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PRINCIPLE AND APPLICATION OF COMPLEMENTARY ENERGY METHOD  
FOR THIN HOMOGENEOUS AND SANDWICH PLATES AND  
SHELLS WITH FINITE DEFLECTIONS

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PRINCIPLE AND APPLICATION OF COMPLEMENTARY ENERGY METHOD  
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SUMMARY

The principle of complementary energy in the nonlinear elasticity theory is shown to be derivable from the principle of potential energy by a Legendre type of transformation. In particular, the expression of the complementary energy is derived for homogeneous and sandwich plates and shells with large deflections. By the method of complementary energy, the stress-strain relations are derived for homogeneous shells, sandwich plates, and sandwich shells. Without the use of this method much lengthier calculations would be necessary.

INTRODUCTION

In the theory of elasticity, the most important variational principle is perhaps the principle of potential energy, which states that of all displacements satisfying given boundary conditions those that satisfy the equilibrium conditions make the potential energy a stationary value. For stable equilibrium, the stationary value may be shown to be a minimum (reference 1). The potential energy is defined as the difference between the strain energy and the potential or virtual work which the surface stresses do over that portion of the surface on which the surface stresses are prescribed. This principle is capable of general application as it holds true no matter what the law connecting load and deformation may be (reference 2). With the relationships between stresses, strains, and displacements known, the differential equations defining the equilibrium conditions may be derived from the variational principle by the methods of the calculus of variations.

The principle of potential energy was obtained by comparing the strain energy  $U$  of the equilibrium state, characterized by displacements  $u$ ,  $v$ , and  $w$ , with the strain energy  $U + \Delta U$  of a neighboring displacement state  $u + \Delta u$ ,  $v + \Delta v$ , and  $w + \Delta w$ . A corresponding

variational principle may be derived by varying the stresses rather than the displacements. This results in the so-called principle of complementary energy which states that of all stress states satisfying the conditions of equilibrium in the interior and on that portion of the surface on which the surface forces are prescribed the actual state of stress is such that the complementary energy is a stationary value. Again, for stable equilibrium, this stationary value is a minimum. For a material that obeys Hooke's law and has linear strain-displacement relationships, the complementary energy is the difference between the strain energy and the work which the surface stresses do over that portion of the surface where the displacements are prescribed. However, for nonlinear problems, it seems that the expression of the complementary energy has not been given previously.

When the equilibrium equations are known in terms of stresses, the stress-displacement or stress-strain relations may be derived from the principle of complementary energy by methods of the calculus of variation. In some elasticity problems this approach has been found to be convenient. For example, in the linear case Trefftz (reference 3) has used the method for the derivation of the stress-displacement relationships in the case of thin homogeneous shells and Reissner has recently derived the stress-displacement relationships of sandwich plates (reference 4) and shells (reference 5) by the same method. Without resource to this method these derivations may require much lengthier calculations.

In this report the principle of complementary energy is derived for thin plates and shells with large deflections. The plates and shells may be either homogeneous or of sandwich-type construction. The application of the principle to the derivation of the stress-displacement relations for homogeneous and sandwich shells as well as sandwich plates is given.

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#### SYMBOLS

|   |                      |
|---|----------------------|
| A | total area of domain |
| C | boundary of domain   |

|                    |  |
|--------------------|--|
| D                  | flexural rigidity of plates and shells ( $Eh^3/12(1 - \nu^2)$ )  |
| e                  | effective transverse normal strain for sandwich plates and shells  |
| E                  | Young's modulus of elasticity  |
| G                  | modulus of rigidity in shear   |
| h                  | thickness of homogeneous plates and shells; also thickness of core layer in the case of sandwich-type construction |
| I                  | potential energy   |
| J                  | complementary energy   |
| $M_x, M_y, M_{xy}$ | resultant bending and twisting moments in plates and shells; unit in moment per unit length                        |
| $M_{nx}, M_{ny}$   | x- and y-components of the resultant moment on boundary  |
| $N_x, N_y, N_{xy}$ | stress resultant in middle plane of plate or in middle surface of shell; unit in force per unit length             |
| p                  | external lateral pressure  |
| q                  | difference in p acting on upper face and lower face of sandwich plates and shells ( $\frac{1}{2}(p_u - p_l)$ )     |
| $Q_x, Q_y$         | shearing stress resultant in plate or shell, normal to middle surface, unit in force per unit length               |
| $Q_n$              | resultant Q on boundary  |
| R                  | radius of curvature at any point in shell  |
| t                  | thickness of face layer of sandwich shell  |
| U                  | work done by stress resultants due to large deflection   |
| V                  | strain energy  |
| W                  | virtual work done by external forces and moments   |
| x, y, z            | coordinates; y is curvilinear coordinate in direction of circumference in the case of a shell                      |

|  |   |
|--|---|
| $X_n, Y_n$   | x- and y-components of stress resultant acting on boundary                  |
| $u, v, w$  | displacements in x-, y-, and z-directions, respectively                     |
| $\beta_x, \beta_y$   | components of change of shape of normal to middle surface of plate or shell |
| $\gamma$   | shear strain  |
| $\delta$   | first variation   |
| $\epsilon_x, \epsilon_y$   | strain in x- and y-directions, respectively                                 |
| $\nu$  | Poisson's ratio   |
| $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$ | Lagrangian multipliers  |
| $\sigma_z$   | component of transverse normal stress in core layer                         |
| $\tau_x, \tau_y$   | transverse shearing stress in core layer                                    |
| $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5$                                | arbitrary parameters  |
| $\chi$   | change of curvature at any point in shell                                   |
| Subscripts   |   |
| c  | core layer of sandwich plate or shell                                       |
| d  | portion of boundary over which displacements are prescribed                 |
| f  | face layer of sandwich plate or shell                                       |
| l  | lower face of sandwich plate or shell                                       |
| s  | portion of boundary over which stresses are prescribed                      |
| o  | given values on boundary  |
| u  | upper face of sandwich plate or shell                                       |
| m  | values of stress components at middle surface<br>$z = 0$                    |

EXPRESSION OF COMPLEMENTARY ENERGY FOR  
THIN HOMOGENEOUS PLATES WITH LARGE DEFLECTIONS

The principle of complementary energy can be derived from the principle of potential energy by means of a Legendre type of transformation. Since the principle of potential energy is also valid for nonlinear strain-displacement relationships, the potential energy

$$I = V - W$$

is stationary or  $\delta I = 0$  when  $I$  is varied with respect to the displacement components  $u$ ,  $v$ , and  $w$ . In the above expression  $V$  is the strain energy and  $W$  is the virtual work. In the case of a thin elastic plate with large deflection the strain energy consists of two parts, one due to the stretching of the plate and the other due to bending. Consequently, the potential energy may be written as follows

$$\begin{aligned}
 I = & \frac{1}{2Eh} \iint_A \left[ N_x^2 + N_y^2 - 2\nu N_x N_y + 2(1 + \nu) N_{xy}^2 \right] dx dy + \\
 & \frac{1}{2(1 - \nu^2)D} \iint_A \left[ M_x^2 + M_y^2 - 2\nu M_x M_y + 2(1 + \nu) M_{xy}^2 \right] dx dy - \\
 & \iint_A p w dx dy - \int_{C_S} \left[ X_n u + Y_n v + \left( X_n \frac{\partial w}{\partial x} + Y_n \frac{\partial w}{\partial y} + Q_n \right) w + \right. \\
 & \left. M_{nx} \frac{\partial w}{\partial x} + M_{ny} \frac{\partial w}{\partial y} \right] ds \tag{1}
 \end{aligned}$$

where  $A$  is the area of the plate and  $C_S$  is that portion of the boundary where the surface forces are prescribed;  $I$  has a stationary value when it is varied with respect to  $u$ ,  $v$ , and  $w$ , subject to the following stress-displacement relations:

$$\left. \begin{aligned}
 N_x &= \frac{Eh}{1 - \nu^2} \left[ \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 + \nu \frac{\partial v}{\partial y} + \frac{\nu}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right] \\
 N_y &= \frac{Eh}{1 - \nu^2} \left[ \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 + \nu \frac{\partial u}{\partial x} + \frac{\nu}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right] \\
 N_{xy} &= \frac{Eh}{2(1 + \nu)} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) \\
 M_x &= D \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \\
 M_y &= D \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \\
 M_{xy} &= D(1 - \nu) \frac{\partial^2 w}{\partial x \partial y} \\
 Q_x &= D \frac{\partial}{\partial x} \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) \\
 Q_y &= D \frac{\partial}{\partial y} \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)
 \end{aligned} \right\} \quad (2)$$

Expressions (2) are the usual stress-displacement relations when the deflection of the plate is large compared with its thickness but is still small enough to allow the use of simplified formulas for curvature. The boundary conditions are: On that portion of the surface where the surface forces are not prescribed, boundary displacements  $u$ ,  $v$ , and  $w$  are equal to the given values  $u_0$ ,  $v_0$ , and  $w_0$ .

