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## GUST-RESPONSE ANALYSIS OF AN AIRPLANE INCLUDING WING BENDING FLEXIBILITY

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# GUST-RESPONSE ANALYSIS OF AN AIRPLANE INCLUDING WING BENDING FLEXIBILITY[1]

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## SUMMARY

An analysis is made of the gust response (including bending moment) of an airplane having the degrees of freedom of vertical motion and wing bending flexibility and basic parameters are established. A convenient, but accurate, numerical solution of the response equations is developed which is very well suited for making trend studies. An example treated shows results which are in very good agreement with the results obtained by a more precise but more lengthy method.

The method of determining a gust causing a known response is indicated and a procedure is given for determining the response of an airplane directly from the known response of another airplane by eliminating the common gust condition.

## INTRODUCTION

In the design of aircraft the condition of gust encounter has become critical in more and more instances, mainly because of the ever-increasing flight speeds. Aircraft designers have therefore placed greater emphasis on obtaining rational methods for accurately predicting the stresses that develop. As a result, the number of papers dealing with the prediction of stresses in an aircraft traversing a vertical gust has significantly increased. (See, for example, refs. 1 to 9.) Many of the papers have treated the airplane as a rigid body and in so doing have dealt with either the degree of freedom of vertical motion alone (refs. 5 to 8) or with the degrees of freedom of vertical motion and pitch (refs. 7 and 9).

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[1]This paper is a revision and extension of a paper entitled "The Determination of the Response Due to Gusts of One Airplane From the Known Response of Another Airplane" published as TN No. Structures 40, British R.A.E., June 1949, which was completed by Mr. Houbolt during a temporary tour of duty with the Royal Aircraft Establishment. Since the present paper is complete in itself, no further reference to the earlier paper is necessary.

Of greater concern in the consideration of gust penetration, however, is the influence that wing flexibility has on structural response. This concern has two main aspects: (1) that including wing flexibility may lead to the calculation of higher stresses than would be obtained by rigid-body treatment of the problem and (2) that wing flexibility may introduce some error when the airplane is used as an instrument for measuring gust intensity. Thus, there are also many recent papers which treat the airplane as an elastic body. In most of these papers the approach used involves the development of the structural response in terms of the natural modes of vibration of the airplane (refs. 2 to 4). Others used a more unusual approach, as for example, reference 1, which deals with the simultaneous treatment of the conditions of equilibrium between aerodynamic forces and structural deformation at a number of points along the wing span. Whatever the approach, however, the main disadvantage of these elastic-body analyses is that they are not very well suited for making trend studies without excessive computation time.

In the present paper, the case of the gust penetration of an airplane having the degrees of freedom of vertical motion and wing bending is considered. Wing bending was chosen because designers have expressed greater concern about the influence of this flexibility on gust response than they have about other types of flexibility. The paper has the objective of trying to establish some of the basic parameters that are involved when wing bending flexibility is included and of developing a method of solution which is fairly well suited for trend studies without excessive computation time. Such a procedure would be useful in evaluation studies which are intended, for example, to evaluate the effect of such factors as forward speed, spanwise mass distribution, gust length, and gust shape.

The equations for response (including accelerations, displacements, and bending moments) are derived and the basic parameters outlined. An easy numerical solution for the response which is readily handled either by manual or machine methods is then given. The inverse of the response problem is considered briefly; that is, the method of determining the gust causing a given response is indicated. Finally, on this basis, a procedure is outlined whereby the response of one airplane may be found directly from the known response of another airplane without the necessity of establishing the gust causing the known response.

## SYMBOLS

$a$ slope of lift curve

$a_n$ deflection coefficient for nth mode, function of time alone

| | |
|---|---|
| $A$ | aspect ratio of wing |
| $b$ | span of wing |
| $c$ | chord of wing |
| $c_o$ | chord of wing midspan |
| $E$ | Young's modulus of elasticity |
| $f(s)$ | nondimensional gust force, $\int_0^s \frac{u'}{U} \psi(s - \sigma)d\sigma$ |
| $F$ | external applied load per unit span |
| $g$ | acceleration due to gravity |
| $H$ | distance to gust peak, chords |
| $I$ | bending moment of inertia |
| $K_j$ | nondimensional bending-moment factor $\left(M_j = K_j \frac{a}{2} \rho UVM_{c_0}\right)$ |
| $L_v$ | aerodynamic lift per unit span of wing due to vertical motion of the airplane |
| $L_g$ | aerodynamic lift per unit span of wing due to gust |
| $m$ | mass per unit span of wing |
| $M_j$ | net incremental bending moment at wing station j |
| $M_{c_0}$ | moment of wing area about spanwise station under consideration |
| $M_n$ | generalized mass of nth mode |
| $\Delta n$ | incremental number of g acceleration |
| $p$ | load intensity per unit spanwise length |
| $s,\sigma$ | distance traveled, $\frac{2V}{c_o} t$, half-chords |
| $S$ | wing area |

| | |
|---|---|
| $t,\tau$ | time, zero at beginning of gust penetration |
| $u$ | vertical velocity of gust |
| $U$ | maximum vertical velocity of gust |
| $V$ | forward velocity of flight |
| $W$ | total weight of airplane |
| $y$ | distance along wing measured from airplane center line |
| $w$ | deflection of elastic axis of wing, positive upward |
| $w_n$ | deflection of elastic axis in nth mode, given in terms of a unit tip deflection |
| $z_n$ | response coefficient based on $a_n$, $\frac{V}{Uc_o} a_n$ |
| $\alpha$ | second derivative of $z_0$ with respect to $s$ |
| $\beta$ | second derivative of $z_1$ with respect to $s$ |
| $\gamma_M$ | bending-moment response factor, ratio of bending moment obtained for airplane considered flexible to bending moment obtained for airplane considered rigid |
| $\epsilon$ | distance interval, half-chords |
| $\lambda$ | reduced-frequency parameter, $\frac{\omega_1 c_o}{2V}$ |
| $\mu_n$ | nondimensional relative-density parameter, $\frac{8M_n}{a\rho c_o S}$ |
| $\eta_n$ | nondimensional bending-moment parameter, $\frac{8M_{m_n}}{a\rho c_o M_{c_0}}$ |
| $\rho$ | mass density of air |
| $\psi$ | function which denotes growth of lift on rigid wing entering a sharp-edge gust (Küssner function) |

$\omega_n$ natural circular frequency of vibration of nth mode

$1 - \Phi$ function which denotes growth of lift on an airfoil following a sudden change in angle of attack (Wagner function)

$\theta = 1 - \Phi$

Subscripts:

j spanwise station

n natural modes of vibration

m number of distance intervals traveled

Notation:

$| \; |$ column matrix

$[ \; ]$ square matrix

Dots are used to denote derivatives with respect to time; primes denote derivatives with respect to $s$ or $\sigma$.

## ANALYSIS

### Equations for Structural Response

Equations of motion.- Consider an airplane flying horizontally into vertical gusts, and suppose that it is desired to include wing bending flexibility in determining the stresses induced by these gusts. The problem is actually one of determining the response of an elastic wing subject to dynamic forces. For dynamic forces of intensity $F$ per unit length, the differential equation for wing bending is, if structural damping is neglected,

$$\frac{\partial^2}{\partial y^2} EI \frac{\partial^2 w}{\partial y^2} = -m\ddot{w} + F \tag{1}$$

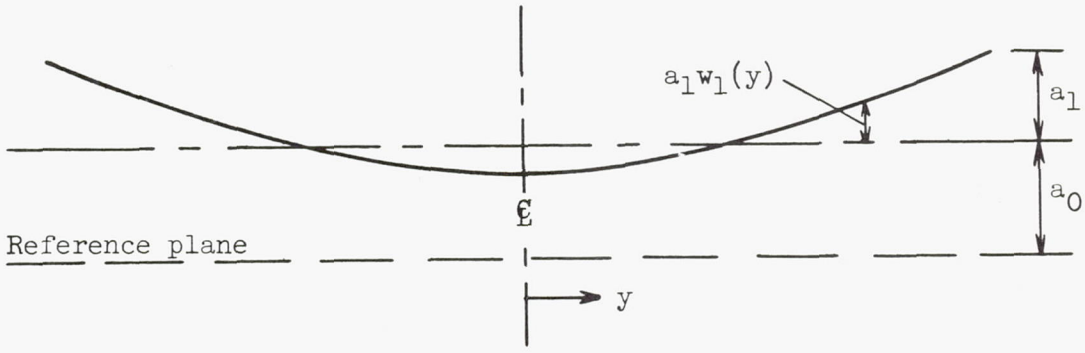
where $w$ is the deflection of the elastic axis referred to a fixed reference plane. The task of determining the deflection that results

from the applied forces $F$ may be handled conveniently by expressing the deflection in terms of the natural free-free vibration modes of the wing. With regard to the flight of an airplane through gusts, examination of a number of acceleration and strain records that have been taken in normal flight with several different aircraft shows that the response to gusts is composed primarily of a rigid-body vertical translation and fundamental-bending-mode excitation of the wing. Thus, the assumption is made in the present analysis that the response may be given with fair accuracy by considering only these two degrees of freedom. This assumption is probably invalid when the airplane is flying near the flutter speed, for then a large amount of coupled bending-torsion displacement may occur. (See ref. 3.)

