NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2655

CRITICAL STUDY OF INTEGRAL METHODS IN

COMPRESSIBLE LAMINAR BOUNDARY LAYERS

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SUMMARY

A number of the most promising integral methods for solving approximately the compressible-laminar-boundary-layer equations are investigated in order to determine a computationally convenient and sufficiently accurate method of calculating boundary-layer characteristics. The chief methods considered are: (a) The one-parameter Kármán-Pohlhausen method, with three different assumptions for the velocity profiles, and (b) the two-parameter method, first applied by Sutton, with two different assumptions for the velocity profiles. After the methods are explicitly described in general terms for the case of zero pressure gradient and for the case of a pressure gradient in the direction of flow with zero heat transfer, they are applied to the calculation of the compressible laminar boundary layer over a surface with zero pressure gradient, with and without heat transfer at the surface, for the purpose of establishing the accuracy of the methods. Comparison of the results is made with those of known exact solutions for skin-friction and heat-transfer coefficients, velocity profiles, velocity derivatives, and especially laminar-boundary-layer stability. From this comparison it is found that the Kármán-Pohlhausen method with a sixth-degree polynomial as the velocity profile is the most suitable for many practical purposes.

INTRODUCTION

It is well-known that the differential equations of two-dimensional compressible-laminar-boundary-layer flow are difficult to solve exactly. Stewartson (reference 1) and Illingworth (reference 2) have recently shown that if the Prandtl number is unity and the viscosity coefficient is proportional to the temperature, then the equations for the compressible heat-insulated boundary layer with a given pressure gradient can be transformed into the equations for an incompressible boundary layer with a different pressure gradient; however, this principle appears at present tedious to apply in practice.

The most frequently used and most fruitful methods of solving the boundary-layer equations approximately are the integral methods, in which the partial differential equations are integrated over the boundary-layer thickness, and are hence satisfied only "in the average." By assuming definite forms for the velocity profiles as functions of the normal distance, ordinary differential equations are obtained, with distance along the surface as the independent variable. Any integral method may be regarded as either of two types: (a) The single-integral type, in which the partial differential equations are integrated once across the boundary-layer thickness, and the profiles contain a single parameter to be determined by the resulting ordinary differential equation; (b) the multiple-integral type, in which several (say m) integral equations are used, and the assumed velocity profiles contain m parameters to be determined by the m resulting ordinary differential equations.

The best-known integral method is the Kármán-Pohlhausen, which is of type (a) with fourth-degree profiles. This method has been found quite useful for incompressible flow (cf., e.g., Dryden, reference 3) but it has two important limitations. It fails to predict the separation point accurately in an adverse pressure gradient and it often does not give sufficiently accurate results for stability calculations based on criteria developed by Lin and Lees (references 4 and 5).[1] For these purposes refinements in the usual Kármán-Pohlhausen method must be made.

In the methods of type (a) the refinements usually consist of assuming types of profiles which satisfy more boundary conditions than the fourth-degree profiles. Schlichting and Ulrich (reference 7), for example, have used sixth-degree profiles for incompressible flow, satisfying an additional boundary condition at the wall and also at the outer edge of the boundary layer. Satisfactory results were obtained, except for flow in the vicinity of a stagnation point. Weil (reference 8) has recently applied this method to compressible flow with zero heat transfer. However, no investigation was made here of the expected accuracy of the results, although it was pointed out by Weil that the use of sixth-degree profiles is expected to yield satisfactory results for stability calculations since these calculations involve first and second derivatives of the velocity, and the velocity profiles are made to satisfy additional conditions involving the rate of change of second derivatives (viz, third derivatives).

Timman (reference 9) has suggested the use of exponential profiles based on exact profiles derived from the solutions of Von Kármán and

[1]Lees (reference 6) has recently applied the Kármán-Pohlhausen method to an investigation of the stability of compressible laminar boundary layers with favorable pressure gradients, but this investigation was considered by its author to be essentially qualitative.

Millikan (reference 10). Although the calculations are thereby made more tedious, it is believed that no significant improvement over polynomial profiles should be expected, since the latter type of profile usually satisfies a fairly large number of conditions at the outer boundary-layer edge anyway, and since primary interest usually lies in the region near the wall. Moreover, Yuan (reference 11) and Lew (reference 12) have also used exponential profiles with no evident improvement in accuracy. In cases of adverse pressure gradients, Timman has suggested the use of a special profile satisfying an additional condition involving the fourth derivative of velocity at the separation point. Considerable improvement in the accuracy of prediction of the separation point was thereby obtained (reference 9).

Loitsianskii (reference 13) has suggested a modification of the Kármán-Pohlhausen method based on multiplying the momentum equation by a small variation of velocity and then integrating across the boundary-layer thickness. A velocity profile with a single undetermined parameter is, as usual, assumed. The skin friction is subsequently calculated by means of the Kármán momentum equation. The method was applied in reference 13 for several cases of incompressible flow, but the results did not seem to indicate superiority of this method over the usual Kármán-Pohlhausen method with a fourth-degree profile.

In the integral methods of type (b) the Kármán-Pohlhausen method is extended by deriving more than one integral equation. This can be done by multiplying the momentum partial differential equation by a series of different factors, and then by integrating the resulting equations over the boundary-layer thickness. The factors which have usually been chosen are integral powers of either the velocity (Leibenson, reference 14; Golubev, reference 15; Sutton, reference 16; and Wieghardt, reference 17) or the normal distance (Whitehead, reference 18). A further possibility, suggested and applied by Whitehead, is successive integration of the momentum equation. The integral equations obtained by these procedures are to some extent analogous to those which would be obtained by the method of moments, and an infinite number of such equations would be equivalent to the original partial differential equation. Because of the elaborate nature of the calculations required in such a procedure, however, only the first two of the infinite set of equations have usually been considered.

The multiple-integral methods have thus far been developed and applied only for incompressible flow. In the present investigation only the use of powers of velocity as factors will be considered. In the application made by Sutton (reference 16) a fourth-degree velocity profile was assumed with two undetermined parameters. However, one of the boundary conditions at the wall ordinarily satisfied in the Kármán-Pohlhausen method was not satisfied. Wieghardt (reference 17) has also used this two-parameter method, but he has assumed eleventh-degree velocity profiles satisfying additional conditions at the wall (in fact, the same as those satisfied by the sixth-degree profiles of references 7 and 8) and at the outer boundary-layer edge. Although results of comparatively high accuracy can thereby be obtained, Wieghardt's method can be quite tedious in practice. This, in fact, is one of the general disadvantages of the multiple-integral methods.

The general case of heat transfer in a compressible laminar boundary layer with a pressure gradient is complicated by the fact that in this case there is no known solution of the energy partial differential equation giving the temperature explicitly as a function of the velocity. One means of treating such a case is by transforming the energy, as well as the momentum, equation into a differential-integral equation, and assuming a profile not only for the velocity but also for the temperature. Kalikhman (reference 19) has investigated this case by this means using, analogously to the ordinary Kármán-Pohlhausen method, fourth-degree profiles for both velocity and stagnation enthalpy. Although important useful results are thus obtained, their accuracy is subject to the limitations of the Kármán-Pohlhausen method previously discussed.

The aim of the present study is to investigate the practical feasibility of the most promising integral methods (single-integral and double-integral methods with various polynomial velocity profiles) from the point of view of simultaneous accuracy and ease of computation. The approach here is primarily a posteriori. The implications of each of the methods considered are developed for only the simplest case, that of a surface with zero axial pressure gradient in a subsonic and in a supersonic stream, and these are compared with the corresponding implications of an exact solution. In particular, results for skin friction and heat transfer at the surface, velocity profiles, velocity derivatives, and laminar stability based on the work of Lees (reference 5) have been considered. The comparison of stability is particularly critical, since it is here that the largest errors are incurred in the approximate methods, and that the greatest differences among the results of the various methods appear.

The following points are among the distinguishing features of this study: (a) The two-parameter method involving two differential-integral equations is developed and applied to compressible flow, and (b) the implications of the various methods are directly compared with respect to stability criteria; this, moreover, is the main basis here for judging the relative accuracy of the methods for practical purposes.

Although it does not necessarily follow, without further investigation, that the conclusions drawn from the analysis of the solutions for the flow with zero axial pressure gradient will be valid, without modification, for flows with axial pressure gradients, it is believed that the comparisons and resulting conclusions should nevertheless serve as an indication of the relative merits of the various methods in more general cases.

It may be noted that Mangler's tranformation (reference 20), which brings the boundary-layer equations of axially symmetric flow into the form of the two-dimensional equations, further extends the usefulness of two-dimensional-flow solutions. Moreover, axially symmetric flows of constant pressure thereby lead to equations analogous to those of two-dimensional flow with zero pressure gradient.

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SYMBOLS

a_j	coefficients appearing in velocity profile
C	factor of proportionality in equation $\frac{\mu}{\mu_\infty} = C\frac{T}{T_\infty}$
C_f	coefficient of average skin-friction drag for surface of length x
c_p	specific heat at constant pressure
c_v	specific heat at constant volume
F_j	integrals defined in equations (22)
h	local heat-transfer coefficient
k	thermal conductivity of fluid
M	Mach number
N_{Nu}	local Nusselt number $\left(hx/k_\infty\right)$

N_{Pr}	Prandtl number of gas $\left(\mu c_p/k\right)$
p	pressure
R	gas constant $\left(c_p - c_v\right)$
R_{cr}	critical Reynolds number
R_x	Reynolds number based on length x $\left(U_\infty x/\nu_\infty\right)$
R_θ	Reynolds number based on momentum thickness θ $\left(U_\infty \theta/\nu_\infty\right)$
S	Sutherland's constant
t	transformation variable defined in equation (15)
T	absolute temperature
U_1	velocity at outer edge of boundary layer
U_∞	free-stream velocity
u,v	velocity components in x- and y-directions, respectively
x,y	coordinates parallel and perpendicular to surface, respectively
ϵ	variable defined in equation (35)
ρ	mass density
μ	absolute coefficient of viscosity
γ	ratio of specific heats $\left(c_p/c_v\right)$
ν	kinematic viscosity (μ/ρ)
δ	thickness of boundary-layer velocity profile in xy-plane
δ_t	thickness of boundary-layer velocity profile in xt-plane
τ	transformation variable $\left(t/\delta_t\right)$

Subscripts:

∞	free-stream conditions for subsonic flow; uniform conditions behind shock for supersonic flow
1	local conditions at outer edge of boundary layer
o	conditions at surface
c	values at critical coordinate for stability

BASIC EQUATIONS

The following equations describe the steady, two-dimensional, laminar-boundary-layer flow of a compressible gas along a surface whose radius of curvature is large compared with the boundary-layer thickness:

Momentum equation in x-direction:

$$\rho u(\partial u/\partial x) + \rho v(\partial u/\partial y) = -(\partial p/\partial x) + (\partial/\partial y)\left[\mu(\partial u/\partial y)\right] \qquad (1)$$

Momentum equation in y-direction:

$$\partial p/\partial y = 0 \qquad (2)$$

Equation of continuity:

$$(\partial/\partial x)(\rho u) + (\partial/\partial y)(\rho v) = 0 \qquad (3)$$

Equation of state:

$$p = \rho RT \qquad (4)$$

Energy equation:

$$\rho uc_p(\partial T/\partial x) + \rho vc_p(\partial T/\partial y) = u(\partial p/\partial x) + (\partial/\partial y)\left[k(\partial T/\partial y)\right] + \mu(\partial u/\partial y)^2 \qquad (5)$$

As a consequence of equation (2) the static pressure p at any axial position x in the boundary layer is equal to the corresponding static pressure in the potential flow, in which conditions are assumed to be known.