The principle of complementary energy may be obtained directly by applying a Legendre type transformation to the principle of potential energy. This procedure is known as "Friedrich's Method" (reference 6) in the calculus of variation. Let  $\lambda_1$ ,  $\lambda_2$ ,  $\lambda_3$ ,  $\lambda_4$ ,  $\lambda_5$ , and  $\lambda_6$  be the Lagrangian multipliers and  $\mu_1$ ,  $\mu_2$ ,  $\mu_3$ ,  $\mu_4$ , and  $\mu_5$  be similar

parameters. Then the variation problem, equation (1), is equivalent to varying the integral H with respect to  $N_x$ ,  $N_y$ ,  $N_{xy}$ ,  $M_x$ ,  $M_{xy}$ ,  $u$ ,  $v$ , and  $w$  independently, where

$$\begin{aligned}
 H = & \frac{1}{2Eh} \iint_A \left[ N_x^2 + N_y^2 - 2vN_xN_y + 2(1+v)N_{xy}^2 \right] dx dy + \\
 & \frac{1}{2(1-v^2)D} \iint_A \left[ M_x^2 + M_y^2 - 2vM_xM_y + 2(1+v)M_{xy}^2 \right] dx dy + \\
 & \iint_A \lambda_1 \left\{ N_x - \frac{Eh}{1-v^2} \left[ \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 + v \frac{\partial v}{\partial y} + \frac{v}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right] \right\} dx dy + \\
 & \iint_A \lambda_2 \left\{ N_y - \left[ \frac{Eh}{1-v^2} \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 + v \frac{\partial u}{\partial x} + \frac{v}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right] \right\} dx dy + \\
 & \iint_A \lambda_3 \left[ N_{xy} - \frac{Eh}{2(1+v)} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) \right] dx dy + \\
 & \iint_A \lambda_4 \left[ M_x - D \left( \frac{\partial^2 w}{\partial x^2} + v \frac{\partial^2 w}{\partial y^2} \right) \right] dx dy + \\
 & \iint_A \lambda_5 \left[ M_y - D \left( \frac{\partial^2 w}{\partial y^2} + v \frac{\partial^2 w}{\partial x^2} \right) \right] dx dy + \\
 & \iint_A \lambda_6 \left[ M_{xy} - D(1-v) \frac{\partial^2 w}{\partial x \partial y} \right] dx dy - \iint_A p w dx dy - \\
 & \int_{C_s} \left[ X_n u + Y_n v + \left( X_n \frac{\partial w}{\partial x} + Y_n \frac{\partial w}{\partial y} + Q_n \right) w + M_{xn} \frac{\partial w}{\partial x} + M_{yn} \frac{\partial w}{\partial y} \right] ds + \\
 & \int_{C_d} \left[ \mu_1 (u - u_0) + \mu_2 (v - v_0) + \mu_3 (w - w_0) + \mu_4 \left( \frac{\partial w}{\partial y} - \frac{\partial w_0}{\partial x} \right) + \right. \\
 & \left. \mu_5 \left( \frac{\partial w}{\partial y} - \frac{\partial w_0}{\partial y} \right) \right] ds \tag{3}
 \end{aligned}$$

The condition  $\delta H = 0$  leads to:

$$\delta H = 0$$

$$\begin{aligned}
 &= \iint_A \left( \left( \frac{N_x - \nu N_y}{Eh} + \lambda_1 \right) \delta N_x + \left( \frac{N_y - \nu N_x}{Eh} + \lambda_2 \right) \delta N_y + \right. \\
 &\quad \left[ \frac{2(1 + \nu)N_{xy}}{Eh} + \lambda_3 \right] \delta N_{xy} + \left[ \frac{M_x - \nu M_y}{(1 - \nu^2)D} + \lambda_4 \right] \delta M_x + \\
 &\quad \left[ \frac{M_y - \nu M_x}{(1 - \nu^2)D} + \lambda_5 \right] \delta M_y + \left[ \frac{2M_{xy}}{(1 - \nu)D} + \lambda_6 \right] \delta M_{xy} + \\
 &\quad \left[ \frac{\partial}{\partial x} \frac{Eh}{1 - \nu} (\lambda_1 + \nu \lambda_2) + \frac{\partial}{\partial y} \frac{Eh \lambda_3}{2(1 + \nu)} \right] \delta u + \\
 &\quad \left[ \frac{\partial}{\partial x} \frac{Eh \lambda_3}{2(1 + \nu)} + \frac{\partial}{\partial y} \frac{Eh}{1 - \nu^2} (\lambda_2 + \nu \lambda_1) \right] \delta v + \\
 &\quad \left\{ D \nabla \nabla w + \frac{Eh(\lambda_1 + \nu \lambda_2)}{1 - \nu^2} \frac{\partial^2 w}{\partial x^2} + \frac{Eh \lambda_3}{(1 + \nu)} \frac{\partial^2 w}{\partial x \partial y} + \right. \\
 &\quad \left. \frac{Eh(\lambda_2 + \nu \lambda_1)}{1 - \nu^2} \frac{\partial^2 w}{\partial y^2} + \frac{\partial w}{\partial x} \left[ \frac{\partial}{\partial x} \frac{Eh(\lambda_1 + \nu \lambda_2)}{1 - \nu^2} + \frac{\partial}{\partial y} \frac{Eh \lambda_3}{2(1 + \nu)} \right] + \right. \\
 &\quad \left. \frac{\partial w}{\partial y} \left[ \frac{\partial}{\partial x} \frac{Eh \lambda_3}{2(1 + \nu)} + \frac{\partial}{\partial y} \frac{Eh(\lambda_2 + \nu \lambda_1)}{1 - \nu^2} \right] - p \right\} \delta w \, dx \, dy - \\
 &\quad \iint_A \left\{ \left[ \frac{\partial}{\partial x} \frac{Eh}{1 - \nu^2} (\lambda_1 + \nu \lambda_2) \delta u + \frac{\partial}{\partial y} \frac{Eh \lambda_3 \delta u}{2(1 + \nu)} \right] \right. \\
 &\quad \left[ \frac{\partial}{\partial x} \frac{Eh \lambda_3 \delta v}{2(1 + \nu)} + \frac{\partial}{\partial y} \frac{Eh}{1 - \nu^2} (\lambda_2 + \nu \lambda_1) \delta v \right] + \\
 &\quad \left. \left[ \frac{\partial}{\partial x} \frac{Eh}{1 - \nu^2} (\lambda_1 + \nu \lambda_2) \frac{\partial w}{\partial x} \delta w + \frac{\partial}{\partial y} \frac{Eh \lambda_3}{2(1 + \nu)} \frac{\partial w}{\partial x} \delta w \right] + \right.
 \end{aligned}$$

$$\begin{aligned}
& \left[ \frac{\partial}{\partial x} \frac{Eh\lambda_3}{2(1+\nu)} \frac{\partial w}{\partial y} \delta w + \frac{\partial}{\partial y} \frac{Eh}{1-\nu^2} (\lambda_1 + \nu\lambda_2) \frac{\partial w}{\partial y} \delta w \right] + \\
& \left[ \frac{\partial}{\partial x} D(\lambda_4 + \nu\lambda_5) \delta \frac{\partial w}{\partial x} + \frac{\partial}{\partial y} \frac{D(1-\nu)}{2} \lambda_6 \delta \frac{\partial w}{\partial x} \right] + \\
& \left[ \frac{\partial}{\partial x} \frac{D(1-\nu)}{2} \lambda_6 \delta \frac{\partial w}{\partial y} + \frac{\partial}{\partial y} D(\lambda_5 + \nu\lambda_4) \delta \frac{\partial w}{\partial y} \right] + \\
& \left. \left[ \frac{\partial}{\partial x} \delta w D \frac{\partial}{\partial x} \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + \frac{\partial}{\partial y} \delta w D \frac{\partial}{\partial y} \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) \right] \right\} dx dy - \\
& \int_{C_s} \left[ X_n \delta u + Y_n \delta v + \left( X_n \frac{\partial w}{\partial x} + Y_n \frac{\partial w}{\partial y} + Q_n \right) \delta w + \right. \\
& \left. M_{xn} \delta \frac{\partial w}{\partial x} + M_{yn} \delta \frac{\partial w}{\partial y} \right] ds + \\
& \int_{C_d} \left( \mu_1 \delta u + \mu_2 \delta v + \mu_3 \delta w + \mu_4 \delta \frac{\partial w}{\partial x} + \mu_5 \delta \frac{\partial w}{\partial y} \right) ds \tag{4}
\end{aligned}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

In carrying out the above calculation, integration by parts has been used, an example of which is

$$\begin{aligned}
& - \iint \left[ \frac{Eh}{1-\nu^2} (\lambda_1 + \nu\lambda_2) \frac{\partial \delta u}{\partial x} + \frac{Eh\lambda_3}{2(1+\nu)} \frac{\partial \delta u}{\partial y} \right] dx dy = \\
& - \iint \left[ \frac{\partial}{\partial x} \frac{Eh}{1-\nu^2} (\lambda_1 + \nu\lambda_2) \delta u + \frac{\partial}{\partial y} \frac{Eh\lambda_3}{2(1+\nu)} \delta u \right] dx dy + \\
& \iint \left[ \frac{\partial}{\partial x} \frac{Eh}{1-\nu^2} (\lambda_1 + \nu\lambda_2) + \frac{\partial}{\partial y} \frac{Eh\lambda_3}{2(1+\nu)} \right] \delta u dx dy \tag{5}
\end{aligned}$$