The wing deflection is thus assumed to be given by the equation

$$w = a_0 + a_1 w_1 \tag{2}$$

where $w_1$ is the deflection given in terms of a unit tip deflection along the elastic axis of the wing for the fundamental mode, and $a_0$ and $a_1$ are functions of time alone. In this form $a_0$ denotes the free-body vertical displacement of the airplane (in this case the displacement of the nodal points) and $a_1$ is the part of the wing-tip deflection which is associated with the fundamental mode, as illustrated in the following sketch:




The use of symmetrical modes implies that only the symmetrical gust is to be considered hereinafter.

Substitution of equation (2) into equation (1) yields

$$a_1 \frac{\partial^2}{\partial y^2} EI \frac{\partial^2 w_1}{\partial y^2} = -m\left(\ddot{a}_0 + \ddot{a}_1 w_1\right) + F \tag{3}$$

From the following relation which expresses the condition for natural fundamental-mode vibration

$$\frac{\partial^2}{\partial y^2} EI \frac{\partial^2 w_1}{\partial y^2} = \omega_1{}^2 m w_1$$

equation (3) may be written

$$a_1 \omega_1{}^2 m w_1 = -m\left(\ddot{a}_0 + \ddot{a}_1 w_1\right) + F \tag{4}$$

where $\omega_1$ is a natural circular frequency of vibration of the fundamental mode. If this equation is integrated over the wing span and use is made of the following known orthogonality condition of the free-body and fundamental modes:

$$\int_{-b/2}^{b/2} m w_1 \, dy = 0 \tag{5}$$

the following equation results:

$$M_0 \ddot{a}_0 = \int_{-b/2}^{b/2} F \, dy \tag{6}$$

where $M_0$ is the airplane mass. Now, if equation (4) is first multiplied through by $w_1$ and then integrated over the wing span and use is made of equation (5), the following equation is obtained:

$$M_1 \ddot{a}_1 + \omega_1{}^2 M_1 a_1 = \int_{-b/2}^{b/2} F w_1 \, dy \tag{7}$$

where $M_1$ is the generalized mass for the fundamental mode, that is, $M_1 = \int_{-b/2}^{b/2} m w_1{}^2 dy$. Equations (6) and (7) represent, respectively, the equations for free-body motion and fundamental wing bending and can be solved if the forces $F$ are known.

For the present case of the airplane flying through a gust, the force $F$ is composed of two parts: a part designated by $L_V$ due to the vertical motion of the airplane (including both rigid-body and bending displacements) and a part $L_g$ resulting directly from the gust (this latter part is the gust force which would develop on the wing considered rigid and restrained against vertical motion). These two parts are defined (see refs. 1 and 3) in the equation of $F$ as follows:

$$F = L_V + L_g = -\frac{a}{2}\rho cV\int_0^t \ddot{w}\left[1 - \Phi(t - \tau)\right]d\tau + \frac{a}{2}\rho cV\int_0^t \dot{u}\psi(t - \tau)d\tau \tag{8}$$

where $1 - \Phi(t)$ is a function (commonly referred to as the Wagner function) which denotes the growth of lift on a wing following a sudden change in angle of attack and for two-dimensional incompressible flow is given by the approximation

$$\left[1 - \Phi(t)\right]_{A=\infty} = 1 - 0.165e^{-0.09\frac{V}{c}t} - 0.335e^{-0.6\frac{V}{c}t} \tag{9}$$

and $\psi(t)$ is a function (commonly referred to as the Küssner function) which denotes the growth of lift on a rigid wing penetrating a sharp-edge gust and for two-dimensional incompressible flow is given by the approximation

$$\left[\psi(t)\right]_{A=\infty} = 1 - 0.5e^{-0.26\frac{V}{c}t} - 0.5e^{-2\frac{V}{c}t} \tag{10}$$

An additional term which involves the apparent air mass should be included in equation (8); this mass term is inertial in character and may be included with the structural mass (see ref. 1) although it is usually small in comparison. The lift-curve slope $a$ may be chosen so as to include approximate over-all corrections for aspect ratio and compressibility effects.

If $w$ as given by equation (2) is substituted into equation (8) and the resulting equation for $F$ is substituted into equations (6) and (7), the following two equations are obtained for the case of a uniform spanwise gust:

$$\frac{2M_0}{a\rho VS}\ddot{a}_0 = -\int_0^t\left(\ddot{a}_0 + \frac{S_1}{S}\ddot{a}_1\right)\left[1 - \Phi(t - \tau)\right]d\tau + \int_0^t \dot{u}\psi(t - \tau)d\tau \tag{11}$$

$$\frac{2M_1}{a\rho VS}\ddot{a}_1 + \frac{\omega_1^2 2M_1}{a\rho VS}a_1 = -\int_0^t\left(\frac{S_1}{S}\ddot{a}_0 + \frac{S_2}{S}\ddot{a}_1\right)\left[1 - \Phi(t - \tau)\right]d\tau +$$

$$\frac{S_1}{S}\int_0^t \dot{u}\psi(t - \tau)d\tau \tag{12}$$

where (because of mode symmetry)

$$\left.\begin{aligned} S &= 2\int_0^{b/2} c\,dy \\ S_1 &= 2\int_0^{b/2} cw_1\,dy \\ S_2 &= 2\int_0^{b/2} cw_1^2 dy \end{aligned}\right\} \tag{13}$$

Equations (11) and (12) may be put in convenient nondimensional form by introducing the notation

$$s = \frac{2V}{c_o}t \qquad \text{or} \qquad \sigma = \frac{2V}{c_o}\tau \tag{14}$$

$$z_n = \frac{V}{Uc_o}a_n \tag{15}$$

where $c_o$ is the midspan chord of the wing and $U$ is the maximum vertical velocity of the gust. With this notation, equations (11) and (12) may be written

$$\mu_0 z_0'' = -2\int_0^s\left(z_0'' + r_1 z_1''\right)\left[1 - \Phi(s - \sigma)\right]d\sigma + \int_0^s \frac{u'}{U}\psi(s - \sigma)d\sigma \tag{16}$$

$$\mu_1 z_1'' + \mu_1 \lambda^2 z_1 = -2\int_0^s \left(r_1 z_0'' + r_2 z_1''\right)\left[1 - \Phi(s - \sigma)\right] d\sigma +$$

$$r_1 \int_0^s \frac{u'}{U} \psi(s - \sigma) d\sigma \tag{17}$$

where

$$\left.\begin{aligned} \mu_0 &= \frac{8M_0}{a\rho c_0 S} \\ \mu_1 &= \frac{8M_1}{a\rho c_0 S} \\ \lambda &= \frac{\omega_1 c_0}{2V} \\ r_1 &= \frac{S_1}{S} \\ r_2 &= \frac{S_2}{S} \end{aligned}\right\} \tag{18}$$

and a prime denotes a derivative with respect to $\sigma$. Equations (16) and (17) are the basic response equations in the present analysis. The five parameters appearing in these equations and given by equations (18) depend upon the forward velocity, air density, lift-curve slope, and the airplane physical characteristics: the wing plan form, wing bending stiffness, and wing mass distribution. Experience has shown that variations in the physical characteristics cause significant variations in the first three of the five parameters, while the last two vary only to a minor extent. The first three are therefore the most basic parameters; $\mu_0$ is a relative-density factor, frequently referred to as a mass parameter, and is associated with vertical free-body motion of the airplane; $\mu_1$, similar to $\mu_0$, is the mass parameter associated with the fundamental mode; and $\lambda$ by its nature may be interpreted as a reduced-frequency parameter similar to that used in flutter analysis.