From the momentum (Bernoulli) equation in potential flow, the pressure p can be expressed in terms of the potential-flow velocity distribution, thus:

$$\begin{aligned} \partial p/\partial x &= dp/dx \\ &= -\rho_1 U_1 U_1' \end{aligned} \tag{6}$$

where the prime denotes differentiation with respect to x, and the subscript 1, conditions at the outer edge of the boundary layer. As is usual in aeronautical problems, the coefficients of viscosity and heat conductivity are taken to be known functions of the temperature only, while the coefficient of specific heat at constant pressure will be assumed constant.

The treatment of these equations is greatly simplified if use is made of well-known algebraic velocity-temperature relations which are exactly valid for a Prandtl number of unity. These relations are for the case of zero heat transfer with or without an axial pressure gradient

$$\left(u^2/2\right) + c_p T = A \tag{7}$$

and for the case of zero axial pressure gradient with heat transfer

$$\left(u^2/2\right) + c_p T = A + Bu \tag{8}$$

where A and B are arbitrary constants determined from wall and free-stream conditions. With the use of these conditions, equation (7) becomes

$$T/T_1 = 1 + (\gamma - 1)\left(M_1{}^2/2\right)\left[1 - \left(u/U_1\right)^2\right] \tag{9}$$

while equation (8) becomes

$$T/T_1 = \left(T_o/T_1\right) - \left\{\left(T_o/T_1\right) - \left[1 + (\gamma - 1)\left(M_1{}^2/2\right)\right]\right\} \left(u/U_1\right) - (\gamma - 1)\left(M_1{}^2/2\right)\left(u/U_1\right)^2 \tag{10}$$

The use of equations (9) and (10) which are valid only for a Prandtl number of 1 in lieu of the more general energy equation (5) is justified primarily by the simplifications introduced in the analysis. However, it has been found that momentum boundary-layer characteristics such as skin friction are not significantly affected by a change in Prandtl number from unity to the actual value for air, which is between 0.65 and 0.76. Heat-transfer coefficients are affected approximately 10 percent by such a change.

Before discussing the approximate solution of the basic equations by means of integral methods, it is convenient to present several subsidiary relations for later use. Since the pressure within the boundary layer at any given value of x is a constant, equation (4) gives

$$\rho/\rho_1 = T_1/T \tag{11}$$

The most accurate representation of the variation of the coefficient of viscosity of gases with temperature is usually considered to be Sutherland's formula, namely,

$$\mu/\mu_\infty = \left(T/T_\infty\right)^{3/2}\left(T_\infty + S\right)\Big/(T + S) \tag{12}$$

where S is a constant which for air is 216° R and where the subscript ∞ denotes reference conditions taken in the undisturbed free stream for subsonic flow and in the uniform flow behind any shock wave associated with the leading edge for supersonic flow. Although it is possible in principle to use equation (12) in the methods employed here, it is inconvenient to do so since certain integrals must then be evaluated numerically. It is thus desirable that some acceptable simplifying approximation to equation (12) be made. In reference (21) Chapman and Rubesin suggested that the mathematical advantages of assuming μ and T to be linearly related can be obtained along with satisfactory accuracy if it is assumed that

$$\mu/\mu_\infty = C\left(T/T_\infty\right) \tag{13}$$

where C is a factor which is chosen so that the Sutherland viscosity-temperature relation is exactly satisfied at the wall temperature T_o. Thus,

$$C = \left(T_o/T_\infty\right)^{1/2}\left(T_\infty + S\right)\Big/\left(T_o + S\right) \tag{14}$$

Equation (13) will be used throughout this report.

It is convenient in the method used here to apply the Dorodnitzyn transformation. Thus a new variable t is defined so that for a given value of x the physical variable y is given by the equation

$$y = \int_0^t \left(T/T_1\right) dt \tag{15}$$

Correspondingly the thickness δ_t of the boundary layer in the xt-plane is defined as the value of t when $y = \delta$, and therefore

$$\delta = \int_0^{\delta_t}\left(T/T_1\right) dt \tag{16}$$

With the additional relations given by equations (11), (13), (14), and (15), equation (1) can be recast into an infinite set of integral-differential equations. This can be done by multiplying equation (1) by $u^n\,dy(n = 1,2,3, \ldots)$ and by integrating from $y = 0$ to $y = \delta$ with the boundary conditions that at $y = 0$

$$u = v = 0$$

and at $y = \delta$

$$\left.\begin{aligned} u &= U_1 \\ \partial u/\partial y &= 0 \end{aligned}\right\} \tag{17}$$

After transforming to the xt-plane, nondimensionalizing, and introducing $\tau \equiv t/\delta_t$, the following set of equations is obtained:

$$\left[(n + 1)\rho_1 U_1{}^{n+2}\right]^{-1} \delta_t (d/dx) \left\{\rho_1 U_1{}^{n+2} \delta_t \int_0^1 (u/U_1)\left[1 - (u/U_1)^{n+1}\right] d\tau\right\} -$$

$$(dU_1/dx)\left(\delta_t{}^2/U_1\right) \int_0^1 \left[u/U_1 - (T/T_1)(u/U_1)^n\right] d\tau =$$

$$\begin{cases} nC\mu_\infty T_1\left(\rho_1 U_1 T_\infty\right)^{-1} \int_0^1 (u/U_1)^{n-1}\left[(\partial/\partial\tau)(u/U_1)\right]^2 d\tau & n \geqq 1 \\ C\mu_\infty T_1\left(\rho_1 U_1 T_\infty\right)^{-1}\left[(\partial/\partial\tau)(u/U_1)\right]_0 & n = 0 \end{cases} \tag{18}$$

The application of the infinite set of equations represented by equations (18) will now be discussed. By means of equations (9) and (10) the temperature can be expressed in terms of the velocity component u. Now if it were assumed that

$$u/U_1 = \sum_{j=0}^{\infty} a_j(x)\tau^j \tag{19}$$

then several of the a_j coefficients could be determined in terms of the other coefficients so that certain boundary conditions, at least those in the xt-plane corresponding to those given by equations (17) in the xy-plane, would be satisfied. The rest of the coefficients and δ_t could be calculated from the infinite set of first-order ordinary differential equations represented by equations (18). Such a solution would in principle be exact and would be equivalent to a solution of the original partial differential equations.

In actuality, however, since the attainment of such a solution would be quite difficult, it will usually be considered sufficient for practical purposes to obtain approximate solutions to the infinite set by taking only the first few differential equations of this set and correspondingly only the first few terms of the power series in equation (19). Indeed, much information of engineering importance for the flat-plate case has been obtained in the past by using only the equation corresponding to $n = 0$, that is, to the Von Kármán momentum integral (e.g., references 3, 11, and 12). Additional accuracy may be obtainable by the use of a second equation, that is, by the use of equations (18) for $n = 0$ and $n = 1$. Therefore, these two equations will now be explicitly written.

For $n = 0$:

$$\left(F_1/2\right)\left(\delta_t^{2}\right)' + \left[F_1' + F_1\left(\log_e \rho_1 U_1^{2}\right)' - F_2\left(\log_e U_1\right)'\right]\delta_t^{2} = \left(F_3\mu_\infty C T_1\right)\Big/\left(\rho_1 U_1 T_\infty\right) \tag{20}$$

and for $n = 1$:

$$\left(F_4/4\right)\left(\delta_t^{2}\right)' + \left[\left(F_4'/2\right) + \left(F_4/2\right)\left(\log_e \rho_1 U_1^{3}\right)' - F_5\left(\log_e U_1\right)'\right] \delta_t^{2} = \left[\left(F_6\mu_\infty C\right)\Big/\left(\rho_1 U_1\right)\right]\left(T_1/T_\infty\right) \tag{21}$$

where the prime denotes differentiation with respect to x and where

$$\left.\begin{aligned}
F_1 &= \int_0^1 \left(u/U_1\right)\left[1 - \left(u/U_1\right)\right] d\tau \\
F_2 &= \int_0^1 \left[\left(u/U_1\right) - \left(T/T_1\right)\right] d\tau \\
F_3 &= \left[(\partial/\partial\tau)\left(u/U_1\right)\right]_0 \\
F_4 &= \int_0^1 \left(u/U_1\right)\left[1 - \left(u/U_1\right)^2\right] d\tau \\
F_5 &= \int_0^1 \left(u/U_1\right)\left[1 - \left(T/T_1\right)\right] d\tau \\
F_6 &= \int_0^1 \left[(\partial/\partial\tau)\left(u/U_1\right)\right]^2 d\tau
\end{aligned}\right\} \tag{22}$$

In these F_j integrals either equation (9) or (10), as the case may be, is used to express T/T_1.

In the actual application of either or both equations (20) and (21), u/U_1 is assumed as some function of τ. For convenience a polynomial in τ is here assumed, that is

$$u/U_1 = \sum_{j=0}^{N} a_j(x)\tau^j \tag{23}$$

where N is the degree of the assumed polynomial containing (N + 1) of the a_j's.

The a_j's are determined from the boundary conditions and from the differential equations (20) and (21). The boundary conditions in the xt-plane corresponding to those given by equations (17) must be satisfied. Furthermore, additional accuracy can be obtained if the a_j's are selected so that the approximate solution given by this method has the same value of lower derivatives at $y = 0$ and $y = \delta$ as an exact solution to the partial differential equations would have. These values may be obtained by differentiating equation (1) one or more times with respect to y or t. For completeness the useful boundary conditions in the x,t coordinates will now be listed for flows having zero axial pressure gradient with or without heat transfer or having an axial pressure gradient but no heat transfer:

$$\left.\begin{aligned}
&\text{At } t = 0: && u/U_1 = 0 \\
& && \mu_\infty C\left(T_1/T_\infty\right)\left(T_1/T_o\right)\left(\partial^2/\partial\tau^2\right)\left(u/U_1\right) = -\rho_1 U_1{}'\delta_t{}^2 \\
& && \left(\partial^3/\partial\tau^3\right)\left(u/U_1\right) = 0 \\
&\text{At } t = \delta_t: && u/U_1 = 1 \\
& && \left(\partial^m/\partial\tau^m\right)\left(u/U_1\right) = 0(m = 1,2,3, \ldots)
\end{aligned}\right\} \tag{24}$$

It may be pointed out that although additional conditions at $y = 0$ can be derived they become impractically cumbersome to apply.

