NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2540

APPLICATION OF RESPONSE FUNCTION TO CALCULATION OF FLUTTER CHARACTERISTICS OF A WING CARRYING CONCENTRATED MASSES

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Washington
November 1951

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SUMMARY

Concepts involved in the harmonic-response-function method, such as the direct or conjugate characteristic modes, are illustrated by application of the method to the calculation of the change in flutter characteristics of a wing due to adding concentrated masses. The main purpose of the numerical procedures which are given is to illustrate the scope of the method and to make some of its abstract formulations more specific, rather than to stress the immediate and current practical usefulness. The rigid body is first idealized as a point mass, then as a distributed mass. An appendix is given which contains some of the essential theoretical background.

INTRODUCTION

In recent years, a number of papers dealing with certain special phases of aeroelasticity, such as flutter, divergence, and control reversal problems, have been published. Each such problem has usually been treated as a special case so that there exist today a number of procedures which have little relation one to another.

There has been in existence, however, a general approach (reference 1) to aeroelastic phenomena. Although the underlying theory is couched in abstract terms, the method has the advantages of conceptual simplicity and physical interpretation; the abstract formulation, in fact, permits application to a number of apparently different problems.

Engineers who have worked with vibration mountings are familiar with the significance of the natural frequencies and modes of an elastic system in reference to forced oscillations. The response of the system to an external force cannot be properly appreciated without relating characteristics of that force to the stability characteristics (resonances) of the system. The method described below is simply the logical extension of this framework of ideas to the case of nonconservative forces as encountered in aerodynamic phenomena.

The method of the harmonic response function is here applied to a special problem, namely, the effect on flutter of the attachment of a rigid mass to the airplane. Specifically, it is supposed that a flutter analysis has been carried out on the original airplane in n degrees of freedom. A mass is then attached. The problem is: What are the flutter characteristics of the modified airplane?

The problem is important in airplane design because (a) mass balances are frequently attached to a control surface for flutter prevention and (b) power-plant changes result in the addition of concentrated weights.

The results of reference 2 showed that in some instances a Rayleigh type analysis may show no flutter when a mass is attached, whereas a more rigorous analysis (based on the continuous system) would show flutter. The problem raised, as to how many degrees of freedom are required for a Rayleigh type of analysis, is one of the difficult, unsolved problems that do not fall within the scope of this report.

A solution of the problem by means of the response function depends on the accuracy with which the system is described by the n degrees; in fact, the method of the present paper is exact to the same extent as the n degrees of freedom describe the deflections of the system with and without mass.

In the case where there are an infinite number of degrees of freedom (as is the case for a continuous structure), the method of Rayleigh approximates the motion of the system by restricting the motion to a finite number of degrees. If these finite degrees are chosen to favor the representation of the system without mass, then the representation of the system with mass will be less favorable because the mass will introduce discontinuities in shear which normally would not appear in any of the approximating modes.

Therefore, for continuous systems one may expect that a representation of the original system by a finite number of degrees will not serve so well for the system with a concentrated mass. However, the present analysis shows that the results of this approximation yield as favorable results as have been obtained by the more conventional methods of analysis.

This work was done at Purdue University under the sponsorship and with the financial assistance of the National Advisory Committee for Aeronautics.

SYMBOLS

a abscissa of wing elastic axis measured from wing midchord point, positive rearward, in units of b

a_1 real part of λ_1

a_{rs} coefficients of inertia-plus-aerodynamic virtual work

A_{rs} intensity factor (see text)

$A(\vec{y}, \vec{x}) = \sum_{r,s} y_r a_{rs} x_s$

b semichord of wing

b_1 imaginary part of λ_1

b_{rs} coefficients of elastic virtual work

$B(\vec{y}, \vec{x}) = \sum_{r,s} y_r b_{rs} x_s$

B_{rs} intensity factor (see text)

c constant

e_w distance between center of gravity of attached mass and elastic axis, positive rearward

$\vec{e}$ direct characteristic mode (normalized)

$\vec{f}$ conjugate characteristic mode (normalized)

$f_h(\xi)$ bending deflection of wing

$f_\alpha(\xi)$ torsional deflection of wing

F amplitude of oscillating force

$\vec{g}$ direct free mode

$\vec{h}$ conjugate free mode

I moment of inertia of attached mass, relative to elastic axis

$\bar{I}$ moment of inertia of attached mass, relative to its center of gravity

$k = \omega b/v$

K constant

l number of elastic degrees of freedom; semispan

m attached mass

n number of degrees of freedom

q_r amplitude of generalized force

Q_r generalized force corresponding to x_r

T amplitude of oscillating torque

v free-stream velocity; flutter speed

v_o flutter speed with zero attached mass

$w_r = -q_r/\omega^2$

W virtual work

$\vec{x}$ direct mode $(x_1, \ldots, x_n)$

$\vec{y}$ conjugate mode $(y_1, \ldots, y_n)$

$\vec{x}^\nu$ direct characteristic mode (nonnormalized)

$\vec{y}^\nu$ conjugate characteristic mode (nonnormalized)

x displacement of a point of the system

x' displacement of reference point on mass, positive in a direction opposite to x

EI bending stiffness of wing

GJ torsional stiffness of wing

$\vec{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)$ geometric parameters defining x

θ torsional deflection of wing, positive nose up

θ' angular deflection of body, positive opposite to θ

$\lambda = (\omega_r/\omega)^2$

λ_ν characteristic number

ν dummy index

ξ spanwise coordinate of wing

ξ_r scalar coefficient of direct characteristic vector

η_r scalar coefficient of conjugate characteristic vector

ω frequency at flutter

ω_o frequency of spring-mounted mass

ω_h bending frequency

ω_r reference frequency

ω_α torsional frequency

HARMONIC RESPONSE FUNCTION

In accordance with practice, the deflection of a system from some initial position of equilibrium is described by n coordinates x_1, x_2, . . ., x_n. Some of these symbols may represent rigid-body displacements of the system of nonelastic displacements, such as free control-surface motions. These are called free-body displacements. The others, involving elastic deformation, are called elastic displacements.

The set of displacements x_1, x_2, . . . x_n arranged in any specific way may be conveniently regarded as a vector. The theory of reference 1 is essentially a geometrical theory based on the vector concept of displacements and forces.

In accordance with classical mechanics, the selection of coordinates defines a system of generalized forces $Q_1, Q_2, \ldots, Q_n$ so that the work performed in moving these forces through a virtual displacement $x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n$ is given by

$$W = y_1Q_1 + y_2Q_2 + \ldots + y_nQ_n \tag{1}$$

The symbol on the right is conveniently regarded as an "inner product" of the two vectors $\vec{y} = (y_1, y_2, \ldots, y_n)$ and $\vec{Q} = (Q_1, Q_2, \ldots, Q_n)$ and written $(\vec{y}, \vec{Q})$.

Now suppose the system, here an airplane, is subjected to a system of harmonic forces and let the symbols $q_1, q_2, \ldots, q_n$ now represent the amplitudes of the corresponding generalized forces and let $x_1, x_2, \ldots, x_n$ represent the corresponding amplitudes of oscillation. Then the equations of motion take the form

$$\omega^2 \sum_s a_{rs}x_s - \omega_r{}^2 \sum_s b_{rs}x_s + q_r = 0 \qquad r = 1, 2, \ldots, n \tag{2}$$

where the matrix $\omega^2(a_{rs})$ is the matrix of aerodynamic-plus-inertia forces, ω_r is a reference frequency, and $-\omega_r{}^2(b_{rs})$ is the matrix of elastic forces. The coefficients a_{rs} depend on the reduced frequency $k = \frac{\omega b}{v}$, where b is a reference length (usually the semichord) and v the flight speed; the coefficients a_{rs} are obtained from aerodynamic analysis of oscillating flow.

Dividing equation (2) by ω^2 and replacing $(\omega_r/\omega)^2$ by λ and $-q_r/\omega^2$ by w_r, one has the set of equations

$$\sum_s a_{rs}x_s - \lambda \sum_s b_{rs}x_s = w_r \qquad r = 1, 2, \ldots, n \tag{3}$$

When the external forces are zero, then $w_r = 0$ for $r = 1, 2, \ldots, n$ and the nonhomogenous set (3) is replaced by

$$\sum_s a_{rs}x_s - \lambda \sum_s b_{rs}x_s = 0 \tag{4}$$

The characteristic values λ_ν associated with equation (4) are the zeros of the characteristic determinant $\left|a_{rs} - \lambda b_{rs}\right|$. To each such zero value of λ_ν there are associated two characteristic vectors or modes, one, $\vec{x}^\nu = \left(x_1{}^\nu, \ldots, x_n{}^\nu\right)$ called the <u>direct characteristic mode</u>, and the other, $\vec{y}^\nu = \left(y_1{}^\nu, \ldots, y_n{}^\nu\right)$ called the <u>conjugate characteristic mode</u>. The direct mode is a solution of equation (4) with $\lambda = \lambda_\nu$

$$\sum_s a_{rs}x_s{}^\nu - \lambda_\nu \sum_s b_{rs}x_s{}^\nu = 0 \qquad r = 1, 2, \ldots, n \tag{5a}$$

whereas the conjugate mode satisfies a similar set of equations

$$\sum_r y_r{}^\nu a_{rs} - \lambda_\nu \sum_r y_r{}^\nu b_{rs} = 0 \qquad s = 1, 2, \ldots, n \tag{5b}$$

The vectors $\vec{x}^\nu$ and $\vec{y}^\nu$ are not defined uniquely because any scalar multiple of these vectors would also satisfy equations (5a) and (5b), respectively. In the response function introduced below, a "normalization factor" appears which makes it immaterial which solutions of equations (5a) and (5b) are used. This factor is

$$B\left(\vec{y}^\nu, \vec{x}^\nu\right) = \sum_{r,s} y_r{}^\nu b_{rs}x_s{}^\nu \tag{6}$$

and represents the work done by the elastic forces set up in the deformation $\vec{x}^\nu$ when these forces are subjected to the displacement $\vec{y}^\nu$.

When $a_{rs} = a_{sr}$ and $b_{rs} = b_{sr}$, the system is <u>conservative</u> and the vectors $\vec{x}^\nu$ and $\vec{y}^\nu$ may be taken as identical. Aerodynamic systems are nonconservative in general; the difference between the vectors $\vec{x}^\nu$

and $\vec{y}^{\nu}$, when the two vectors are made comparable by choosing the same component to be unity, is therefore associated with the energy-absorbing or energy-producing characteristic of the system.

In case λ_ν is positive, $\vec{x}^{\nu}$ represents the deflection of the system when vibrating at the frequency $\omega = \omega_r/\sqrt{\lambda_\nu}$. When λ_ν is not positive, the mode $\vec{x}^{\nu}$ is sometimes regarded as a similar deflection of the system, modified by a suitable damping coefficient.

One may, on the basis of equation (5b), regard the conjugate characteristic mode as the direct characteristic mode of a hypothetical system with coefficient matrices a_{rs}' and b_{rs}' which are the adjoints of the original matrices a_{rs} and b_{rs}, respectively. Another interpretation of the conjugate modes will be made below.

The <u>harmonic response function</u> is the deflection of the system under the action of harmonic external forces represented by w_1, w_2, . . ., w_n. The response function may then be regarded as the deflection of the airplane in flight under the action of vibrators. The use of the response-function concept has become increasingly important in recent years both in flight-testing, as is done for classical dynamic stability, and in theory (reference 3).

The harmonic response function may be expressed as the sum of (a) the response of the system as a free body under the action of the external forces and (b) the sum of the elastic responses in each degree of freedom. The result derived in reference 1 states[1] that the harmonic response $\vec{x}$ is

$$\vec{x} = \sum_{\nu=1}^{l} \frac{\left(\vec{y}^{\nu},\vec{w}\right)}{\left(\lambda_\nu - \lambda\right)B\left(\vec{y}^{\nu},\vec{x}^{\nu}\right)}\vec{x}^{\nu} + \vec{x}^{l+1} \tag{7}$$

Here the integer l represents the number of distinct elastic modes; that is, $n - l$ is the number of independent free-body degrees of freedom. The vector $\vec{x}^{l+1}$ is the free-body response referred to in

[1]For the benefit of the reader who does not have access to reference 1, a derivation of equation (7) has been included in appendix A.

item (a) above; the terms in the summation are the direct characteristic vectors, each appearing with a scalar coefficient

$$\frac{\left(\vec{y}^{\nu},\vec{w}_{r}\right)}{\left(\lambda_{\nu} - \lambda\right)B\left(\vec{y}^{\nu},\vec{x}^{\nu}\right)} = \frac{\sum_{r} y_{r}^{\nu} w_{r}}{\left(\lambda_{\nu} - \lambda\right)\sum_{r,s} y_{r}^{\nu} b_{rs} x_{s}^{\nu}} \tag{8}$$

The denominator of this coefficient contains a term depending on the frequency of the forcing oscillation, that is,

$$\lambda_{\nu} - \lambda = \lambda_{\nu}\left[1 - \frac{1}{\lambda_{\nu}}\left(\frac{\omega_{r}}{\omega}\right)^{2}\right]$$

and is of the form of a magnification factor normally appearing in the theory of vibration in one degree of freedom.

The numerator in equation (8) represents the work fed by the external forces into the conjugate mode. Here lies the importance of the conjugate mode; the intensity with which the direct mode $\vec{x}^{\nu}$ appears in the response is proportional to the work absorbed by the conjugate mode, not by the direct mode. This shows that the true nature of the couplings due to additional degrees of freedom is to be found from the relation of these degrees to the conjugate modes of the original system.

ADDITION OF A POINT MASS

Consider an airplane flying at speed v. Suppose that a complete flutter analysis has been made so that the following data are available: The characteristic numbers λ_{ν}, the direct modes $\vec{x}^{\nu}$, and the conjugate modes $\vec{y}^{\nu}$.

