free boundary layer between two parallel streams of semi-infinite extent in plane flow. In the analysis, the fluid medium is considered incompressible. After the steady-state flow configuration is determined, the stability of that flow configuration is investigated. The solution is carried out for only the case of flow in which one of the streams is considered at rest.

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SYMBOLS

The following symbols are used in the theoretical development:

x	positional coordinate in direction of principal flow
y	positional coordinate perpendicular to principal flow
t	time coordinate
u	velocity component in x-direction
v	velocity component in y-direction
Ψ	stream function
ζ	vorticity

ρ	mass density of fluid medium
μ	absolute viscosity of fluid medium
$\nu = \mu/\rho$	kinematic viscosity of fluid medium
δ	characteristic measure of boundary-layer thickness
U_1	free-stream velocity of one of the parallel streams (taken as characteristic velocity)
U_2	free-stream velocity of other parallel stream, $U_1 > U_2$
$R = \delta U_1/\nu$	Reynolds number of boundary-layer flow
η	$y/\sqrt{\nu x/U_1}$
$f(\eta)$	function defining form of boundary-layer stream function for time-independent flow
$x^* = x/\delta$	dimensionless positional coordinate in direction of principal flow
$y^* = y/\delta$	dimensionless positional coordinate perpendicular to principal flow
$t^* = t\,\frac{U_1}{\delta}$	dimensionless time coordinate
$u^* = u/U_1$	dimensionless velocity component in x^*-direction
$v^* = v/U_1$	dimensionless velocity component in y^*-direction
Ψ^*	dimensionless stream function $\Psi^* = \Phi(y^*) + \varphi(y^*)e^{i\alpha(x^*-ct^*)}$
$\Phi(y^*)$	steady-state part of dimensionless stream function
$\varphi(y^*)$	amplitude of disturbance part of dimensionless stream function
$\varphi'(y^*) =$	$\frac{d\varphi}{dy^*}$

$f'(\eta) =$	$\frac{df}{d\eta}$
α	wave number of disturbance (always real)
α_s	eigenvalue of α when $R \rightarrow \infty$
c	dimensionless phase velocity of disturbance
c_s	eigenvalue of c when $R \rightarrow \infty$
i	$\sqrt{-1}$
Erfc $z =$	$\int_z^\infty e^{-t^2}\,dt$
k_1, k_2, k_3	constants of integration
Im	imaginary part of
exp	base of Napierian logarithmic system e raised to power in parentheses following exp

THEORETICAL CONSIDERATIONS

The Steady-State, Laminar, Boundary-Layer Flow between Parallel Streams

The equations for the steady-state incompressible boundary-layer flow with no body forces and no pressure gradient over the flow field are

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \tag{1}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2}$$

From continuity considerations (equation (2)), a stream function Ψ can be so introduced that

$$u = \frac{\partial \Psi}{\partial y}$$

$$v = -\frac{\partial \Psi}{\partial x}$$

If the form of Ψ be specified as

$$\Psi = \sqrt{\nu x U_1}\, f(\eta)$$

$$\eta = y/\sqrt{\nu x/U_1}$$

then

$$u = U_1\, f'(\eta)$$

$$v = \frac{1}{2}\sqrt{\nu U_1/x}\,(\eta f' - f)$$

and equation (1) simplifies to

$$ff'' + 2f''' = 0 \tag{3}$$

From the physical flow configuration (fig. 1), the boundary conditions are

$$y \rightarrow +\infty \qquad u \rightarrow U_1$$

$$y \rightarrow -\infty \qquad u \rightarrow U_2$$

or, in terms of the variables of equation (3),

$$\eta \rightarrow +\infty \qquad f' \rightarrow 1$$

$$\eta \rightarrow -\infty \qquad f' \rightarrow \frac{U_2}{U_1}$$

Because equation (3) is of the third order, a third boundary condition must be specified. The third boundary condition is arbitrarily selected as

$$\eta = \eta_0 \qquad f = 0$$

It is of interest to discuss the asymptotic behavior of the boundary-layer equation (equation (3)).

When

$$\eta \rightarrow \infty$$

$$f'(\eta) \rightarrow 1$$

and

$$f(\eta) \longrightarrow \eta$$

Therefore, as a first approximation for large positive values of η, equation (3) can be written as

$$\eta f'' + 2f''' \cong 0$$

and it therefore follows that

$$1 - f' \cong k_1 \operatorname{Erfc}\left(\frac{\eta}{2}\right) \tag{4}$$

Similarly, for large negative values of η,

$$f' - \frac{U_2}{U_1} \cong - k_2 \operatorname{Erfc}\left(-\sqrt{\frac{U_2}{U_1}}\,\frac{\eta}{2}\right) \tag{5}$$

Equation (5) degenerates for $U_2/U_1 = 0$. For that case, if it is assumed that when

$$\eta \longrightarrow -\infty$$

$$f'(\eta) \longrightarrow 0$$

and

$$f(\eta) \longrightarrow -a$$

then

$$f'(\eta) \longrightarrow k_3\, e^{\frac{a}{2}\eta}$$

It is also of interest to discuss the position of the boundary between the fluid of both streams. This boundary must be a streamline and also pass through the point $(x = 0,\ y = 0)$. From the form of the stream function Ψ, it can then be seen that $\Psi = 0$ is the boundary streamline. In order for this condition to hold at any value of x, it is necessary that $f(\eta) = 0$ along the streamline. The boundary streamline is therefore

$$\eta_0 = \frac{y}{\sqrt{\frac{\nu x}{U_1}}}$$

The significance of the η_0 line can also be demonstrated from momentum considerations. If, for a fixed value of x, the boundary streamline is located at y_1, the momentum principle can be formulated as follows:

$$\int_{y_1}^{\infty} \rho u(U_1-u)\ dy + \int_{-\infty}^{y_1} \rho u(U_2-u)\ dy = 0 \tag{6}$$

If f and η are then substituted into equation (6), the following equation is obtained:

$$\int_{\eta_1}^{\infty} f'(1-f')\ d\eta + \int_{-\infty}^{\eta_1} f'\left(\frac{U_2}{U_1} - f'\right) d\eta = 0$$

where η_1 is the boundary between the two streams.