According to the well-known rules of the calculus of variations, the Euler equations of the variational problem are

$$\frac{N_x - \nu N_y}{Eh} + \lambda_1 = 0 \quad (6)$$

$$\frac{N_y - \nu N_x}{Eh} + \lambda_2 = 0 \quad (7)$$

$$\frac{2(1 + \nu)N_{xy}}{Eh} + \lambda_3 = 0 \quad (8)$$

$$\frac{M_x - \nu M_y}{(1 - \nu^2)D} + \lambda_4 = 0 \quad (9)$$

$$\frac{M_y - \nu M_x}{(1 - \nu^2)D} + \lambda_5 = 0 \quad (10)$$

$$\frac{2M_{xy}}{(1 - \nu)D} + \lambda_6 = 0 \quad (11)$$

$$\frac{\partial}{\partial x} \frac{Eh}{1 - \nu^2} (\lambda_1 + \nu \lambda_2) + \frac{\partial}{\partial y} \frac{Eh \lambda_3}{2(1 - \nu)} = 0 \quad (12)$$

$$\frac{\partial}{\partial x} \frac{Eh \lambda_3}{2(1 + \nu)} + \frac{\partial}{\partial y} \frac{Eh}{1 + \nu^2} (\lambda_2 + \nu \lambda_1) = 0 \quad (13)$$

$$\begin{aligned} D \nabla \nabla w + \frac{Eh(\lambda_1 + \nu \lambda_2)}{1 - \nu^2} \frac{\partial^2 w}{\partial x^2} + \frac{Eh \lambda_3}{(1 + \nu)} \frac{\partial^2 w}{\partial x \partial y} + \\ \frac{Eh(\lambda_2 + \nu \lambda_1)}{1 - \nu^2} \frac{\partial^2 w}{\partial y^2} + \frac{\partial w}{\partial x} \left[ \frac{\partial}{\partial x} \frac{Eh(\lambda_1 + \nu \lambda_2)}{1 - \nu^2} + \frac{\partial}{\partial y} \frac{Eh \lambda_3}{2(1 + \nu)} \right] + \\ \frac{\partial w}{\partial y} \left[ \frac{\partial}{\partial x} \frac{Eh \lambda_3}{2(1 + \nu)} + \frac{\partial}{\partial y} \frac{Eh(\lambda_1 + \nu \lambda_2)}{1 - \nu^2} \right] - p = 0 \end{aligned} \quad (14)$$

With the Lagrangian multipliers  $\lambda_1, \dots, \lambda_6$  determined by equations (6) to (11), equations (12) to (14) become identical with the equilibrium equations; namely

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \quad (12a)$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \quad (13a)$$

$$D\nabla\nabla w + N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} - p = 0 \quad (14a)$$

In writing down equation (14a) the relations (12a) and (13a) have been included.

Next, substitute the relations (6) to (11) into the second group of surface integrals in equation (4). By Gauss' theorem these surface integrals may be transformed into line integrals as follows:

$$\begin{aligned} & - \iint_A \left[ \frac{\partial}{\partial x} \frac{Eh}{1-\nu^2} (\lambda_1 + \nu\lambda_2) \delta u + \right. \\ & \left. \frac{\partial}{\partial y} \frac{Eh\lambda_3 \delta u}{2(1+\nu)} \right] dx dy = \iint_A \left( \frac{\partial}{\partial x} N_x \delta u + \frac{\partial}{\partial y} N_{xy} \delta u \right) dx dy \\ & = \int_C X_n \delta u ds \\ & - \iint_A \left[ \frac{\partial}{\partial x} \frac{Eh\lambda_3}{2(1+\nu)} \delta v + \frac{\partial}{\partial y} \frac{Eh}{1-\nu^2} (\lambda_2 + \nu\lambda_1) \delta v \right] dx dy = \int_C Y_n \delta u ds \end{aligned}$$

$$- \iint_A \left[ \frac{\partial}{\partial x} \frac{Eh}{1-\nu^2} (\lambda_1 + \nu\lambda_2) \frac{\partial w}{\partial x} \delta w + \frac{\partial}{\partial y} \frac{Eh\lambda_3}{2(1+\nu)} \frac{\partial w}{\partial x} \delta w \right] dx dy = \int_C X_n \frac{\partial w}{\partial x} \delta w ds$$

$$- \iint_A \left[ \frac{\partial}{\partial x} \frac{Eh\lambda_3}{2(1+\nu)} \frac{\partial w}{\partial y} \delta w + \frac{\partial}{\partial y} \frac{Eh}{1-\nu^2} (\lambda_2 + \nu\lambda_1) \frac{\partial w}{\partial y} \delta w \right] dx dy = \int_C Y_n \frac{\partial w}{\partial y} \delta w ds$$

$$- \iint_A \left[ \frac{\partial}{\partial x} D(\lambda_4 + \nu\lambda_5) \delta \frac{\partial w}{\partial x} + \frac{\partial}{\partial y} \frac{D(1-\nu)}{2} \lambda_6 \delta \frac{\partial w}{\partial x} \right] dx dy = \int_C M_{nx} \delta \frac{\partial w}{\partial x} ds$$

$$- \iint_A \left[ \frac{\partial}{\partial x} \frac{D(1-\nu)}{2} \lambda_6 \delta \frac{\partial w}{\partial y} + \frac{\partial}{\partial y} D(\lambda_5 + \nu\lambda_4) \delta \frac{\partial w}{\partial y} \right] dx dy = \int_C M_{ny} \delta \frac{\partial w}{\partial y} ds$$

$$- \iint_A \left[ \frac{\partial}{\partial x} \delta w D \frac{\partial}{\partial x} \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + \frac{\partial}{\partial y} \delta w D \frac{\partial}{\partial y} \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) \right] dx dy = \int_C Q_n \delta w ds$$

Combining these integrals with the line integrals in equation (4), the condition that the line integrals vanish requires that

$$\mu_1 = -X_n$$

$$\mu_2 = -Y_n$$

$$\mu_3 = - \left( X_n \frac{\partial w}{\partial x} + Y_n \frac{\partial w}{\partial y} + Q_n \right)$$

$$\mu_4 = -M_{nx}$$

$$\mu_5 = -M_{ny} \tag{15}$$

on  $C_d$ .

Substituting relations (6) to (15) into equation (3), carrying out integration by parts, and applying Gauss' theorem to transform surface integrals into line integrals wherever possible, one obtains finally

$$\begin{aligned}
 H = & - \frac{1}{2Eh} \iint \left[ N_x^2 + N_y^2 - 2\nu N_x N_y + 2(1 + \nu) N_{xy}^2 \right] dx dy - \\
 & \frac{1}{2(1 - \nu^2)D} \iint \left[ M_x^2 + M_y^2 - 2\nu M_x M_y + 2(1 + \nu) M_{xy}^2 \right] dx dy - \\
 & \frac{1}{2} \iint \left[ N_x \left( \frac{\partial w}{\partial x} \right)^2 + 2N_{xy} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} + N_y \left( \frac{\partial w}{\partial y} \right)^2 \right] dx dy + \\
 & \int_{C_d} \left[ X_n u + Y_n v + \left( X_n \frac{\partial w}{\partial x} + Y_n \frac{\partial w}{\partial y} + Q_n \right) w + \right. \\
 & \left. M_{nx} \frac{\partial w}{\partial x} + M_{ny} \frac{\partial w}{\partial y} \right] ds \tag{16}
 \end{aligned}$$

The complementary energy  $J$  is defined as negative  $H$  or

$$J = -H = V + U - W \tag{17}$$

where  $V$  is the strain energy,  $U$  represents the work done by the forces in the middle plane of the plate due to large deflection, and  $W$  is the work done by the surface stresses over that portion of the boundary  $C_d$  where the displacements are prescribed.

## APPLICATION OF PRINCIPLE OF COMPLEMENTARY ENERGY

## Thin Homogeneous Cylindrical Shells

Assume that the complementary energy for the circular cylindrical shell may be written in the same form as in the case of a flat plate. With the equilibrium equations derived, the stress-displacement relations may be obtained from the principle of complementary energy as follows:

Consider the case where  $w$  is large in comparison with  $u$  and  $v$ . Retaining the terms consisting of products of derivatives of  $w$  times the stress resultants and couples, the equilibrium equations for an element of a cylindrical shell with arbitrary cross section (fig. 1) can be put in the following form:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + \frac{N_y}{R} \frac{\partial w}{\partial x} = 0 \quad (18)$$

$$\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} - \frac{N_{xy}}{R} \frac{\partial w}{\partial x} - \frac{Q_y}{R} = 0 \quad (19)$$

$$\begin{aligned} \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial}{\partial x} \left( N_x \frac{\partial w}{\partial x} + N_{xy} \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial y} \left( N_{xy} \frac{\partial w}{\partial x} + N_y \frac{\partial w}{\partial y} \right) + \\ \frac{Q_y}{R} \frac{\partial w}{\partial y} + \frac{N_y}{R} + p = 0 \end{aligned} \quad (20)$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} + \frac{M_y}{R} \frac{\partial w}{\partial x} - Q_x = 0 \quad (21)$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - \frac{M_{xy}}{R} \frac{\partial w}{\partial x} - Q_y = 0 \quad (22)$$

where  $x$  is in the direction of the generator and  $y$  is in the direction of the circumference. These equations may be obtained following a similar derivation as given by Timoshenko (reference 7) with slight modification. Let  $\lambda_1, \lambda_2, \lambda_3, \lambda_4,$  and  $\lambda_5$  be Lagrangian multipliers. By varying the stress resultants and couples independently in the complementary energy equation subject to the equations of equilibrium, one finds that the process is equivalent to formulating a variational integral  $L$  and setting its first variation equal to zero; namely,

$$\begin{aligned} \delta L = & \delta \frac{1}{2Eh} \iint_A \left[ N_x^2 + N_y^2 - 2\nu N_x N_y + 2(1 + \nu) N_{xy}^2 \right] dx dy + \\ & \delta \frac{1}{2(1 - \nu^2)D} \iint_A \left[ M_x^2 + M_y^2 - 2\nu M_x M_y + 2(1 + \nu) M_{xy}^2 \right] dx dy + \\ & \delta \frac{1}{2} \iint_A \left[ N_x \left( \frac{\partial w}{\partial x} \right)^2 + 2N_{xy} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} + N_y \left( \frac{\partial w}{\partial y} \right)^2 \right] dx dy + \\ & \iint_A \left\{ \lambda_1 \delta \left( \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + \frac{N_y}{R} \frac{\partial w}{\partial x} \right) + \lambda_2 \delta \left( \frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} - \frac{N_{xy}}{R} \frac{\partial w}{\partial x} - \frac{Q_y}{R} \right) + \right. \\ & \lambda_3 \delta \left[ \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial}{\partial x} \left( N_x \frac{\partial w}{\partial x} + N_{xy} \frac{\partial w}{\partial y} \right) + \right. \\ & \left. \frac{\partial}{\partial y} \left( N_{xy} \frac{\partial w}{\partial x} + N_y \frac{\partial w}{\partial y} \right) + \frac{Q_y}{R} \frac{\partial w}{\partial x} + \frac{N_y}{R} + p \right] + \\ & \left. \lambda_4 \delta \left( \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} + \frac{M_y}{R} \frac{\partial w}{\partial x} - Q_x \right) + \lambda_5 \delta \left( \frac{\partial M_{xy}}{\partial y} + \frac{\partial M_y}{\partial x} - \frac{M_{xy}}{R} \frac{\partial w}{\partial x} - Q_y \right) \right\} dx dy - \\ & \delta \int_{C_d} \left[ X_n u + Y_n v + \left( X_n \frac{\partial w}{\partial x} + Y_n \frac{\partial w}{\partial y} + Q_n \right) w + M_{nx} \beta_x + M_{ny} \beta_y \right] ds = 0 \end{aligned} \tag{23}$$

where the surface integrals are to be extended over the entire cylindrical surface,  $\beta_x$  is the angle through which  $M_{nx}$  turns, and  $\beta_y$  is the angle through which  $M_{ny}$  turns.

Carrying out the first variation, regrouping the terms, and eliminating the variations of derivatives by integration by parts, one obtains:

$$\begin{aligned}
 \delta L = & \iint_A \left\{ \left[ \frac{N_x - vN_y}{Eh} - \frac{\partial \lambda_1}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 - \frac{\partial w}{\partial x} \frac{\partial \lambda_3}{\partial x} \right] \delta N_x + \right. \\
 & \left[ \frac{N_y - vN_x}{Eh} - \frac{\partial \lambda_2}{\partial y} + \frac{\lambda_3}{R} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 - \frac{\partial w}{\partial y} \frac{\partial \lambda_3}{\partial y} + \frac{\lambda_1}{R} \frac{\partial w}{\partial x} \right] \delta N_y + \\
 & \left[ \frac{2(1+\nu)}{Eh} N_{xy} - \frac{\partial \lambda_1}{\partial y} - \frac{\partial \lambda_2}{\partial x} - \frac{\lambda_2}{R} \frac{\partial w}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} - \frac{\partial w}{\partial x} \frac{\partial \lambda_3}{\partial y} - \frac{\partial w}{\partial y} \frac{\partial \lambda_3}{\partial x} \right] \delta N_{xy} + \\
 & \left( -\lambda_4 - \frac{\partial \lambda_3}{\partial x} \right) \delta Q_x + \left( -\lambda_5 - \frac{\partial \lambda_3}{\partial y} - \frac{\lambda_2}{R} + \frac{\lambda_3}{R} \frac{\partial w}{\partial x} \right) \delta Q_y + \\
 & \left[ \frac{M_x - vM_y}{(1-\nu^2)D} - \frac{\partial \lambda_4}{\partial x} \right] \delta M_x + \left[ \frac{M_y - vM_x}{(1-\nu^2)D} - \frac{\partial \lambda_5}{\partial y} + \frac{\lambda_4}{R} \frac{\partial w}{\partial x} \right] \delta M_y + \\
 & \left. \left[ \frac{2M_{xy}}{(1-\nu)D} - \frac{\partial \lambda_4}{\partial y} - \frac{\partial \lambda_5}{\partial x} - \frac{\lambda_5}{R} \frac{\partial w}{\partial x} \right] \delta M_{xy} \right\} dx dy + \\
 & \iint_A \left[ \left( \frac{\partial \lambda_1 \delta N_x}{\partial x} + \frac{\partial \lambda_1 \delta N_{xy}}{\partial y} \right) + \left( \frac{\partial \lambda_2 \delta N_{xy}}{\partial x} + \frac{\partial \lambda_2 \delta N_y}{\partial y} \right) + \right. \\
 & \left( \frac{\partial}{\partial x} \lambda_3 \frac{\partial w}{\partial x} \delta N_x + \frac{\partial}{\partial y} \lambda_3 \frac{\partial w}{\partial x} \delta N_{xy} \right) + \left( \frac{\partial}{\partial x} \lambda_3 \frac{\partial w}{\partial y} \delta N_{xy} + \frac{\partial}{\partial y} \lambda_3 \frac{\partial w}{\partial y} \delta N_y \right) + \\
 & \left( \frac{\partial \lambda_3 \delta Q_x}{\partial x} - \frac{\partial \lambda_3 \delta Q_y}{\partial y} \right) + \left( \frac{\partial \lambda_4 \delta M_x}{\partial x} + \frac{\partial \lambda_4 \delta M_{xy}}{\partial y} \right) + \\
 & \left. \left( \frac{\partial \lambda_5 \delta M_{xy}}{\partial x} + \frac{\partial \lambda_5 \delta M_y}{\partial x} \right) \right] dx dy - \int_{C_d} \left[ u \delta X_n + v \delta Y_n + \right. \\
 & \left. \left( \frac{\partial w}{\partial x} \delta X_n + \frac{\partial w}{\partial y} \delta Y_n + \delta Q_n \right) w + \beta_x \delta M_{nx} + \beta_y \delta M_{ny} \right] ds = 0 \quad (24)
 \end{aligned}$$

Reducing the second surface integral in equation (24) to a line integral by Gauss' theorem, the final boundary integral of equation (24) becomes

$$\int_C \left[ \lambda_1 \delta X_n + \lambda_2 \delta Y_n + \lambda_3 \left( \frac{\partial w}{\partial x} \delta X_n + \frac{\partial w}{\partial y} \delta Y_n + \delta Q_n \right) + \lambda_4 \delta M_{nx} + \lambda_5 \delta M_{ny} \right] ds -$$

$$\int_{C_d} \left[ u \delta X_n + v \delta Y_n + w \left( \frac{\partial w}{\partial x} \delta X_n + \frac{\partial w}{\partial y} \delta Y_n + \delta Q_n \right) + \beta_x \delta M_{nx} + \beta_y \delta M_{ny} \right] ds \quad (25)$$

The first line integral vanishes on the part of the boundary where surface stresses are prescribed. It therefore has a nonzero value only on that portion of the boundary  $C_d$  where the displacements are prescribed.

Since expression (25) must be zero, one obtains on the boundary

$$\left. \begin{aligned} \lambda_1 &= u \\ \lambda_2 &= v \\ \lambda_3 &= w \\ \lambda_4 &= \beta_x \\ \lambda_5 &= \beta_y \end{aligned} \right\} \quad (26)$$

As equation (24) also holds for any part of the cylinder, if the boundary displacements referring to this part are identified with the displacement occurring in the actual solution of the problem, it follows that the Lagrangian multipliers throughout the shell are related to the generalized displacements in the interior of the boundary. It follows that equations (26) also hold true now in the interior.