Now suppose that at a point P of the airplane a concentrated, harmonically oscillating force $Fe^{i\omega t}$ is applied. Let x represent the displacement of the point P in the direction of the force F. Then, within the range of linearity,

$$x = \beta_{1}x_{1} + \beta_{2}x_{2} + \ldots + \beta_{n}x_{n}$$

where $\beta_1, \beta_2, \ldots, \beta_n$ are suitable geometric constants. The relation may be conveniently written as an inner product of the vector $\vec{\beta} = \left(\beta_1, \beta_2, \ldots, \beta_n\right)$ and the vector $\vec{x} = \left(x_1, x_2, \ldots, x_n\right)$

$$x = \left(\vec{\beta}, \vec{x}\right) \tag{9}$$

Let $\vec{q} = \left(q_1, \ldots, q_n\right)$ represent the vector with components corresponding to the generalized forces developed by the exciting force F. In order to obtain the components of $\vec{q}$, consider the work developed by the force F under each of the virtual displacements $\delta x_1, \delta x_2, \ldots, \delta x_n$. Each such virtual displacement δx_r results in a displacement $\beta_r \delta x_r$ of the point P, according to equation (9); therefore the force F does work $F\beta_r \delta x_r$. Hence,

$$q_r = F\beta_r$$

$$\vec{q} = F\vec{\beta}$$

$$\vec{w} = -\frac{F}{\omega^2}\vec{\beta}$$

The harmonic response, equation (7), yields for the deflection x

$$x = -\frac{F}{\omega^2}\sum_{\nu=1}^{l}\frac{\left(\vec{y}^{\nu}, \vec{\beta}\right)\left(\vec{\beta}, \vec{x}^{\nu}\right)}{\left(\lambda_\nu - \lambda\right)B\left(\vec{y}^{\nu}, \vec{x}^{\nu}\right)} - \frac{F}{\omega^2}\left(\vec{\beta}, \vec{x}^{l+1}\right) \tag{10}$$

The force F may, for instance, be the interaction force produced by a mass mounted on the airplane at P. In order to make the results more general, suppose that the mass m be connected to the airplane at P by a spring (fig. 1(a)). Let ω_o represent the frequency of the mass on the spring, P held fixed. Then the deflection x' of the end of the spring under the reaction F is

$$x' = \frac{F}{m}\left(-\frac{1}{\omega^2} + \frac{1}{\omega_o^2}\right) \tag{11}$$

Since $x = -x'$, comparison of equations (10) and (11) gives

$$\frac{F}{m}\left(-\frac{1}{\omega^2} + \frac{1}{\omega_o{}^2}\right) = \frac{F}{\omega^2}\sum_{\nu=1}^{l} \frac{\left(\vec{y}^\nu, \vec{\beta}\right)\left(\vec{\beta}, \vec{x}^\nu\right)}{\left(\lambda_\nu - \lambda\right)B\left(\vec{y}^\nu, \vec{x}^\nu\right)} + \frac{F}{\omega^2}\left(\vec{\beta}, \vec{x}^{l+1}\right)$$

Dividing through by F/ω^2,

$$\frac{1}{m}\left(-1 + \frac{\omega^2}{\omega_o{}^2}\right) = \sum_{\nu=1}^{l} \frac{\left(\vec{y}^\nu, \vec{\beta}\right)\left(\vec{\beta}, \vec{x}^\nu\right)}{\left(\lambda_\nu - \lambda\right)B\left(\vec{y}^\nu, \vec{x}^\nu\right)} + \frac{F}{\omega^2}\left(\vec{\beta}, \vec{x}^{l+1}\right) \tag{12a}$$

In the case where there are no free modes, $l = n$, and equation (12a) is replaced by

$$\frac{1}{m}\left(-1 + \frac{\omega^2}{\omega_o{}^2}\right) = \sum_{\nu=1}^{n} \frac{\left(\vec{y}^\nu, \vec{\beta}\right)\left(\vec{\beta}, \vec{x}^\nu\right)}{\left(\lambda_\nu - \lambda\right)B\left(\vec{y}^\nu, \vec{x}^\nu\right)} \tag{12b}$$

In case the mass is rigidly attached to the point P, ω_o is infinite and the term $\omega^2/\omega_o{}^2$ is zero.

Consider the case in which the mass is attached with a spring to the wing. Suppose, in addition, that the deflection of the wing is described by two degrees of freedom, one bending and represented by $f_h(\xi)$, and one torsion represented by $f_\alpha(\xi)$. Then if e_w represents the chordwise coordinate of the point P relative to the elastic axis, positive rearward

$$\vec{\beta} = \left(f_h(\xi),\ e_w f_\alpha(\xi)\right) \tag{13}$$

The right sides of equations (12) are functions only of $v/\omega b$ and ω/ω_r. Hence equation (12) may be solved, in principle, for m and $\omega^2/\omega_o{}^2$ for each value of $v/\omega b$. In practice, it appears desirable to carry out the solution by a graphical construction. In this procedure, one regards the right side of equation (12), for each value of $v/\omega b$, as a function of the real parameter $\lambda = \omega_r^2/\omega^2$. The function is plotted in the complex plane and the abscissa of intersection of the resulting plot on the axis of reals is equal to

$$\frac{1}{m}\left(-1 + \frac{\omega^2}{\omega_o^2}\right) = \frac{1}{m}\left(-1 + \frac{1}{\lambda}\frac{\omega_r^2}{\omega_o^2}\right) \tag{14}$$

for which flutter will occur. The parameter λ at the intersection defines the frequency of flutter and the value of m corresponding to equation (14) the mass required to maintain flutter. As in conventional practice (reference 4), the mass is to be interpreted as the nondimensional mass $m/\pi\rho b^3$.

The graphical construction is facilitated by noting that each of the summands in equation (12) has the form $c/(\lambda_1 - \lambda)$ where c is a constant and λ_1 is a characteristic value. Both c and $\lambda_1 = a_1 + ib_1$ are complex numbers in general. A plot of the fraction $c/(\lambda_1 - \lambda)$, with λ a real parameter varying from minus infinity to plus infinity, yields a circle passing through the origin of the complex plane. To construct the circle in the complex plane:

(a) Form the diametral vector c/ib and draw the corresponding circle through the origin (fig. 2). The diametral vector represents the complex number of largest modulus given by $c/(a_1 + ib_1 - \lambda)$, attained when $\lambda = a_1$.

(b) Give a sense to the direction of motion around the circle as λ varies from minus infinity to plus infinity according to the rule:

If $b_1 > 0$, the circle is traversed clockwise

If $b_1 < 0$, the circle is traversed counterclockwise

(c) Calibrate the circle. Referring to figure 2, use any convenient scale, lay out distance $|b_1|$ along the diametral vector, and draw line PP perpendicular to the diametral vector. Give the same sense to points on PP as has been established on the circle. To point Q, attach the value $\lambda = a_1$. To any other point Q' on line PP, attach the value $\lambda = a_1 \pm |QQ'|$, where $|QQ'|$ is the distance from Q to Q' measured to the scale chosen above for $|b_1|$, and the plus or minus sign is selected to correspond with the sense of increasing or decreasing values of λ. Draw the radial vector from the origin through Q'. The value of λ at Q' is then attached to the intersection R of the radial vector on the circle.

The procedure is carried out for each constituent of the response function. The resulting vectors OR, corresponding to the same values of λ, are added vectorially.[2] The locus of the sum in the complex plane is a plot of the harmonic response which appears on the right side of equation (12b).

The analysis described above is illustrated in appendix B.

ADDITION OF A DISTRIBUTED MASS

In the preceding section, the mass was regarded as concentrated in a point; that is, the moment of inertia $\overline{I}$ of the mass relative to its center of gravity is zero. In this section, the equation corresponding to equation (12) but for a distributed mass ($\overline{I} \neq 0$), rigidly attached to the original system, will be derived.

There are now two conditions required that the deflection of the mass and the system be compatible. Two equations of compatibility are therefore derived.

If the original system is described only by two degrees of freedom, then the analysis of the mass effect as described below is not a convenient device because the conditions of compatibility are equal in number to the original equations of flutter. On the other hand, if a large number of degrees of freedom were required to describe the original system, then the numerical work required to treat the two equations of compatibility may be significantly less than a reworking of the entire flutter analysis to include the mass effect.

In order to make the analysis definite, suppose the mass is attached to an airplane wing which can vibrate only transverse to the plane of the wing. Let P be a point common to the body and mass (see fig. 1(b)) and

x vertical displacement of wing at P

x' vertical displacement of body at P

θ torsional displacement of wing at P

θ' torsional displacement of body at P

The sign convention is such that x' and θ' are positive in directions opposite to x and θ.

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[2]In order to perform the vectorical addition graphically, it is desirable that the individual circles, and therefore the diametral vectors, be drawn to the same scale.

There are now geometric parameters $\vec{\beta}^1$ and $\vec{\beta}^2$ such that[3]

$$x = \left(\vec{\beta}^1, \vec{x}\right)$$

$$\theta = \left(\vec{\beta}^2, \vec{x}\right)$$

At point P, apply a harmonic force F and torque T. Let

x_F, x_T vertical displacements of wing at point P due to F and T, respectively

θ_F, θ_T torsional displacements of wing due to F and T, respectively

x'_F, x'_T vertical displacements of body at point P due to reversed F and T, respectively

θ'_F, θ'_T torsional displacements of body due to reversed F and T, respectively

Under the action of F, the forcing vector $\vec{w}$ is $\vec{w}' = -\frac{F}{\omega^2}\vec{\beta}^1$. Under the action of T, the forcing vector $\vec{w}^2$ is $-\frac{T}{\omega^2}\vec{\beta}^2$. Therefore,

$$\left.\begin{aligned}
x_F &= -\frac{F}{\omega^2}\sum_r \frac{\left(\vec{y}^\nu, \vec{\beta}^1\right)\left(\vec{\beta}^1, \vec{x}^\nu\right)}{\left(\lambda_\nu - \lambda\right)B\left(\vec{y}^\nu, \vec{x}^\nu\right)} - \frac{F}{\omega^2}\left(\vec{\beta}^1, \vec{x}^{l+1}\right) \\
x_T &= -\frac{T}{\omega^2}\sum_r \frac{\left(\vec{y}^\nu, \vec{\beta}^2\right)\left(\vec{\beta}^1, \vec{x}^\nu\right)}{\left(\lambda_\nu - \lambda\right)B\left(\vec{y}^\nu, \vec{x}^\nu\right)} - \frac{I}{\omega^2}\left(\vec{\beta}^1, \vec{x}^{l+1}\right) \\
\theta_F &= -\frac{F}{\omega^2}\sum_r \frac{\left(\vec{y}^\nu, \vec{\beta}^1\right)\left(\vec{\beta}^2, \vec{x}^\nu\right)}{\left(\lambda_\nu - \lambda\right)B\left(\vec{y}^\nu, \vec{x}^\nu\right)} - \frac{F}{\omega^2}\left(\vec{\beta}^2, \vec{x}^{l+1}\right)
\end{aligned}\right\} \tag{15}$$

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[3]In the particular case considered here, of wing bending and wing torsion, $\vec{\beta}^1 = \left(f_h(\xi),\ e_W f_\alpha(\xi)\right)$ and $\vec{\beta}^2 = \left(0, f_\alpha(\xi)\right)$.

$$\left.\begin{aligned}
\theta_T &= -\frac{T}{\omega^2}\sum_{\nu}\frac{\left(\vec{y}^{\nu},\vec{\beta}^{2}\right)\left(\vec{\beta}^{2},\vec{x}^{\nu}\right)}{\left(\lambda_{\nu}-\lambda\right)B\left(\vec{y}^{\nu},\vec{x}^{\nu}\right)} - \frac{T}{\omega^2}\left(\vec{\beta}^{2},\vec{x}^{l+1}\right) \\
x_F' &= -\frac{1}{\omega^2}\frac{I}{m\bar{\bar{I}}}F \qquad\qquad \theta_F' = \frac{be}{\bar{\bar{I}}\omega^2}F \\
x_T' &= \frac{be}{\bar{\bar{I}}\omega^2}T \qquad\qquad \theta_T' = -\frac{1}{\bar{\bar{I}}\omega^2}T
\end{aligned}\right\} \tag{16}$$

The condition of compatibility results in a pair of simultaneous equations

$$\left.\begin{aligned}
x_F + x_F' + x_T + x_T' &= 0 \\
\theta_F + \theta_F' + \theta_T' + \theta_T &= 0
\end{aligned}\right\} \tag{17}$$

These equations can be expressed in terms of the characteristics of the wing and mass by substituting from equations (15) and (16). Restricting the discussion, for simplicity, to the case where there are no free modes present ($l = n$), one has

$$\left.\begin{aligned}
0 &= F\left[-\sum\frac{\left(\vec{y}^{\nu},\vec{\beta}^{1}\right)\left(\vec{\beta}^{1},\vec{x}^{\nu}\right)}{\left(\lambda_{\nu}-\lambda\right)B\left(\vec{y}^{\nu},\vec{x}^{\nu}\right)} - \frac{I}{m\bar{\bar{I}}}\right] + T\left[-\sum\frac{\left(\vec{y}^{\nu},\vec{\beta}^{2}\right)\left(\vec{\beta}^{1},\vec{x}^{\nu}\right)}{\left(\lambda_{\nu}-\lambda\right)B\left(\vec{y}^{\nu},\vec{x}^{\nu}\right)} + \frac{be}{\bar{\bar{I}}}\right] \\
0 &= F\left[-\sum\frac{\left(\vec{y}^{\nu},\vec{\beta}^{1}\right)\left(\vec{\beta}^{2},\vec{x}^{\nu}\right)}{\left(\lambda_{\nu}-\lambda\right)B\left(\vec{y}^{\nu},\vec{x}^{\nu}\right)} + \frac{be}{\bar{\bar{I}}}\right] + T\left[-\sum\frac{\left(\vec{y}^{\nu},\vec{\beta}^{2}\right)\left(\vec{\beta}^{2},\vec{x}^{\nu}\right)}{\left(\lambda_{\nu}-\lambda\right)B\left(\vec{y}^{\nu},\vec{x}^{\nu}\right)} - \frac{1}{\bar{\bar{I}}}\right]
\end{aligned}\right\} \tag{18}$$

where the factor $1/\omega^2$ has been canceled out.

Let Δ represent the determinant of the system (18):

$$\Delta = \begin{vmatrix} \dfrac{A_{11}}{\lambda_1 - \lambda} + \dfrac{B_{11}}{\lambda_2 - \lambda} - \dfrac{I}{m\bar{\bar{I}}} & \dfrac{A_{12}}{\lambda_1 - \lambda} + \dfrac{B_{12}}{\lambda_2 - \lambda} + \dfrac{be}{\bar{\bar{I}}} \\ \dfrac{A_{21}}{\lambda_1 - \lambda} + \dfrac{B_{21}}{\lambda_2 - \lambda} + \dfrac{be}{\bar{\bar{I}}} & \dfrac{A_{22}}{\lambda_1 - \lambda} + \dfrac{B_{22}}{\lambda_2 - \lambda} - \dfrac{1}{\bar{\bar{I}}} \end{vmatrix}$$

where

$$A_{rs} = -\frac{\left(\vec{y}^1,\vec{\beta}^s\right)\left(\vec{\beta}^r,\vec{x}^1\right)}{B\left(\vec{y}^1,\vec{x}^1\right)}$$

$$B_{rs} = -\frac{\left(\vec{y}^2,\vec{\beta}^s\right)\left(\vec{\beta}^r,\vec{x}^2\right)}{B\left(\vec{y}^2,\vec{x}^2\right)}$$

Then the condition for the existence of a nontrivial solution of equation (18) reads

$$\Delta = 0 \tag{19}$$

It will now be assumed that the matrix b_{rs} is the unit matrix and that the direct and conjugate modes have been normalized so that

$$B\left(\vec{y}^\nu,\vec{x}^\nu\right) = 1 \qquad \nu = 1,2 \tag{20}$$

These assumptions represent no additional limitation.