The method involving the simultaneous solution of equations (20) and (21) will be referred to as the "two-parameter method," while that of using only equation (20) will be the "one-parameter method." In the two-parameter method one coefficient a_1 in addition to δ_t is determined from the two differential equations, the rest from the boundary conditions. In the one-parameter method δ_t is determined from the solution of equation (20) and all of the a_j coefficients are determined from boundary conditions.

In carrying out the integration of these differential equations graphical or numerical methods are in general required. The arbitrary constants are usually determined so that the unknowns are either well-behaved or have definite prescribed values at some value of x.

Before applying the methods indicated here to the case of an axial pressure gradient, it is considered desirable to investigate critically the accuracy obtained by using several types of profiles in equation (20) and in both equations (20) and (21) for the compressible flow with no axial pressure gradient. An exact solution for this case has been given by Chapman and Rubesin (reference 21). The accuracy of the results will be measured by two criteria, namely, by the accuracy of the skin-friction and heat-transfer values and by the critical Reynolds number for the stability of the laminar boundary layer.

The application of these integral methods to the flat plate and the results for skin friction and heat transfer will now be discussed in detail.

BOUNDARY LAYER WITH ZERO AXIAL PRESSURE GRADIENT

The one-parameter and two-parameter methods described in the preceding section are here explicitly applied to the compressible flow over a surface with zero axial pressure gradient. Various profiles will be assumed and the results compared with those obtainable by a mathematically exact solution for this case.

Solution of Equations

For the case of zero axial pressure gradient $U_1 \equiv U_\infty$ and equations (20) and (21) become

$$\left(\delta_t^{\,2}\right)' = \left(2\nu_\infty CF_3\right)/\left(U_\infty F_1\right) \tag{25}$$

and

$$\left(\delta_t{}^2\right)' = \left(4\nu_\infty CF_6\right)\Big/\left(U_\infty F_4\right) \tag{26}$$

where the F_j integrals are in this case constants, and not functions of x.

In the two-parameter method equations (25) and (26) are solved for δ_t and the additional a_j coefficient which has not been determined from boundary conditions. With the initial condition $\delta_t = 0$ at $x = 0$ these unknowns are determined by the equations

$$2F_6/F_4 = F_3/F_1 \tag{27}$$

and

$$\delta_t/x = \sqrt{\left(2F_3/F_1\right)\left(C/R_x\right)} \tag{28}$$

where $R_x = U_\infty x/\nu_\infty$. Equation (27) leads to a quartic equation in the unknown coefficient a_i, three roots of which must be rejected on physical grounds. Once the constant value of a_i is known, F_3 and F_1 can be evaluated and substituted into equation (28) to complete the solution.

In the one-parameter method the F_j integrals are completely determined once the velocity profile is chosen so as to satisfy the selected boundary conditions. Thus, only equation (28) is used to determine the single unknown δ_t.

Skin Friction and Heat Transfer

From the solutions outlined in the preceding section, the skin-friction and heat-transfer characteristics are readily determined. The coefficient of average skin-friction drag C_f is defined by the equation

$$C_f = 2\left[\int_0^x (\mu\ \partial u/\partial y)_o\ dx\right]\Big/\left(\rho_\infty U_\infty{}^2 x\right) \tag{29}$$

By use of equation (28) this can be written as

$$C_f = 2\left[2F_1F_3\left(C/R_x\right)\right]^{1/2} \tag{30}$$

Similarly a local heat-transfer coefficient h can be defined as

$$h = -(k\ \partial T/\partial y)_o \Big/ \left(T_o - T_e\right) \tag{31}$$

where T_e is the equilibrium wall temperature for no heat transfer and, for $N_{Pr} = 1$, is given by the equation

$$T_e = T_\infty\left\{1 + \left[(\gamma - 1)\left(M_\infty{}^2/2\right)\right]\right\} \tag{32}$$

By the use of equation (28) the equation for h is found to be

$$N_{Nu} = hx/k$$

$$= \left(F_1F_3R_xC/2\right)^{1/2} \tag{33}$$

Hence, by comparison of equations (30) and (33),

$$N_{Nu} = C_fR_x/4 \tag{34}$$

Velocity Profiles and Results for Skin Friction and Heat Transfer

Several velocity profiles were assumed and the friction and heat transfer given by these profiles were calculated. In the two-parameter method fourth- and fifth-degree polynomials were used, while in the one-parameter method fourth-, fifth-, and sixth-degree polynomials were chosen.

The results obtained here are compared mainly with those of the exact analysis of Chapman and Rubesin (reference 21) which treats a general Prandtl number and variable surface temperature. The exact velocity profiles of reference 21 are derived from the well-known Blasius differential equation, solutions of which are tabulated, for example, in reference 22. With the assumptions $N_{Pr} = 1$ and constant surface temperature for which equations (9) and (10) are valid, exact temperature profiles are thus obtainable from the exact velocity profiles of reference 21. (These profiles, when expressed in terms of the Blasius variable η (cf. equation 14, reference 21), are independent of the surface temperature and the Prandtl number.)

The variation of skin friction with Mach number for an insulated surface is represented in figure 1. Included for comparison are results already calculated in reference 21 for $N_{Pr} = 0.72$, as well as exact results for $N_{Pr} = 1$.

Table I gives values of the skin-friction coefficient and Nusselt number obtained by the one- and two-parameter methods with the various velocity profiles. Values derived from the exact analysis of reference 21 for $N_{Pr} = 1$ and 0.72 are listed for comparison. The free-stream temperature and the wall temperature, which for an insulated surface is a function of the free-stream Mach number, are contained explicitly in the term C. Typical values of $\sqrt{C}$ for an insulated surface are presented in table II in connection with the calculation of the skin-friction drag coefficient.

Table I shows that the values obtained for skin friction and heat transfer by the two-parameter method with either of the two profiles used and by the one-parameter method with a sixth-degree profile differ from the exact values by less than 0.6 percent. The values obtained by the one-parameter method with the fourth-degree profiles are in error by approximately 3 percent. Thus, all methods used here give sufficiently accurate results for engineering purposes for skin-friction and heat-transfer coefficients.

Since in both the exact solutions (reference 21) and the solutions obtained here by the integral methods, the effect of Mach number in the case of an insulated plate with $N_{Pr} = 1$ is given by the same factor, namely $\sqrt{C}$, it follows that (for a linear temperature-viscosity relation) these integral methods lead to the exact effect of Mach number on C_f and N_{Nu}. It may be noted that the Mach number effect depends solely on the assumed temperature-viscosity relation. The effect of the latter can be seen from figure 1, where the results of the various integral methods and of reference 21, all based on a linear temperature-viscosity relation, are shown along with Crocco's results which are based on the Sutherland formula.

From figure 1 it can be seen, incidentally, that the value of the Prandtl number has a small effect on the variation of C_f with Mach number in the case of zero heat transfer. For example, in the range $M = 0$ to $M = 5$, with $T_\infty = 648^\circ$ R, $C_f\sqrt{R_x}$ varies from 1.328 to 0.983 for $N_{Pr} = 0.72$, and from 1.328 to 0.954 for $N_{Pr} = 1$. It can be seen from table I, moreover, that the Prandtl number has a somewhat larger effect on the value of Nusselt number at all Mach numbers, the change from $N_{Pr} = 1$ to $N_{Pr} = 0.72$ being about 10 percent.

Comparison of Velocity Profiles and Derivatives in Physical Plane

For estimating the stability of the laminar boundary layer, the accuracy of the first and second derivatives of the velocity profile, as well as that of the profile itself, are of importance. In this section the profiles and derivatives obtained by the integral methods under discussion are compared with the exact Blasius solution of references 21 and 22. The comparison can be conveniently made in the physical xy-plane by introduction of the dimensionless variable

$$\epsilon = (y/2x)\left(R_x/C\right)^{1/2} \tag{35}$$

The derivatives with respect to ϵ are shown in figures 3 and 4.

It is seen from figures 2, 3, and 4 that, although the velocity profiles obtained by the integral methods agree well with the exact profile, the derivatives, as might be expected, do not correspond so closely. Previous comparison of skin friction and heat transfer, which depend upon the first derivatives at the surface, indicates that each of the integral methods gives good results for these characteristics. Hence each of the integral methods may be considered to predict the value of the first derivative at the surface with sufficient accuracy. Moreover, except for the fourth-degree profiles, the first and second derivatives given by all the approximate solutions are in good qualitative agreement with the exact results throughout the boundary-layer thickness. Examination of figures 3 and 4 for first and second derivatives over the boundary-layer thickness indicates, however, that the one-parameter method with a sixth-degree velocity profile gives on the whole the most satisfactory results. It is to be anticipated that this method will consequently also give satisfactory results for stability calculations.

DETERMINATION OF CRITICAL REYNOLDS NUMBER FOR STABILITY

The experimental work of Schubauer and Skramstad (reference 23) has clearly established that the transition from a laminar to a turbulent boundary layer is due to an instability of the laminar layer, if the turbulence in the free stream is low, and if the surface of the body is aerodynamically smooth and has a large radius of curvature. Thus the laminar-boundary-layer stability theory, which has been developed for incompressible flow by several investigators over the past 25 years and which has been recently extended by Lin and Lees (references 4 and 5) to the practically interesting case of compressible flow, may be used to predict the local Reynolds number at which instability of the laminar layer will first occur, or above which self-propagated disturbances are not damped out. Transition to turbulent flow will take place downstream of the point corresponding to this local Reynolds number, which is termed the critical Reynolds number; the exact distance downstream cannot be predicted by the present stability theory, which is based on small-perturbation methods, but seems to be dependent on the value of the critical Reynolds number, on the magnitude of the small but possibly finite turbulence in the free stream, on the surface roughness, and on the potential-flow pressure gradient.

The exact calculation of the critical Reynolds number for a given velocity profile is tedious. However, the approximate stability rules of Lin and Lees (references 4 and 5) permit a rapid determination of this critical value with a minimum of labor. These rules have been shown by the work of Lin and Lees and that of Hahneman, Freeman, and Finston (reference 24) to give reliable results and have, therefore, been used extensively. It might be further mentioned that because of approximations inherent in the laminar-boundary-layer stability theory, and because of the undetermined relation between boundary-layer neutral stability and transition, there seems to be no practical justification for obtaining greater accuracy than that given by the approximate stability rules.