Then

$$f(\infty) - \left(1 - \frac{U_2}{U_1}\right) f(\eta_1) - \frac{U_2}{U_1} f(-\infty) - \int_{-\infty}^{+\infty} (f')^2\ d\eta = 0 \tag{7}$$

However,

$$\int (f')^2\ d\eta = ff' - \int ff''\ d\eta$$

From equation (3)

$$ff'' = -2f'''$$

Therefore,

$$\int (f')^2\ d\eta = ff' + 2f''$$

When the preceding relation is substituted into equation (7),

$$f(\infty) - \left(1 - \frac{U_2}{U_1}\right) f(\eta_1) - \frac{U_2}{U_1} f(-\infty) - f(\infty)\ f'(\infty) + f(-\infty)\ f'(-\infty) -$$

$$2f''(\infty) + 2f''(-\infty) = 0 \tag{8}$$

However,

$$f''(\infty) = f''(-\infty) = 0$$

Equation (8) therefore reduces to

$$f(\infty)\left[1-f'(\infty)\right] - f(-\infty)\left[\frac{U_2}{U_1} - f'(-\infty)\right] - \left(1 - \frac{U_2}{U_1}\right)f(\eta_1) = 0$$

From the asymptotic behavior of f (equations (4) and (5))

$$f(\infty)\left[1-f'(\infty)\right] = 0$$

$$f(-\infty)\left[\frac{U_2}{U_1} - f'(-\infty)\right] = 0$$

Therefore

$$f(\eta_1) = 0$$

and $\eta_1 = \eta_0$, as previously demonstrated.

The Stability of the Laminar Boundary Layer between Parallel Streams

The criterion used in the investigation of the stability of the laminar-flow configuration is the behavior with respect to time of a small periodic disturbance introduced into the flow field. If the disturbance is damped with respect to time, the flow configuration is said to be stable with respect to that disturbance. If the disturbance grows with respect to time, the flow is said to be unstable. If the disturbance remains unchanged with time, the flow is considered as neutrally stable.

In reference 17, Squire demonstrates that two-dimensional disturbances are more destabilizing than three-dimensional disturbances for two-dimensional parallel flows. Only two-dimensional disturbances are therefore considered in this discussion.

The equation of vorticity for an incompressible viscous fluid in plane flow can be stated as

$$\zeta_t + u\zeta_x + v\zeta_y = \nu\Delta\zeta \tag{9}$$

where

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$\zeta_t = \frac{\partial \zeta}{\partial t} \qquad \zeta_x = \frac{\partial \zeta}{\partial x} \qquad \zeta_y = \frac{\partial \zeta}{\partial y}$$

$$\Delta\zeta = \frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2}$$

Now, let ζ, u, and v consist of steady-state (time-independent) and disturbance (time-dependent) parts.

Then

$$\zeta = \bar{\zeta}(x,y) + \zeta'(x,y,t)$$

$$u = \bar{u}(x,y) + u'(x,y,t)$$

$$v = \bar{v}(x,y) + v'(x,y,t)$$

where

$$\zeta' \ll \bar{\zeta} \qquad u' \ll \bar{u} \qquad v' \ll \bar{v}$$

If the expressions for ζ, u, and v are substituted in equation (9), the steady-state terms cancel out as satisfying themselves. Because the disturbance is considered very small compared with the steady-state flow, all terms nonlinear in the disturbance are neglected. Equation (9) then becomes

$$\zeta'_t + \bar{u}\zeta'_x + u'\bar{\zeta}_x + \bar{v}\zeta'_y + v'\bar{\zeta}_y = \nu\Delta\zeta'$$

It is assumed that u' and v' are of the same order of magnitude and that ζ'_x and ζ'_y are of the same order of magnitude. If it is now considered that the steady-state flow is of the boundary-layer variety and that the principal direction of flow is the x-direction, then

$$\bar{v} \ll \bar{u} \quad \text{and} \quad \bar{\zeta}_x \ll \bar{\zeta}_y$$

Equation (9) then reduces to

$$\zeta'_t + \bar{u}\zeta'_x + v'\bar{\zeta}_y = \nu\Delta\zeta' \tag{10}$$

If a stream function of the form

$$\Psi^* = \Phi(y^*) + \varphi(y^*)\, e^{i\alpha(x^*-ct^*)} \tag{11}$$

is introduced into equation (10) after the dimensionless variables u^*, v^*, x^*, and y^* are substituted for the corresponding dimensional variables, the Orr-Sommerfeld equation is then obtained (references 18 and 19).

$$(\Phi'-c)(\varphi''-\alpha^2\varphi) - \Phi'''\varphi = -\frac{i}{\alpha R}(\varphi^{IV}-2\alpha^2\varphi'' + \alpha^4\varphi) \tag{12}$$

where

$$\Phi' = \frac{d\Phi}{dy^*} = u^* = \frac{u}{U_1} = f'(\eta)$$

A more rigorous and detailed derivation of the Orr-Sommerfeld equation may be found in reference 20.

If the characteristic length δ is set equal to $\sqrt{\frac{\nu x}{U_1}}$, then

$$y^* = \eta$$

and

$$\Phi''' = \frac{d^3\Phi}{dy^{*3}} = \frac{d^3f}{d\eta^3} = f'''$$

The Orr-Sommerfeld equation for the laminar boundary layer therefore becomes

$$(f'-c)(\varphi''-\alpha^2\varphi) - f'''\varphi = -\frac{i}{\alpha R}(\varphi^{IV} - 2\alpha^2\varphi'' + \alpha^4\varphi) \tag{13}$$

For boundary conditions on equation (13), it suffices to specify that φ is bounded throughout the flow field.