It is significant to note that, if any one of the three quantities $z_0$, $z_1$, and $u$ appearing in equations (16) and (17) is specified or known, the other two may be determined. Thus, if the gust is known, the response may be determined, or conversely, if either $z_0$ or $z_1$ is known, the gust may be determined. A useful equation relating $z_0$ and $z_1$ may be found by combining equations (16) and (17) so as to eliminate the integral dealing with the gust. The result is the equation

$$\frac{\mu_1}{r_1}\left(z_1'' + \lambda^2 z_1\right) + 2\left(\frac{r_2}{r_1} - r_1\right)\int_0^s z_1''\left[1 - \Phi(s - \sigma)\right]d\sigma = \mu_0 z_0'' \tag{19}$$

which is used subsequently.

It may also be of interest to note that $\mu_0 z_0''$ in effect defines a frequently used acceleration ratio. From equations (12) and (11), the rigid-body component of the vertical acceleration may be written

$$\ddot{a}_0 = \frac{4VU}{c_0} z_0''$$

or, when expressed in terms of the incremental number of g's,

$$\Delta n = \frac{\ddot{a}_0}{g} = \frac{4VU}{c_0 g} z_0''$$

An acceleration factor $\Delta n_s$ based on quasi-steady flow and peak gust velocity is now introduced according to the definition

$$\Delta n_s W = \frac{a}{2} \rho S V^2 \frac{U}{V}$$

The ratio $\frac{\Delta n}{\Delta n_s}$ is thus found to be

$$\frac{\Delta n}{\Delta n_s} = \mu_0 z_0''$$

Where the gust shape is represented analytically and the unsteady-lift functions are taken in the form given by equations (9) and (10), solution of the response equations may be made by the Laplace transform method, but such a solution is more laborious than desired. Therefore, a numerical procedure which permits a rather rapid solution of the equations has been devised and is presented in a subsequent section. It may be well to mention, however, that the response equations are suitable for solution by some of the analog computing machines.

Bending stresses.- The bending moment and, hence, the bending stresses that develop in the wing due to the gust may be found as follows: The right-hand side of equation (1) defines the loading on the wing; suppose this loading is noted by $p$, then

$$p = -m\ddot{w} + F$$

By use of equations (2) and (8), and the notation of equations (14) and (15), this equation becomes

$$p = -m\frac{4VU}{c_o}\left(z_0'' + z_1''w_1\right) - a\rho cVU\int_0^s \left(z_0'' + z_1''w_1\right)\left[1 - \Phi(s - \sigma)\right]d\sigma +$$

$$\frac{a}{2}\rho cV\int_0^s u'\psi(s - \sigma)d\sigma$$

If the moment of this loading is taken about a given wing station, say $y_j$, the following equation for incremental bending moment at that station would result:

$$M_j = \int_{y_j}^{b/2} p\left(y - y_j\right)dy$$

$$= -\frac{4VU}{c_o}\left(M_{m_0}z_0'' + M_{m_1}z_1''\right) - a\rho VU\int_0^s \left(M_{c_0}z_0'' + M_{c_1}z_1''\right)\left[1 - \Phi(s - \sigma)\right]d\sigma +$$

$$\frac{a}{2}\rho VM_{c_0}\int_0^s u'\psi(s - \sigma)d\sigma \tag{20}$$

where the M's bearing double subscripts are first moments defined as follows:

$$\left.\begin{aligned} M_{m_0} &= \int_{y_j}^{b/2} m(y - y_j)dy \qquad & M_{c_0} &= \int_{y_j}^{b/2} c(y - y_j)dy \\ M_{m_1} &= \int_{y_j}^{b/2} mw_1(y - y_j)dy \qquad & M_{c_1} &= \int_{y_j}^{b/2} cw_1(y - y_j)dy \end{aligned}\right\} \tag{21}$$

and $y_j$ is the station being considered. Division through of equation (20) by the quantity $\frac{a}{2}\rho VUM_{c_0}$ gives the following equation which is considered to define a bending-moment factor $K_j$ at wing station $y_j$

$$K_j = \frac{M_j}{\frac{a}{2}\rho VUM_{c_0}}$$

$$= -\frac{8M_{m_0}}{a\rho c_0 M_{c_0}}\left(z_0'' + \frac{M_{m_1}}{M_{m_0}} z_1''\right) - 2\int_0^s \left(z_0'' + \frac{M_{c_1}}{M_{c_0}} z_1''\right)\left[1 - \Phi(s - \sigma)\right]d\sigma +$$

$$\int_0^s \frac{u'}{U} \psi(s - \sigma)d\sigma \tag{22}$$

The factor $\frac{a}{2}\rho VUM_{c_0}$ may be regarded as the maximum aerodynamic bending moment that would be developed by the gust under conditions of quasi-steady flow and with the wing considered rigid and restrained against vertical motion at the root. The bending-moment factor $K_j$ may thus be seen to be the ratio of the actual dynamic bending moment that occurs to this quasi-steady bending moment and therefore may be regarded as a response or an alleviation factor.

A more convenient form for the bending-moment factor may be obtained by solving equations (16) and (17) simultaneously for the quantities $\int_0^s z_0''\left[1 - \Phi(s - \sigma)\right]d\sigma$ and $\int_0^s z_1''\left[1 - \Phi(s - \sigma)\right]d\sigma$ and substituting these values into equation (22). With these operations the following equation results:

$$K_j = \frac{M_j}{\frac{a}{2}\rho VUM_{c_0}}$$

$$= \left(\frac{r_3 r_1 - r_2}{r_1^2 - r_2}\mu_0 - \eta_0\right)z_0'' + \left(\frac{r_1 - r_3}{r_1^2 - r_2}\mu_1 - \eta_1\right)z_1'' + \frac{r_1 - r_3}{r_1^2 - r_2}\mu_1\lambda^2 z_1 \tag{23}$$

where

$$\left.\begin{aligned} r_3 &= \frac{M_{c_1}}{M_{c_0}} \\ \eta_0 &= \frac{8M_{m_0}}{a\rho c_0 M_{c_0}} \\ \eta_1 &= \frac{8M_{m_1}}{a\rho c_0 M_{c_0}} \end{aligned}\right\} \tag{24}$$

It is seen that, when bending moments are being determined, three additional basic parameters (eqs. (24)) appear. The similarity of $\eta_0$ and $\eta_1$ to $\mu_0$ and $\mu_1$ is to be noted; first moments of masses and areas are involved rather than masses and areas.

Reduction to rigid case.- It may be of interest to show the reduction of the response equation to the case of the airplane considered as

a rigid body. Thus, if $z_1$ is equated to zero in equation (16), the following equation for rigid-body response is obtained:

$$\mu_0 z_0'' = -2\int_0^s z_0''\left[1 - \Phi(s - \sigma)\right]d\sigma + \int_0^s \frac{u'}{U}\psi(s - \sigma)d\sigma \tag{25}$$

If $z_1''$ is set equal to zero in equation (22) and use is made of equation (25), the following equation for the bending-moment parameter for the rigid-body case is obtained

$$K_j = \left(\mu_0 - \eta_0\right)z_0'' \tag{26}$$

where $z_0''$ is the acceleration of the airplane considered as a rigid body.

## Matrix Solution of Response Equations

In this section a rather simple numerical solution of the response equations (16) and (17) is presented. The procedure is readily adapted to either manual or punch-card machine calculations.

The derivation proceeds on the basis that the response due to a given gust is to be determined. The airplane, just before gust penetration, is considered to be in level flight and hence has the initial conditions that the vertical displacement and vertical velocity are both zero. These conditions mean that $z_0$, $z_1$, $z_0'$, and $z_1'$ are all zero at $s = 0$. The gust force can be shown to start from zero and, therefore, the additional initial conditions can be established that $z_0''$ and $z_1''$ are also zero at $s = 0$. By the numerical procedure, solution for the response at successive values of $s$ of increment $\epsilon$ will be made and, for the case being considered, it is found advantageous to solve directly for the accelerations rather than the displacements.