Using equations (26), the Euler equations of  $\delta L = 0$  give the following stress-displacement relations of the cylindrical shells:

$$\begin{aligned}
 \frac{N_x - \nu N_y}{Eh} &= \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \\
 \frac{N_y - \nu N_x}{Eh} &= \frac{\partial v}{\partial y} - \frac{w}{R} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 - \frac{u}{R} \frac{\partial w}{\partial x} \\
 \frac{2(1 + \nu)N_{xy}}{Eh} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} + \frac{v}{R} \frac{\partial w}{\partial x} \\
 \beta_x &= - \frac{\partial w}{\partial x} \\
 \beta_y &= - \frac{v}{R} - \frac{\partial w}{\partial y} + \frac{w}{R} \frac{\partial w}{\partial x} \\
 \frac{M_x - \nu M_y}{(1 - \nu^2)D} &= \frac{\partial \beta_x}{\partial x} \\
 \frac{M_y - \nu M_x}{(1 - \nu^2)D} &= \frac{\partial \beta_y}{\partial y} - \frac{\beta_x}{R} \frac{\partial w}{\partial x} \\
 \frac{2M_{xy}}{(1 - \nu)D} &= \frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x} + \frac{\beta_y}{R} \frac{\partial w}{\partial x}
 \end{aligned}
 \tag{27}$$

By means of Hooke's Law and the relations between moments and changes of curvature, equations (27) can be reduced to give

$$\begin{aligned}
 \epsilon_x &= (N_x - \nu N_y)/Eh \\
 &= \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \\
 \epsilon_y &= (N_y - \nu N_x)/Eh \\
 &= \frac{\partial v}{\partial y} - \frac{w}{R} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 - \frac{u}{R} \frac{\partial w}{\partial x} \\
 \gamma_{xy} &= [2(1 + \nu)N_{xy}]/Eh \\
 &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} + \frac{\nu}{R} \frac{\partial w}{\partial x} \\
 \chi_x &= -(M_x - \nu M_y)/(1 - \nu^2)D \\
 &= \frac{\partial^2 w}{\partial x^2} \\
 \chi_y &= -(M_y - \nu M_x)/(1 - \nu^2)D \\
 &= \frac{1}{R} \frac{\partial v}{\partial y} + \frac{\partial^2 w}{\partial y^2} - \frac{\partial}{\partial y} \left( \frac{w}{R} \frac{\partial w}{\partial x} \right) - \frac{1}{R} \left( \frac{\partial w}{\partial x} \right)^2 \\
 \chi_{xy} &= -M_{xy}/(1 - \nu)D \\
 &= \frac{1}{2R} \frac{\partial v}{\partial x} + \frac{\partial^2 w}{\partial x \partial y} - \frac{1}{2} \frac{\partial}{\partial x} \left( \frac{w}{R} \frac{\partial w}{\partial x} \right) + \frac{1}{2R} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y}
 \end{aligned} \tag{28}$$

It is interesting to note that the above expressions contain terms which have not been taken into account previously. Nor is it obvious from a study of these expressions that these terms are necessarily small compared with those retained.

In the above calculation, the expression for the complementary energy has been assumed to be of the same form as in the case of a thin flat plate. This fact will now be verified in the following manner. The principle of potential energy is known to be applicable in this case. If incorrect stress-displacement relations are used the correct equilibrium equations will not be obtained by means of the principle of potential energy. Suppose the incorrect expression of the complementary energy was used; the stress-strain relations obtained naturally would not be correct. With these incorrect relations, the first variation of the potential energy will not lead to the correct equilibrium equations. However, in the present case, when equations (28) are used, the first variation of the potential energy leads directly to the equilibrium equations (18) to (22), thus confirming the fact that the expression of complementary energy used is correct.

#### Cylindrical Shells and Flat Plates of Sandwich Construction

As in the case of homogeneous thin shells, the stress-displacement relations for sandwich shells may also be obtained from the principle of complementary energy. A sandwich shell consists of a core layer surrounded by two face layers. The face layers are treated like thin shells of thickness  $t$  having negligible bending stiffness about their own middle surface. The loads applied to these face shells are of two kinds: External loads and loads caused by the stresses in the core layer. The core layer of thickness  $h$  is assumed to behave like a three-dimensional elastic continuum in which those stresses which are parallel to the faces are negligible compared with the transverse shear and normal stresses. (See fig. 2.)

Assuming a large deflection, the equilibrium differential equations for the upper face layers can easily be shown to be

$$\frac{\partial N_{xu}}{\partial x} \left( 1 + \frac{h+t}{2R} \right) + \frac{\partial N_{yxu}}{\partial y} + \frac{N_{yu}}{R} \frac{\partial w_u}{\partial x} + \tau_{xu} \left( 1 + \frac{h+t}{2R} \right) = 0 \quad (29)$$

$$\frac{\partial N_{xyu}}{\partial x} \left( 1 + \frac{h+t}{2R} \right) + \frac{\partial N_{yu}}{\partial y} - \frac{N_{xyu}}{R} \frac{\partial w_u}{\partial x} + \tau_{yu} \left( 1 + \frac{h+t}{2R} \right) = 0 \quad (30)$$

$$\begin{aligned}
 & \frac{N_{yu}}{R} + \frac{\partial}{\partial x} \left[ N_{xu} \left( 1 + \frac{h+t}{2R} \right) \frac{\partial w_u}{\partial x} + N_{yxu} \frac{\partial w_u}{\partial y} \right] + \\
 & \frac{\partial}{\partial y} \left[ N_{xyu} \frac{\partial w_u}{\partial x} + \frac{N_{yu}}{1 + \frac{h+t}{2R}} \frac{\partial w_u}{\partial y} \right] + \\
 & p_u \left( 1 + \frac{h+t}{2R} \right) + \sigma_{yu} \left( 1 + \frac{h+t}{2R} \right) + \tau_{xu} \frac{\partial w_u}{\partial x} \left( 1 + \frac{h+t}{2R} \right) + \\
 & \tau_{yu} \frac{\partial w_u}{\partial y} = 0
 \end{aligned} \tag{31}$$

Similarly, for the lower face layer, the equilibrium equations are

$$\frac{\partial N_{xl}}{\partial x} \left( 1 - \frac{h+t}{2R} \right) + \frac{\partial N_{yxl}}{\partial y} + \frac{N_{yl}}{R} \frac{\partial w_l}{\partial x} - \tau_{xl} \left( 1 - \frac{h+t}{2R} \right) = 0 \tag{32}$$

$$\frac{\partial N_{xyl}}{\partial x} \left( 1 - \frac{h+t}{2R} \right) + \frac{\partial N_{yl}}{\partial y} - \frac{N_{xyl}}{R} \frac{\partial w_l}{\partial x} - \tau_{yl} \left( 1 - \frac{h+t}{2R} \right) = 0 \tag{33}$$

$$\begin{aligned}
 & \frac{N_{yl}}{R} + \frac{\partial}{\partial x} \left[ N_{xl} \left( 1 - \frac{h+t}{2R} \right) \frac{\partial w_l}{\partial x} + N_{yxl} \frac{\partial w_l}{\partial y} \right] + \\
 & \frac{\partial}{\partial y} \left[ N_{xyl} \frac{\partial w_l}{\partial x} + \frac{N_{yl}}{\left( 1 - \frac{h+t}{2R} \right)} \frac{\partial w_l}{\partial y} \right] + \\
 & p_l \left( 1 - \frac{h+t}{2R} \right) - \sigma_{zl} \left( 1 - \frac{h+t}{2R} \right) - \tau_{xl} \frac{\partial w_l}{\partial x} \left( 1 - \frac{h+t}{2R} \right) + \tau_{yl} \frac{\partial w_l}{\partial y} = 0
 \end{aligned} \tag{34}$$

where the subscripts *u* and *l* denote the upper and lower surfaces, respectively.

Under the assumption of negligible face-parallel core stresses, the equilibrium equations for the core layer are (reference 5)

$$\frac{\partial}{\partial z} \left[ \left( 1 - \frac{z}{R} \right) \tau_x \right] = 0 \quad (35)$$

$$\frac{\partial}{\partial z} \left[ \left( 1 - \frac{z}{R} \right)^2 \tau_y \right] = 0 \quad (36)$$

$$\frac{\partial}{\partial x} \left( 1 - \frac{z}{R} \right) \tau_x + \frac{\partial}{\partial y} \tau_y + \frac{\partial}{\partial z} \left[ \left( 1 - \frac{z}{R} \right) \sigma_z \right] = 0 \quad (37)$$

Let the values of the three stress components at the middle surface ( $z = 0$ ) be designated by the subscript  $m$ . Integration of equations (35) to (37) results in

$$\left( 1 - \frac{z}{R} \right) \tau_x = \tau_{xm} \quad (38)$$

$$\left( 1 - \frac{z}{R} \right)^2 \tau_y = \tau_{ym} \quad (39)$$

$$(1 - z) \sigma_z = \sigma_{zm} - z \left[ \frac{\partial \tau_y}{\partial y} + \frac{\partial}{\partial x} \left( 1 - \frac{z}{R} \right) \tau_x \right] \quad (40)$$

Let  $Q_x$  and  $Q_y$  be the resultant transverse shear forces. Then

$$Q_x = \int_{-\frac{h+t}{2}}^{\frac{h+t}{2}} \tau_x \left( 1 - \frac{z}{R} \right) dz = (h + t) \tau_{xm} \quad (41)$$

$$Q_y = \int_{-\frac{h+t}{2}}^{\frac{h+t}{2}} \tau_y dz = \frac{(h + t) \tau_{ym}}{1 - \left( \frac{h + t}{2R} \right)^2} \quad (42)$$