In view of equation (18), the expansion of the determinant Δ becomes

$$\Delta = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} \Big/ (\lambda_1 - \lambda)^2 + \begin{vmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{vmatrix} \Big/ (\lambda_2 - \lambda)^2 + \left(\begin{vmatrix} A_{11} & B_{12} \\ A_{21} & B_{22} \end{vmatrix} + \begin{vmatrix} B_{11} & A_{12} \\ B_{21} & A_{22} \end{vmatrix} \right) \Big/ (\lambda_1 - \lambda)(\lambda_2 - \lambda) + \left(\begin{vmatrix} \frac{-I}{m\bar{I}} & A_{12} \\ \frac{be}{\bar{I}} & A_{22} \end{vmatrix} + \begin{vmatrix} A_{11} & \frac{be}{\bar{I}} \\ A_{21} & -\frac{1}{\bar{I}} \end{vmatrix} \right) \Big/ (\lambda_1 - \lambda) + \left(\begin{vmatrix} \frac{-I}{m\bar{I}} & B_{12} \\ \frac{be}{\bar{I}} & B_{22} \end{vmatrix} + \begin{vmatrix} B_{11} & \frac{be}{\bar{I}} \\ B_{21} & -\frac{1}{\bar{I}} \end{vmatrix} \right) \Big/ (\lambda_2 - \lambda) + \begin{vmatrix} \frac{-I}{m\bar{I}} & \frac{be}{\bar{I}} \\ \frac{be}{\bar{I}} & -\frac{1}{\bar{I}} \end{vmatrix} \tag{21}$$

The first three terms may be simplified as follows:

$$\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = \begin{vmatrix} -\left(\vec{y}^1, \vec{\beta}^1\right)\left(\vec{\beta}^1, \vec{x}^1\right) & -\left(\vec{y}^1, \vec{\beta}^2\right)\left(\vec{\beta}^1, \vec{x}^1\right) \\ -\left(\vec{y}^1, \vec{\beta}^1\right)\left(\vec{\beta}^2, \vec{x}^1\right) & -\left(\vec{y}^1, \vec{\beta}^2\right)\left(\vec{\beta}^2, \vec{x}^1\right) \end{vmatrix}$$

$$= \begin{vmatrix} \left(\vec{\beta}^1, \vec{x}^1\right) & 0 \\ 0 & \left(\vec{\beta}^2, \vec{x}^1\right) \end{vmatrix} \times \begin{vmatrix} \left(\vec{y}^1, \vec{\beta}^1\right) & \left(\vec{y}^1, \vec{\beta}^2\right) \\ \left(\vec{y}^1, \vec{\beta}^1\right) & \left(\vec{y}^1, \vec{\beta}^2\right) \end{vmatrix} = 0 \tag{22}$$

in view of the vanishing of the second determinant in the product.

Similarly $|B_{rs}| = 0$. Also,

$$\begin{vmatrix} A_{11} & B_{12} \\ A_{21} & B_{22} \end{vmatrix} + \begin{vmatrix} B_{11} & A_{12} \\ B_{21} & A_{22} \end{vmatrix} = \begin{vmatrix} (\vec{y}^1,\vec{\beta}^1)(\vec{\beta}^1,\vec{x}^1) & (\vec{y}^2,\vec{\beta}^2)(\vec{\beta}^1,\vec{x}^2) \\ (\vec{y}^1,\vec{\beta}^1)(\vec{\beta}^2,\vec{x}^1) & (\vec{y}^2,\vec{\beta}^2)(\vec{\beta}^2,\vec{x}^2) \end{vmatrix} + \begin{vmatrix} (\vec{y}^2,\vec{\beta}^1)(\vec{\beta}^1,\vec{x}^2) & (\vec{y}^1,\vec{\beta}^2)(\vec{\beta}^1,\vec{x}^1) \\ (\vec{y}^2,\vec{\beta}^1)(\vec{\beta}^2,\vec{x}^2) & (\vec{y}^1,\vec{\beta}^2)(\vec{\beta}^2,\vec{x}^1) \end{vmatrix}$$

$$= \begin{vmatrix} (\vec{\beta}^1,\vec{x}^1) & (\vec{\beta}^1,\vec{x}^2) \\ (\vec{\beta}^2,\vec{x}^1) & (\vec{\beta}^2,\vec{x}^2) \end{vmatrix} \begin{vmatrix} (\vec{y}^1,\vec{\beta}^1) & 0 \\ 0 & (\vec{y}^2,\vec{\beta}^2) \end{vmatrix} +$$

$$\begin{vmatrix} (\vec{\beta}^1,\vec{x}^2) & (\vec{\beta}^1,\vec{x}^1) \\ (\vec{\beta}^2,\vec{x}^2) & (\vec{\beta}^2,\vec{x}^1) \end{vmatrix} \begin{vmatrix} (\vec{y}^2,\vec{\beta}^1) & 0 \\ 0 & (\vec{y}^1,\vec{\beta}^2) \end{vmatrix}$$

$$= \begin{vmatrix} (\vec{\beta}^1,\vec{x}^1) & (\vec{\beta}^1,\vec{x}^2) \\ (\vec{\beta}^2,\vec{x}^1) & (\vec{\beta}^2,\vec{x}^2) \end{vmatrix} \left[\begin{vmatrix} (\vec{y}^1,\vec{\beta}^1) & 0 \\ 0 & (\vec{y}^2,\vec{\beta}^2) \end{vmatrix} - \begin{vmatrix} (\vec{y}^2,\vec{\beta}^1) & 0 \\ 0 & (\vec{y}^1,\vec{\beta}^2) \end{vmatrix} \right]$$

$$= \begin{vmatrix} (\vec{\beta}^1,\vec{x}^1) & (\vec{\beta}^1,\vec{x}^2) \\ (\vec{\beta}^2,\vec{x}^1) & (\vec{\beta}^2,\vec{x}^2) \end{vmatrix} \times \begin{vmatrix} (\vec{y}^1,\vec{\beta}^1) & (\vec{y}^2,\vec{\beta}^1) \\ (\vec{y}^1,\vec{\beta}^2) & (\vec{y}^2,\vec{\beta}^2) \end{vmatrix}$$

$$= \begin{vmatrix} \beta_1{}^1 & \beta_2{}^1 \\ \beta_1{}^2 & \beta_2{}^2 \end{vmatrix} \times \begin{vmatrix} x_1{}^1 & x_2{}^1 \\ x_1{}^2 & x_2{}^2 \end{vmatrix} \times \begin{vmatrix} y_1{}^1 & y_1{}^2 \\ y_2{}^1 & y_2{}^2 \end{vmatrix} \times \begin{vmatrix} \beta_1{}^1 & \beta_2{}^1 \\ \beta_1{}^2 & \beta_2{}^2 \end{vmatrix} \tag{23}$$

From the orthogonality property (appendix A) and the normalization (equation (20)) and in view of the fact that matrix B has been assumed to be the unit matrix,

$$B(\vec{y}^1,\vec{x}^1) = (\vec{y}^1,\vec{x}^1) = 1$$

$$B(\vec{y}^2,\vec{x}^2) = (\vec{y}^2,\vec{x}^2) = 1$$

$$B(\vec{y}^1,\vec{x}^2) = (\vec{y}^1,\vec{x}^2) = 0$$

$$B(\vec{y}^2,\vec{x}^1) = (\vec{y}^2,\vec{x}^1) = 0$$

Therefore,

$$\begin{vmatrix} x_1^1 & x_2^1 \\ x_1^2 & x_2^2 \end{vmatrix} \times \begin{vmatrix} y_1^1 & y_1^2 \\ y_2^1 & y_2^2 \end{vmatrix} = \begin{vmatrix} \left(\vec{x}^1,\vec{y}^1\right) & \left(\vec{x}^1,\vec{y}^2\right) \\ \left(\vec{x}^2,\vec{y}^1\right) & \left(\vec{x}^2,\vec{y}^2\right) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

Hence,

$$\begin{vmatrix} A_{11} & B_{12} \\ A_{21} & B_{22} \end{vmatrix} + \begin{vmatrix} B_{11} & A_{12} \\ B_{21} & A_{22} \end{vmatrix} = \begin{vmatrix} \beta_1^1 & \beta_2^1 \\ \beta_1^2 & \beta_2^2 \end{vmatrix}^2 \tag{24}$$

Finally,

$$\begin{vmatrix} -\dfrac{I}{m\overline{I}} & \dfrac{be}{\overline{I}} \\ \dfrac{be}{\overline{I}} & -\dfrac{1}{\overline{I}} \end{vmatrix} = \frac{I}{m\overline{I}^2} - \frac{(be)^2}{\overline{I}^2}$$

$$= \frac{1}{\overline{I}^2}\left[\frac{I}{m} - (be)^2\right] = \frac{1}{\overline{I}m}$$

Combining equations (21) to (24) and multiplying through by $\overline{I}$, the equation of flutter (19) becomes

$$\overline{I}\Delta = \overline{I}\begin{vmatrix} \beta_1^1 & \beta_2^1 \\ \beta_1^2 & \beta_2^2 \end{vmatrix}^2 \Big/ (\lambda_1 - \lambda)(\lambda_2 - \lambda) + \left(\begin{vmatrix} \frac{-I}{m} & A_{12} \\ be & A_{22} \end{vmatrix} + \begin{vmatrix} A_{11} & be \\ A_{21} & -1 \end{vmatrix}\right) \Big/ (\lambda_1 - \lambda) +$$

$$\left(\begin{vmatrix} \frac{-I}{m} & B_{12} \\ be & B_{22} \end{vmatrix} + \begin{vmatrix} B_{11} & be \\ B_{21} & -1 \end{vmatrix}\right) \Big/ (\lambda_2 - \lambda) + \frac{1}{m} = 0 \tag{25}$$

The first term on the right may be separated into partial fractions and combined with the second and third. Equation (25), so modified, has the same form as equation (12b) for the case in which ω_0 is infinite (rigid attachment). Therefore the graphical solution of equation (12b) is directly applicable to equation (25).

Such a graphical solution is illustrated, for the numerical example discussed above, in figure 2(b).

NUMERICAL ILLUSTRATION

The method outlined has been applied to obtain the flutter speeds under figure 4 of reference 2. Here the weight is attached at $e_w = 0.500$ for various span positions. A detailed calculation is presented in appendix B. The results of an analysis are presented in figures 3 to 7.

The specific mass addition with which this report is concerned, that is, $\frac{m}{\pi\rho b^5} = 357$, is represented by a horizontal line in figure 3. The intersection at each span position gives the corresponding value of $\frac{v}{\omega b}$ at which flutter will occur. For these values of $\frac{v}{\omega b}$ the values of λ are determined from figure 4. Table 1 contains the values of $\frac{v}{\omega b}$ and λ read from these curves.

Figure 5 was plotted in a form comparable with figure 4 of reference 2. However, the definition of v_0 differs from that of reference 1. Here v_0 is the flutter speed of the wing without attached mass according to theory or test as the case may be. The curves which are being compared are $\left(\frac{v}{v_0}\right)_{theory}$ and $\left(\frac{v}{v_0}\right)_{test}$.

Figure 6 presents a comparison of the reduced flutter speeds obtained by the method of this report and reference 2. Figure 7 shows a comparison of frequency ratios. In particular it is shown that the effect of $\bar{I}$, in this case, is small.

The effect of flexibility of the suspension is considered in appendix C.

Purdue University
Lafayette, Ind., July 20, 1950

APPENDIX A

DERIVATION OF HARMONIC RESPONSE FUNCTION

In the following paragraphs, a derivation of equation (7) is presented. This derivation is based upon a number of assumptions which are discussed in reference 1. They are:

(a) Let $\vec{e}$ and $\vec{f}$ represent direct and conjugate characteristic vectors associated with a given characteristic value. Then,[4]

$$B\left(\vec{f},\vec{e}\right) \neq 0$$

(b) Let $\vec{g}$ and $\vec{h}$ be free-body direct and conjugate vectors[5] such that

$$\sum_{s} b_{rs}g_{s} = 0$$

$$\sum_{r} h_{r}b_{rs} = 0$$

Then, for each vector $\vec{g}$, there is at least one vector $\vec{h}$ such that

$$\sum_{r,s} h_{r}a_{rs}g_{s} \neq 0$$

(c) The characteristic determinant $|a_{rs} - \lambda b_{rs}|$ does not vanish identically.

Consider the characteristic-value problem

$$\sum_{s} a_{rs}x_{s} - \lambda \sum_{s} b_{rs}x_{s} = 0 \qquad r = 1, 2, \ldots, n \qquad \text{(A1)}$$

---

[4]This condition takes the place of "positive-definiteness" of the energy, required in the analysis of conservative systems. The normalization condition associated with the vectors $\vec{e}$ and $\vec{f}$ is introduced below.

[5]Vectors defining motions with zero elastic potential energy.

Equations (A1) represent the balancing between the inertia-plus-aerodynamic forces represented by the matrix a_{rs} and the elastic forces, represented by $-\lambda(b_{rs})$ developed in the mode $\vec{x} = x_1, x_2, \ldots, x_n$.

Another interpretation of equations (A1) is obtained by multiplying the rth equation by y_r and summing on r. The vector $\vec{y} = y_1, y_2, \ldots, y_n$ represents a virtual displacement. Introducing the notation,

$$\left.\begin{aligned} A(\vec{y},\vec{x}) &= \sum_{rs} y_r a_{rs} x_s \\ B(\vec{y},\vec{x}) &= \sum_{rs} y_r b_{rs} x_s \end{aligned}\right\} \tag{A2}$$

one may write the result in the form

$$A(\vec{y},\vec{x}) - \lambda B(\vec{y},\vec{x}) = 0 \tag{A3}$$

for all values of y. In this form, the equation of dynamic equilibrium states that the sum of the works performed by the inertia-plus-aerodynamic forces and elastic forces, when displaced over an aribitrary virtual displacement, is zero (d'Alembert's principle).