In order to make the presentation more compact, the following notation is introduced:

$$\left.\begin{aligned} \alpha &= z_0'' \\ \beta &= z_1'' \\ \theta &= 1 - \Phi \end{aligned}\right\} \tag{27a}$$

and

$$f(s) = \int_0^s \frac{u'}{U} \psi(s - \sigma)d\sigma \tag{27b}$$

With this notation, equation (16) would appear simply as

$$\mu_0\alpha = -2\int_0^s \left(\alpha + r_1\beta\right)\theta(s - \sigma)d\sigma + f(s) \tag{28}$$

In accordance with numerical-evaluation procedures, the interval between 0 and s is divided into a number of equal stations of interval $\epsilon$ so that $s = m\epsilon$. The product of $\left(\alpha + r_1\beta\right)$ and $\theta(s - \sigma)$ is assumed formed at each station and, with the use of the trapezoidal method for determining areas, the unsteady-lift integral in equation (28) may be written in terms of values of $\alpha$ and $\beta$ at successive stations as follows, where the mth station corresponds to the value s:

$$\int_0^s \left(\alpha + r_1\beta\right)\theta(s - \sigma)d\sigma = \epsilon\left(\theta_{m-1}\alpha_1 + \theta_{m-2}\alpha_2 + \ldots + \theta_1\alpha_{m-1} + \frac{1}{2}\theta_0\alpha_m\right) + \epsilon r_1\left(\theta_{m-1}\beta_1 + \theta_{m-2}\beta_2 + \ldots + \theta_1\beta_{m-1} + \frac{1}{2}\theta_0\beta_m\right) \tag{29}$$

in which $\theta_0$, $\theta_1$, . . . are, respectively, the values of the $1 - \Phi$ function at $s = 0$, $s = \epsilon$, . . . $\left(\alpha_0 \text{ and } \beta_0 \text{ do not appear because of the initial conditions}\right)$. With this equation, equation (28) may be written at various values of s or at successive values of m; the result, for example, for $m = 1$ is

$$\mu_0\alpha_1 = -\epsilon\theta_0\alpha_1 - \epsilon r_1\theta_0\beta_1 + f_1$$

and for $m = 2$,

$$\mu_0\alpha_2 = -\epsilon\left(2\theta_1\alpha_1 + \theta_0\alpha_2\right) - \epsilon r_1\left(2\theta_1\beta_1 + \theta_0\beta_2\right) + f_2$$

where $f_1$ and $f_2$ are the values of the gust-force integral at $s = \epsilon$ and $s = 2\epsilon$. The equations thus formed may be combined in the following matrix equation:

$$\begin{bmatrix} \mu_0 + \theta_0\epsilon & & & & \\ 2\theta_1\epsilon & \mu_0 + \theta_0\epsilon & & & \\ 2\theta_2\epsilon & 2\theta_1\epsilon & \mu_0 + \theta_0\epsilon & & \\ \vdots & \vdots & \vdots & & \\ 2\theta_{m-1}\epsilon & 2\theta_{m-2}\epsilon & \cdots & & \mu_0 + \theta_0\epsilon \end{bmatrix} \begin{vmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_m \end{vmatrix} + \begin{bmatrix} r_1\theta_0\epsilon & & & & \\ 2r_1\theta_1\epsilon & r_1\theta_0\epsilon & & & \\ 2r_1\theta_2\epsilon & 2r_1\theta_1\epsilon & r_1\theta_0\epsilon & & \\ \vdots & \vdots & \vdots & & \\ 2r_1\theta_{m-1}\epsilon & 2r_1\theta_{m-2}\epsilon & \cdots & & r_1\theta_0\epsilon \end{bmatrix} \begin{vmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_m \end{vmatrix} = \begin{vmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_m \end{vmatrix} \tag{30a}$$

which may be abbreviated

$$\left[A\right]\left|\alpha\right| + \left[B\right]\left|\beta\right| = \left|f\right| \tag{30b}$$

The simplicity of the matrices A and B, and all square matrices to follow, is to be noted; the matrices are triangular and all elements in one column are merely the elements in the previous column moved down one row. Thus, only the elements in the first columns have to be known to define completely the matrices.

Now instead of considering directly the second response equation, equation (17), it is expedient to consider equation (19) which is repeated here for convenient reference

$$\frac{\mu_1}{r_1}\left(z_1'' + \lambda^2 z_1\right) + 2\left(\frac{r_2}{r_1} - r_1\right)\int_0^s z_1''\theta(s - \sigma)d\sigma = \mu_0 z_0''$$

According to the derivation presented in the appendix, the value of $z_1$ at $s = m\epsilon$ may be approximated in terms of the past-history value of $z_1''$ by the following equation:

$$z_{1_m} = \epsilon^2\left[(m - 1)\beta_1 + \ldots + 2\beta_{m-2} + \beta_{m-1} + \frac{1}{6}\beta_m\right] \tag{31}$$

where $\beta_1$, $\beta_2$, . . . are the values of $z_1''$ at $s = \epsilon$, $s = 2\epsilon$, . . . . If this equation is used to replace $z_1$ in equation (19) and the unsteady-lift integral is manipulated similarly to the integral in equation (28), equations are obtained for successive values of $m$ which involve only the unknowns $\alpha$ and $\beta$. The results may be combined in the following matrix equation:

$$\begin{bmatrix} \frac{\mu_1}{r_1}\left(1+\frac{\lambda^2\epsilon^2}{6}\right)+\left(\frac{r_2}{r_1}-r_1\right)\theta_0\epsilon & & & & \\ \frac{\mu_1}{r_1}\lambda^2\epsilon^2+2\left(\frac{r_2}{r_1}-r_1\right)\theta_1\epsilon & \frac{\mu_1}{r_1}\left(1+\frac{\lambda^2\epsilon^2}{6}\right)+\left(\frac{r_2}{r_1}-r_1\right)\theta_0\epsilon & & & \\ 2\frac{\mu_1}{r_1}\lambda^2\epsilon^2+2\left(\frac{r_2}{r_1}-r_1\right)\theta_2\epsilon & \frac{\mu_1}{r_1}\lambda^2\epsilon^2+2\left(\frac{r_2}{r_1}-r_1\right)\theta_1\epsilon & \frac{\mu_1}{r_1}\left(1+\frac{\lambda^2\epsilon^2}{6}\right)+\left(\frac{r_2}{r_1}-r_1\right)\theta_0\epsilon & & \\ \vdots & \vdots & \vdots & & \\ (m-1)\frac{\mu_1}{r_1}\lambda^2\epsilon^2+2\left(\frac{r_2}{r_1}-r_1\right)\theta_{m-1}\epsilon & (m-2)\frac{\mu_1}{r_1}\lambda^2\epsilon^2+2\left(\frac{r_2}{r_1}-r_1\right)\theta_{m-2}\epsilon & \cdots & & \frac{\mu_1}{r_1}\left(1+\frac{\lambda^2\epsilon^2}{6}\right)+\left(\frac{r_2}{r_1}-r_1\right)\theta_0\epsilon \end{bmatrix} \begin{vmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_m \end{vmatrix} = \mu_0 \begin{vmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_m \end{vmatrix} \tag{32a}$$

which may be written

$$\left[C\right]\left|\beta\right| = \mu_0\left|\alpha\right| \tag{32b}$$

The square matrix C is seen to be similar to the other square matrices in that it is triangular with all the elements in one column made up of the elements in the previous column moved down one row.

An equation in $|\beta|$ alone is obtained by substituting $|\alpha|$ from this equation into equation (30) to yield

$$\left\{\frac{1}{\mu_0}[A][C] + [B]\right\}|\beta| = [D]|\beta| = |f| \tag{33}$$

which is the basic response equation relating $\beta$ (that is $z_1''$) to the gust force. This equation represents a system of linear simultaneous equations where the order of the matrix is arbitrary; that is, the equations may be written up to any desired value of $s = m\epsilon$. The solution for response can therefore be carried on as far as desired. Fortunately, the equations are of such a nature that simultaneous solution is not required. As mentioned, each of the matrices $[A]$, $[B]$, and $[C]$ is triangular with all elements 0 above the main diagonal and with all elements on the main diagonal of each matrix equal; therefore, the main diagonal elements of $[D]$ will also all have the same value and the elements above this diagonal will be 0. If each element on the main diagonal of $[D]$ is denoted by $d_1$ and $[D_1]$ is the matrix D with the main diagonal elements replaced by 0's, then

$$[D] = d_1[I] + [D_1]$$

With this equation, equation (33) may be written

$$|\beta| = \frac{1}{d_1}|f| - \frac{1}{d_1}[D_1]|\beta| \tag{34}$$

Expanded, this equation has the form

$$\begin{vmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \cdot \\ \cdot \\ \cdot \end{vmatrix} = \frac{1}{d_1} \begin{vmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ \cdot \\ \cdot \\ \cdot \end{vmatrix} - \frac{1}{d_1} \begin{bmatrix} 0 & & & & & \\ d_2 & 0 & & & & \\ d_3 & d_2 & 0 & & & \\ d_4 & d_3 & d_2 & 0 & & \\ d_5 & d_4 & d_3 & d_2 & 0 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{vmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \cdot \\ \cdot \\ \cdot \end{vmatrix} \tag{35}$$

It can be seen that a step-by-step solution for the successive values of $\beta$ may now be made; that is, $\beta_1$ is solved for first then, with $\beta_1$ established, $\beta_2$ is solved for, and so on as far as is desired. With the value of $|\beta|$ thus established, solution for $|\alpha|$ may now be made directly from equations (32). Values of the displacements $z_0$ and $z_1$ may be obtained directly from $\alpha$ and $\beta$; $z_1$ may be obtained from equation (31); and $z_0$ may be obtained from this same equation with $\beta$ replaced by $\alpha$.