Because equation (13) is of the fourth order, a set of four linearly independent solutions to the equation exists.

$$\varphi = c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3 + c_4\varphi_4$$

The nature of the solutions φ_3 and φ_4 can be investigated by introducing the following transformation into equation (13):

$$\varphi = \exp\left(\int g\, d\eta\right)$$

Equation (13) then becomes

$$(f'-c)\left[(g^2+g')-\alpha^2\right]-f''' = -\frac{i}{\alpha R}\left[g^4 + 6g^2g' + 3g'^2 + 4gg'' + g''' - 2\alpha^2\,(g^2 + g') - \alpha^4\right]$$

If g is expanded as follows:

$$g = (\alpha R)^{\frac{1}{2}}\, g_0 + g_1 + (\alpha R)^{-\frac{1}{2}}\, g_2 + \ldots$$

the following set of equations is then obtained by equating like powers of αR.

$$(f'-c)\, g_0^{\,2} = -\, i\, g_0^{\,4}$$

$$(f'-c)(g_0' + 2g_0g_1) = -i(4g_0^{\,3}g_1 + 6g_0^{\,2}g_0')$$

. . . . . .

The successive approximations can then be obtained without integration

$$g_0 = \pm\sqrt{i(f'-c)}$$

$$g_1 = -\frac{5}{2}\frac{g_0'}{g_0}$$

. . . . . .

Finally

$$\varphi_3 \cong (f'-c)^{-\frac{5}{4}} \exp\left[-\int \sqrt{i\ \alpha R(f'-c)}\ dy\right]$$

$$\varphi_4 \cong (f'-c)^{-\frac{5}{4}} \exp\left[+\int \sqrt{i\ \alpha R(f'-c)}\ dy\right].$$

It should be noted that when

$$\eta \longrightarrow -\infty \qquad \varphi_3 \longrightarrow \infty$$

$$\eta \longrightarrow +\infty \qquad \varphi_4 \longrightarrow \infty$$

Because φ_3 and φ_4 are each unbounded somewhere in the flow field, they cannot be considered in the solution of equation (13).

φ_1 and φ_2 can be solved for in the following manner:

Let

$$\varphi = \sum_{k=0}^{n} \left(-\frac{i}{\alpha R}\right)^k \varphi^{(k)} \tag{14}$$

If equation (14) is substituted into equation (13) and like orders of $\left(-\frac{i}{\alpha R}\right)$ are equated, the following system of equations is obtained:

$$(f'-c)(\varphi^{(k)''}-\alpha^2\varphi^{(k)})-f'''\varphi^{(k)}=\varphi^{(k-1)'V}-2\alpha^2\varphi^{(k-1)''}+\alpha^4\varphi^{(k-1)} \tag{15}$$

For $k = 0$, equation (15), after being rearranged, becomes

$$\varphi^{(0)''} - \left(\alpha^2 + \frac{f'''}{f'-c}\right)\varphi^{(0)} = 0 \tag{16}$$

The asymptotic behavior of $\varphi^{(0)}$ can now be investigated. Where

$$\eta \longrightarrow \pm\infty$$

$$\frac{f'''}{f'-c} \longrightarrow 0$$

Therefore,

$$\varphi^{(0)}(\pm\infty) \longrightarrow C_1\ e^{\alpha\eta} + C_2\ e^{-\alpha\eta}$$

Because the solution must remain bounded,

$$C_1 = 0 \qquad \text{when} \quad \eta \longrightarrow +\infty$$

$$C_2 = 0 \qquad\qquad \eta \longrightarrow -\infty$$

In the region where $\varphi^{(0)}$ is exponential, $\varphi^{(1)}$ is also exponential, as can be shown by substitution into equation (15). It can thus be demonstrated that $\varphi^{(k)}$ is also exponential in the aforementioned region. The proper boundary conditions on φ are therefore

$$\eta \longrightarrow -\infty \qquad \varphi' \longrightarrow \alpha\varphi$$

$$\eta \longrightarrow +\infty \qquad \varphi' \longrightarrow -\alpha\varphi$$

From the boundary conditions and the general form of the solution, the secular relation formally stating the eigenvalue problem can be easily obtained.

$$F_{(\alpha,c,R)} = \begin{vmatrix} \varphi_1'(-\infty) - \alpha\varphi_1(-\infty) & \varphi_2'(-\infty) - \alpha\varphi_2(-\infty) \\ \varphi_1'(+\infty) + \alpha\varphi_1(+\infty) & \varphi_2'(+\infty) + \alpha\varphi_2(+\infty) \end{vmatrix} = 0 \qquad (17)$$

From equation (17), it can be seen that only particular combinations of the parameters α, c, and R allow the boundary conditions on φ to be satisfied.

For a rigorous treatment of the asymptotic expansions for φ_1, φ_2, φ_3, and φ_4, see reference 21.

SOLUTION OF THE BOUNDARY LAYER AND ORR-SOMMERFELD EQUATIONS FOR THE LAMINAR BOUNDARY LAYER BETWEEN PARALLEL STREAMS

The general plan of solution of the boundary layer and Orr-Sommerfeld equations is to carry the analytical methods only far enough to reduce the problem to a routine numerical solution. In the problem under consideration, it is necessary to investigate the asymptotic behavior of the solutions so that boundary conditions at finite values of the independent variable are known in terms of the infinite boundary conditions. After boundary conditions at finite points are obtained, the solution can be continued by ordinary numerical methods.

Asymptotic Behavior of Steady-State Boundary-Layer Equation

The following transformation of equation (3) $ff'' + 2f''' = 0$ is permissible without altering the form of the equation:

$$q(x) = \frac{1}{a} f(\eta)$$

$$x = a\eta + b$$

If the discussion is limited to the case where $U_2/U_1 = 0$, it is then possible to expand a solution for the transformed equation as follows:

$$q = A_0 + A_1 e^{\frac{1}{2}x} + A_2 e^{x} + A_3 e^{\frac{3}{2}x} + \ldots$$

The foregoing expansion is convergent for negative values of x. If a numerical solution is started from inside the area of convergence of the series solution, the asymptotic value of $q'(\infty)$ can be evaluated. Because the asymptotic value of $f'(\infty)$ is 1, the scaling factor a is given by

$$a = \frac{1}{\sqrt{q'(\infty)}}$$

The scaling factor b can be fixed by designating

$$f(\eta_0) = 0$$

The asymptotic form of f is then given by

$$f = a_0 + a_1 e^{\frac{1}{2}a\eta} + a_2 e^{a\eta} + a_3 e^{\frac{3}{2}a\eta} + \ldots$$

and, since f' (starting) $< c$,

$$\frac{f'''}{f'-c} = b_1 e^{\frac{1}{2}a\eta} + b_2 e^{a\eta} + b_3 e^{\frac{3}{2}a\eta}$$

The various coefficients of the expansion can easily be evaluated by elementary methods.