In the special case where b_{rs} is the unit matrix,

$$B(\vec{y},\vec{x}) = \sum_{r=1} y_r x_r$$

Then $B(\vec{y},\vec{x})$ will be written simply as $(\vec{y},\vec{x})$. In this case, $B(\vec{y},\vec{x}) = (\vec{y},\vec{x})$ has the interpretation as an "inner product" between two vectors $\vec{y}$ and $\vec{x}$. In two and three dimensions, $(\vec{y},\vec{x})$ is then equal to the product of the lengths of $\vec{x}$ and $\vec{y}$ by the cosine of the included angle. In the case of arbitrary values of b_{rs}, a similar interpretation will be made; that is, $B(\vec{y},\vec{x})$ will be regarded as an inner product between the conjugate mode $\vec{y}$ and the direct mode $\vec{x}$. Further, $\vec{y}$ and $\vec{x}$ will be orthogonal with regard to b_{rs} if

$$B(\vec{y},\vec{x}) = 0 \tag{A4}$$

It is to be emphasized that, in using this concept of an inner product, the direct vector must appear in the second argument and the conjugate vector in the first argument.

Similarly $A(\vec{y},\vec{x})$ defines an inner product. It is conceivable that a vector $\vec{y}$ may be orthogonal to $\vec{x}$ with respect to b_{rs} but not orthogonal with respect to a_{rs} and vice versa. But equation (A3) states that, if $\vec{x} = \vec{e}^1$ is a direct characteristic mode corresponding to a characteristic root $\lambda = \lambda_1$, then

$$A\left(\vec{y},\vec{e}^1\right) = \lambda_1 B\left(\vec{y},\vec{e}^1\right) \tag{A5}$$

for all values of $\vec{y}$. Therefore, if $\vec{y}$ is orthogonal to $\vec{e}^1$ with respect to b_{rs}, then the right side of equation (A5) is zero and so also is the left; then $\vec{y}$ is orthogonal to $\vec{e}^1$ also with respect to a_{rs}.

One may, along with equations (A1), consider the set

$$\sum_r y_r a_{rs} - \lambda \sum_r y_r b_{rs} = 0 \qquad s = 1, 2, \ldots, n \tag{A6}$$

Multiplying the sth equation by x_s and summing on s, one has again equation (A3) which must be satisfied for arbitrary values of $\vec{x}$. Since $\lambda = \lambda_1$ is a characteristic root of equation (A6), it follows that there is a nontrivial solution $y = \vec{f}^1$ of equation (A6) for $\lambda = \lambda_1$. Therefore, analogous to equation (A5),

$$A\left(\vec{f}^1,\vec{x}\right) = \lambda_1 B\left(\vec{f}^1,\vec{x}\right) \tag{A7}$$

for all values of $\vec{x}$.

Whereas in the classical theory of conservative systems the vectors $\vec{e}^1$ and $\vec{f}^1$ may be taken as identical, the nonconservative character of aerodynamic forces requires that $\vec{e}^1$ and $\vec{f}^1$ be treated separately.

Let $\vec{x}$ be an arbitrary vector representing a direct mode. Then resolve $\vec{x}$ into components "parallel" and orthogonal with reference to b_{rs}, to $\vec{f}^1$. That is, decompose $\vec{x}$ in the form

$$\vec{x} = \xi_1\vec{e}^1 + \vec{x}^{(2)} \tag{A8}$$

where ξ is a scalar and $\vec{x}^{(2)}$ is such that

$$B\left(\vec{f}^1,\vec{x}^{(2)}\right) = 0 \tag{A9}$$

Such a decomposition is possible because equation (A9) is equivalent to

$$\left.\begin{aligned} &B\left(\vec{f}^1,\vec{x} - \xi,\vec{e}^1\right) = 0 \\ &B\left(\vec{f}^1,\vec{x}\right) - \xi_1 B\left(\vec{f}^1,\vec{e}^1\right) = 0 \\ &\xi_1 = B\left(\vec{f}^1,\vec{x}^1\right)\Big/B\left(\vec{f}^1,\vec{e}^1\right) \end{aligned}\right\} \tag{A10}$$

At the same time, one may resolve $\vec{y}$ into components, parallel and orthogonal, with reference to b_{rs}, to $\vec{e}^1$.

$$\vec{y} = \eta_1\vec{f}^1 + \vec{y}^{(2)} \tag{A11}$$

$$B\left(\vec{y}^{(2)},\vec{e}^1\right) = 0 \tag{A12}$$

$$\eta_1 = B\left(\vec{y},\vec{e}^1\right)\Big/B\left(\vec{f}^1,\vec{e}^1\right) \tag{A13}$$

Since $B\left(\vec{f}^1,\vec{e}^1\right) \neq 0$ (Assumption (a)), one may assume, for convenience, that

$$B\left(\vec{f}^1,\vec{e}^1\right) = 1 \tag{A14}$$

a condition ("normalization condition") which may be met by multiplication of $\vec{e}^1$ or $\vec{f}^1$ by a suitably chosen scalar.

The orthogonality relations (A9) and (A12) with respect to b_{rs} imply orthogonality with respect to a_{rs}, as stated above,

$$A\left(\vec{f}^1,\vec{x}^{(2)}\right) = 0 \tag{A15}$$

$$A\left(\vec{y}^{(2)},\vec{e}^{1}\right) = 0 \tag{A16}$$

so that there is zero coupling between the first characteristic modes and the orthogonal modes.

In view of equations (A9), (A12), (A15), and (A16), $\vec{e}^{1}$ and $\vec{f}^{1}$ are particularly well-suited for this construction of a set of coordinate vectors, one for the direct modes and one for the conjugate modes, with reference to which $A(\vec{y},\vec{x})$ and $B(\vec{y},\vec{x})$ take simple forms. For,

$$\begin{aligned}
A(\vec{y},\vec{x}) &= A\left(\eta_1\vec{f}^{1} + \vec{y}^{(2)},\xi_1\vec{e}^{1} + \vec{x}^{(2)}\right) \\
&= \eta_1\xi_1A\left(\vec{f}^{1},\vec{e}^{1}\right) + \eta_1A\left(\vec{f}^{1},\vec{x}^{(2)}\right) + \xi_1A\left(\vec{y}^{(2)},\vec{e}^{1}\right) + A\left(\vec{y}^{(2)},\vec{x}^{(2)}\right) \\
&= \eta_1\xi_1A\left(\vec{f}^{1},\vec{e}^{1}\right) + A\left(\vec{y}^{(2)},\vec{x}^{(2)}\right)
\end{aligned} \tag{A17}$$

Similarly, using equation (A14)

$$\begin{aligned}
B(\vec{y},\vec{x}) &= \eta_1\xi_1B\left(\vec{f}^{1},\vec{e}^{1}\right) + B\left(\vec{y}^{(2)},\vec{x}^{(2)}\right) \\
&= \eta_1\xi_1 + B\left(\vec{y}^{(2)},\vec{x}^{(2)}\right)
\end{aligned} \tag{A18}$$

Substituting $\vec{y} = \vec{f}^{1}$ in equation (A5) and using equation (A14),

$$A\left(\vec{f}^{1},\vec{e}^{1}\right) = \lambda_1 \tag{A19}$$

and equation (A17) takes the form

$$A(\vec{y},\vec{x}) = \lambda_1\eta_1\xi_1 + A\left(\vec{y}^{(2)},\vec{x}^{(2)}\right) \tag{A20}$$

In virtue of the orthogonality conditions (A9) and (A12), the vectors $\vec{x}^{(2)}$ and $\vec{y}^{(2)}$ each have $n - 1$ degrees of freedom. Every value of $\vec{x}^{(2)}$ may then be represented as a linear combination of

$n - 1$ vectors $\vec{x}^{21}$, $\vec{x}^{22}$, . . ., $\vec{x}^{2,n-1}$. Similarly, every value of $\vec{y}^{(2)}$ is a linear combination of vectors $\vec{y}^{21}$, $\vec{y}^{22}$, . . ., $\vec{y}^{2,n-1}$. Then an arbitrary direct mode $\vec{x}$ may be written

$$\vec{x} = \xi_1\vec{e}^1 + \xi_{21}\vec{x}^{21} + \ldots + \xi_{2,n-1}\vec{x}^{2,n-1}$$

and an arbitrary mode $\vec{y}$

$$\vec{y} = \eta_1\vec{f}^1 + \eta_{21}\vec{y}^{21} + \ldots + \eta_{2,n-1}\vec{y}^{2,n-1}$$

With the aid of equations (A18) and (A20), equation (A3) may now be written

$$\lambda_1\eta_1\xi_1 + \sum_{r,s=1}^{n-1} \eta_{2,r}A\left(\vec{y}^{2,r},\vec{x}^{2,s}\right)\xi_{2,s} - \lambda\left[\eta_1\xi_1 + \sum_{r,s=1}^{n-1} \eta_{2,r}B\left(\vec{y}^{2,r},\vec{x}^{2,s}\right)\xi_{2,s}\right] = 0 \qquad \text{(A21)}$$

Putting the coefficients of η_1 and $\eta_{2,r}$ equal to zero, a set of linear equations in ξ_1 and $\xi_{2,r}$ is obtained. The characteristic equation is now

$$0 = \begin{vmatrix} \lambda_1 - \lambda & 0 & 0 & \cdots & 0 \\ 0 & A\left(\vec{y}^{2,1},\vec{x}^{2,1}\right) - \lambda B\left(\vec{y}^{2,1},\vec{x}^{2,1}\right) & A\left(\vec{y}^{2,1},\vec{x}^{2,1}\right)_1 - \lambda B\left(\vec{y}^{2,1},\vec{x}^{2,2}\right) & \cdots & A\left(\vec{y}^{2,1},\vec{x}^{2,n-1}\right) - \lambda B\left(\vec{y}^{2,1},\vec{x}^{2,n-1}\right) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & A\left(\vec{y}^{2,n-1},\vec{x}^{2,1}\right) - \lambda B\left(\vec{y}^{2,n-1},\vec{x}^{2,1}\right) & \cdots & \cdots & A\left(\vec{y}^{2,n-1},\vec{x}^{2,n-1}\right) - \lambda B\left(\vec{y}^{2,n-1},\vec{x}^{2,n-1}\right) \end{vmatrix}$$

It can now be seen that the characteristic roots of the original problem in n degrees are identical with λ_1 together with the set of the (n - 1)-degree problem corresponding to $A\left(\vec{y}^{(2)},\vec{x}^{(2)}\right) - \lambda B\left(\vec{y}^{(2)},\vec{x}^{(2)}\right)$.

Since the assumptions (a) to (c), assumed valid for the n-degree problem, hold for the (n - 1)-degree problem, one may continue the reduction. Namely, to a characteristic root λ_2 of the characteristic determinant associated with $A\left(\vec{y}^{(2)},\vec{x}^{(2)}\right) - \lambda B\left(\vec{y}^{(2)},\vec{x}^{(2)}\right)$ there is a direct characteristic mode $\vec{e}^2$ and a conjugate characteristic mode $\vec{f}^2$. Assuming that they are normalized,

$$B\left(\vec{f}^2,\vec{e}^2\right) = 1$$

then it is possible to resolve $\vec{x}^{(2)}$ and $\vec{y}^{(2)}$ in the forms

$$\vec{x}^2 = \xi_2\vec{e}^2 + \vec{x}^{(3)} \qquad \vec{y}^{(2)} = \eta_2\vec{f}^{(2)} + \vec{y}^{(3)}$$

$$B\left(\vec{f}^2,\vec{x}^{(3)}\right) = 0 \qquad B\left(\vec{y}^{(3)},\vec{e}^2\right) = 0$$

$$\xi_2 = B\left(\vec{f}^2,\vec{x}^{(2)}\right) \qquad \eta_2 = B\left(\vec{y}^{(2)},\vec{e}^2\right)$$

At the same time,

$$A\left(\vec{y}^{(2)},\vec{x}^{(2)}\right) = \lambda_2\eta_2\xi_2 + A\left(\vec{y}^{(3)},\vec{x}^{(3)}\right)$$

$$B\left(\vec{y}^{(2)},\vec{x}^{(2)}\right) = \eta_2\xi_2 + B\left(\vec{y}^{(3)},\vec{x}^{(3)}\right)$$

The characteristic equation then takes the form

$$0 = \left|\begin{array}{cc|ccc} \lambda_1 - \lambda & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 - \lambda & 0 & \cdots & 0 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & \multicolumn{3}{c}{\left[A\left(\vec{y}^{(3)},\vec{x}^{(3)}\right) - \lambda B\left(\vec{y}^{(3)},\vec{x}^{(3)}\right)\right]} \\ 0 & 0 & & & \end{array}\right|$$

One may continue the reduction process until all the roots of the characteristic equation have been used. If the reduction has been made n times, then,

$$\vec{x} = \sum \xi_r \vec{e}^r$$

$$\vec{y} = \sum \eta_s \vec{f}^s$$

$$A(\vec{y},\vec{x}) = \sum_{r=1}^{n} \lambda_r \eta_r \xi_r$$

$$B(\vec{y},\vec{x}) = \sum_{1}^{n} \eta_r \xi_r$$

and the characteristic equation is

$$0 = \begin{vmatrix} \lambda_1 - \lambda & 0 & \dots & 0 \\ 0 & \lambda_2 - \lambda & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n - \lambda \end{vmatrix}$$

On the other hand, if after l reductions the characteristic determinant is a constant ($\neq 0$, by assumption (c)), then it can be shown (reference 1) that the reduced b_{rs} is the zero matrix. Hence,

$$\left.\begin{aligned} \vec{x} &= \sum_{1}^{l} \xi_r \vec{e}^r + \vec{x}^{l+1} \\ \vec{y} &= \sum_{s} \eta_s \vec{f}^s + \vec{y}^{l+1} \\ A(\vec{y},\vec{x}) &= \sum_{1}^{l} \lambda_r \eta_r \xi_r + A\left(\vec{y}^{l+1},\vec{x}^{l+1}\right) \\ B(\vec{y},\vec{x}) &= \sum_{r=1}^{l} \eta_r \xi_r \end{aligned}\right\} \qquad \text{(A22)}$$

The orthogonality conditions[6] prevail:

$$\left.\begin{aligned}
&A\left(\vec{f}^{r},\vec{e}^{s}\right) = B\left(\vec{f}^{r},\vec{e}^{s}\right) = 0 && r \neq s \\
&A\left(\vec{f}^{r},\vec{e}^{l+1}\right) = B\left(\vec{f}^{r},\vec{x}^{l+1}\right) = 0 && r = 1, 2, \ldots, l \\
&A\left(\vec{y}^{l+1},\vec{e}^{s}\right) = B\left(\vec{y}^{l+1},\vec{e}^{s}\right) = 0 && s = 1, 2, \ldots, l
\end{aligned}\right\} \tag{A23}$$

The characteristic equation is

$$0 = \left|\begin{array}{cccc|ccc}
\lambda_1 - \lambda & 0 & \ldots & 0 & & & \\
0 & \lambda_2 - \lambda & \ldots & 0 & & 0 & \\
\ldots & \ldots & \ldots & \ldots & & & \\
0 & 0 & \ldots & \lambda_l - \lambda & & & \\
\hline
 & & & & * & * & * \\
 & 0 & & & * & * & * \\
 & & & & \underbrace{*}_{} & \underbrace{*}_{n - l} & \underbrace{*}_{}
\end{array}\right| \left.\begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \end{array}\right\} n - l$$

where the terms indicated with asterisks are independent of λ.