Some mention should be made with regard to the selection of the time interval $\epsilon$. A rough guide to use in selecting $\epsilon$ can be obtained by considering $\lambda$, which appears as the characteristic frequency in most response calculations. The period based on this frequency would be $T_s = \frac{2\pi}{\lambda}$. Experience has shown that a time interval in the neighborhood of 1/12 of this period yields very good results (in general less than 1 percent error); accordingly, a reasonable guide in choosing $\epsilon$ would be the equation $\epsilon \approx \frac{1}{2\lambda}$. Some convenient value near that given by this equation should be satisfactory; in general, it will be found that $\epsilon$ may be 1 or greater.

The procedure thus outlined provides a rather rapid evaluation of the response due to a prescribed gust. With the response thus evaluated the bending moment at any value of $s$ or the complete time history of

bending moment may be found by application of equation (23). It should be evident that, if response values for either $z_0''$ or $z_1''$ are known, the gust causing this response can be found by suitable manipulation of equations (30) and (32). Thus, if $z_0''$ is known, $\beta$ in equations (30b) and (32b) may be eliminated to give the equation

$$\left\{[A] + \mu_0[B][C]^{-1}\right\}|\alpha| = |f|$$

Direct substitution of $z_0''$ in this equation allows $|f|$ to be determined. In most practical cases the second term in equation (30b) contributes only a small amount and may be dropped with little resulting error in the gust force. The equation for $|f|$ is then simply

$$[A]|\alpha| = |f|$$

## Determination of Response of One Airplane From Known Response of Another Airplane

In general, a given gust condition produces different responses either for two different airplanes or for the same airplane with different loading conditions or forward velocity. It would be expected, however, that the response of the two airplanes could be correlated through the common gust condition. This correlation may be demonstrated quite easily by means of the equations given in the preceding section. The case to be treated is as follows: The time history of bending moment due to a gust is assumed to have been measured in one airplane and it is desired to calculate directly from this time history what the bending moment due to the same gust would have been in another airplane. Although the derivation is presented in terms of bending moment, a similar derivation could be made in terms of either accelerations or displacements.

If use is made of equation (31) to write the successive values of the displacement $z_1$ in terms of the accelerations, the bending-moment factor, equation (23), may be written in terms of the accelerations alone and the following matrix equation for $K$ may be formulated:

$$\begin{vmatrix} K_1 \\ K_2 \\ K_3 \\ \cdot \\ \cdot \\ \cdot \\ K_m \end{vmatrix} = d \begin{vmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_m \end{vmatrix} + e \begin{vmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \cdot \\ \cdot \\ \cdot \\ \beta_m \end{vmatrix} + h\epsilon^2 \begin{bmatrix} \frac{1}{6} & & & \\ 1 & \frac{1}{6} & & \\ 2 & 1 & \frac{1}{6} & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ m - 1 & \cdot & \cdot \cdot & \frac{1}{6} \end{bmatrix} \begin{vmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \cdot \\ \cdot \\ \cdot \\ \beta_m \end{vmatrix} \tag{36a}$$

where

$$d = \frac{r_1 r_3 - r_2}{r_1^2 - r_2} \mu_0 - \eta_0$$

$$e = \frac{r_1 - r_3}{r_1^2 - r_2} \mu_1 - \eta_1$$

$$h = \frac{r_1 - r_3}{r_1^2 - r_2} \mu_1 \lambda^2$$

With the use of equations (32) this equation may be written

$$|K| = [E]|\beta| \tag{36b}$$

where

$$[E] = \left[\frac{d}{\mu_0}[C] + e[I] + h\epsilon^2[G]\right] \tag{37}$$

in which $[I]$ is the identity matrix and

$$[G] = \begin{bmatrix} \frac{1}{6} & & & \\ 1 & \frac{1}{6} & & \\ 2 & 1 & \frac{1}{6} & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ m-1 & \cdot & \cdot\ \cdot & \frac{1}{6} \end{bmatrix}$$

Substitution of the value of $\beta$ as obtained from equations (36) into equation (33) gives the following relation between the gust forces and the bending-moment parameter:

$$|f| = [D][E]^{-1}|K_m| \tag{38}$$

The gust-force matrix $|f|$ (see eq. (27b)) may be expressed in terms of the gust velocity by the following process: It is assumed that the initial vertical velocity of the gust is zero and that successive values of gust velocity of increment $\epsilon$ are designated by $u_1$, $u_2$, $u_3$ . . . . First-order difference equations are used to approximate the slope of the gust velocity, so that, in general,

$$u'_m = \frac{u_{m+1} - u_{m-1}}{2\epsilon}$$

If this equation is used and the integral equation (27b) is handled by the trapezoidal integration method similar to that used for equation (29), the gust force may be written in terms of the successive values of gust gradient so as to form the following matrix equation:

$$\begin{vmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{vmatrix} = \frac{1}{2} \begin{bmatrix} \psi_1 & & & \\ \psi_2 & \psi_1 & & \\ \psi_3 - \psi_1 & \psi_2 & \psi_1 & \\ \psi_4 - \psi_2 & \psi_3 - \psi_1 & \psi_2 & \psi_1 \end{bmatrix} \begin{vmatrix} \frac{u_1}{U} \\ \frac{u_2}{U} \\ \frac{u_3}{U} \\ \frac{u_4}{U} \end{vmatrix} \tag{39}$$

where $\psi_1$, $\psi_2$, $\psi_3$ . . . are successive values of the $\psi$ function. Substitution of this equation into equation (38) allows for the solution of $\left|\frac{u}{U}\right|$ in terms of the parameter K as

$$\left|\frac{u}{U}\right| = [\psi]^{-1}[D][E]^{-1}\left|K_m\right| \tag{40}$$

where $[\psi]$ is the square matrix in equation (39). Different airplanes flying through the same gust will experience the same vertical gust velocity for equal absolute distances of gust penetration; that is,

$$(Vt)_{\text{airplane 1}} = (Vt)_{\text{airplane 2}}$$

From equation (14), then, the following conditions must prevail:

$$\left.\begin{aligned} (sc_o)_1 &= (sc_o)_2 \\ (\epsilon c_o)_1 &= (\epsilon c_o)_2 \end{aligned}\right\} \tag{41}$$

where the subscripts 1 and 2 denote airplane 1 and 2, respectively. Satisfaction of the latter condition insures that the gust velocity as given by equation (40) would be the same for the two airplanes being

compared. This common gust condition may therefore be eliminated to yield the result

$$\left([\psi]^{-1}[D][E]^{-1}|K|\right)_1 = \left([\psi]^{-1}[D][E]^{-1}|K|\right)_2$$

If it is assumed that K for airplane 1 is known, then K for airplane 2 may be written

$$|K|_2 = \left[[E][D]^{-1}[\psi]\right]_2\left[[\psi]^{-1}[D][E]^{-1}|K|\right]_1 \tag{42}$$

where again the time interval chosen for the two airplanes must satisfy relation (41). Thus, if the bending moment due to a given gust sequence is known for one airplane, the bending moment that would develop in another airplane encountering the same gust sequence can be determined from this known bending moment by the use of equation (42). If the mid-chords of the two airplanes are equal, the time interval may be taken equal and equation (42) reduces to

$$|K|_2 = \left[[E][D]^{-1}\right]_2\left[[D][E]^{-1}|K|\right]_1 \tag{43}$$

## SUMMARY OF CALCULATION PROCEDURE

As a convenience, a summary of the basic steps necessary for calculating the response of an airplane to a gust is given as follows:

For accelerations and displacements:

(1) With the use of the fundamental mode, wing plan form, and mass distribution, calculate the quantities $\mu_0$, $\mu_1$, $\lambda$, $r_1$, and $r_2$ as given by equations (18).

(2) Choose the time interval $\epsilon$. A convenient rule of thumb is $\epsilon \approx \frac{1}{2\lambda}$, but for most cases $\epsilon = 1$ should give satisfactory results.