Asymptotic Behavior of Expanded Orr-Sommerfeld Equation

If the asymptotic form of $f'''/(f'-c)$ is inserted into equation (16), the following equation is obtained:

$$\varphi^{(0)''} - \left(\alpha^2 + b_1 e^{\frac{1}{2}a\eta} + b_2 e^{a\eta} + b_3 e^{\frac{3}{2}a\eta} + \ldots\right)\varphi^{(0)} = 0$$

and the form of $\varphi^{(0)}$ follows:

$$\varphi^{(0)} = e^{a\eta} + h_{10} e^{(a + \frac{1}{2}a)\eta} + h_{20} e^{(\alpha + a)\eta} + h_{30} e^{(\alpha + \frac{3}{2}a)\eta} + \ldots \tag{18}$$

If $\varphi^{(0)}$ is substituted into equation (15) to obtain the next order term $\varphi^{(1)}$, the following equation is obtained:

$$\varphi^{(1)''} - \left(\alpha^2 + b_1 e^{\frac{1}{2}a\eta} + b_2 e^{a\eta} + b_3 e^{\frac{3}{2}a\eta} + \ldots\right) \varphi^{(1)}$$

$$= l_1 e^{(\alpha + \frac{1}{2}a)\eta} + l_2 e^{(\alpha + a)\eta} + l_3 e^{(\alpha + \frac{3}{2}a)\eta}$$

and

$$\varphi^{(1)} = e^{a\eta} + h_{11} e^{(\alpha + \frac{1}{2}a)\eta} + h_{21} e^{(\alpha + a)\eta} + h_{31} e^{(\alpha + \frac{3}{2}a)\eta} + \ldots \tag{19}$$

From equations (18) and (19), the boundary conditions can be obtained for integrating the stability equations starting at a finite η.

A tabulation of the values for the coefficients of the first few terms of the expansions is given in the appendix.

Integration of Boundary-Layer Equation

The boundary-layer equation is

$$ff'' + 2f''' = 0 \tag{3}$$

or

$$f''' = -\frac{1}{2} ff''$$

Differentiation once gives

$$f^{IV} = -\frac{1}{2} (ff''' + f'f'')$$

and twice,

$$f^{V} = -\frac{1}{2} (ff^{IV} + 2f'f''' + f''^2)$$

Expansion in a Taylor's series gives

$$f(\eta+w) = f(\eta)+wf'(\eta)+\frac{w^2}{2}f''(\eta)+\frac{w^3}{6}f'''(\eta)+\frac{w^4}{24}f^{IV}(\eta)+\frac{w^5}{120}f^{V}(\eta)$$

$$f'(\eta+w) = f'(\eta)+wf''(\eta)+\frac{w^2}{2}f'''(\eta)+\frac{w^3}{6}f^{IV}(\eta)+\frac{w^4}{24}f^{V}(\eta)+\ldots$$

$$f''(\eta+w) = f''(\eta)+wf'''(\eta)+\frac{w^2}{2}f^{IV}(\eta)+\frac{w^3}{6}f^{V}(\eta)+\frac{w^3}{24}\delta f^{V}(\eta)+\ldots$$

where

$$\delta f^{V}(\eta) = f^{V}(\eta) - f^{V}(\eta-w)$$

The values of $f'''(\eta+w)$, $f^{IV}(\eta+w)$, $f^{V}(\eta+w)$, and $\delta f^{V}(\eta+w)$ can now be algebraically computed as indicated and the integration carried forth for the next interval. The process can be started by evaluating δf^{V} from the asymptotic form of the solution.

Integration of Expanded Orr-Sommerfeld Equation

Consider the equations for the first two terms of the expansion (equation (15)) of φ:

$$(f'-c)\left(\varphi^{(0)''}-\alpha^2\varphi^{(0)}\right)-f'''\varphi^{(0)} = 0$$

$$(f'-c)\left(\varphi^{(1)''}-\alpha^2\varphi^{(1)}\right) - f'''\varphi^{(1)} = \left(\varphi^{(0)IV} - 2\alpha^2\varphi^{(0)''} + \alpha^4\varphi^{(0)}\right)$$

Expansion of $\varphi^{(0)}$ in a Taylor's series gives

$$\varphi^{(0)}(\eta+w) = \varphi^{(0)}(\eta)+w\varphi^{(0)'}(\eta)+\frac{w^2}{2}\varphi^{(0)''}(\eta)+\frac{w^3}{6}\varphi^{(0)'''}(\eta)+\frac{w^4}{24}\varphi^{(0)IV}(\eta)+\ldots$$

$$\varphi^{(0)'}(\eta+w) = \varphi^{(0)'}(\eta)+w\varphi^{(0)''}(\eta)+\frac{w^2}{2}\varphi^{(0)'''}(\eta)+\frac{w^3}{6}\varphi^{(0)IV}(\eta)+\frac{w^3}{24}\delta\varphi^{(0)IV}(\eta)+\ldots$$

The values of $\varphi^{(0)''}(\eta+w)$, $\varphi^{(0)'''}(\eta+w)$, $\varphi^{(0)'v}(\eta+w)$, and $\delta\varphi^{(0)'v}(\eta+w)$ can now be computed in a manner similar to the case of the boundary-layer equation and the integration of $\varphi^{(0)}$ can be carried forth. As before, $\delta\varphi^{(0)'v}$ can be evaluated in starting.