From equations (28), (39), (41), and (42) the following relations may be obtained:

$$\left. \begin{aligned} \left(1 + \frac{h+t}{2R}\right)\tau_{xu} - \left(1 - \frac{h+t}{2R}\right)\tau_{xl} &= 0 \\ \left(1 + \frac{h+t}{2R}\right)\tau_{yu} - \left(1 - \frac{h+t}{2R}\right)\tau_{yl} &= -\frac{Q_y}{R} \\ \frac{h+t}{2} \left[ \left(1 + \frac{h+t}{2R}\right)\tau_{xu} + \left(1 - \frac{h+t}{2R}\right)\tau_{xl} \right] &= Q_x \\ \frac{h+t}{2} \left[ \left(1 + \frac{h+t}{2R}\right)\tau_{yu} + \left(1 - \frac{h+t}{2R}\right)\tau_{yl} \right] &= Q_y \end{aligned} \right\} \quad (43)$$

For thin sandwich shells  $\frac{h+t}{2R} \ll 1$ . In such cases a combination of equations (40), (41), and (42) gives

$$\left(1 + \frac{h+t}{2R}\right)\sigma_{zu} - \left(1 - \frac{h+t}{2R}\right)\sigma_{zl} = \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} \quad (44)$$

In view of the fact that all face-parallel core stresses are neglected, the face-parallel stress resultants and couples of the composite shell are due to the stresses in the face layers only and may be obtained as follows:

$$\begin{aligned}
 N_x &= N_{xu} \left( 1 + \frac{h+t}{2R} \right) + N_{xl} \left( 1 - \frac{h+t}{2R} \right) \\
 N_y &= N_{yu} + N_{yl} \\
 N_{xy} &= N_{xyu} \left( 1 + \frac{h+t}{2R} \right) + N_{xyl} \left( 1 - \frac{h+t}{2R} \right) \\
 N_{yx} &= N_{yxu} + N_{yxl} \\
 M_x &= \frac{h+t}{2} \left[ -N_{xu} \left( 1 + \frac{h+t}{2R} \right) + N_{xl} \left( 1 - \frac{h+t}{2R} \right) \right] \\
 M_y &= \frac{h+t}{2} (-N_{yu} + N_{yl}) \\
 M_{xy} &= \frac{h+t}{2} \left[ -N_{xyu} \left( 1 + \frac{h+t}{2R} \right) + N_{xyl} \left( 1 - \frac{h+t}{2R} \right) \right] \\
 M_{yx} &= \frac{h+t}{2} (-N_{yxu} + N_{yxl}) \\
 p &= p_u \left( 1 + \frac{h+t}{2R} \right) + p_l \left( 1 - \frac{h+t}{2R} \right) \\
 q &= \frac{1}{2} \left[ p_u \left( 1 + \frac{h+t}{2R} \right) - p_l \left( 1 - \frac{h+t}{2R} \right) \right] \\
 \sigma_{zm} &= \frac{1}{2} \left[ \sigma_{zu} \left( 1 + \frac{h+t}{2R} \right) + \sigma_{zl} \left( 1 - \frac{h+t}{2R} \right) \right]
 \end{aligned} \tag{45}$$

Since  $N_{yxu} = N_{xyu}$ ,  $N_{yxl} = N_{xyl}$ , and  $\frac{h+t}{2R} \ll 1$ , one may write  $N_{xy} = N_{yx}$  and  $M_{xy} = M_{yx}$ .

Define the deformations of the composite shell to be  $w = (w_u + w_l)/2$  and  $e = (w_u - w_l)/(h + t)$  where  $w$  represents the effective transverse deflection of the middle surface and  $e$  represents the effective transverse normal strain for the composite shell. From these definitions,  $w_u$  and  $w_l$  may be written in terms of  $w$  and  $e$  as follows:

$$\begin{aligned}w_u &= w + \frac{(h + t)e}{2} \\w_l &= w - \frac{(h + t)e}{2}\end{aligned}\quad (46)$$

With equations (43), (44), (45), and (46), the following equations may be obtained by carrying out addition as well as subtraction of equations (29) and (32) and (30) and (33).

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + \frac{N_y}{R} \frac{\partial w}{\partial x} - \frac{M_y}{R} \frac{\partial e}{\partial x} = 0 \quad (47)$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} - \frac{N_{xy}}{R} \frac{\partial w}{\partial x} + \frac{M_{xy}}{R} \frac{\partial e}{\partial x} - \frac{Q_y}{R} = 0 \quad (48)$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} + \frac{M_y}{R} \frac{\partial w}{\partial x} - \frac{(h + t)^2 N_y}{4R} \frac{\partial e}{\partial x} - Q_x = 0 \quad (49)$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - \frac{M_{xy}}{R} \frac{\partial w}{\partial x} + \frac{(h + t)^2 N_{xy}}{4R} \frac{\partial e}{\partial x} - Q_y = 0 \quad (50)$$

From equations (31) and (34) the following two relations may be derived:

$$\begin{aligned}&\frac{N_y}{R} + \frac{\partial}{\partial x} \left( N_x \frac{\partial w}{\partial x} + N_{xy} \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial y} \left( N_{xy} \frac{\partial w}{\partial x} + N_y \frac{\partial w}{\partial y} \right) - \\&\frac{\partial}{\partial x} \left( M_x \frac{\partial e}{\partial x} + M_{xy} \frac{\partial e}{\partial y} \right) - \frac{\partial}{\partial y} \left( M_{xy} \frac{\partial e}{\partial x} + M_y \frac{\partial e}{\partial y} \right) + \\&p + \left( \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} \right) - \frac{Q_y}{R} \frac{\partial w}{\partial y} + \left( Q_x \frac{\partial e}{\partial x} + Q_y \frac{\partial e}{\partial y} \right) = 0\end{aligned}\quad (51)$$

$$\begin{aligned} & \frac{M_y}{R} + \frac{\partial}{\partial x} \left( M_x \frac{\partial w}{\partial x} + M_{xy} \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial y} \left( M_{xy} \frac{\partial w}{\partial x} + M_y \frac{\partial w}{\partial y} \right) - \\ & \frac{(h+t)^2}{4} \frac{\partial}{\partial x} \left( N_x \frac{\partial e}{\partial x} + N_{xy} \frac{\partial e}{\partial y} \right) - \frac{(h+t)^2}{4} \frac{\partial}{\partial y} \left( N_{xy} \frac{\partial e}{\partial x} + N_y \frac{\partial e}{\partial y} \right) - \\ & (h+t)(\sigma_{zm} + q) - \left( Q_x \frac{\partial w}{\partial x} + Q_y \frac{\partial w}{\partial y} \right) + \frac{(h+t)^2}{4} \frac{Q_y}{R} = 0 \end{aligned} \quad (52)$$

When the radius of curvature  $R$  becomes infinite, equations (47) to (52) reduce to the same equations obtained by Reissner (references 8 and 9). When the effective transverse normal strain  $e$  for the composite shell is neglected, equations (47) to (51) reduce to the usual equations of force and moment equilibrium. Equation (52) has no counterpart in the theory of homogeneous shells, in the sense that the corresponding equation for the homogeneous shell contains information that is not of practical interest and is therefore never formulated. This equation gives the local change of thickness of the shell caused directly by the external loads by way of the nonlinear terms having stress resultants and couples as factors.

Denote the properties of face layers and the core by the subscripts  $f$  and  $c$ , respectively. The strain energy of the face layers and the core may be written as (reference 5)

$$\begin{aligned} V_f &= \frac{1}{2} \iint \frac{1}{2tE_f} \left[ N_x^2 + N_y^2 - 2\nu_f N_x N_y + 2(1 + \nu_f) N_{xy}^2 \right] \times \\ & \quad \frac{2}{t(h+t)^2 E_f} \left[ M_x^2 + M_y^2 - 2\nu_f M_x M_y + 2(1 + \nu_f) M_{xy}^2 \right] dx dy \\ V_c &= \frac{1}{2} \iint \left\{ \frac{Q_x^2 + Q_y^2}{(h+t)G_c} + \frac{h+t}{E_c} \left[ \sigma_{zm}^2 + \frac{1}{12} \left( \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} \right)^2 \right] \right\} dx dy \end{aligned}$$

The work done by the stress resultants due to large deflection is

$$\begin{aligned} U &= \frac{1}{2} \iint \left[ N_{xu} \left( \frac{\partial w_u}{\partial x} \right)^2 + 2N_{xyu} \frac{\partial w_u}{\partial x} \frac{\partial w_u}{\partial y} + N_y u \left( \frac{\partial w_u}{\partial y} \right)^2 \right] \left( 1 + \frac{h+t}{2R} \right) dx dy + \\ & \quad \frac{1}{2} \iint \left[ N_{xl} \left( \frac{\partial w_l}{\partial x} \right)^2 + 2N_{xyl} \frac{\partial w_l}{\partial x} \frac{\partial w_l}{\partial y} + N_y l \left( \frac{\partial w_l}{\partial y} \right)^2 \right] \left( 1 - \frac{h+t}{2R} \right) dx dy \end{aligned}$$