(3) Determine values of the unsteady-lift function $\theta = 1 - \Phi$ at successive multiple intervals of $\epsilon$. (See fig. 1.) Also determine corresponding values of the gust-force integral $f(s)$, equation (27b). As an aid, curves for $f(s)$ are presented in figure 1 for the sharp-edge

gust and in figure 2 for various-length sine gusts, sine$^2$ gusts, and triangular gusts. (The curves in fig. 1 have been obtained from eqs. (9) and (10). These approximations, although rather accurate for the lower values of $s$, are noted to cross; actually, they should not cross and are known to have the same asymptotic approach to unity.)

(4) From the following definitions:

$$A_1 = \mu_0 + \epsilon\theta_0$$

$$A_m = 2\epsilon\theta_{m-1} \qquad (m > 1)$$

$$B_1 = r_1\epsilon\theta_0$$

$$B_m = 2r_1\epsilon\theta_{m-1} \qquad (m > 1)$$

$$C_1 = \frac{\mu_1}{r_1}\left(1 + \frac{\epsilon^2\lambda^2}{6}\right) + \left(\frac{r_2}{r_1} - r_1\right)\epsilon\theta_0$$

$$C_m = (m - 1)\frac{\mu_1}{r_1}\epsilon^2\lambda^2 + 2\left(\frac{r_2}{r_1} - r_1\right)\epsilon\theta_{m-1} \qquad (m > 1)$$

set up the following matrices:

$$[A] = \begin{bmatrix} A_1 & & & & \\ A_2 & A_1 & & & \\ A_3 & A_2 & A_1 & & \\ A_4 & A_3 & A_2 & A_1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$$[B] = \begin{bmatrix} B_1 & & & & \\ B_2 & B_1 & & & \\ B_3 & B_2 & B_1 & & \\ B_4 & B_3 & B_2 & B_1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$$[C] = \begin{bmatrix} C_1 & & & & \\ C_2 & C_1 & & & \\ C_3 & C_2 & C_1 & & \\ C_4 & C_3 & C_2 & C_1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

Then calculate the matrix

$$[D] = \frac{1}{\mu_0}[A][C] + [B]$$

(5) Solve for the values of $\beta$ $\left(\text{which equals } z_1''\right)$ from equation (33), by the method outlined following equation (33). (See eq. (34).) The values of $z_1$ and $\alpha$ $\left(\text{which equals } z_0''\right)$ can then be calculated from equations (31) and (32).

For bending moment:

(6) In order to compute bending moment, determine $r_3$, $\eta_0$, and $\eta_1$ as given by equations (24), where $M_{m_0}$, $M_{m_1}$, $M_{c_0}$, and $M_{c_1}$ in these equations depend on the particular wing station being considered and are given by equations (21).

(7) Determine bending moment by use of equation (23) with the values of response already established. This equation may be applied directly to any desired time value. Maximum bending moment usually occurs very close to the time when $z_1$ is a maximum.

## EXAMPLES

Example A.- In order to provide an illustration and give an idea of the accuracy of the present analysis, the response to a sharp-edge gust of the two-engine-airplane example considered in reference 1 was determined. The weight distribution over the semispan, the wing-chord distribution, and the fundamental bending mode are shown in figures 3, 4, and 5. The frequency and deflection of the fundamental mode were calculated by the method given in reference 10. The solution is made for a forward velocity of 210 miles per hour and a gust velocity of 10 feet per second.

The lift-curve slope used in reference 1 was 5.41; to be consistent, the same value was used here. Furthermore, the unsteady-lift function used for a change in angle of attack in the example presented in reference 1 was given by the equation

$$(1 - \Phi)_{A=6} = 1 - 0.361e^{-0.381s}$$

rather than by equation (9). Thus, this equation was also used here. The gust unsteady-lift function used was that given by equation (10).

The various physical constants and the basic response and bending-moment parameters are given in table 1; the values of the unsteady-lift function and the values of the gust force are listed in table 2. The matrices $[A]$, $[B]$, and $[C]$ used in the solution are given in table 3.

The solution for response is shown in figure 6(a) where the deflection coefficients $a_0$ and $a_1$ in inches are plotted against distance traveled in half-chords. The corresponding deflection quantities for the example given in reference 1 were determined and, for comparison, are also shown in the figure. A similar comparison is made in figure 6(b) for bending stresses at the fuselage and engine stations, stations 0 and 1 from reference 1. The agreement is seen to be good.

Example B.- A second example is included in order to illustrate one means by which the method may be used to evaluate the influence of bending flexibility upon the response to a gust. The physical characteristics for the airplane considered in this example are listed also in table 1, and

equation (9) is used for the function $1 - \Phi$ instead of the values given in table 2. Maximum values of the bending moment that develops at the fuselage station during flights through sine gusts of various lengths have been determined, both for the airplane considered flexible and for the airplane considered rigid. The results are shown in figure 7(a) where maximum values of the bending-moment parameter $K$ are plotted against gust-gradient distance $H$. The difference between the two curves represents the increase in bending moment due to effects of wing bending flexibility. By taking the ratio of $K$ for the flexible case to $K$ for the rigid case, a type of dynamic response factor is formulated which gives a direct measure of the influence of wing flexibility. This ratio is designated $\gamma_M$ and is shown in figure 7(b). As an example of the significance of this plot, the value of $\gamma_M = 1.16$ at $H = 5$ means that flexibility results in a 16-percent dynamic overshoot in the stress from the value that would be obtained at $H = 5$ on the basis of a rigid-body analysis. It may be seen also that the value of $\gamma_M$ is approximately unity for values of $H = 10$ and greater; therefore, in this range of gust-gradient distances a rigid-body treatment would be sufficient for this airplane.

## DISCUSSION

The derivation presented herein is intended to provide a convenient engineering method for calculating the response of an airplane to a gust where wing bending flexibility is included. The method is believed to be well-suited for making trend studies which evaluate, for example, the effect on response of such factors as mass distribution, speed, and altitude. Although the unsteady-lift functions for two-dimensional unsteady flow are presented, the method is general enough so that the unsteady-lift functions for finite aspect ratio, for subsonic compressible flow, and for supersonic flow may be used as well. (See refs. 7 and 11 to 15.)

Since the numerical method is based on an integration procedure, it possesses the desirable feature that a fairly large time interval may be used and good accuracy still obtained. As an accuracy test, solutions of equations (16) and (17) were made for several cases by the exact Laplace transform method as well as by the numerical process, in which process the time interval was selected according to the rule of thumb suggested. When the results were plotted to three figures, the difference between the two solutions was barely discernible.

Additional bending modes could be included in the analysis but this refinement is really not warranted. Some calculations made with additional modes gave results which differed only slightly from the results

obtained when only the fundamental mode was used. The good agreement of results obtained in example A with the results obtained by the more precise method given in reference 1 also illustrates this point. Furthermore, if additional degrees of freedom are to be used, it would appear more important to include wing torsion and airplane pitch. The extent to which torsion influences the results is probably governed most by the nearness to the flutter speed. The importance of airplane pitch is probably governed most by the gust length; some investigations dealing with pitch have indicated that except for very light wing loadings the pitch of the airplane does not influence the results appreciably until gust lengths of from 20 to 30 chords or larger are involved. Thus, the present analysis, although limited to the degrees of freedom of vertical motion and wing bending, should probably be sufficiently satisfactory for speeds near the cruising speed and for gust-gradient distances up to approximately 10 chords.

The analysis may be useful in assessing the significance of wing flexibility in the technique of measuring gust intensity by means of an airplane. In this technique gust severity is usually measured by means of an accelerometer placed at the center line of the airplane. In order to obtain a rough idea of whether flexibility may have some effect on this measurement, calculations for the maximum accelerations at the center line and for the maximum acceleration at the nodal points (the true center-of-gravity acceleration) may be made for various assumed gust lengths. A comparison of these computed maximum acceleration values should give some idea as to the extent to which wing flexibility may alter the measurements in actual flight.

## CONCLUDING REMARKS

The analysis presented herein for the response of an airplane to a gust should provide a useful means for evaluating the effects of wing flexibility. A convenient, but accurate, numerical solution of the response equations is developed which is well-suited for trend studies such as the evaluation of the effects of mass distribution, speed, altitude, and similar factors.

As indicated by an example, the method gives good agreement with the results of the more precise but more lengthy recurrence matrix method of NACA Rep. 1010.

The method permits the evaluation of a gust causing a known response. A procedure is given wherein the known gust response of one airplane may

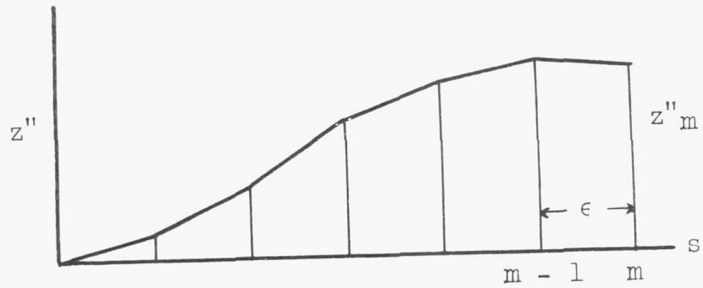
be used directly to determine what the response would be for another airplane flying through the same gust condition.