Expansion of $\varphi^{(1)}$ in a Taylor's series gives

$$\varphi^{(1)}(\eta+w) = \varphi^{(1)}(\eta)+w\varphi^{(1)'}(\eta)+\frac{w^2}{2}\varphi^{(1)''}(\eta)+\frac{w^3}{6}\varphi^{(1)'''}(\eta) +$$

$$\frac{w^4}{24}\varphi^{(1)'v}(\eta) + \frac{w^5}{120}\varphi^{(1)v}(\eta) + \frac{w^6}{720}\varphi^{(1)v'}(\eta) + \ldots$$

Now $\delta^2\varphi^{(1)}(\eta) = \delta\varphi^{(1)}(\eta+w) - \delta\varphi^{(1)}(\eta)$

therefore

$$\delta^2\varphi^{(1)}(\eta) = \varphi^{(1)}(\eta+w) + \varphi^{(1)}(\eta-w) - 2\,\varphi^{(1)}(\eta)$$

$$= w^2\varphi^{(1)''}(\eta) + \frac{w^4}{12}\varphi^{(1)'v}(\eta) + \frac{w^6}{360}\varphi^{(1)v'}(\eta) + \ldots$$

and

$$\delta^2\varphi^{(1)''}(\eta) = w^2\varphi^{(1)'v}(\eta) + \frac{w^4}{12}\varphi^{(1)v'}(\eta) + \ldots$$

Therefore,

$$\delta^2\left[\varphi^{(1)}(\eta) - \frac{w^2}{12}\varphi^{(1)''}(\eta)\right] \cong w^2\varphi^{(1)''}(\eta)$$

The quantity $\varphi^{(1)}$ can be integrated as indicated in a step-by-step manner. From the asymptotic form of $\varphi^{(1)}$, $\delta\left(\varphi^{(1)} - \frac{w^2}{12}\varphi^{(1)''}\right)$ can be evaluated in starting.

When $\varphi^{(0)}$ and $\varphi^{(1)}$ have been integrated to a large positive value of η, the boundary condition at $\eta = +\infty$ can be imposed to evaluate R:

$$\varphi \cong \varphi^{(0)} + \left(\frac{-i}{\alpha R}\right)\varphi^{(1)}$$

$$\varphi'(+\infty) + \alpha\varphi(+\infty) = 0$$

Therefore,

$$R = \frac{i}{\alpha}\left[\left(\frac{\varphi^{(1)'}(+\infty) + \alpha\varphi^{(1)}(+\infty)}{\varphi^{(0)'}(+\infty) + \alpha\varphi^{(0)}(+\infty)}\right)\right] \tag{20}$$

Although the foregoing method of integration was used in the solution of the problem, it was not the most efficient scheme. Originally, the problem was programmed for integration along a rectangular path through the complex plane; complex integration was necessary because of the singularity at the point where $f' = c$ in equation (15). It later became apparent that the rectangular path was disadvantageous and that a better choice was a path starting in the third quadrant of the complex plane and travelling in a straight line to a point on the positive, real axis (fig. 2). Most of the original program could be used for the straight-line path. In order to expedite the solution, the equations were therefore integrated along the new path in the manner previously indicated. The interval of integration w was taken as $0.2 + 0.05i$.

A more advantageous scheme for the straight-line path of integration of equation (15) is included herewith for completeness.

$$\left.\begin{aligned} \delta^2\left(\varphi^{(k)} - \frac{w^2}{30}\varphi^{(k)''}\right) &= w^2\varphi^{(k)''} + \frac{w^4}{20}\varphi^{(k)'v} \\ \delta^2\left(\varphi^{(k)''} - \frac{w^2}{12}\varphi^{(k)'v}\right) &= w^2\varphi^{(k)'v} \end{aligned}\right\} \tag{21}$$

Method of Solution of Eigenvalue Problem

If sets of eigenvalues of α, c, and R are to have any physical significance, R must be real; α is taken real and c may be complex. The process then to be followed is:

1. For a fixed value of α and a set of values of c, integrate the equations and obtain the corresponding values of R. These values of R are usually complex.

2. For the value of R that is real, the corresponding c is the desired eigenvalue.

3. Repeat the process until the secular relation (equation (17)) is explored.

RESULTS AND DISCUSSION

The secular relation (equation (17)) for the case of the "free" boundary layer was explored for damped, neutrally stable, and amplified disturbances. The stable disturbance corresponds to $\text{Im } c < 0$, the neutrally stable disturbance corresponds to $\text{Im } c = 0$, and the unstable disturbance corresponds to $\text{Im } c > 0$ (equation (11)). The disturbance equations were solved for values of $\text{Im } c = 0$, $\text{Im } c = 0.05$, $\text{Im } c = 0.10$, and $\text{Im } c = -0.05$ (table I). The results were then interpolated for $\text{Im } R = 0$ to give the neutral stability curve and curves of equivalent degrees of damping and amplification, (table II and fig. 3).

The curves obtained were continuous, which indicates a low effect of rounding errors in the numerical techniques employed. Some integrations were redone using an interval of half that used throughout the problem. The results of both integrations were the same to the fifth significant figure and indicated the low truncation error. Because the parameter of expansion of the eigenfunction φ (equation (14)) was $(-i/\alpha R)$ and only the first two terms of that expansion were used, the determination of the eigenvalues of α, c, and R was inaccurate at low values of αR. For the same reason, it was impossible to obtain the lower branch of the curve of neutral stability. For purposes of comparison with the free boundary layer, Lin's curve of neutral stability for Blasius-type flow is included in figure 3. (See references 5 and 6.)

A comparison of the stability characteristics of the free boundary layer with Blasius-type flow reveals certain basic differences. First, the free boundary layer is unstable for some disturbances at an infinite Reynolds number (tantamount to "inviscid" flow), whereas the Blasius-type flow is completely stable in that range. This type of instability is caused by the presence of a point of inflection in the velocity profile. Some implications of the presence of a point of inflection in the velocity profile were first investigated by Rayleigh (reference 22). Another salient difference between the two types of flow is the presence of a highly oscillatory solution (φ_3) in the general solution for the disturbance in Blasius-type flow, whereas no such solution exists for the free boundary layer. This difference arises from the doubly-infinite boundary conditions for the free boundary layer as opposed to the singly-infinite boundary conditions for Blasius flow. The foregoing is apparent from the asymptotic forms of the solutions φ_3 and φ_4. As shown by the form of the solutions, viscosity has a second-order effect on φ_1 and φ_2, whereas it has a first-order

effect on φ_3 and φ_4. The inclusion of φ_3 in the general solution of the disturbance for Blasius flow therefore indicates a greater effect of viscosity on the disturbance than for the case of the free boundary layer. As could be inferred from the foregoing, therefore, the effect of viscosity on the stability characteristics of the free boundary layer is apparent only at very low Reynolds number, whereas the Blasius flow stability characteristics are much more affected even at higher Reynolds numbers.