The results of stability calculations are sensitive to the accuracy of the profiles. Thus the critical Reynolds numbers based on approximate solutions to the boundary-layer equations have not in general been in good agreement with those based on exact solutions. The reason for this discrepancy is clear when it is considered that the stability calculation depends strongly on first and second derivatives of the velocity profile. Approximate solutions may be expected to give fairly accurate velocity profiles, but unless special care is exercised the first and second derivatives of the profile throughout the entire boundary-layer thickness might be quite in error. This has been shown in the previous section where it has been pointed out that the frequently used Kármán-Pohlhausen method with a fourth-degree velocity profile gives reasonably accurate values for the skin friction, the heat transfer, and the velocity profile but yields quite inaccurate values for the derivatives. It may be expected, therefore, that a crucial test of the accuracy of approximate solutions to the laminar-boundary-layer equations would be a comparison of the critical Reynolds numbers predicted by these solutions with those predicted by exact solutions.

In this section the critical Reynolds numbers of the profiles obtained in the previous section for the zero-heat-transfer case are obtained and compared with those of the exact solution of Chapman and Rubesin.

With the prime denoting partial differentiation with respect to ϵ, Lees' criterion modified for the viscosity-temperature relation used here is given as follows:

$$\begin{aligned} R_{cr} &= 2\left(R_{x,cr}C\right)^{1/2} \\ &= 25\left(u/U_1\right)_o{}'C\left(T/T_1\right)^2\left(u/U_1\right)^{-4}\left\{1 - M_1{}^2\left[1 - \left(u/U_1\right)_c\right]^2\right\}^{-1/2} \end{aligned} \tag{36}$$

where the subscript c denotes values at the point $\epsilon = \epsilon_c$ at which the following relations are satisfied:

$$J = -\pi\left(T_1/T_o\right)\left(u/U_1\right)_o{}'\left(u/U_1\right)\left[\left(u/U_1\right)'\right]^{-3}\left[\left(T/T_1\right)\left(u/U_1\right)'' - \left(u/U_1\right)'\left(T/T_1\right)'\right] \tag{37}$$

and

$$0.580/J = 1 - 2\left[\left(u/U_1\right)_o{}'\epsilon\left(u/U_1\right) - 1\right] \tag{38}$$

It will be noted from equation (37) that the first and second derivatives of the velocity and temperature profiles in the critical region around ϵ_c are influential in the determination of R_{cr}. The procedure for obtaining R_{cr} is to calculate J from equation (37) for various values of ϵ, and to find the values of $\epsilon = \epsilon_c$ for which equation (38) is satisfied. After the values of $\left(u/U_1\right)_c$ and $\left(T/T_1\right)_c$ are determined at this point, R_{cr} and $R_{x,cr}$ follow from equation (36).

For comparison with previous results it is convenient to calculate the critical Reynolds number $R_{\theta,cr}$ based on momentum thickness. Since θ is defined by

$$\theta = \int_0^{\delta} (\rho/\rho_1)(u/U_1)\left[1 - (u/U_1)\right] dy \tag{39}$$

it is found, for the integral methods, that

$$R_{\theta,cr} = \left(F_1F_3/2\right)^{1/2}R_{cr} \tag{40}$$

and for the profiles of reference (21)

$$R_{\theta,cr} = 0.332R_{cr} \tag{41}$$

Calculations of the minimum critical Reynolds numbers have been made for the flow over a flat insulated surface at free-stream Mach numbers of 0, 1, and 2. The results are tabulated in table III and plotted in figure 5. Primarily, comparison of the integral methods presented here should be made with the results derived from the exact profiles of reference 21 since identical temperature-viscosity relations are used in both. The results of Lees, given in figure 5 of reference 5, are also presented for comparison. Note should be taken that Lees' approximate values are based on still a further approximation to equations (36), (37), and (38) presented here.

Figure 5 indicates that the one-parameter sixth-degree polynomial method gives the best general agreement with results of the exact solutions over the range of Mach numbers studied. The two-parameter, fifth-degree method is next best, while in order of decreasing accuracy, the one-parameter fifth-degree, two-parameter fourth-degree, and one-parameter fourth-degree polynomials follow. The two-parameter methods seem to give better general agreement than the one-parameter methods for equal degree of the assumed polynomial for the velocity profile. (Because of the inaccuracies introduced by the approximation in this stability criterion, as well as by the required graphical method involved in its application, the derived results are not reliable to more than two significant figures.)

The stability analysis given here has thus indicated that integral methods of solution of the boundary-layer equations may be used to predict the critical Reynolds number provided that either additional integral equations or higher-degree polynomials and additional boundary conditions with a single integral equation are used. Thus the choice of method to be used in the more general cases of pressure gradient with and without heat transfer is either the two-parameter method or the one-parameter method with higher-degree polynomials, particularly of the sixth degree.

While these two methods lead to results which are of nearly the same accuracy, use of the one-parameter method has the advantage of simplicity of calculation, which becomes significant in cases involving pressure gradient. It is thus concluded that the one-parameter method with sixth- or higher-degree polynomials is the most satisfactory and promising method for extension to general compressible boundary-layer analyses. Sixth-degree polynomials will probably prove to be satisfactory in many cases, but use of a seventh-degree profile satisfying an additional condition at the separation point (cf. reference 9) may increase the accuracy of determining the separation point in adverse-pressure-gradient cases. Moreover, a sixth-degree profile may not be satisfactory for flow near a stagnation point (cf. reference 7).

CONCLUSIONS

The laminar-boundary-layer equations for compressible flow can be converted into one, two, or more integral-differential equations. Approximate solutions can be obtained by assuming special forms for the velocity profiles satisfying various boundary conditions and containing, in practice, either one or two parameters to be determined by these equations.

From a comparison of the integral methods discussed here for the compressible flow over a surface with no axial pressure gradient, the following conclusions can be drawn:

1. All of the methods predict the values of the skin-friction and heat-transfer coefficients at the wall as well as the over-all velocity profiles with satisfactory accuracy for engineering purposes. The two-parameter method and the one-parameter method with a sixth-degree profile are particularly accurate.

2. The one-parameter method with a sixth-degree velocity profile gives on the whole the most accurate results for velocity profiles and their derivatives throughout the boundary-layer region.

3. The critical Reynolds number for laminar-boundary-layer stability is predicted with qualitative accuracy by all methods. Moreover, reasonably good quantitative accuracy is obtained with the two-parameter method and with the one-parameter method with a sixth-degree profile.

4. Because of computational simplicity and equality of accuracy, the one-parameter method with sixth- or higher-degree polynomials appears to be the most satisfactory and promising method of investigating the more general case of laminar compressible boundary layer with axial pressure gradient and heat transfer.

Polytechnic Institute of Brooklyn
New York, N. Y., November 6, 1950

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TABLE I.- SKIN-FRICTION DRAG COEFFICIENT

AND NUSSELT NUMBER

Method	Degree of polynomial	Boundary conditions satisfied (cf. equations (24))	$\frac{1}{2}C_f\left(\frac{R_x}{C}\right)^{1/2}$	$2N_{Nu}(R_xC)^{1/2}$
Two parameter	4 5	1,4,5(m = 1,2) 1,2,4,5(m = 1,2)	0.663 .660	0.663 .660
One parameter	4 5 6	1,2,4,5(m = 1,2) 1,2,3,4,5(m = 1,2) 1,2,3,4,5(m = 1,2,3)	0.685 .644 .661	0.685. .644 .661
Reference 21 T_o = Constant N_{Pr} = 1			0.664	0.664
Reference 21 T_o = Constant N_{Pr} = 0.72			0.664	0.592

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TABLE II.- VALUES OF $\sqrt{C}$ FOR AN

INSULATED SURFACE FOR $N_{Pr} = 1$

M_∞	T_∞	
	72° F absolute	648° F absolute
0	1.0	1.0
1	1.021	.976
2	1.057	.916
3	1.074	.844
4	1.067	.776
5	1.043	.718

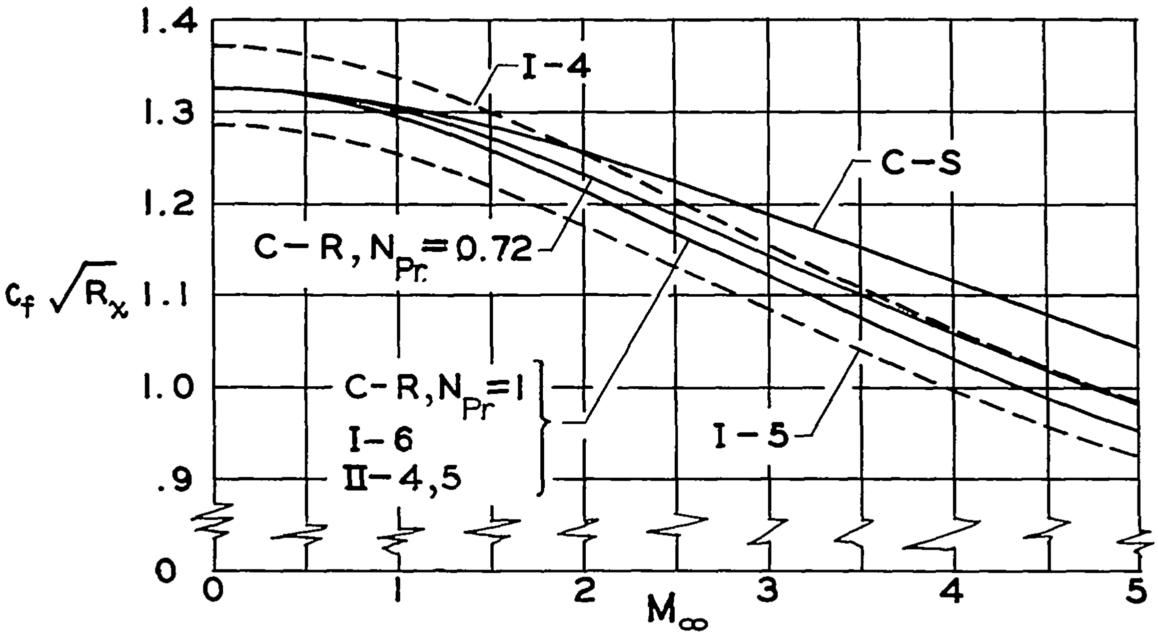
TABLE III.- MINIMUM CRITICAL REYNOLDS NUMBERS FOR STABILITY