From the last set of equations (A23), it follows that $\vec{x}^{l+1}$ and $\vec{y}^{l+1}$ are free-body modes.

With the reduction shown in equation (A22), it becomes a simple matter to solve equations (3) which are more conveniently treated in the form of an equation of virtual work

$$A(\vec{y},\vec{x}) - \lambda B(\vec{y},\vec{x}) = (\vec{y},\vec{w}) \tag{A24}$$

for all values of $\vec{y}$.

[6]The orthogonality described here is sometimes referred to as "bi-orthogonality." For instance, see reference 5 where there is presented a method of analysis, similar to that in reference 1, for calculating the effects on flutter of small changes in the parameters.

Substituting from equation (A22),

$$\sum_{1}^{l} \lambda_r \eta_r \xi_r + A\left(\vec{y}^{l+1}, \vec{x}^{l+1}\right) - \lambda \sum_{1}^{l} \eta_r \xi_r = \left(\sum \eta_s \vec{f}^s + \vec{y}^{l+1}, \vec{w}\right)$$

$$= \sum_{1}^{l} \eta_s \left(\vec{f}^s, \vec{w}\right) + \left(\vec{y}^{l+1}, \vec{w}\right)$$

for all values of η_r and all values of $\vec{y}^{l+1}$. Hence,

$$(\lambda_r - \lambda)\xi_r = \left(\vec{f}^r, \vec{w}\right) \tag{A25}$$

and, for all values of $\vec{y}^{l+1}$,

$$A\left(\vec{y}^{l+1}, \vec{x}^{l+1}\right) = \left(\vec{y}^{l+1}, \vec{w}\right) \tag{A26}$$

From equation (A25),

$$\xi_r = \left(\vec{f}^r, \vec{w}\right) \Big/ \left(\lambda_2 - \lambda\right) \tag{A27}$$

Equation (A26) defines the free-body motion (elastic degrees regarded as frozen) under the action of the external forces and may be reduced to a set of linear equations. Substituting from equation (A27) to equation (A22),

$$\vec{x} = \sum \frac{\left(\vec{f}^r, \vec{w}\right)}{\lambda_r - \lambda} \vec{e}^r + \vec{x}^{l+1} \tag{A28}$$

In case the normalization condition (A14) does not hold, then it can be shown that equation (A28) is modified as shown in equation (7).

APPENDIX B

SAMPLE CALCULATION

In this appendix, a sample calculation is carried out for the example discussed in reference 2. The case considered corresponds to $e_w = 0.500$ and $\omega_o = \infty$ (rigid attachment).

Homogenous Solution

The flutter determinant to be solved in two degrees of freedom, according to the notation of reference 3, pages 62 to 63, is

$$\begin{vmatrix} A & B \\ D & E \end{vmatrix} = \begin{vmatrix} a_{11} - \lambda b_{11} & a_{12} \\ a_{21} & a_{22} - \lambda b_{22} \end{vmatrix} = 0$$

For a wing with the following properties:

Chord, ft	2/3
Length, ft	4
Aspect ratio	6
Taper ratio	1
Airfoil section	NACA 16-010
γ, slugs/ft	0.02702
I_α, slug-ft^2/ft	0.00080
a	-0.126
EI, lb-ft^2	977.08
GJ, lb-ft^2	480.56
m, slugs	0.0988
I, slugs	0.00452
ω_h, radians/sec	41.1
ω_α, radians/sec	299.6

The shapes of the deflection curves in bending $f_h(\xi)$ and torsion $f_\alpha(\xi)$ along the span are

$$f_h(\xi) = \frac{1}{2}\left(\cosh 1.87 \frac{\xi}{l} - \cos 1.87 \frac{\xi}{l}\right) + 0.734\left(\sin 1.87 \frac{\xi}{l} - \sinh 1.87 \frac{\xi}{l}\right)$$

$$f_\alpha(\xi) = \sin \frac{\pi\xi}{2l}$$

so that

$$\int_0^l f_h^2(\xi)d\xi = 1$$

$$\int_0^l f_\alpha^2(\xi)d\xi = 2$$

$$\int_0^l f_h(\xi)f_\alpha(\xi)d\xi = 1.4114$$

Consider as an example $\frac{v}{\omega b} = 8.33$. Define $\lambda = \frac{\omega_\alpha^2}{\omega^2}$. The values of the terms in the determinant are

$$\begin{vmatrix} a_{11} - \lambda b_{11} & a_{12} \\ a_{21} & a_{22} - \lambda b_{22} \end{vmatrix} = \begin{vmatrix} (91.6484 - 40.3155i) - \lambda(1.841762) & (-476.7902 + 34.8984i) \\ (9.97095 + 21.2750541) & (308.417315 - 68.551392i) - \lambda(52.049447) \end{vmatrix} = 0$$

Solving a quadratic equation for the basic characteristic value λ, the values obtained for the two modes are

$$\lambda_1 = 5.9882 + 1.0438i$$

$$\lambda_2 = 49.6991 - 24.2515i$$

Enough information is now available to compute the direct and conjugate characteristic modes from the virtual-work equations thus:

$$\delta \text{ Work} = 0 = \left(y_1 y_2\right)\begin{pmatrix} a_{11} - \lambda b_{11} & a_{12} \\ a_{21} & a_{22} - \lambda b_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Let

$$y_1{}^{\nu} = x_1{}^{\nu} = 1$$

Then,

$$x_2{}^{\nu} = -\frac{\left(a_{11} - \lambda_{\nu} b_{11}\right)}{a_{12}} = -\frac{a_{21}}{\left(a_{22} - \lambda_{\nu} b_{22}\right)}$$

$$y_2{}^{\nu} = -\frac{\left(a_{11} - \lambda b_{11}\right)}{a_{21}} = -\frac{a_{12}}{\left(a_{22} - \lambda_{\nu} b_{22}\right)}$$

Table 2 has been compiled of all the characteristic modes thus computed.

Response Function - Mass Addition

The response function developed in appendix A when expanded is

$$\text{R.F.} = \sum_{\nu} \frac{\left(\overrightarrow{y^{\nu},\beta}\right)\left(\overrightarrow{\beta,x^{\nu}}\right)}{\left(\lambda_{\nu} - \lambda\right) B\left(\overrightarrow{y^{\nu},x^{\nu}}\right)}$$

$$= -\frac{1}{m/\pi\rho b^3}$$

$$= \frac{\left[y_1{}^1 f_h(\xi) + y_2{}^1 e_w f_\alpha(\xi)\right]\left[x_1{}^1 f_h(\xi) + x_2{}^1 e_w f_\alpha(\xi)\right]}{\left(\lambda_1 - \lambda\right) B\left(y^1, x^1\right)} +$$

$$\frac{\left[y_2{}^1 f_h(\xi) + y_2{}^2 e_w f_\alpha(\xi)\right]\left[x_2{}^1 f_h(\xi) + x_2{}^2 e_w f_\alpha(\xi)\right]}{\left(\lambda_2 - \lambda\right) B\left(y^2, x^2\right)}$$

Since $y_1{}^{\nu} = x_1{}^{\nu} = 1$,

$$\text{R.F.} = \frac{\left[f_h(\xi) + y_2{}^1 e_w f_\alpha(\xi)\right]\left[f_h(\xi) + x_2{}^1 e_w f_\alpha(\xi)\right]}{\left(\lambda_1 - \lambda\right)B\left(y^1, x^1\right)} + \frac{\left[f_h(\xi) + y_2{}^2 e_w f_\alpha(\xi)\right]\left[f_h(\xi) + x_2{}^2 e_w f_\alpha(\xi)\right]}{\left(\lambda_2 - \lambda\right)B\left(y^2, x^2\right)}$$

$$= \frac{K_1}{\lambda_1 - \lambda} + \frac{K_2}{\lambda_2 - \lambda}$$

Consider again $\frac{v}{\omega b} = 8.33$ and a span position of 75 percent $(\xi = 3')$; then K_1 and K_2 are computed from the following data:

$$f_h(3) = 0.655045$$

$$e_w f_\alpha(3) = 0.461940$$

Substituting,

$$\text{R.F.} = \frac{\left[0.655045 + (0.172 + 3.868i)(0.461940)\right]\left[0.655045 + (0.175 - 0.0757i)(0.46194)\right]}{\left(\lambda_1 - \lambda\right)(18.6357 + 34.4883i)} +$$

$$\frac{\left[0.655045 + (-0.171 - 0.0740i)(0.461940)\right]\left[0.655045 + (-0.000405 + 0.009301i)(0.461940)\right]}{\left(\lambda_2 - \lambda\right)(1.8817 - 0.08104i)}$$

$$= \frac{0.0362119 + 0.00211654i}{\lambda_1 - \lambda} + \frac{0.200690 - 0.00193856i}{\lambda_2 - \lambda}$$

$$K_1 = 0.0362119 + 0.00211654i$$

$$K_2 = 0.200690 - 0.00193856i$$

The values of the constants K_1 and K_2 are shown in table 3.

Having found K_1 and K_2, the values of λ and $\frac{1}{m/\pi\rho b^3}$ are determined as shown graphically in figure 2. The data calculated in this manner are compiled in table 1. For these computations natural frequencies ω_h and ω_α found in test and corrected for the effect of apparent mass have been used (table 4).

APPENDIX C

FLEXIBILITY OF MASS SUSPENSION

The numerical example, figures 1 and 3 to 6, has been carried out for the case in which the mass m is rigidly attached to the wing. The results plotted in figures 3 and 4 are still valid if one replaces m by

$$m \longrightarrow m \frac{1}{1 - \frac{{\omega_0}^2}{\omega^2}} = m \frac{1}{1 - \frac{1}{\lambda} \frac{\omega^2}{{\omega_0}^2}}$$

Then, for a given value of m, the data of figures 1 and 3 may be cross-plotted to give a relation between $(\omega_\alpha/\omega_0)^2$ and the flutter speed v.

This has been done for the case where the mass is attached at 50-percent span and the results are presented in figure 8. When the stiffness of the mass suspension decreases, the flutter speed decreases.

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3. Flax, A. H.: Aeroelastic Problems at Supersonic Speed. Paper presented at R.A.S. - I.A.S. Conference, New York City, May 25, 1949.

4. Smilg, Benjamin, and Wasserman, Lee S.: Application of Three-Dimensional Flutter Theory to Aircraft Structures. A.C.T.R. No. 4798, Materiel Div., Army Air Corps, July 9, 1942.

5. Van de Vooren, A. I.: A Method to Determine the Change in Flutter Speed Due to Small Changes in the Mechanical System. Report V. 1366, Nationaal Luchtvaartlaboratorium, 1947.

TABLE 1

RESULTS

$\left[m = 0;\ v_o = 312 \text{ fps};\ \frac{v}{\omega b} = 5.85;\ \lambda = 3.5\right]$

	25-percent span	50-percent span	75-percent span	100-percent span
$\frac{v}{\omega b}$	5.9	6.65	9.4	13.0
λ	3.7	6.75	14.1	26.5
ω_α^2	90,000	90,000	90,000	90,000
$\omega^2 = \frac{\omega_\alpha^2}{\lambda}$	24,300	13,400	6,400	3,400
ω	156	116	80	58.5
v	306	256	250	254
$\frac{v}{v_o}$	0.98	0.82	0.80	0.815

TABLE 2

λ_ν DIRECT AND CONJUGATE MODES

v/ωb		λ_ν	x_1^ν	x_2^ν	y_1^ν	y_2^ν	$B(y^\nu, x^\nu)$
5.00	λ_1	2.8155 - 0.1484i	1.0	0.583 - 0.142i	1.0	-2.440 + 6.055i	-27.5064 + 201.8602i
	λ_2	51.3996 - 12.2745i	1.0	0.00203 + 0.00503i	1.0	-0.0573 - 0.0140i	1.83937 - 0.016492i
6.25	λ_1	3.8861 + 0.08094i	1.0	0.345 - 0.0994i	1.0	-1.099 + 5.076i	8.3356 + 96.9019i
	λ_2	50.5943 - 16.4381i	1.0	0.00144 + 0.00666i	1.0	-0.0946 - 0.0273i	1.8441 - 0.03489i
8.33	λ_1	5.9882 + 1.0438i	1.0	0.175 - 0.0757i	1.0	0.172 + 3.868i	18.6357 + 34.4883i
	λ_2	49.6991 - 24.2515i	1.0	-0.000405 + 0.00930i	1.0	-0.171 - 0.0740i	1.8812 - 0.08104i
10.00	λ_1	717401 + 2.3562i	1.0	0.113 - 0.0671i	1.0	0.764 + 3.196i	17.4791 + 16.1315i
	λ_2	49.5819 - 31.2309i	1.0	-0.00250 + 0.0105i	1.0	-0.231 - 0.138i	1.9470 - 0.1082i
12.50	λ_1	9.7604 + 4.7369i	1.0	0.0618 - 0.0580i	1.0	1.296 + 2.514i	13.5918 + 4.1564i
	λ_2	51.0777 - 42.2572i	1.0	-0.00573 + 0.0111i	1.0	-0.302 - 0.262i	2.0834 - 0.09671i
16.67	λ_1	11.6265 + 8.7735i	1.0	0.035 - 0.0469i	1.0	1.737 + 1.850i	9.5504 - 8.7815i
	λ_2	56.7962 - 61.0220i	1.0	-0.00954 + 0.0102i	1.0	-0.361 - 0.484i	2.2775 + 0.04928i

TABLE 3

VALUES OF CONSTANTS K_1 AND K_2

$$\left[\text{R.F.} = \frac{K_1}{\lambda_1 - \lambda} + \frac{K_2}{\lambda_2 - \lambda}\right]$$