The inaccuracies due to small values of the parameter αR can be avoided by direct integration of the Orr-Sommerfeld equation for those cases. The asymptotic solution of the entire disturbance function could easily be developed as was done for the expanded disturbance function, and the numerical technique of integration after starting from the asymptotic solution would correspond to equation (21). These further solutions should be performed as soon as more high-speed computing-machine service can be obtained.

CONCLUSIONS

It is concluded that the laminar boundary layer between parallel streams is an unstable-flow configuration except at low Reynolds numbers. The method of calculation of stability characteristics is successful for small absolute values of the parameter $(-i/\alpha R)$.

In comparison with Blasius-type flow against a flat plate, instability occurs at much lower Reynolds numbers for the free boundary layer than for the boundary layer against a flat plate with no pressure gradient.

Lewis Flight Propulsion Laboratory,
National Advisory Committee for Aeronautics,
Cleveland, Ohio, March 21, 1949.

APPENDIX - COEFFICIENTS FOR ASYMPTOTIC SOLUTIONS OF φ_0 AND φ_1

$$a = 1.23849316$$

$$a_0 = -1.23849316$$

$$a_1 = 1.23849316$$

$$a_2 = -.30962329$$

$$a_3 = .08600647$$

$$b_1 = -\frac{a^3}{8c} a_1$$

$$b_2 = -\frac{a^3}{c}\left(a_2 + \frac{a}{16c} a_1^2\right)$$

$$b_3 = -\frac{a^3}{c}\left(\frac{27}{8} a_3 + \frac{5a}{8c} a_1 a_2 + \frac{a^2}{32c^2} a_1^3\right)$$

$$d_0 = -\frac{1}{c}$$

$$d_1 = -\frac{a}{2c^2} a_1$$

$$d_2 = -\frac{a}{c^2}\left(a_2 + \frac{a}{4c} a_1^2\right)$$

$$h_{10} = \frac{b_1}{\alpha a + \left(\frac{a}{2}\right)^2}$$

$$h_{20} = \frac{b_1 h_{10} + b_2}{2\alpha a + a^2}$$

$$h_{30} = \frac{b_1 h_{20} + b_2 h_{10} + b_3}{3\alpha a + \frac{3}{2} a^2}$$

$$K_1 = a^2\left(\alpha + \frac{1}{4} a\right)^2 h_{10}$$

$$K_2 = (2a)^2 \left(\alpha + \frac{1}{2} a\right)^2 h_{20}$$

$$K_3 = (3a)^2 \left(\alpha + \frac{3}{4} a\right)^2 h_{30}$$

$$l_1 = d_0 K_1$$

$$l_2 = d_0 K_2 + d_1 K_1$$

$$l_3 = d_0 K_3 + d_1 K_2 + d_2 K_1$$

$$h_{11} = \frac{b_1 + l_1}{\alpha a + \left(\frac{a}{2}\right)^2}$$

$$h_{21} = \frac{b_1 h_{11} + b_2 + l_2}{2\alpha a + a^2}$$

$$h_{31} = \frac{b_1 h_{21} + b_2 h_{11} + b_3 + l_3}{3\alpha a + \left(\frac{3}{2} a\right)^2}$$

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TABLE I - EIGENVALUES OF α, c, AND R

[Inviscid solution ($R \rightarrow \infty$); $c_s = 0.58727198$; and $\alpha_s = 0.39537$.]

α	c	R	α	c	R
		Im c, 0			
0.390	0.584000	292.54 + 34.12i	0.355	0.571000	44.25 - 0.01i
.390	.584550	305.89 + 1.96i	.345	.568600	36.19 + .04i
.390	.584581	307.06 - .01i	.335	.566000	30.83 + .62i
.390	.584600	307.27 - 1.25i	.335	.567000	30.76 + .12i
.390	.585000	312.08 - 28.00i	.335	.567300	30.74 + .02i
.390	.588000	210.66 - 190.29i	.335	.568000	30.68 - .37i
.390	.600000	3.96 - 88.13i	.335	.570000	30.45 - 1.35i
.385	.581500	156.95 + 11.59i	.325	.565000	26.92 + .68i
.385	.582190	160.64 + .25i	.325	.567000	26.75 - .02i
.385	.582204	160.76 + .01i	.315	.550000	24.35 + 4.66i
.385	.582500	161.85 - 5.17i	.315	.555000	24.40 + 3.39i
.380	.579960	109.78 - .01i	.315	.563000	24.09 + 1.29i
.380	.580000	109.83 - .32i	.315	.568000	23.63 + 0 i
.380	.583000	109.18 - 25.38i	.315	.567500	22.68 - 1.69i
.375	.575000	79.73 + 11.71i	.305	.570000	21.12 + .16i
.375	.577700	83.69 + .68i	.305	.570900	21.00 + 0 i
.375	.577850	83.81 + .01i	.305	.571000	21.00 - .02i
.375	.578000	83.94 - .67i	.295	.576200	18.67 + .02i
.375	.580000	84.40 - 10.05i	.295	.576400	18.67 + 0 i
.365	.573600	57.40 + .98i	.295	.576500	18.63 - .02i
.365	.573900	57.49 + .38i	.295	.577100	18.56 - .09i
.365	.574050	57.52 + .07i	.285	.585000	16.38 + .10i
.365	.574100	57.55 + .01i	.285	.586000	16.26 + .02i
.355	.567300	43.68 + 4.08i	.285	.586300	16.25 + 0 i
.355	.570000	44.18 + 1.13i	.285	.586500	16.21 - .02i

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TABLE I - EIGENVALUES OF α, c, AND R - Concluded