IN LAMINAR-BOUNDARY-LAYER FLOW OVER AN INSULATED

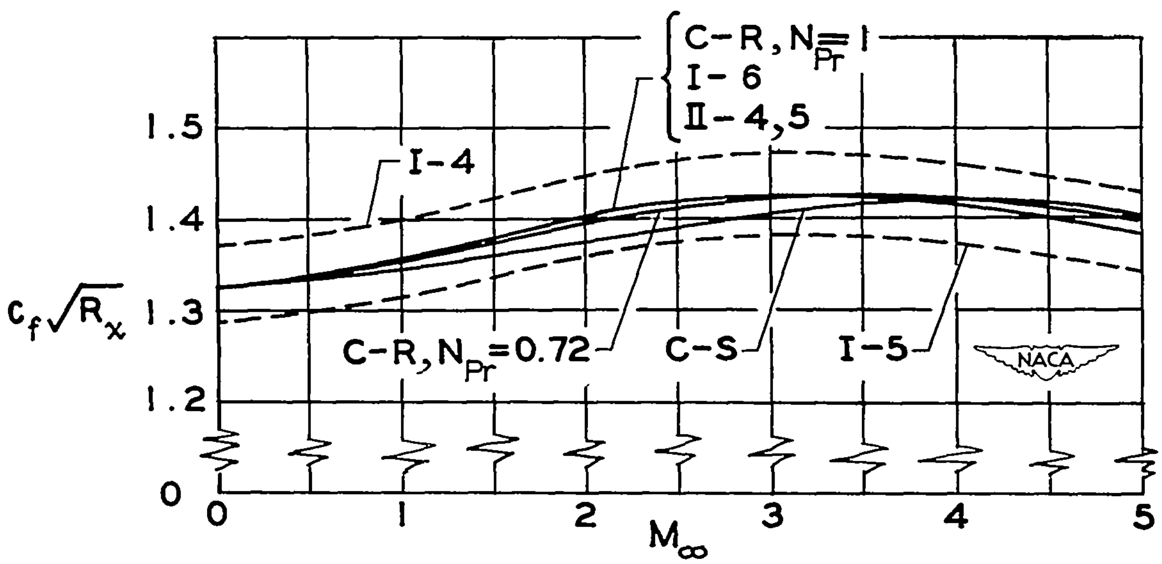
FLAT SURFACE

Method (1)	$M_\infty = 0$					$M_\infty = 1$					$M_\infty = 2$				
	ϵ_c	$\left(\frac{u}{U_1}\right)_c$	$\frac{R_{cr}}{C}$	$\frac{R_{\theta,cr}}{C}$	$\frac{R_{x,cr}}{C}$	ϵ_c	$\left(\frac{u}{U_1}\right)_c$	$\frac{R_{cr}}{C}$	$\frac{R_{\theta,cr}}{C}$	$\frac{R_{x,cr}}{C}$	ϵ_c	$\left(\frac{u}{U_1}\right)_c$	$\frac{R_{cr}}{C}$	$\frac{R_{\theta,cr}}{C}$	$\frac{R_{x,cr}}{C}$
Lees' exact stability calculation				150	51,000				110	27,000					
Lees' approximate stability criterion		0.4186		195	86,000		0.499		119	32,000					
Chapman-Rubesin	0.64	.419	540	180	73,000	0.945	.517	290	96	21,000	1.97	0.724	77	26	1480
2 parameter-4	.56	.370	890	290	198,000	.862	.47	460	152	53,000	2.05	.75	62	21	960
2 parameter-5	.59	.381	780	260	152,000	.958	.51	310	102	24,000	2.17	.773	50	17	630
1 parameter-4	.52	.345	1210	420	370,000	.829	.46	520	178	68,000	2.05	.73	77	26	1480
1 parameter-5	.69	.436	450	145	51,000	1.067	.56	201	65	10,100	2.20	.788	44	14	480
1 parameter-6	.61	.394	690	230	119,000	.963	.52	280	93	19,600	2.07	.75	62	21	960

[1]For example, 2 parameter-4 means two-parameter method with a fourth-degree profile.

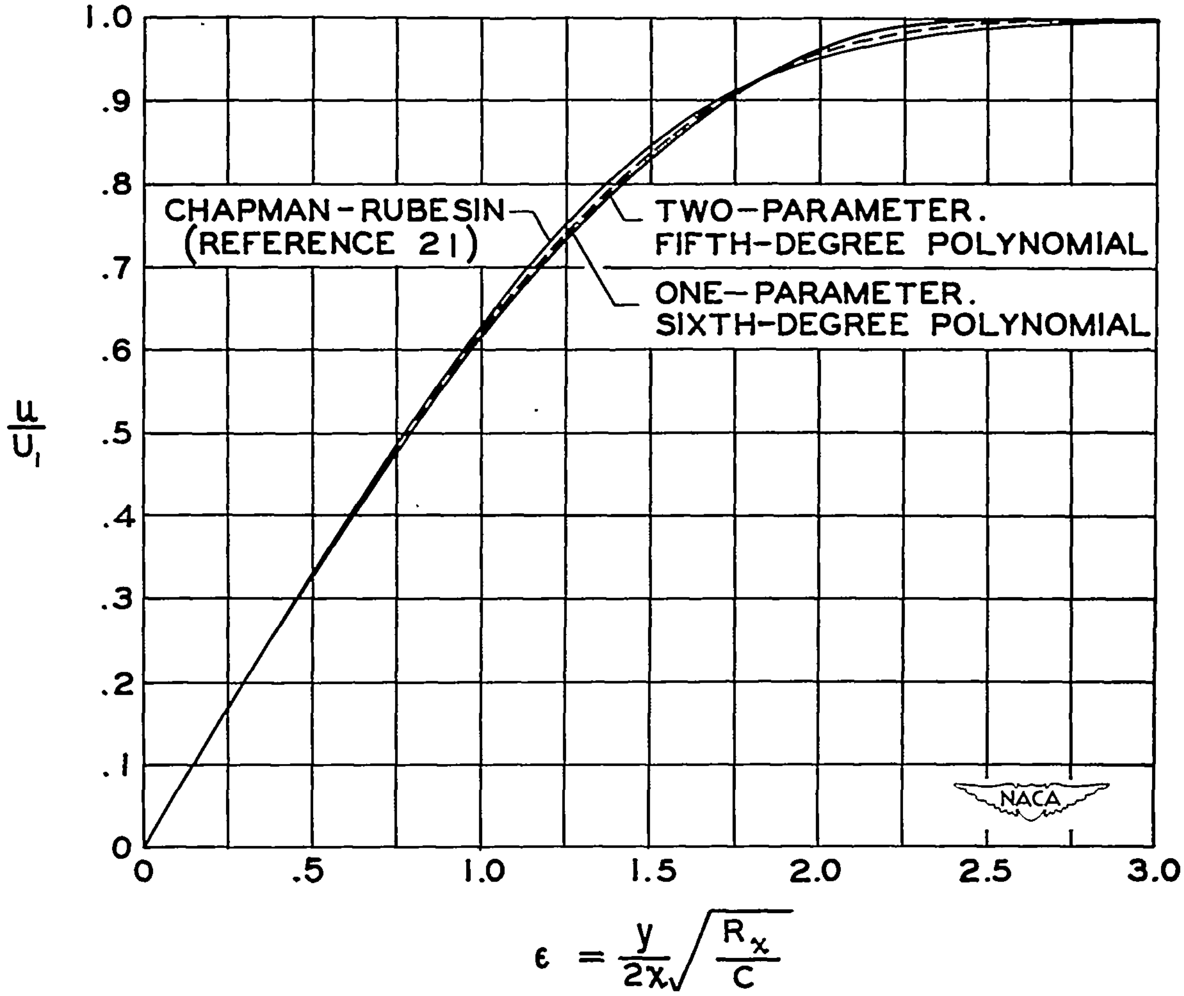


(a) $T_\infty = 648^o$ F absolute.



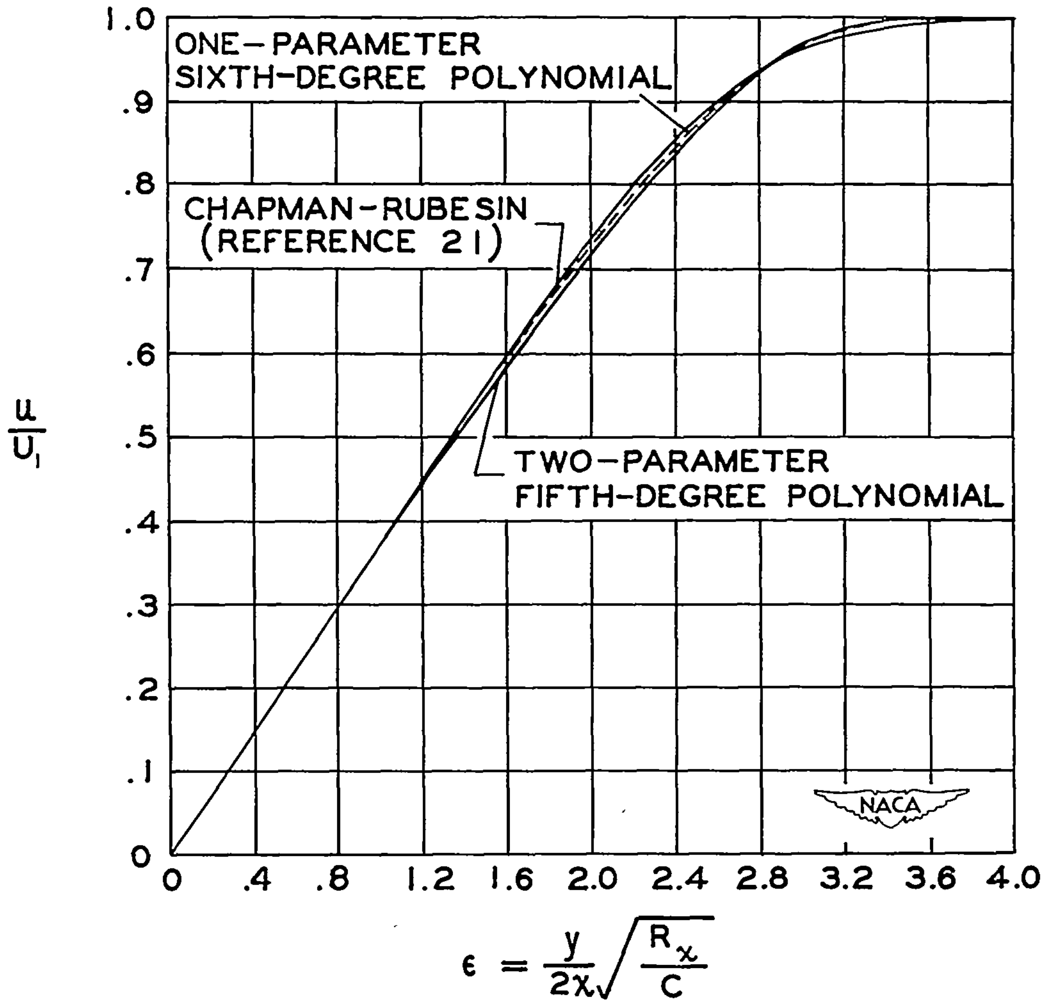
(b) $T_\infty = 72^o$ F absolute.

Figure 1.- Variation of skin-friction drag coefficient with free-stream Mach number for insulated flat surface. I, one-parameter method; II, two-parameter method; 4, 5, 6, degree of polynominals; C-R, Chapman-Rubesin (reference 21); C-S, Crocco's calculation based on Sutherland's temperature-viscosity relation (cf. reference 21).