Langley Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., May 21, 1952.

# APPENDIX

## DERIVATION OF EQUATION RELATING DISPLACEMENT TO PREVIOUS SUCCESSIVE VALUES OF ACCELERATION

In this appendix, a derivation is given of equation (31) which gives the value of displacement in terms of successive past-history values of acceleration. Suppose that the second derivative (acceleration) of a function is approximated by a succession of straight-line segments as shown in the following sketch:




where the segments cover equal intervals $\epsilon$ of the abscissa $s$ and the initial condition that $z_0'' = 0$ is assumed to apply. If a dummy origin is now considered at the station $m - 1$, the segment between stations $m - 1$ and $m$ may be represented by the equation

$$z'' = z''_{m-1} + \frac{z''_m - z''_{m-1}}{\epsilon} s$$

Two successive integrations give the relations for $z'_m$ and $z_m$ as follows:

$$z' = z''_{m-1}s + \frac{z''_m - z''_{m-1}}{2\epsilon} s^2 + z'_{m-1}$$

$$z = z''_{m-1} \frac{s^2}{2} + \frac{z''_m - z''_{m-1}}{6\epsilon} s^3 + z'_{m-1}s + z_{m-1}$$

where the constants of integration $z'_{m-1}$ and $z_{m-1}$ (initial conditions for the interval) have been introduced. If $s$ is set equal to $\epsilon$ in these two equations, the following equations result:

$$z'_m = \frac{\epsilon}{2}\left(z''_m + z''_{m-1}\right) + z'_{m-1} \tag{A1}$$

$$z_m = \frac{\epsilon^2}{6} z''_m + \frac{\epsilon^2}{3} z''_{m-1} + z'_{m-1}\epsilon + z_{m-1} \tag{A2}$$

From these two equations the values of $z'_m$ and $z_m$ at any time interval may be given in terms of the second derivative at all previous time intervals. For example, with initial conditions of $z''_0 = z'_0 = 0$, equation (A1) becomes for $m = 1$

$$z'_1 = \frac{\epsilon}{2} z''_1 \tag{A3}$$

and for $m = 2$

$$z'_2 = \frac{\epsilon}{2}\left(z''_2 + z''_1\right) + z'_1$$

Combining this equation and equation (A3) results in the relation

$$z'_2 = \epsilon\left(z''_1 + \frac{1}{2} z''_2\right)$$

This process may be carried through for each of the time stations to yield the following general equation for $z'_m$:

$$z'_m = \epsilon\left(z''_1 + z''_2 + z''_3 + \ldots + z''_{m-1} + \frac{1}{2} z''_m\right) \tag{A4}$$

which, of course, is the trapezoidal approximation of the area under the $z''$-curve. Equation (A2) for $z_m$ may be treated similarly and it is found that the general equation for $z_m$ may be written

$$z_m = \epsilon^2\left[(m - 1)z''_1 + (m - 2)z''_2 + \ldots + 2z''_{m-2} + z''_{m-1} + \frac{1}{6} z''_m\right] \tag{A5}$$

This equation thus gives the displacement at any time station in terms of the accelerations at all previous time stations.

It may be noted that, if higher-order segments (parabolic or cubic) had been used instead of straight-line segments to approximate the second derivative, equations similar in form to equations (A4) and (A5) would also result. For most practical purposes, however, the accuracy of equation (A5) is sufficiently good as long as the interval $\epsilon$ is chosen so that the straight-line segments roughly approximate the second derivative.

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TABLE 1.- PHYSICAL CHARACTERISTICS FOR EXAMPLE AIRPLANES

| | Example A | Example B |
|---|---|---|
| W, lb | 37,450 | 33,450 |
| S, $ft^2$ | 870 | 870 |
| b, in. | 1120 | 1120 |
| $c_o$, in. | 154 | 164 |
| $\rho$, $lb/ft^3$ | 0.0765 | 0.0765 |
| V, ft/sec | 308 | 374 |
| U, ft/sec | 10 | ------- |
| $\epsilon$ sec | 0.0208 | 0.0183 |
| $\epsilon$ half-chords | 1.0 | 1.0 |
| a | 5.41 | 6.28 |
| $\mu_0$ | 64.16 | 46.8 |
| $\mu_1$ | 0.9045 | 0.748 |
| $\lambda$ | 0.4353 | 0.392 |
| $r_1$ | 0.2181 | 0.225 |
| $r_2$ | 0.1358 | 0.143 |
| $r_3$ fuselage station | 0.452 | 0.457 |
| $r_3$ engine station | 0.547 | ------- |
| $\eta_0$ fuselage station | 23.49 | 15.94 |
| $\eta_0$ engine station | 10.19 | ------- |
| $\eta_1$ fuselage station | 3.665 | 2.555 |
| $\eta_1$ engine station | 3.391 | ------- |
| $^*\frac{z}{I}$, $in.^{-3}$ fuselage station | 0.00537 | 0.00537 |
| $^*\frac{z}{I}$, $in.^{-3}$ engine station | 0.00669 | ------- |

*z here denotes distance from neutral axis to extreme fiber.

TABLE 2.- $1 - \Phi$ ORDINATES AND GUST-FORCE ORDINATES

FOR SHARP-EDGE GUST, $\epsilon = 1.0$

| m | $\theta_m$ or $(1 - \Phi)_{A=6}$ | f or $\psi$ |
|---|---|---|
| 0 | 0.6390 | 0 |
| 1 | .7534 | .377 |
| 2 | .8315 | .547 |
| 3 | .8849 | .635 |
| 4 | .9214 | .692 |
| 5 | .9463 | .736 |
| 6 | .9633 | .771 |
| 7 | .9749 | .798 |
| 8 | .9829 | .821 |
| 9 | .9883 | .845 |

TABLE 3.- MATRICES USED IN EXAMPLE A

A Matrix

$$\begin{bmatrix}
64.799 & & & & & & & & & \\
1.5068 & 64.799 & & & & & & & & \\
1.6630 & 1.5068 & 64.799 & & & & & & & \\
1.7698 & 1.6630 & 1.5068 & 64.799 & & & & & & \\
1.8428 & 1.7698 & 1.6630 & 1.5068 & 64.799 & & & & & \\
1.8926 & 1.8428 & 1.7698 & 1.6630 & 1.5068 & 64.799 & & & & \\
1.9266 & 1.8926 & 1.8428 & 1.7698 & 1.6630 & 1.5068 & 64.799 & & & \\
1.9498 & 1.9266 & 1.8926 & 1.8428 & 1.7698 & 1.6630 & 1.5068 & 64.799 & & \\
1.9658 & 1.9498 & 1.9266 & 1.8926 & 1.8428 & 1.7698 & 1.6630 & 1.5068 & 64.799 & \\
1.9766 & 1.9658 & 1.9498 & 1.9266 & 1.8926 & 1.8428 & 1.7698 & 1.6630 & 1.5068 & 64.799
\end{bmatrix}$$

B Matrix

$$\begin{bmatrix}
0.1394 & & & & & & & & & \\
.3286 & 0.1394 & & & & & & & & \\
.3627 & .3286 & 0.1394 & & & & & & & \\
.3860 & .3627 & .3286 & 0.1394 & & & & & & \\
.4019 & .3860 & .3627 & .3286 & 0.1394 & & & & & \\
.4128 & .4019 & .3860 & .3627 & .3286 & 0.1394 & & & & \\
.4202 & .4128 & .4019 & .3860 & .3627 & .3286 & 0.1394 & & & \\
.4252 & .4202 & .4128 & .4019 & .3860 & .3627 & .3286 & 0.1394 & & \\
.4287 & .4252 & .4202 & .4128 & .4019 & .3860 & .3627 & .3286 & 0.1394 & \\
.4311 & .4287 & .4252 & .4202 & .4128 & .4019 & .3860 & .3627 & .3286 & 0.1394
\end{bmatrix}$$