α	c	R
	Im c, 0.05	
0.315	0.562500 + 0.05i	72.57 + 0.95i
.315	.562800 + .05i	72.68 - .07i
.285	.540000 + .05i	31.19 + 8.00i
.285	.555000 + .05i	32.55 + .25i
.285	.570000 + .05i	28.87 - 7.54i
.285	.580000 + .05i	24.47 - 10.72i
.265	.548000 + .05i	28.89 + 2.46i
.265	.550000 + .05i	24.79 + 1.90i
.265	.560000 + .05i	23.74 - .79i
.245	.560000 + .05i	19.47 + 1.11i
.245	.570000 + .05i	18.26 - .15i
	Im c, 0.10	
0.265	0.550000 + 0.10i	79.17 + 7.14i
.265	.551700 + .10i	81.12 + .12i
.265	.552000 + .10i	58.03 - 1.29i
.245	.545000 + .10i	39.83 + .88i
.245	.548000 + .10i	39.72 - 1.88i
.245	.550000 + .10i	39.45 - 3.73i
.225	.542000 + .10i	27.74 + 1.48i
.225	.544000 + .10i	27.58 + .75i
.225	.546000 + .10i	27.32 + .03i
.205	.550000 + .10i	21.06 + 1.09i
.205	.554000 + .10i	20.58 + .45i
.205	.557000 + .10i	20.18 + .01i
	Im c, -0.05	
0.400	0.500000 - 0.05i	16.25 + 14.25i
.400	.550000 - .05i	26.91 + 13.76i
.400	.580000 - .05i	34.34 + 1.64i
.400	.590000 - .05i	33.20 - 4.56i
.400	.600000 - .05i	29.59 - 9.72i
.420	.580000 - .05i	46.72 + 10.09i
.420	.590000 - .05i	49.92 - 3.10i

TABLE II - EIGENVALUES OF α, c FOR R REAL

α	c	R	α	c	R
Im c, 0			Im c, 0.05		
0.395	0.587	∞	0.315	0.563 + 0.05i	72.6
.390	.585	307.1	.285	.555 + .05i	32.6
.385	.582	160.8	.265	.557 + .05i	24.1
.380	.580	109.8	.245	.569 + .05i	18.4
.375	.578	83.8	Im c, 0.10		
.365	.574	57.6			
.355	.571	44.3	0.265	0.552 + 0.10i	81.1
.345	.569	36.2	.245	.546 + .10i	39.8
.335	.567	30.7	.225	.546 + .10i	27.3
.325	.567	26.8	.205	.557 + .10i	20.2
.315	.568	23.6	Im c, -0.05		
.305	.571	21.0			
.295	.576	18.7	0.400	0.583 - 0.05i	34.0
.285	.586	16.3	.420	.588 - .05i	48.9

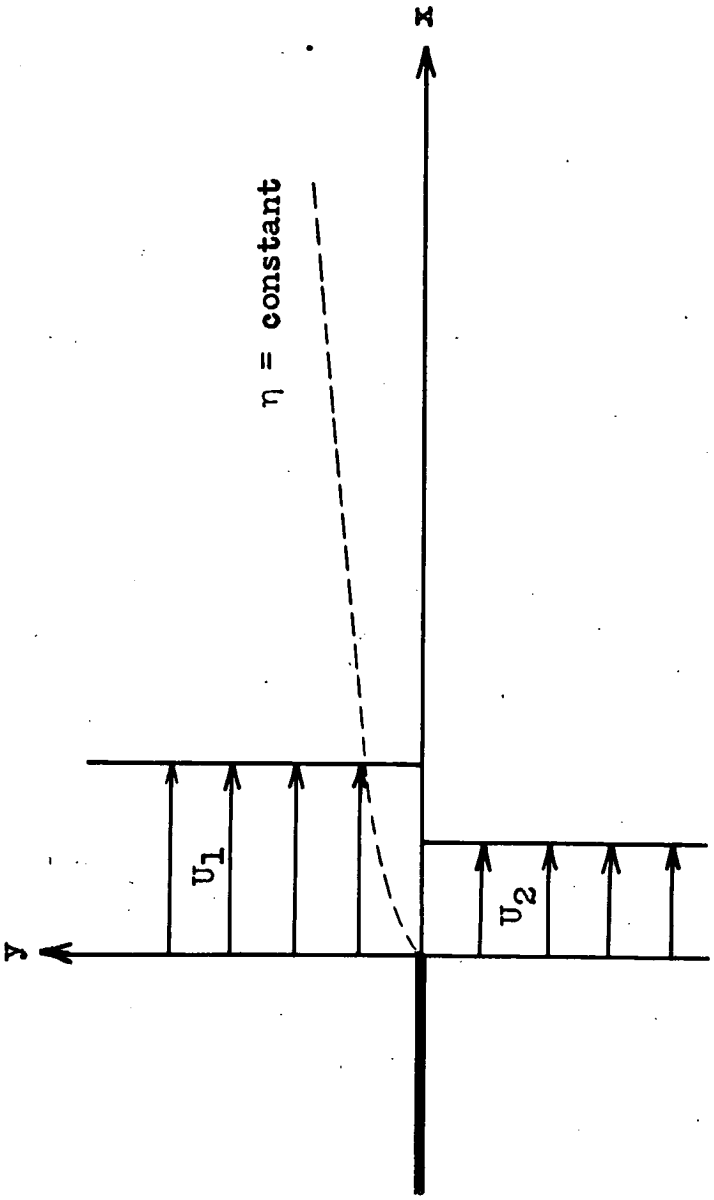

Figure 1. - Flow configuration.