With the aid of equations (45) and (46), U becomes

$$\begin{aligned}
 U = & \frac{1}{2} \iint \left[ N_x \left( \frac{\partial w}{\partial x} \right)^2 + 2N_{xy} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} + N_y \left( \frac{\partial w}{\partial y} \right)^2 \right] dx dy + \\
 & \frac{(h+t)^2}{8} \iint \left[ N_x \left( \frac{\partial e}{\partial x} \right)^2 + 2N_{xy} \frac{\partial e}{\partial x} \frac{\partial e}{\partial y} + N_y \left( \frac{\partial e}{\partial y} \right)^2 \right] dx dy - \\
 & \iint \left[ M_x \frac{\partial e}{\partial x} \frac{\partial w}{\partial x} + M_{xy} \left( \frac{\partial e}{\partial x} \frac{\partial w}{\partial y} + \frac{\partial e}{\partial y} \frac{\partial w}{\partial x} \right) + M_y \frac{\partial e}{\partial y} \frac{\partial w}{\partial y} \right] dx dy
 \end{aligned}$$

The work done by the surface forces over that portion of the surface where the displacements are prescribed is

$$\begin{aligned}
 W = & \int_{C_d} \left[ X_{nu} u_u + Y_{nu} v_u + \left( X_{nu} \frac{\partial w_u}{\partial x} + Y_{nu} \frac{\partial w_u}{\partial y} \right) w_u \right] ds + \\
 & \int_{C_d} \left[ X_{nl} u_l + Y_{nl} v_l + \left( X_{nl} \frac{\partial w_l}{\partial x} + Y_{nl} \frac{\partial w_l}{\partial y} \right) w_l \right] + \\
 & \int_{C_d} Q_n w ds + \int_{C_d} \frac{h+t}{12E_c} \left( \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} \right) Q_n ds
 \end{aligned}$$

where the last two integrals are the work done by the shear stress resultant in the core. The first one of these two is the work done due to the displacement of the middle surface and the second, due to compression of the core.

Define the displacement of the middle surface and the resultant load on the boundary of the composite shell to be as follows:

$$\begin{aligned}
 u &= \frac{1}{2}(u_u + u_l) & v &= \frac{1}{2}(v_u + v_l) \\
 \beta_x &= \frac{1}{h+t}(-u_u + u_l) & \beta_y &= \frac{1}{h+t}(-v_u + v_l) \\
 X_n &= X_{nu} + X_{nl} & Y_n &= Y_{nu} + Y_{nl} \\
 M_{nx} &= \frac{h+t}{2}(-X_{nu} + X_{nl}) & M_{ny} &= \frac{h+t}{2}(-Y_{nu} + Y_{nl})
 \end{aligned}$$

The work  $W$  becomes

$$W = \int_{C_d} \left[ X_n u + Y_n v + \left( X_n \frac{\partial w}{\partial x} + Y_n \frac{\partial w}{\partial y} + Q_n \right) w + \right. \\ \left. M_{nx} \beta_x + M_{ny} \beta_y - \left( M_{nx} \frac{\partial e}{\partial x} + M_{ny} \frac{\partial e}{\partial y} \right) w - \left( M_{nx} \frac{\partial w}{\partial x} + M_{ny} \frac{\partial w}{\partial y} \right) e + \right. \\ \left. \frac{(h+t)^2}{4} \left( X_n \frac{\partial e}{\partial x} + Y_n \frac{\partial e}{\partial y} \right) e + \frac{h+t}{12E_c} \left( \frac{\partial Q_x}{\partial y} + \frac{\partial Q_y}{\partial x} \right) Q_n \right] ds$$

Introducing the Lagrangian multipliers  $\lambda_1$  to  $\lambda_6$  and carrying out the first variation with respect to  $N_x$ ,  $N_y$ ,  $N_{xy}$ ,  $M_x$ ,  $M_y$ ,  $M_{xy}$ ,  $Q_x$ ,  $Q_y$ , and  $\sigma_{zm}$  independently, the relation

$$\delta L = \delta V_f + \delta V_c + \delta U + \delta W + \delta \iint \left\{ \lambda_1 \left( \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + \frac{N_y}{R} \frac{\partial w}{\partial x} - \frac{M_y}{R} \frac{\partial e}{\partial x} \right) + \right. \\ \lambda_2 \left( \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} - \frac{N_{xy}}{R} \frac{\partial w}{\partial x} + \frac{M_{xy}}{R} \frac{\partial e}{\partial x} - \frac{Q_y}{R} \right) + \lambda_3 \left[ \frac{N_y}{R} + \frac{\partial}{\partial x} \left( N_x \frac{\partial w}{\partial x} \right) + \right. \\ \left. \frac{\partial}{\partial y} \left( N_{xy} \frac{\partial w}{\partial x} \right) + \frac{\partial}{\partial x} \left( N_{xy} \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial y} \left( N_y \frac{\partial w}{\partial y} \right) - \frac{\partial}{\partial x} \left( M_x \frac{\partial e}{\partial x} \right) - \frac{\partial}{\partial y} \left( M_{xy} \frac{\partial e}{\partial x} \right) - \right. \\ \left. \frac{\partial}{\partial x} \left( M_{xy} \frac{\partial e}{\partial y} \right) - \frac{\partial}{\partial y} \left( M_y \frac{\partial e}{\partial y} \right) + p + \left( \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} \right) + \left( Q_x \frac{\partial e}{\partial x} + Q_y \frac{\partial e}{\partial y} \right) - \frac{Q_y}{R} \frac{\partial w}{\partial y} \right] + \\ \lambda_4 \left[ \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} + \frac{M_y}{R} \frac{\partial w}{\partial x} - \frac{(h+t)^2}{4} \frac{N_y}{R} \frac{\partial e}{\partial x} - Q_x \right] + \lambda_5 \left[ \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - \right. \\ \left. \frac{M_{xy}}{R} \frac{\partial w}{\partial x} + \frac{(h+t)^2}{4} \frac{N_{xy}}{R} \frac{\partial e}{\partial x} - Q_y \right] + \lambda_6 \left[ \frac{M_y}{R} + \frac{\partial}{\partial x} \left( M_x \frac{\partial w}{\partial x} + M_{xy} \frac{\partial w}{\partial y} \right) + \right. \\ \left. \frac{\partial}{\partial y} \left( M_{xy} \frac{\partial w}{\partial x} + M_y \frac{\partial w}{\partial y} \right) - \frac{(h+t)^2}{4} \frac{\partial}{\partial x} \left( N_x \frac{\partial e}{\partial x} + N_{xy} \frac{\partial e}{\partial y} \right) - \frac{(h+t)^2}{4} \frac{\partial}{\partial y} \left( N_{xy} \frac{\partial e}{\partial x} + \right. \right. \\ \left. \left. N_y \frac{\partial e}{\partial y} \right) - (h+t)(\sigma_{zm} + q) - \left( Q_x \frac{\partial w}{\partial x} + Q_y \frac{\partial w}{\partial y} \right) + \frac{(h+t)^2}{4} \frac{Q_y}{R} \right] \left. \right\} dx dy = 0$$

leads to the conditions  $\lambda_1 = u$ ,  $\lambda_2 = v$ ,  $\lambda_3 = w$ ,  $\lambda_4 = \beta_x$ ,  $\lambda_5 = \beta_y$ , and  $\lambda_6 = -e$  and the following Euler equations

$$\left. \begin{aligned} \frac{N_x - \nu_f N_y}{2tE_f} &= \frac{\partial u}{\partial x} + \frac{1}{2} \left[ \left( \frac{\partial w}{\partial x} \right)^2 + \frac{(h+t)^2}{4} \left( \frac{\partial e}{\partial x} \right)^2 \right] \\ \frac{N_y - \nu_f N_x}{2tE_f} &= \frac{\partial v}{\partial y} - \frac{w}{R} + \frac{1}{2} \left[ \left( \frac{\partial w}{\partial y} \right)^2 + \frac{(h+t)^2}{4} \left( \frac{\partial e}{\partial y} \right)^2 \right] - \frac{u}{R} \frac{\partial w}{\partial x} + \frac{(h+t)^2}{4} \frac{\beta_x}{R} \frac{\partial e}{\partial x} \\ \frac{2(1 + \nu_f)N_{xy}}{2tE_f} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} + \frac{(h+t)^2}{4} \frac{\partial e}{\partial x} \frac{\partial e}{\partial y} + \frac{v}{R} \frac{\partial w}{\partial x} - \frac{(h+t)^2}{4} \frac{\beta_y}{R} \frac{\partial e}{\partial x} \\ \frac{M_x - \nu_f M_y}{t(h+t)^2 E_f / 2} &= \frac{\partial \beta_x}{\partial x} - \frac{\partial w}{\partial x} \frac{\partial e}{\partial x} \\ \frac{M_y - \nu_f M_x}{t(h+t)^2 E_f / 2} &= \frac{\partial \beta_y}{\partial y} - \frac{\partial w}{\partial y} \frac{\partial e}{\partial y} + \frac{u}{R} \frac{\partial e}{\partial x} - \frac{\beta_x}{R} \frac{\partial w}{\partial x} + \frac{e}{R} \\ \frac{4(1 + \nu_f)M_{xy}}{t(h+t)^2 E_f} &= \frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x} - \frac{\partial w}{\partial x} \frac{\partial e}{\partial y} - \frac{\partial w}{\partial y} \frac{\partial e}{\partial x} - \frac{v}{R} \frac{\partial e}{\partial x} + \frac{\beta_y}{R} \frac{\partial w}{\partial x} \\ \frac{Q_x}{(h+t)G_c} - \frac{(h+t)}{12E_c} \frac{\partial}{\partial x} \left( \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} \right) &= \beta_x + \frac{\partial w}{\partial x} - \frac{\partial e}{\partial x} w - \frac{\partial w}{\partial x} e \\ \frac{Q_y}{(h+t)G_c} - \frac{(h+t)}{12E_c} \frac{\partial}{\partial y} \left( \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} \right) &= \beta_y + \frac{\partial w}{\partial y} - \frac{\partial e}{\partial y} w - e \frac{\partial w}{\partial y} + \frac{v}{R} + \frac{w}{R} \frac{\partial w}{\partial y} + \\ &\quad \frac{(h+t)^2}{4} \frac{e}{R} - e = \sigma_{zm} / E_c \end{aligned} \right\} (53)$$

In the case of a flat sandwich plate ( $R \rightarrow \infty$ ) the stress-strain relations (53) reduce to exactly those obtained by Reissner (reference 8). In the case of homogeneous shells ( $G_c = E_c \rightarrow \infty$ ,  $e \rightarrow 0$ ) equations (53) then reduce to the stress-strain relations (27) for homogeneous shells.