	25-percent		50-percent span		75-percent span		100-percent span	
	K_1	K_2	K_1	K_2	K_1	K_2	K_1	K_2
5.00	0.00115529 + 0.0000685931i	0.00456109 - 0.0000562521i	0.00591099 + 0.0000763931i	0.0588210 - 0.0000801211i	0.0128929 - 0.000504001i	0.225415 - 0.00109631i	0.0191386 - 0.00226521i	0.531505 + 0.00227061i
6.25	0.00164188 + 0.000141284i	0.00414816 - 0.000142087i	0.00850877 + 0.00032131i	0.0559078 - 0.00032495i	0.0196935 - 0.00065526i	0.217386 - 0.000648692i	0.0308311 - 0.00410058i	0.516939 - 0.00409345i
8.33	0.00245659 + 0.000526659i	0.00332996 - 0.000523865i	0.0144091 + 0.00210173i	0.0499653 - 0.00206070i	0.0362119 + 0.00211654i	0.200690 - 0.00193865i	0.0614892 - 0.00314115i	0.485967 + 0.00353109i
10.00	0.00310723 + 0.00109658i	0.00268998 - 0.001024i	0.0196925 + 0.00543078i	0.0447656 - 0.00547038i	0.0523205 + 0.00966276i	0.184858 - 0.00977027i	0.0902713 + 0.0106246i	0.454338 - 0.00776097i
12.50	0.00387492 + 0.00237754i	0.00194353 - 0.00217261i	0.0267443 + 0.0141499i	0.0380415 - 0.0127346i	0.0753990 + 0.0328794i	0.163083 - 0.0290599i	0.142438 + 0.0482659i	0.408613 - 0.0414687i
16.67	0.000986428 + 0.00434855i	0.00115146 - 0.00386265i	0.00838717 + 0.0301149i	0.0306178 - 0.0254684i	0.0282219 + 0.0828786i	0.138531 - 0.0658594i	0.0621975 + 0.151468i	0.356243 - 0.111623i

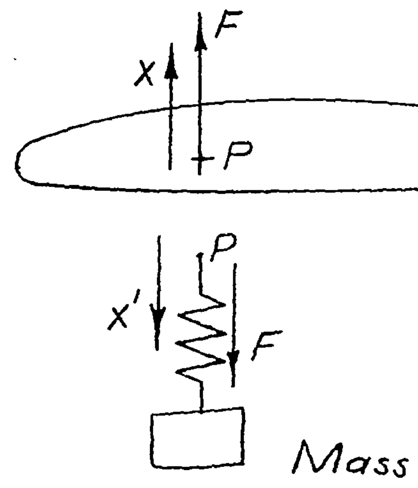
TABLE 4

COMPARISON OF CALCULATED AND TEST FREQUENCIES

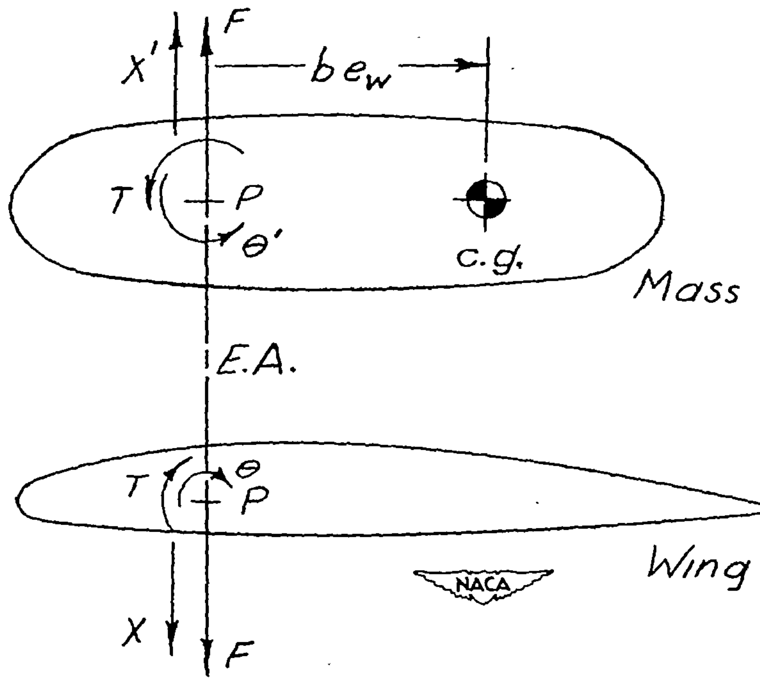
	Calculated	Test	Test (corrected) (1)
First bending (radians/sec)	41.6	40.5	41.1
Second bending (radians/sec)	262	246	249.8
First torsion (radians/sec)	304.4	297.2	299.6
Second torsion (radians/sec)	914	-----	914

[1]Test data are corrected for the effect of apparent mass.

NACA

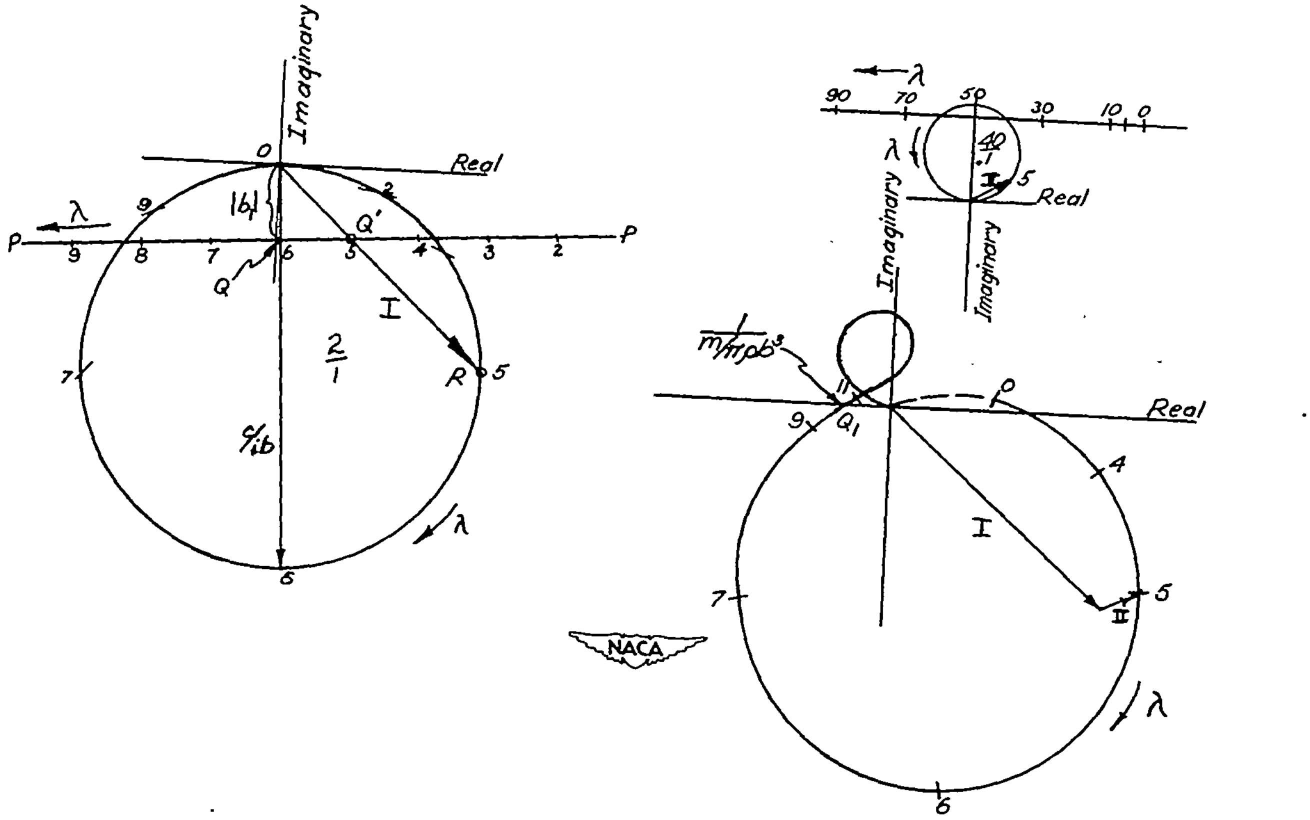


(a) Point mass, elastically attached.



(b) Distributed mass, rigidly attached.

Figure 1.- Symbols for relative deflection of mass and wing.

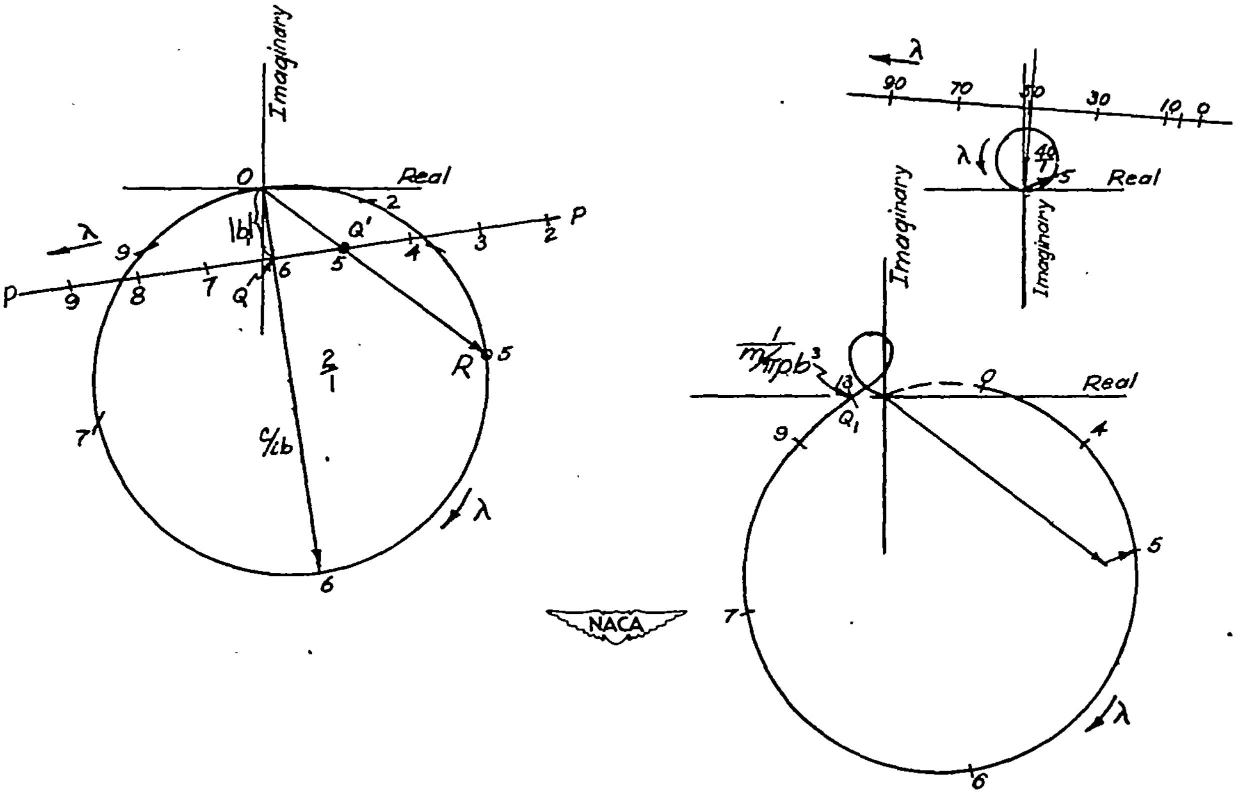


(a) Point mass; $\bar{I} = 0$.

Figure 2.- Response function. $\frac{V}{\omega b} = 8.33$; 75-percent span. Fractions represent scale of calibration line.

$$\frac{0.0362119 + 0.00211654i}{5.9882 + 1.0438i - \lambda} + \frac{0.200690 - 0.00193856i}{49.6991 - 24.2515i - \lambda} = -\frac{1}{m/\pi\rho b^3}.$$

$\lambda = 6.0$, diam. $= 0.00203 - 0.0347i$; $\lambda = 49.7$, diam. $= 0.00008 + 0.008275i$. 1 inch $= 0.012$ for circles.

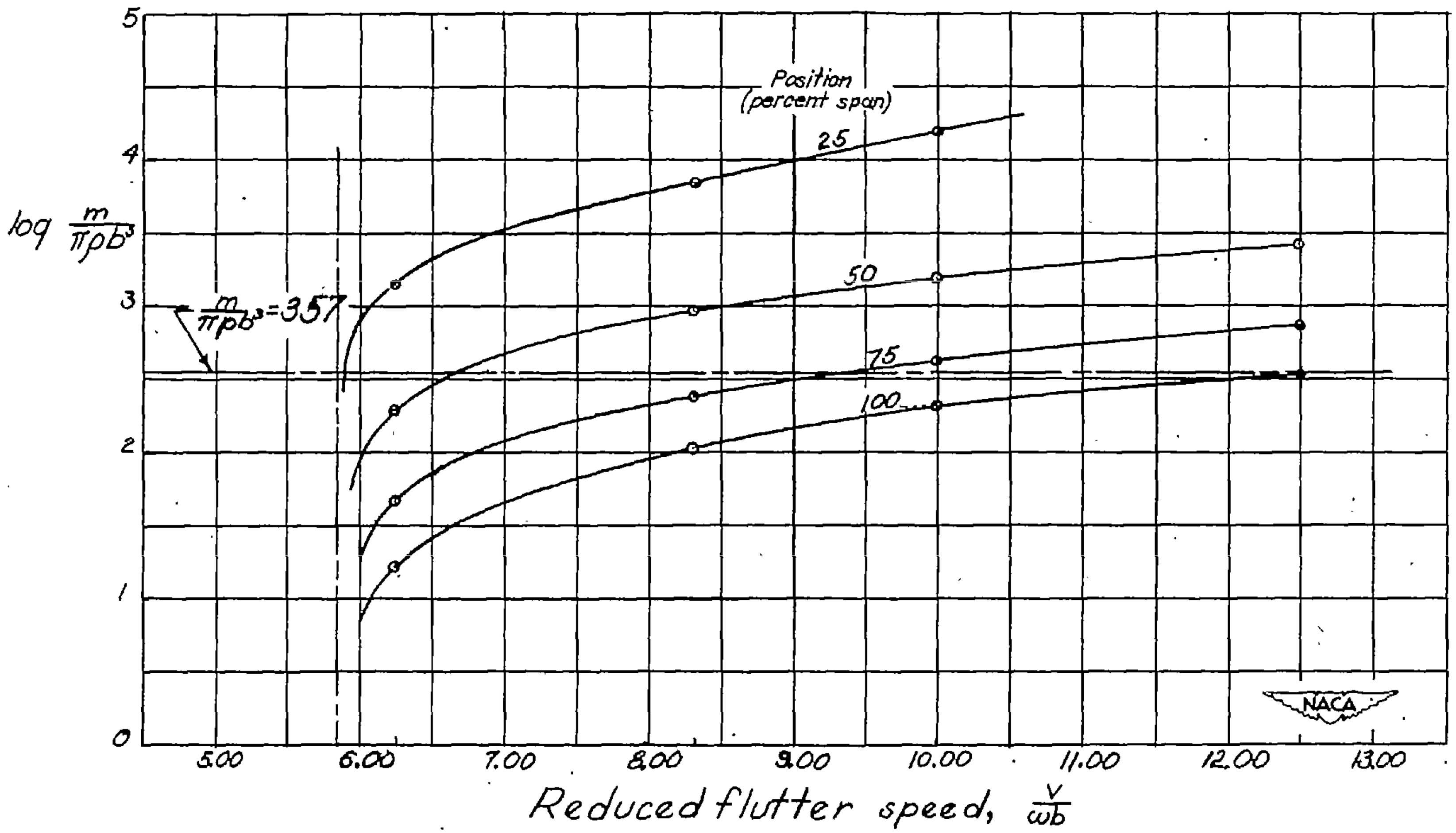


(b) Distributed mass; $\bar{I} = 0.00452$.