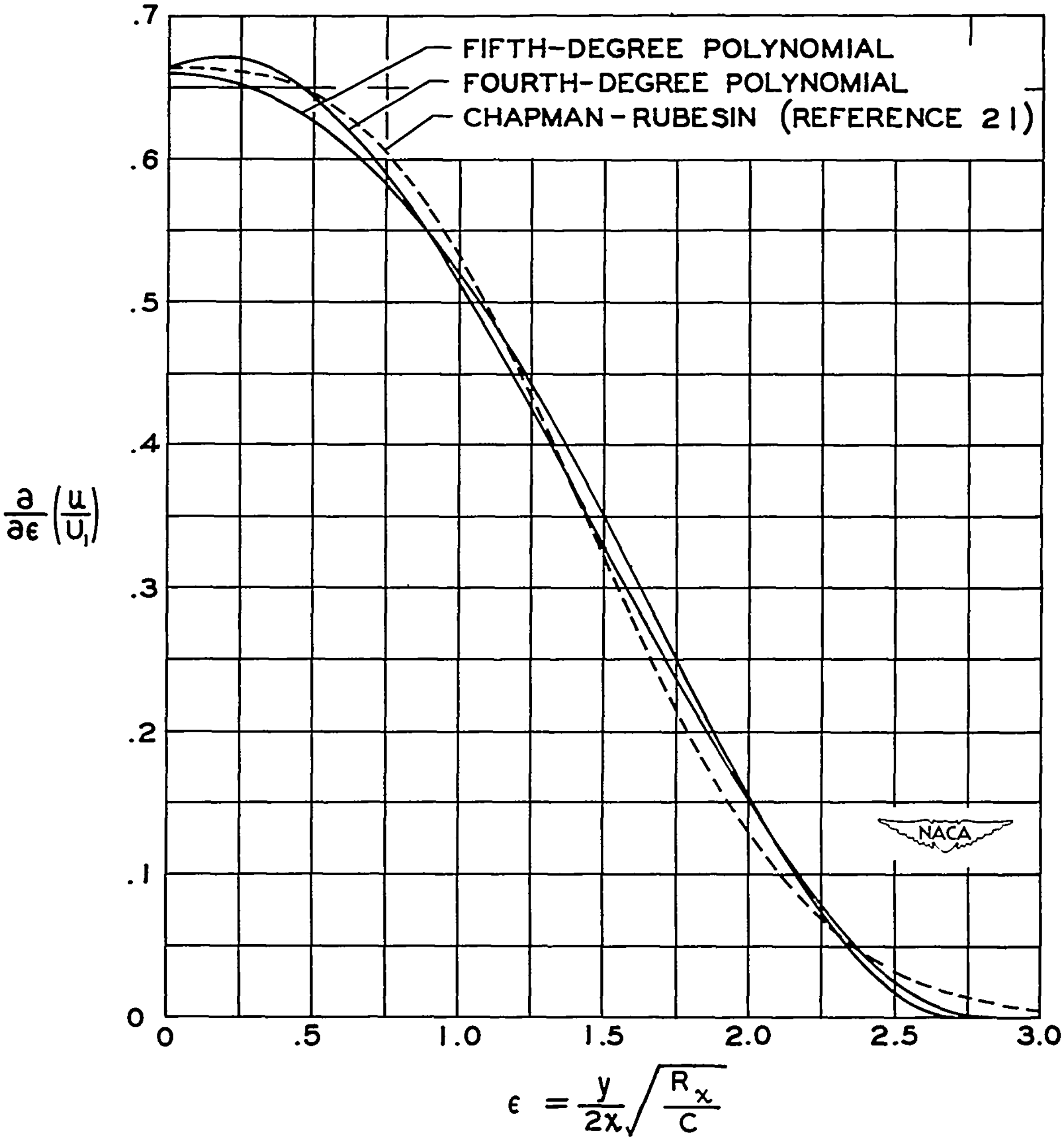
(a) $M_\infty = 0$.

Figure 2.- Comparison of velocity profiles for flow over insulated flat surface.



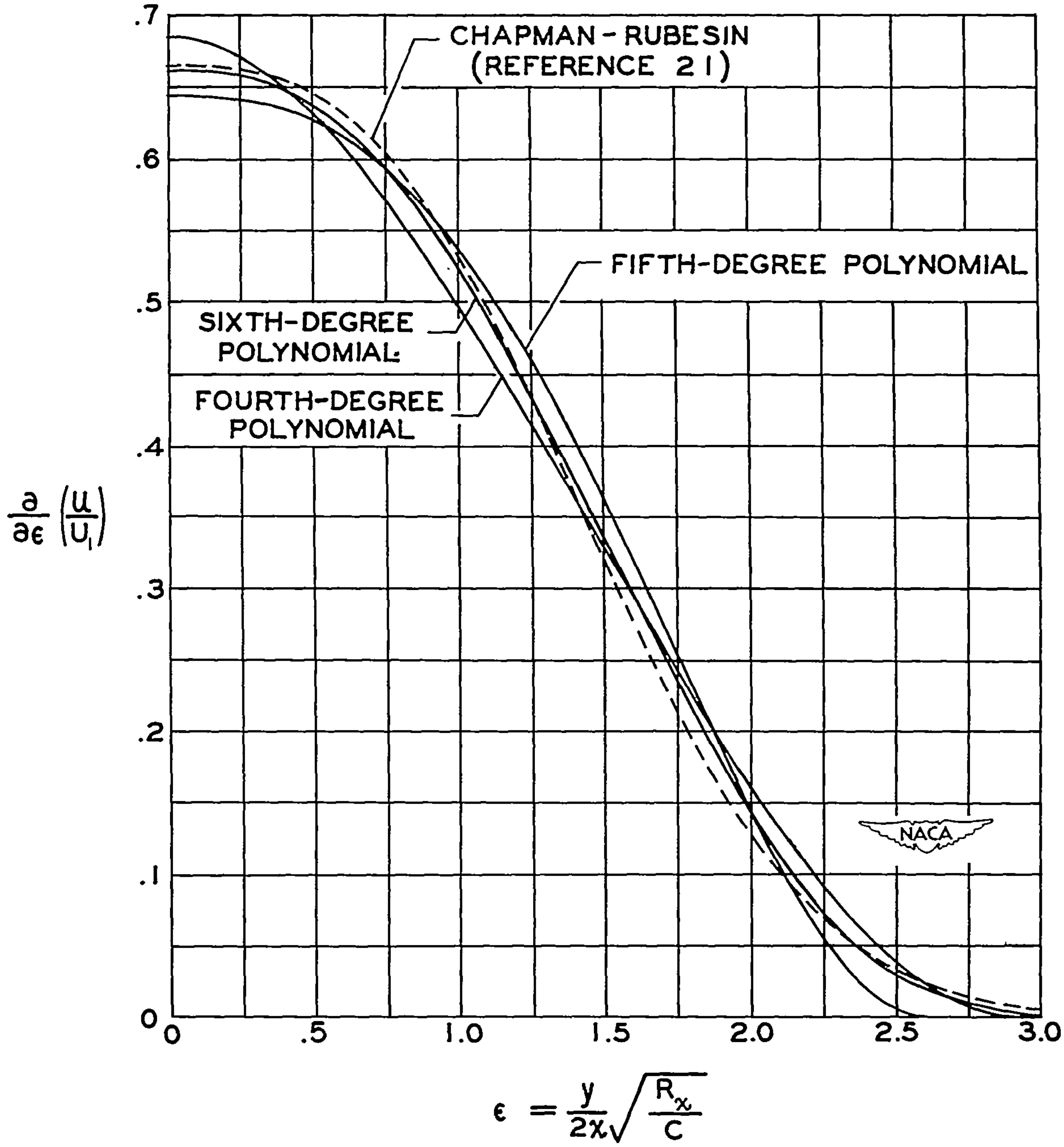
(b) $M_\infty = 2$.

Figure 2.- Concluded.



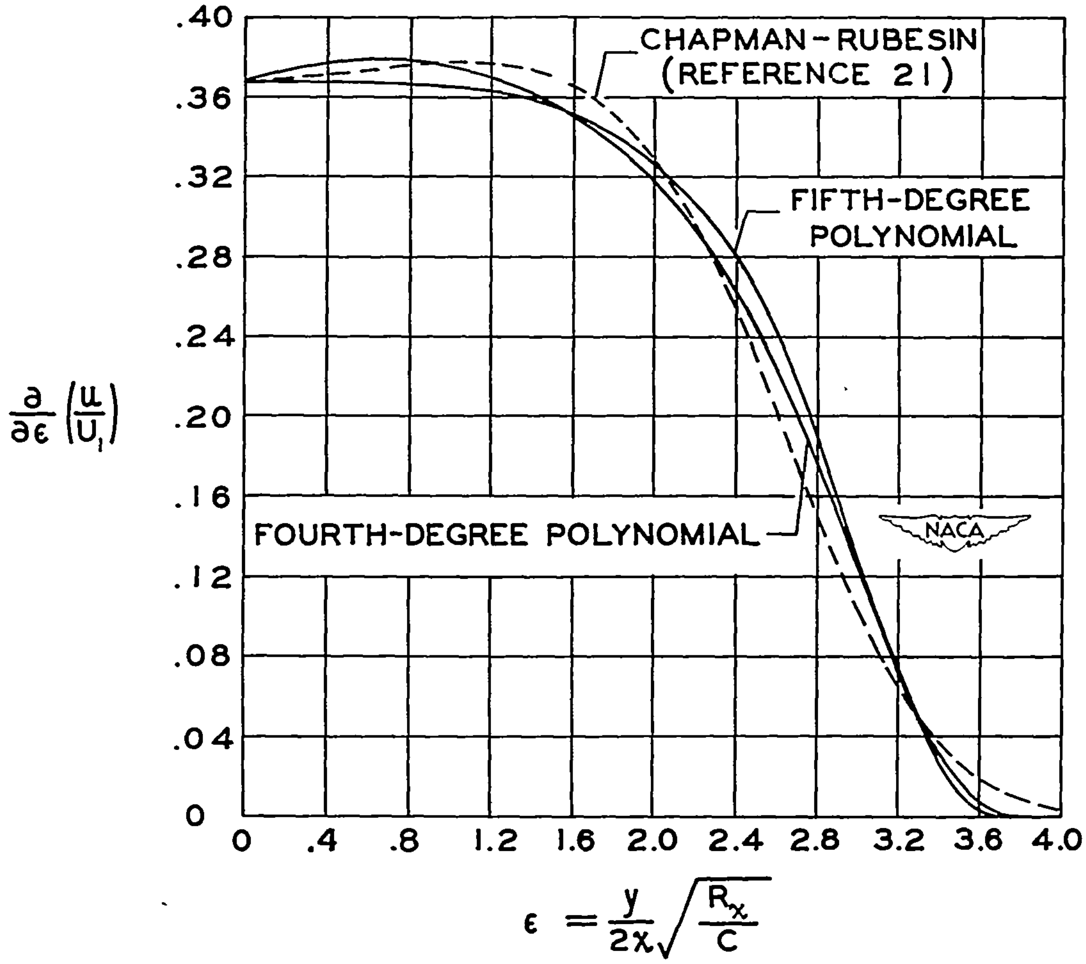
(a) Two-parameter method; $M_\infty = 0$.