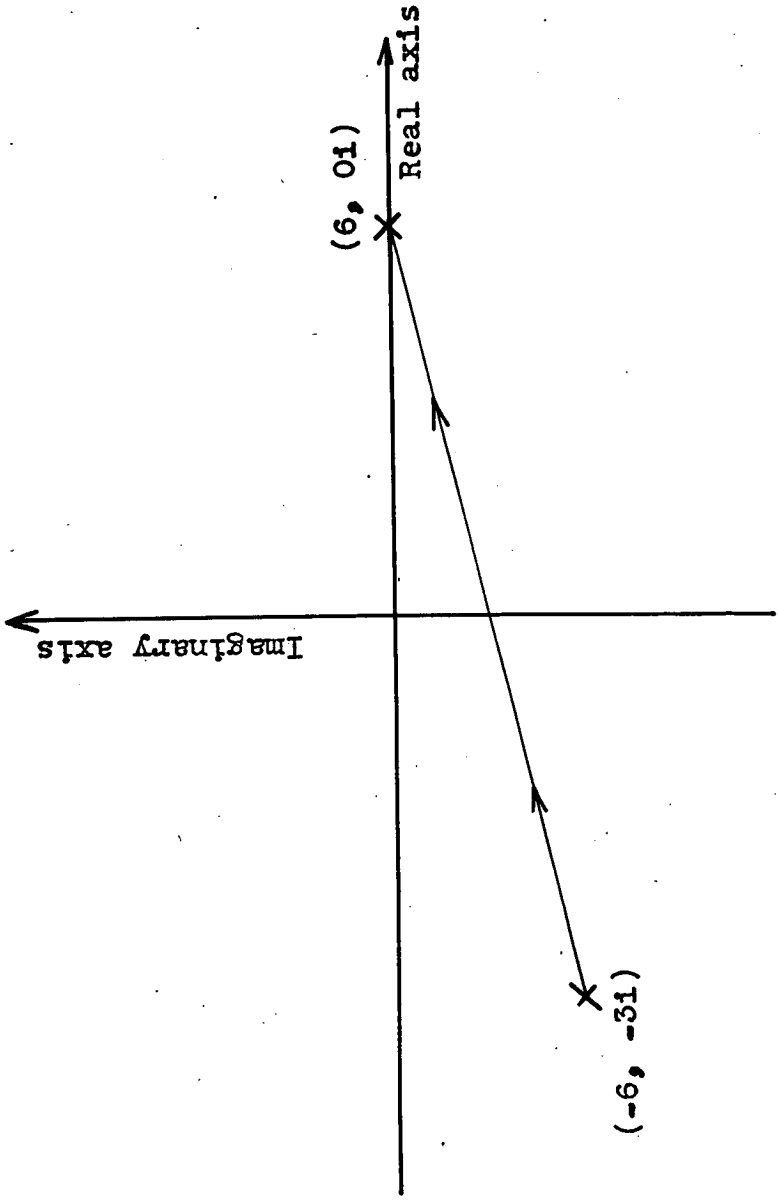

Figure 2. - Path of numerical integration of boundary-layer and disturbance equations in complex η-plane.

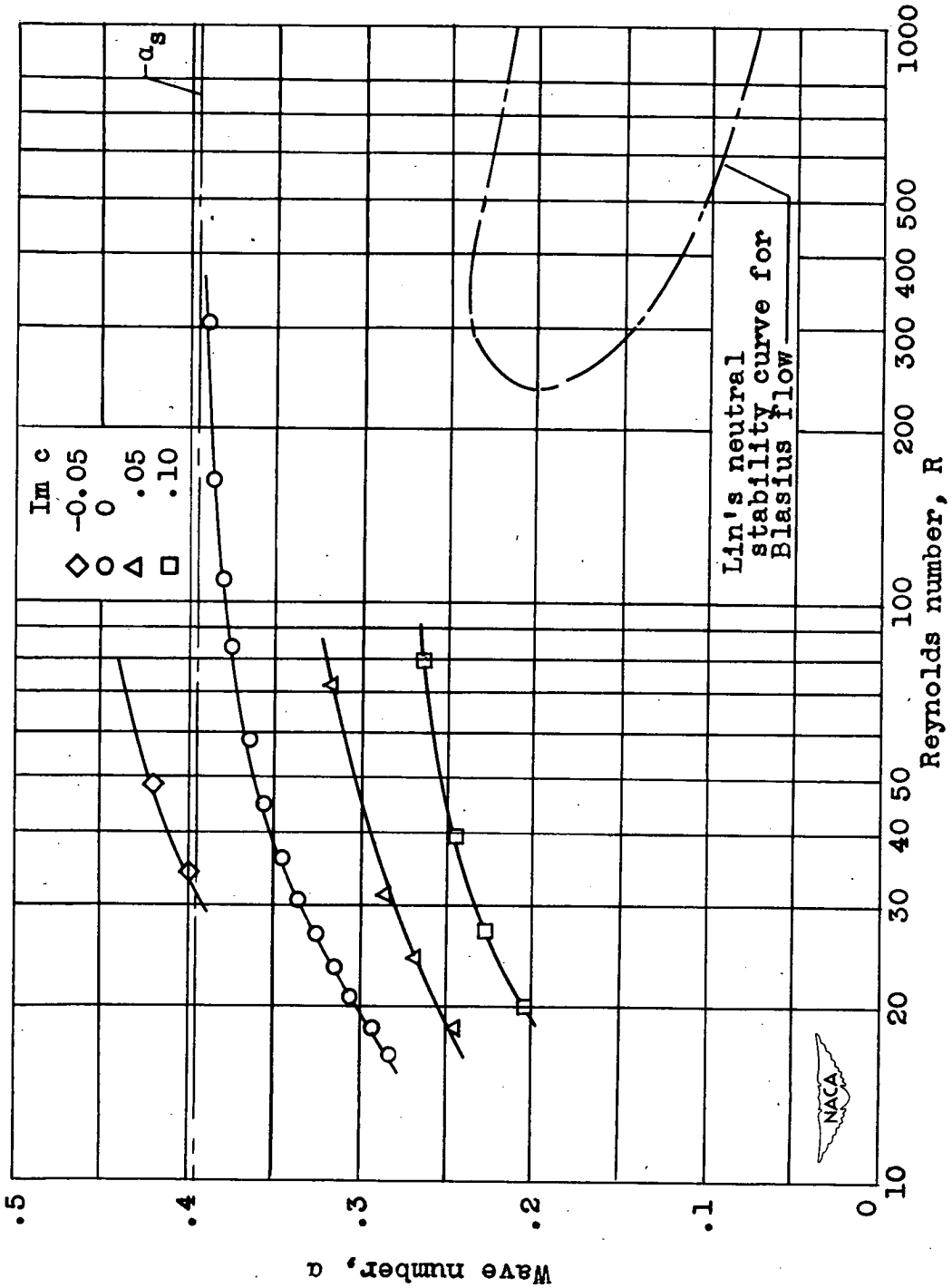

Figure 3. - Curves of equivalent amplification and damping for free boundary layer.