$$\frac{0.052486 + 0.0079551i}{5.9882 + 1.0438i - \lambda} + \frac{0.191216 - 0.0077951i}{49.6991 - 24.2515i - \lambda} = -\frac{1}{m/\pi\rho b^3}.$$

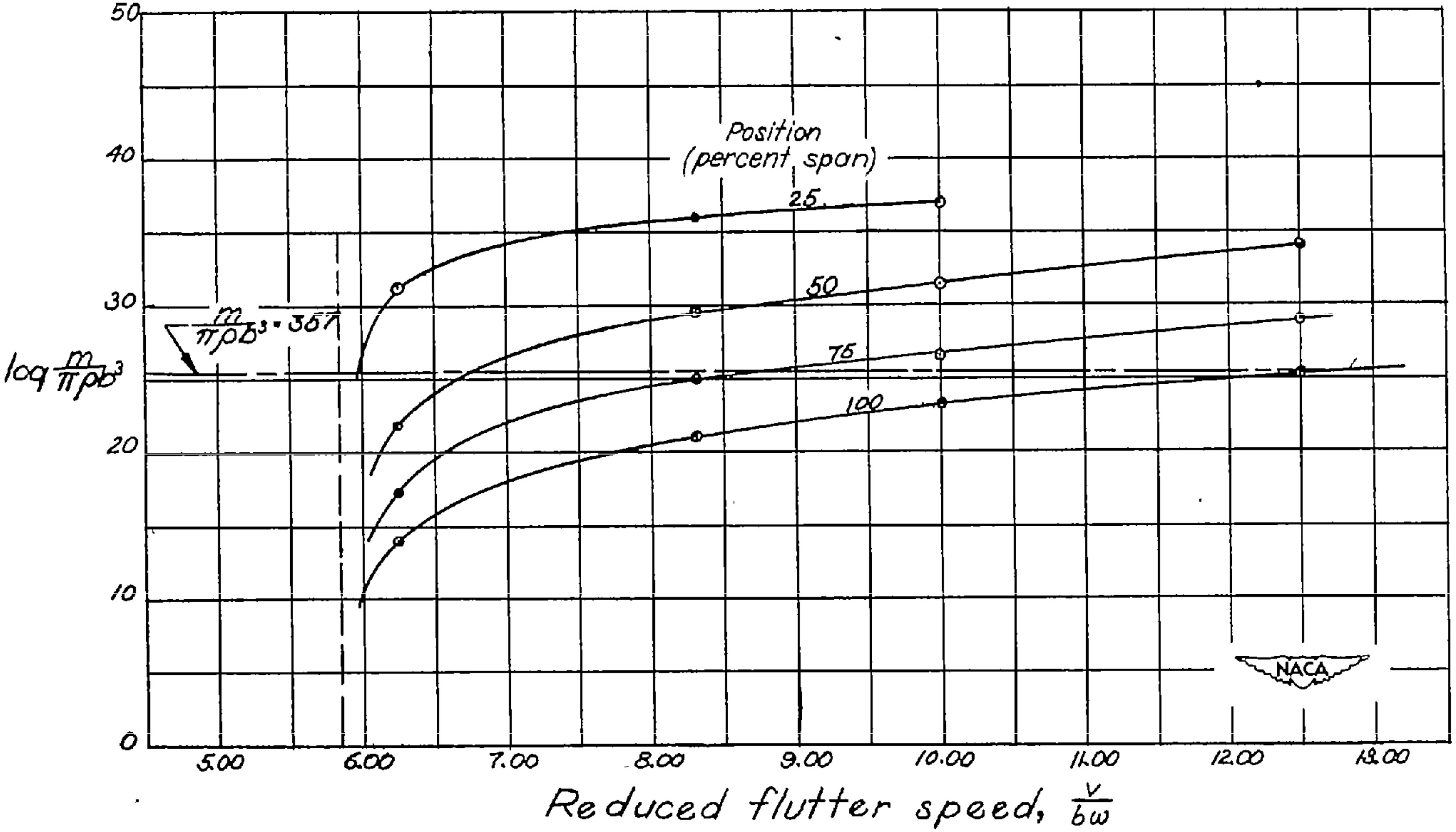
$\lambda = 6.0$, diam. $= 0.007621 - 0.05028i$; $\lambda = 49.7$,
diam. $= 0.0003214 + 0.007885i$. 1 inch $= 0.018$ for circles.

Figure 2.- Concluded.



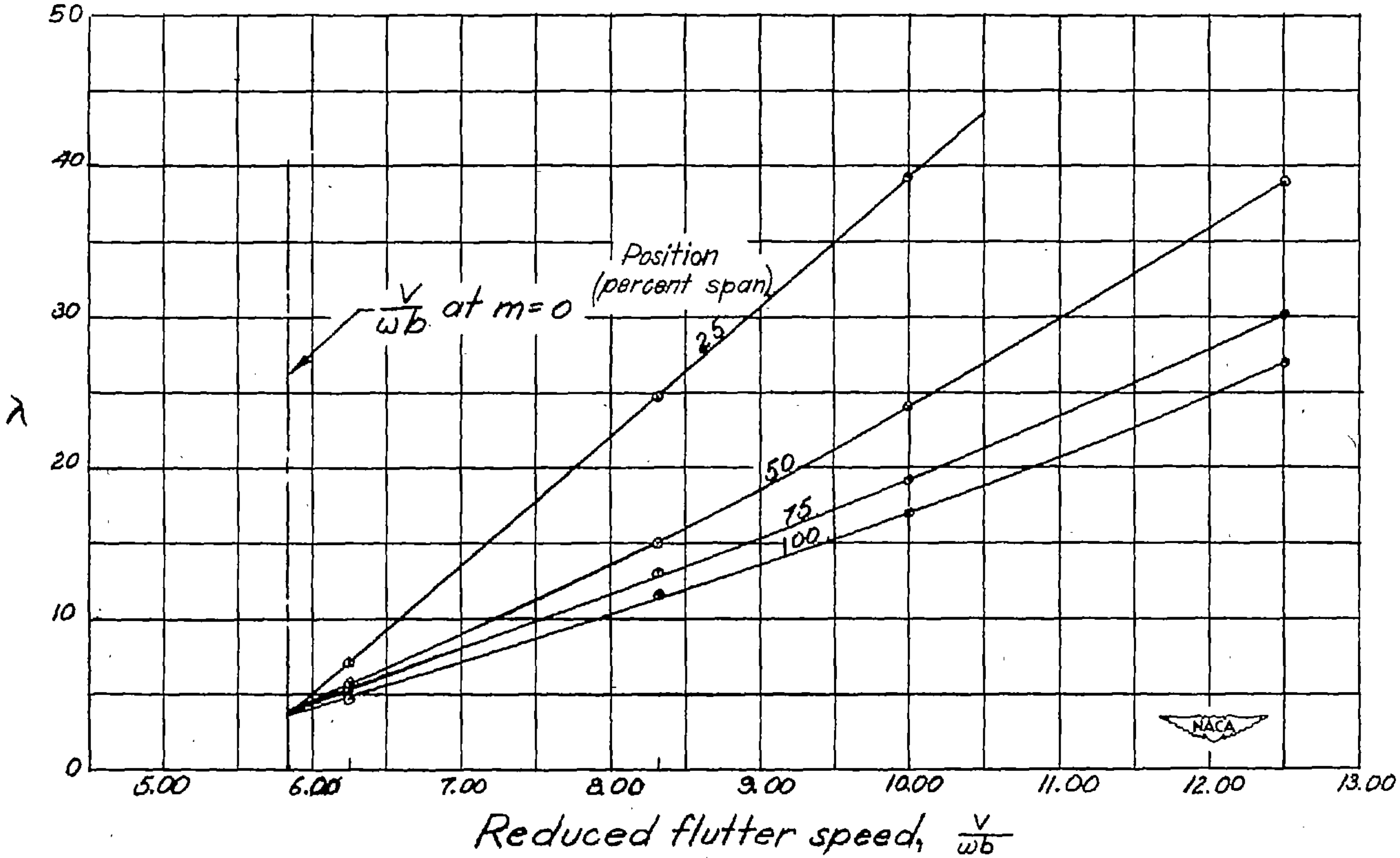
(a) Point mass; $\overline{I} = 0$.

Figure 3.- Mass against reduced speed.



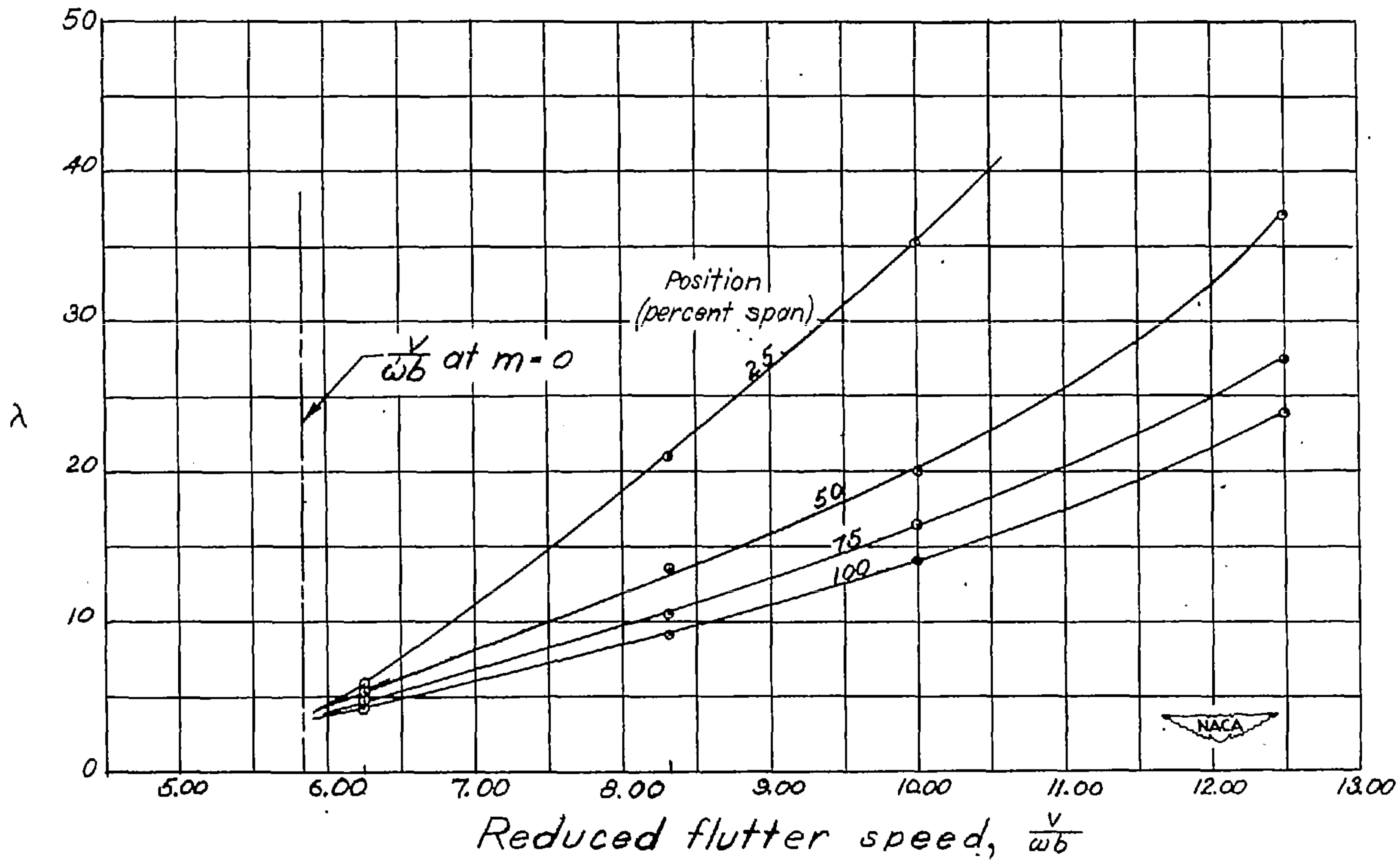
(b) Distributed mass; $\bar{I}$ = 0.00452.

Figure 3.- Concluded.



(a) Point mass; $\overline{I} = 0$.

Figure 4.- Frequency ratio against reduced speed.



(b) Distributed mass; $\overline{I}$ = 0.00452.

Figure 4.- Concluded.