Figure 3.- Comparison of first derivatives of velocity profiles for flow over insulated flat surface.



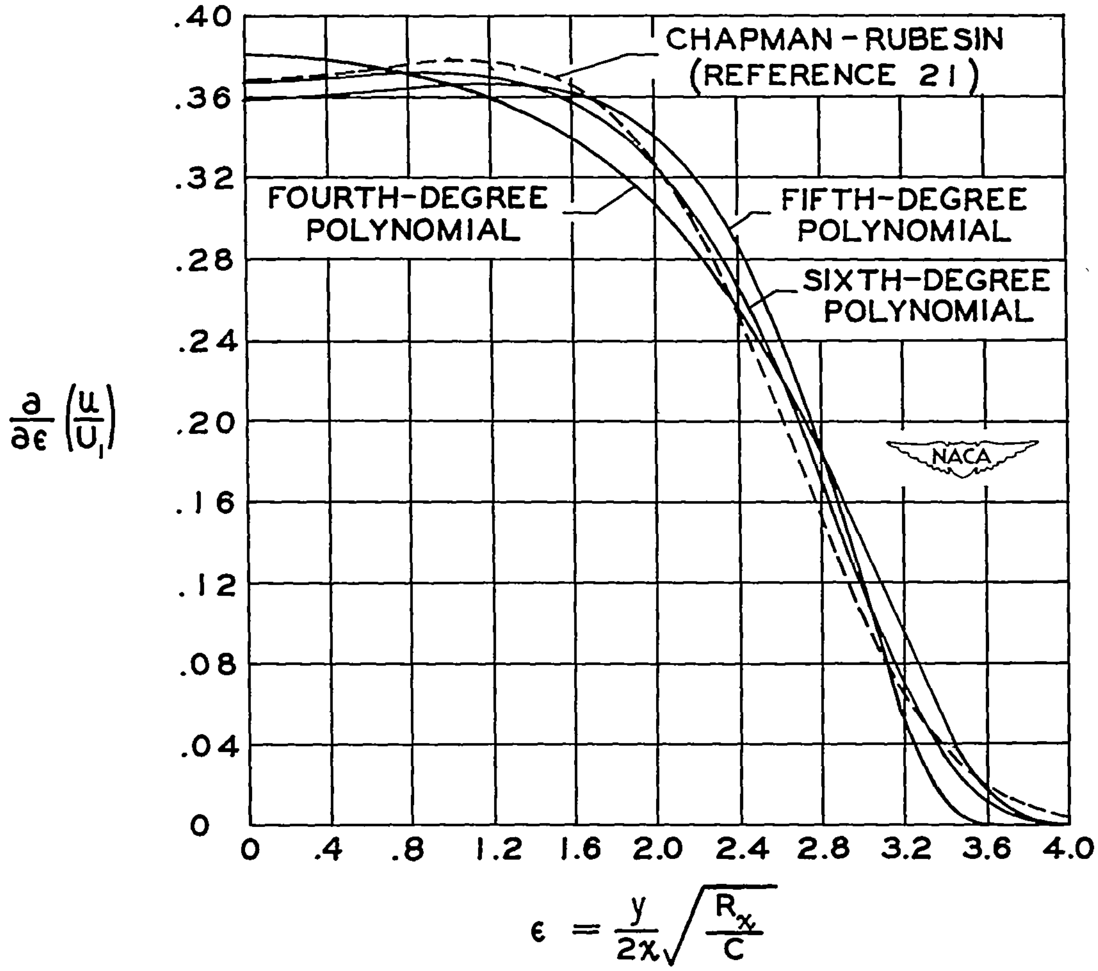
(b) One-parameter method; $M_\infty = 0$.

Figure 3.- Continued.



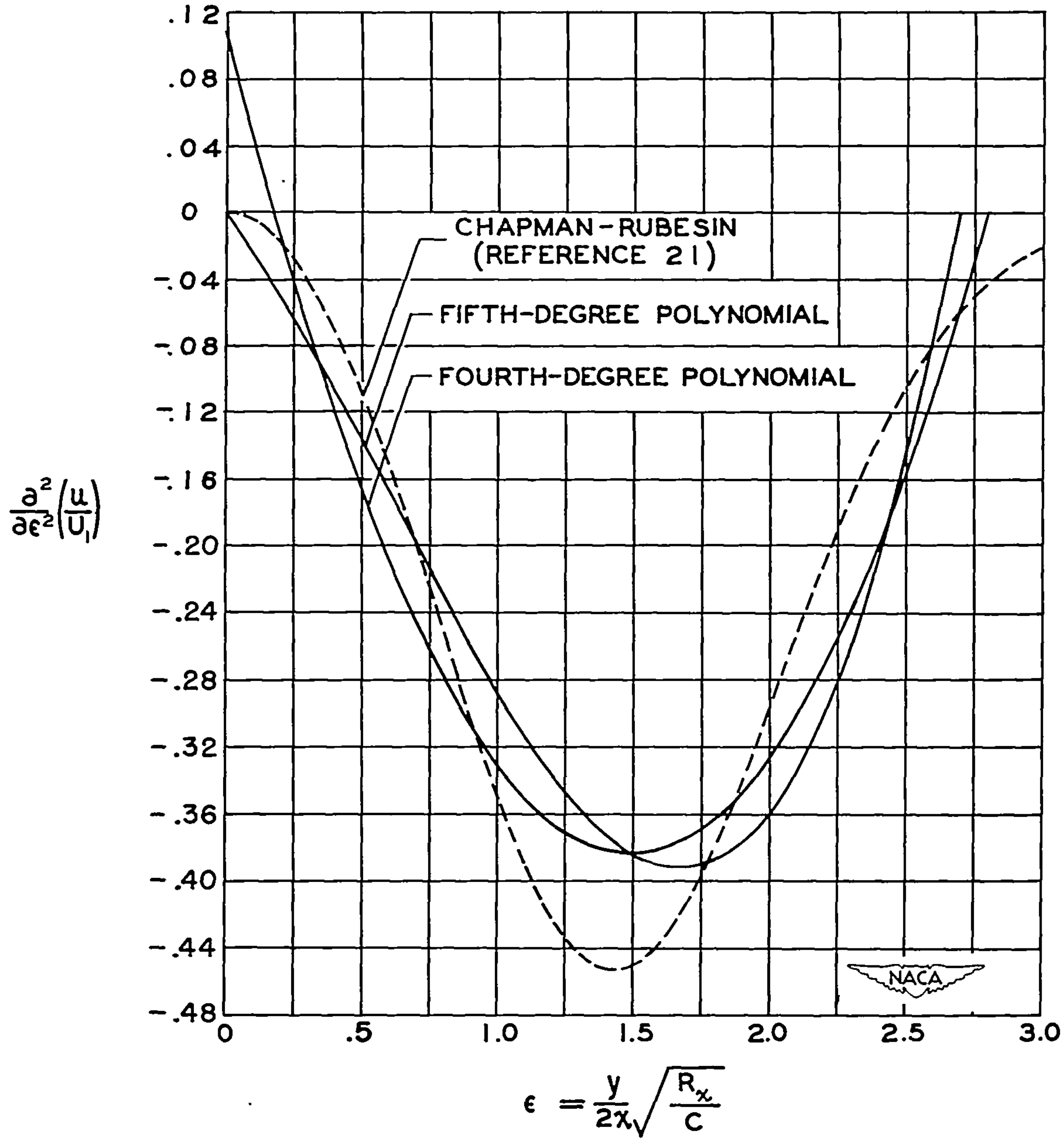
(c) Two-parameter method; $M_\infty = 2$.

Figure 3.- Continued.



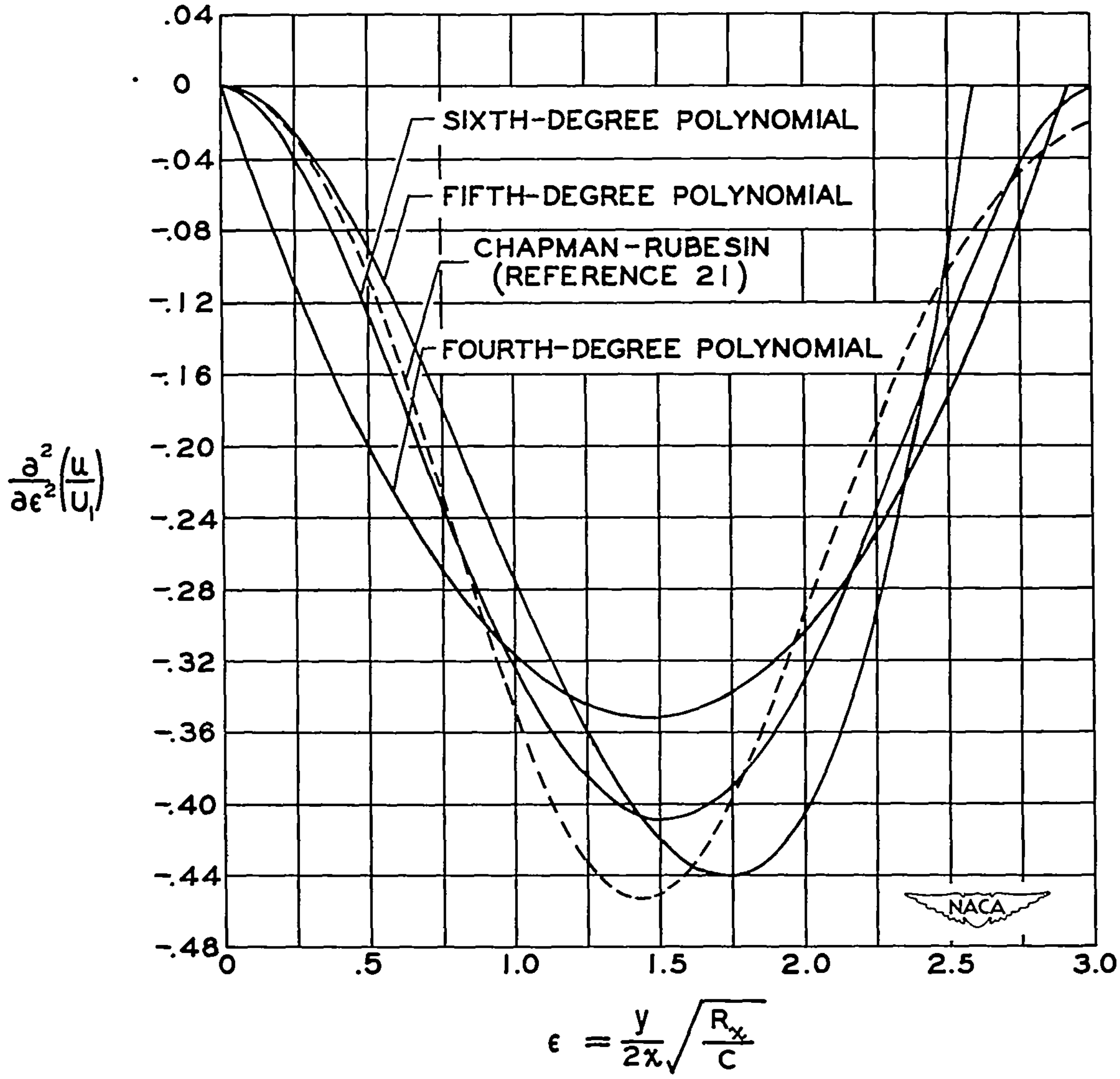
(d) One-parameter method; $M_\infty = 2$.