#### CONCLUDING REMARKS

In this report the principle of complementary energy is derived for thin plates and shells with large deflections. The plates and shells may be either homogeneous or of sandwich-type construction. The application of the principle to the derivation of the stress-displacement relations for homogeneous and sandwich shells as well as sandwich plates is given.

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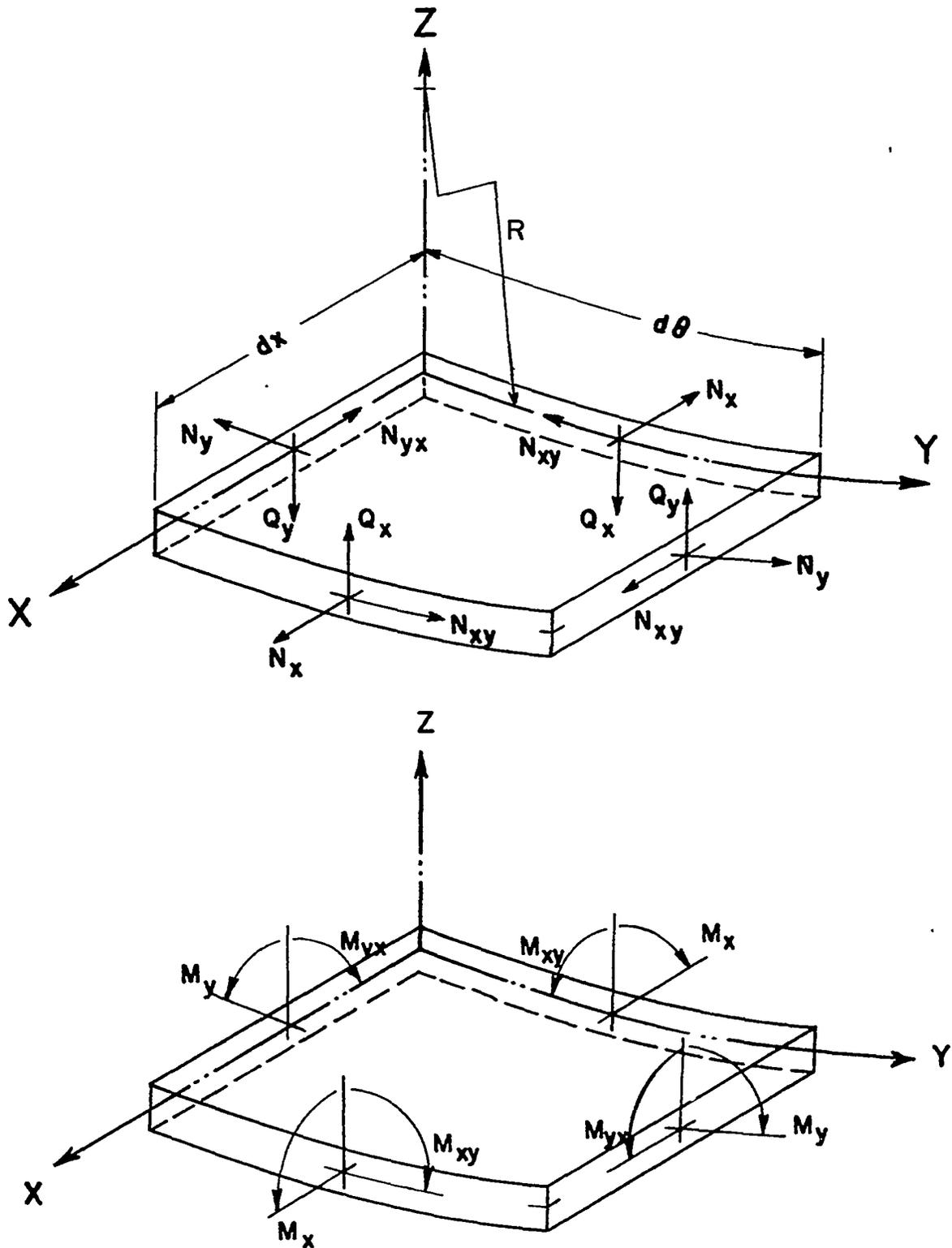


Figure 1.- Element of cylindrical shell with arbitrary cross section.

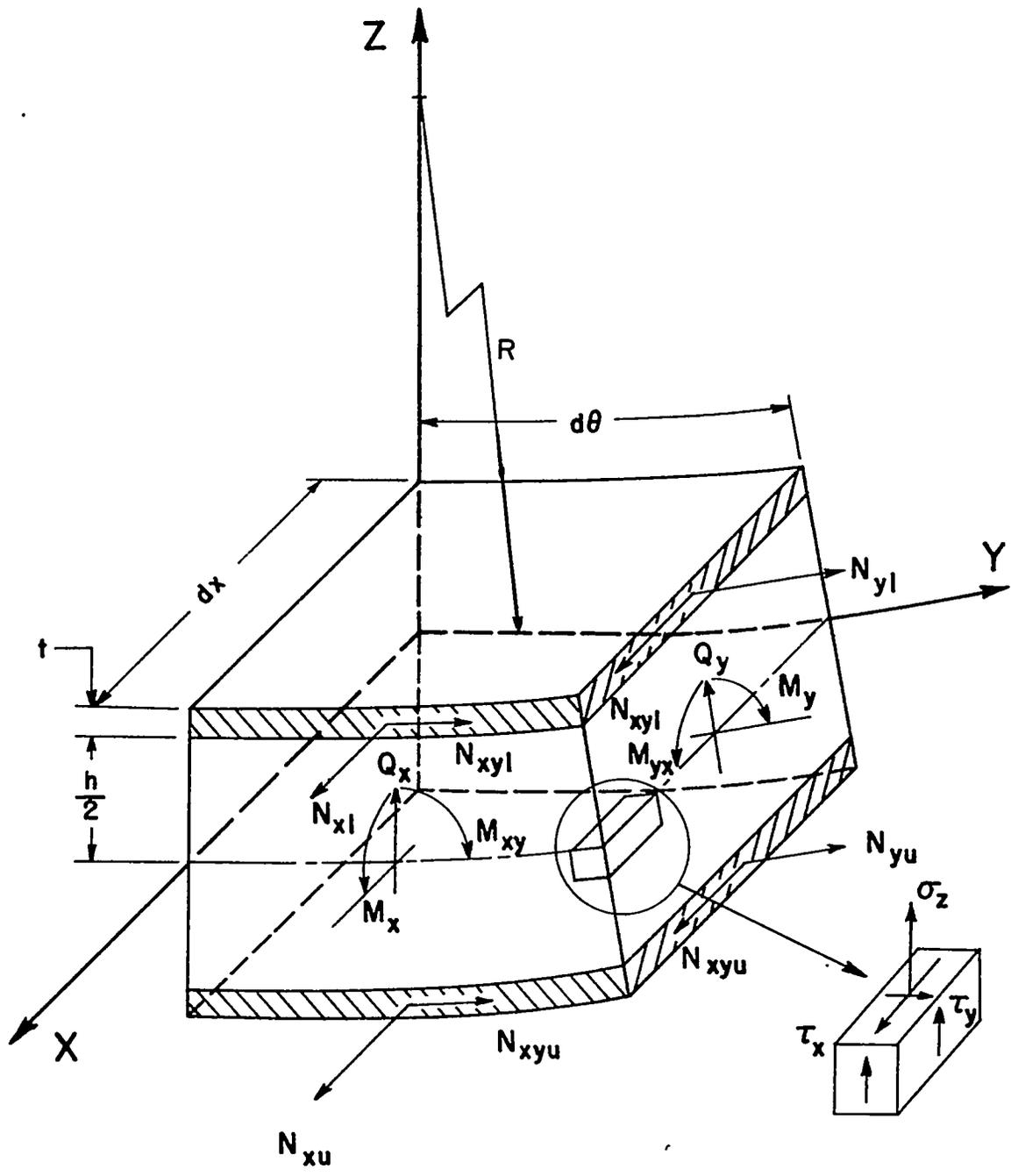


Figure 2.- Element of sandwich shell.