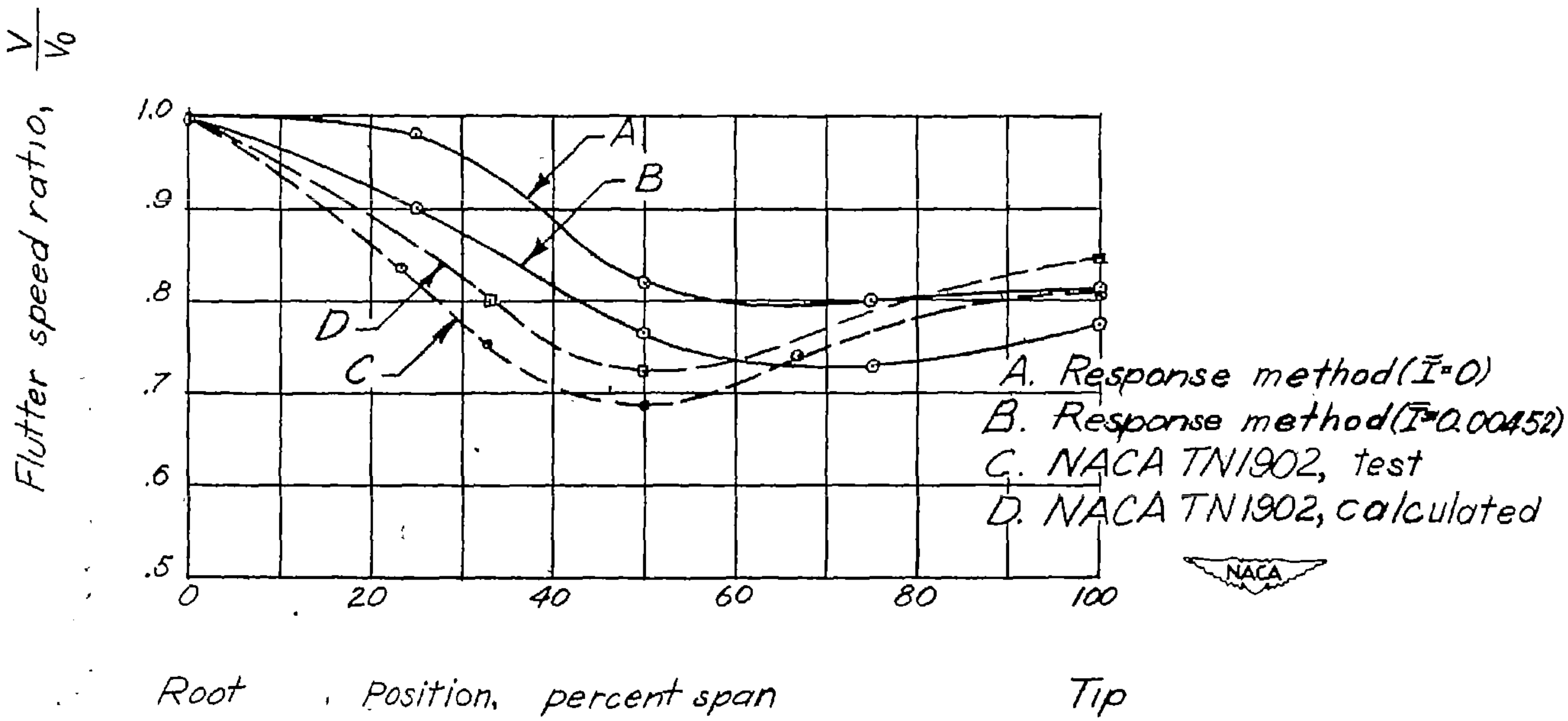


Figure 5.- Flutter speed.

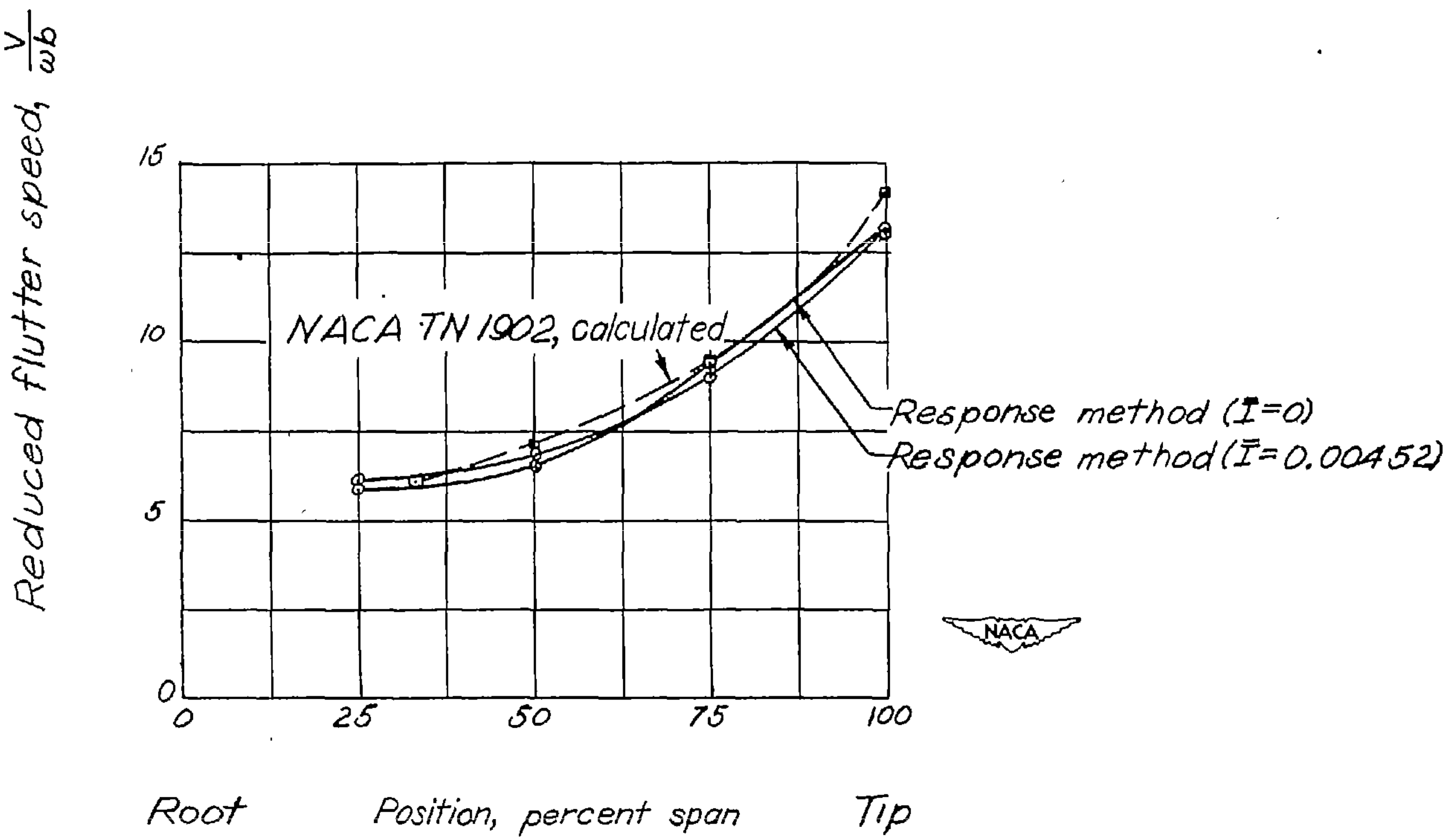


Figure 6.- Reduced flutter speed.

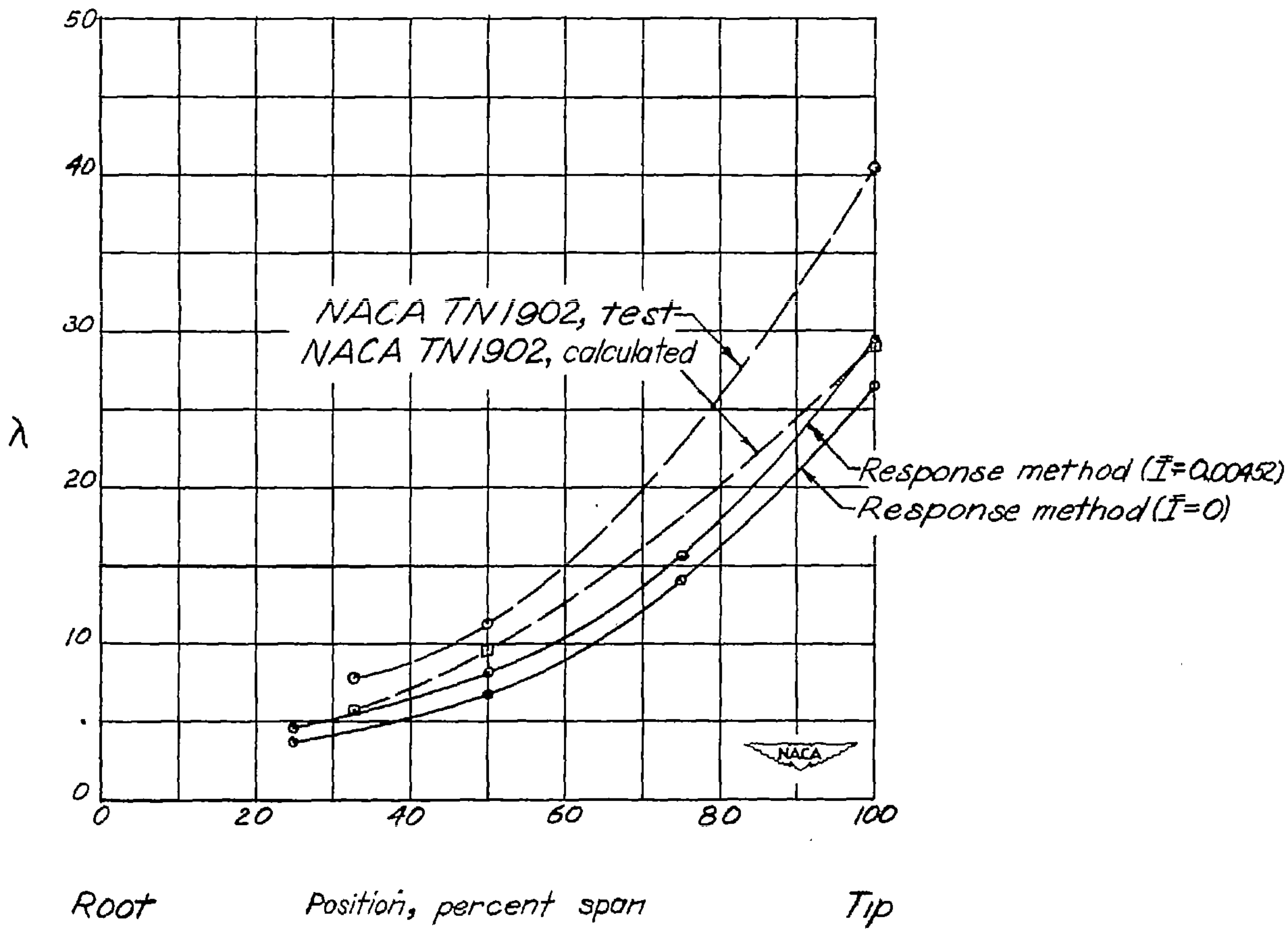


Figure 7.- Frequency ratio.

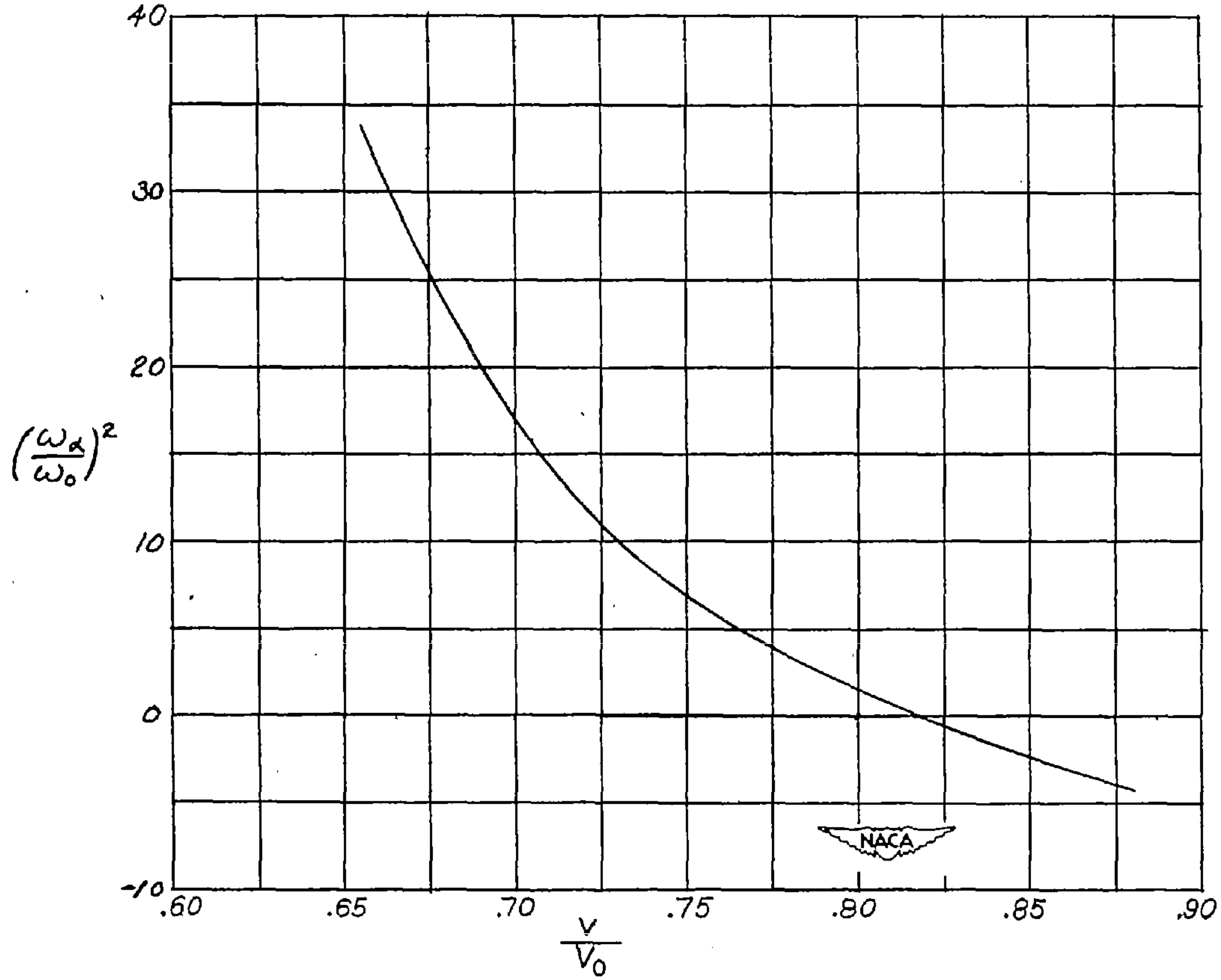


Figure 8.- Effect of rigidity of mass support on flutter speed. Mass at 50-percent span.