Figure 3.- Concluded.



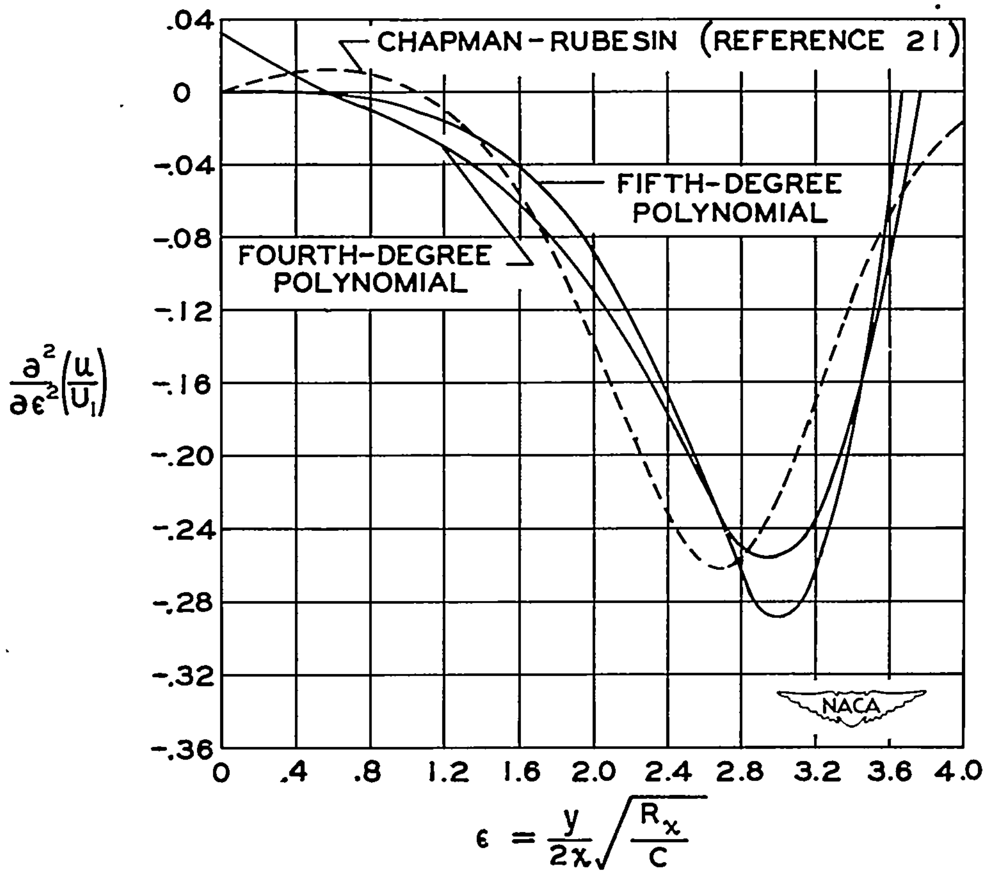
(a) Two-parameter method; $M_\infty = 0$.

Figure 4.- Comparison of second derivatives of velocity profiles for flow over insulated flat surface.



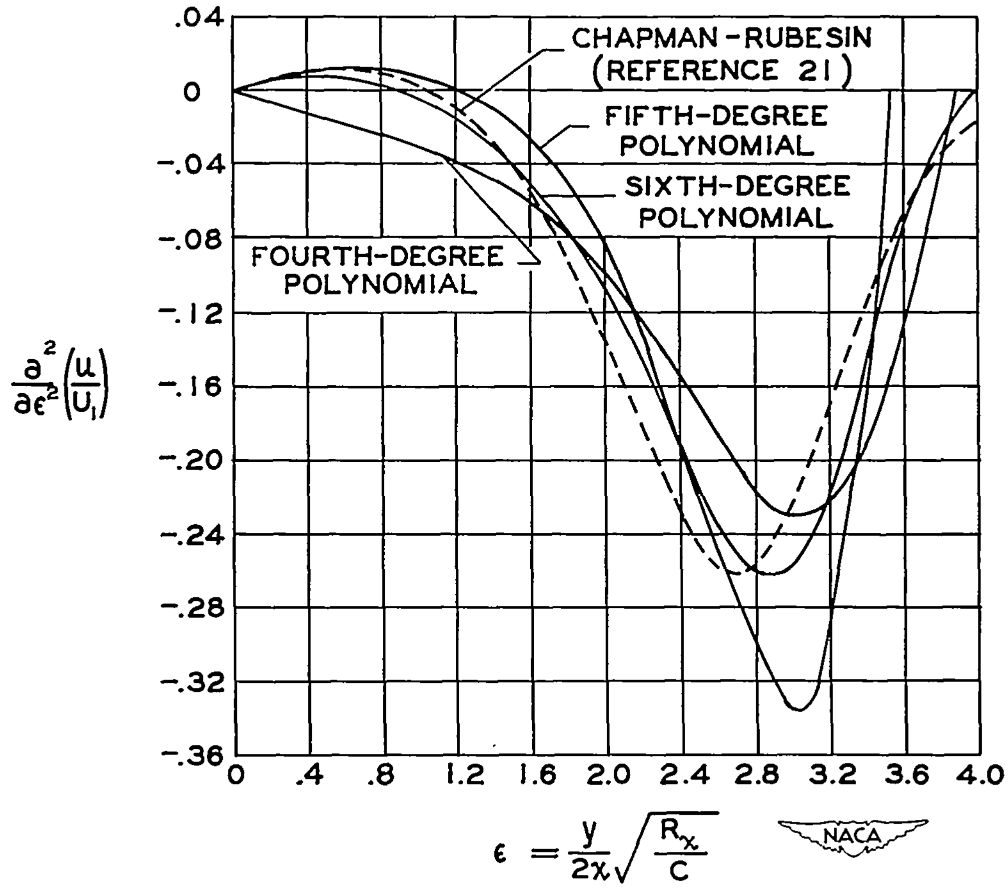
(b) One-parameter method; $M_\infty = 0$.

Figure 4.- Continued.



(c) Two-parameter method; $M_\infty = 2$.

Figure 4.- Continued.



(d) One-parameter method; $M_\infty = 2$.

Figure 4.- Concluded.

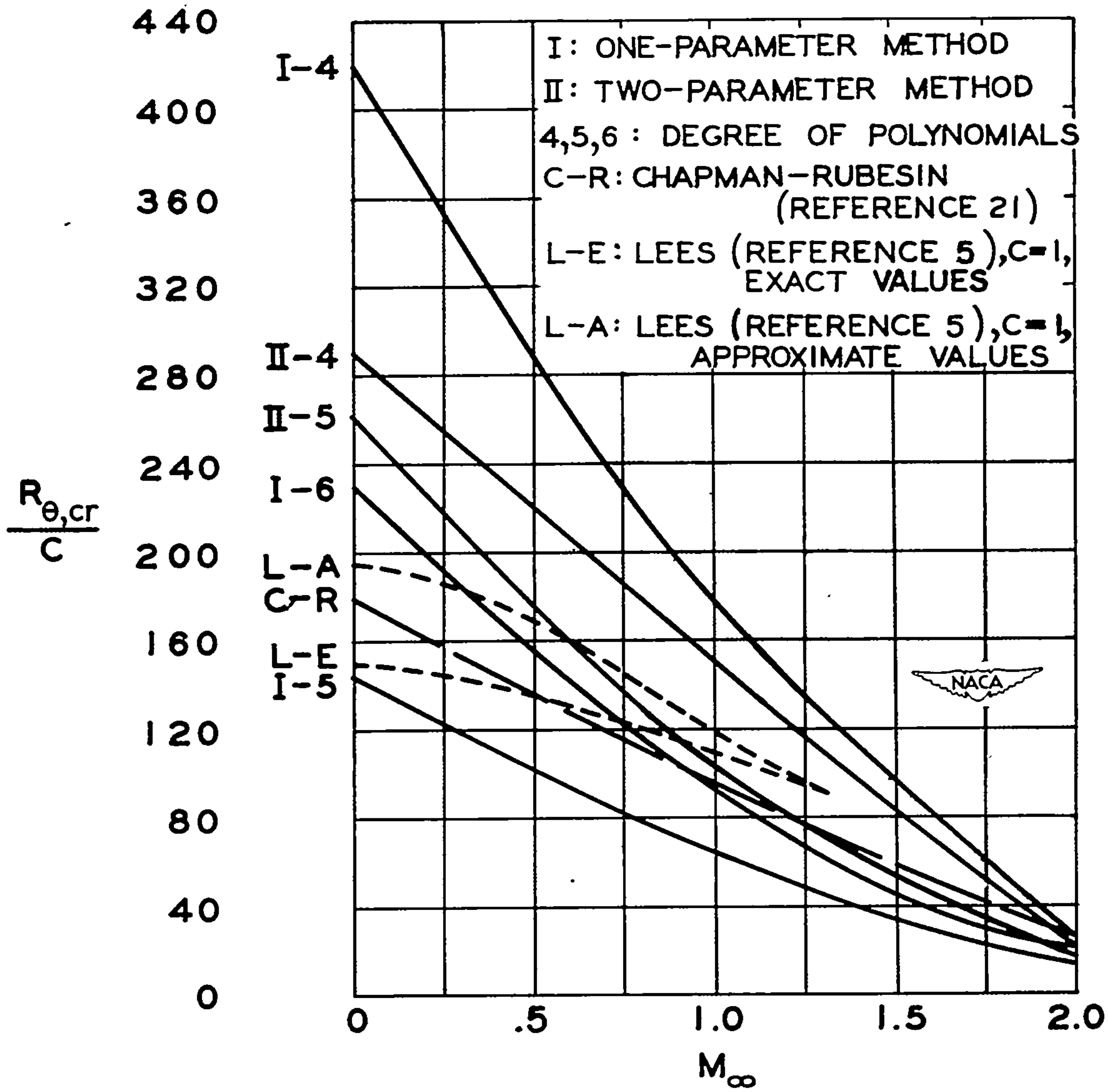


Figure 5.- Variation of minimum critical Reynolds number with Mach number for laminar boundary layer flow over insulated flat surface with $N_{Pr} = 1$.