C Matrix

$$\begin{bmatrix}
4.5367 & & & & & & & & & \\
1.3954 & 4.5367 & & & & & & & & \\
2.2445 & 1.3954 & 4.5367 & & & & & & & \\
3.0735 & 2.2445 & 1.3954 & 4.5367 & & & & & & \\
3.8889 & 3.0735 & 2.2445 & 1.3954 & 4.5367 & & & & & \\
4.6949 & 3.8889 & 3.0735 & 2.2445 & 1.3954 & 4.5367 & & & & \\
5.4947 & 4.6949 & 3.8889 & 3.0735 & 2.2445 & 1.3954 & 4.5367 & & & \\
6.2900 & 5.4947 & 4.6949 & 3.8889 & 3.0735 & 2.2445 & 1.3954 & 4.5367 & & \\
7.0824 & 6.2900 & 5.4947 & 4.6949 & 3.8889 & 3.0735 & 2.2445 & 1.3954 & 4.5367 & \\
7.8726 & 7.0824 & 6.2900 & 5.4947 & 4.6949 & 3.8889 & 3.0735 & 2.2445 & 1.3954 & 4.5367
\end{bmatrix}$$

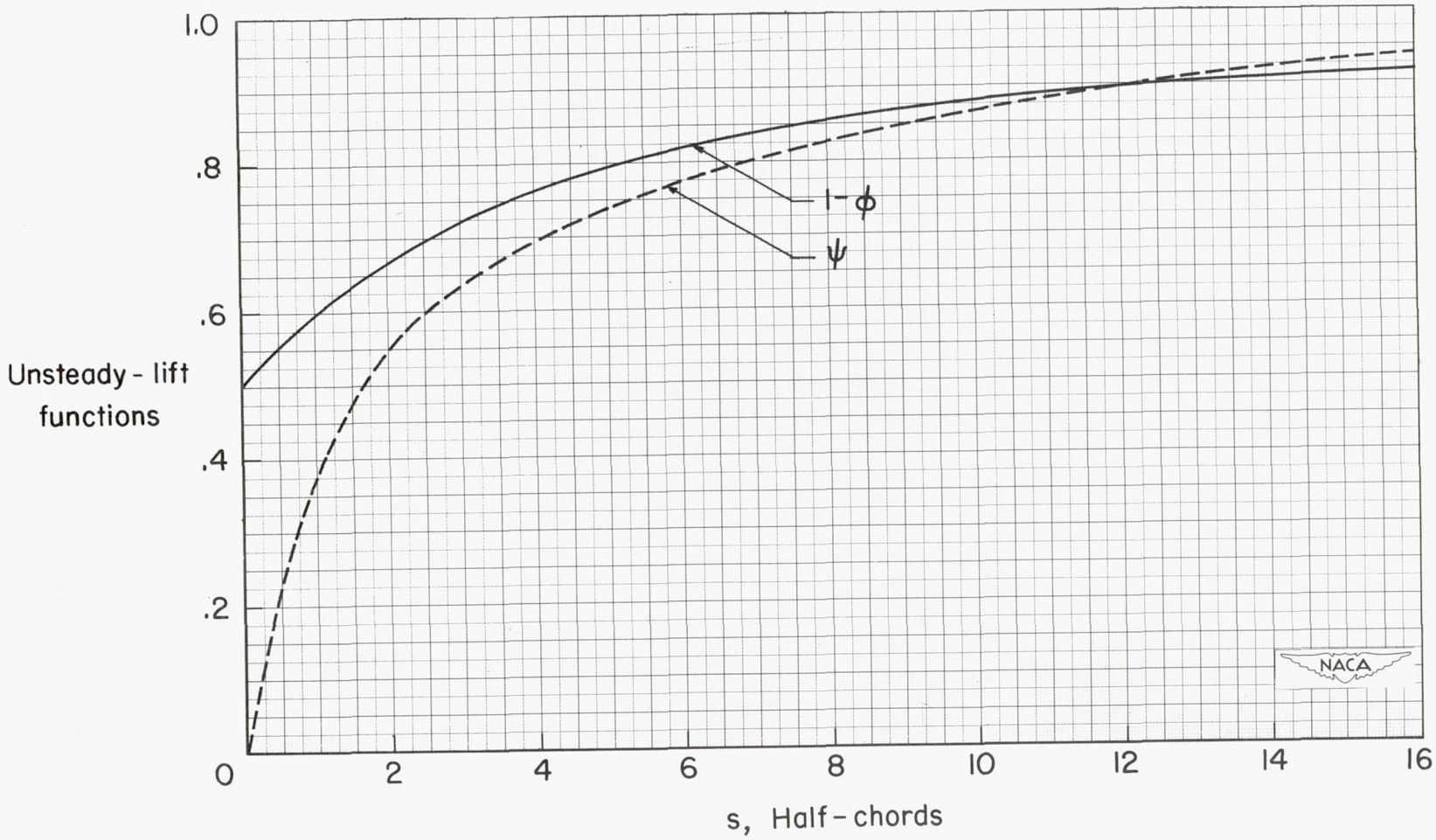
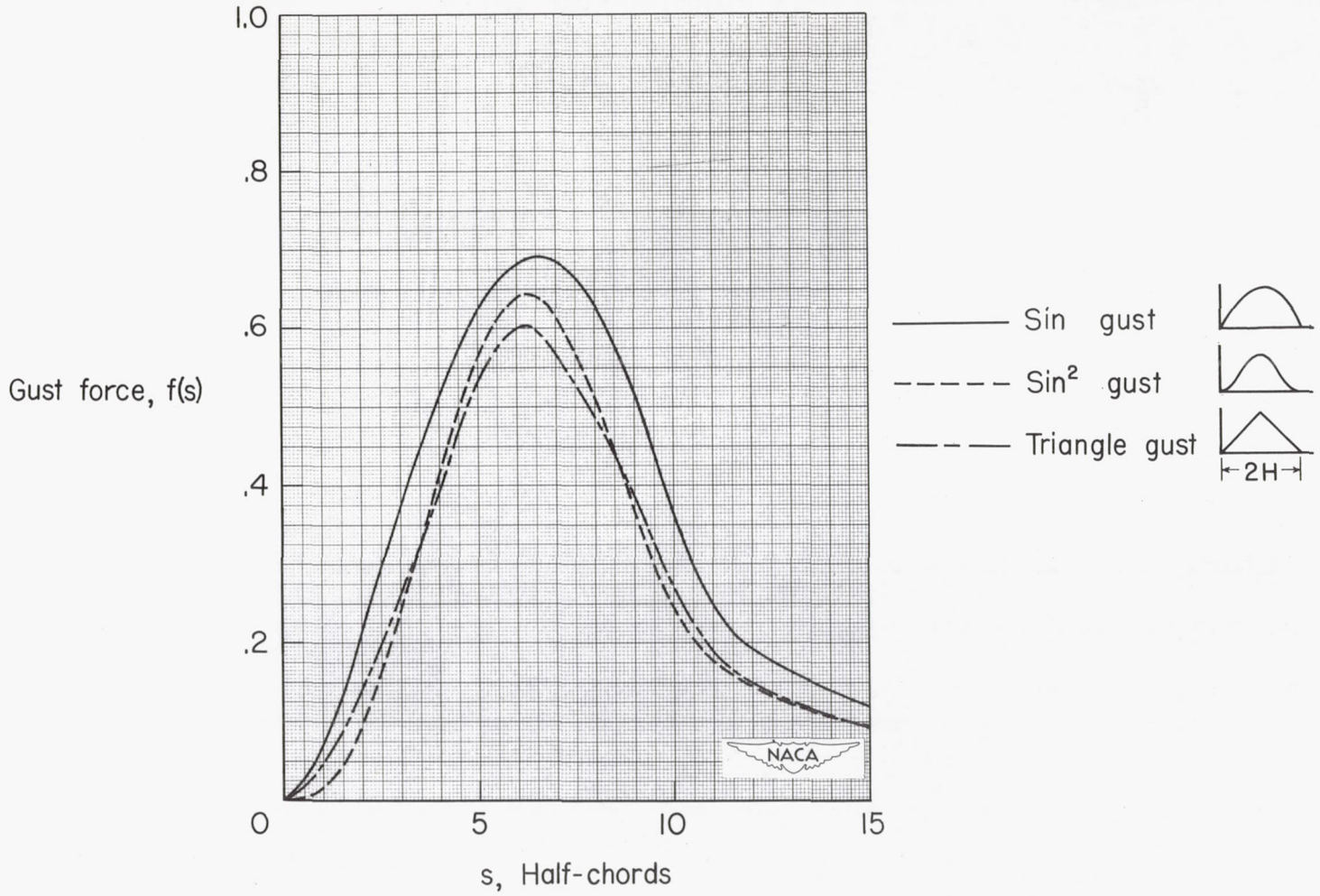
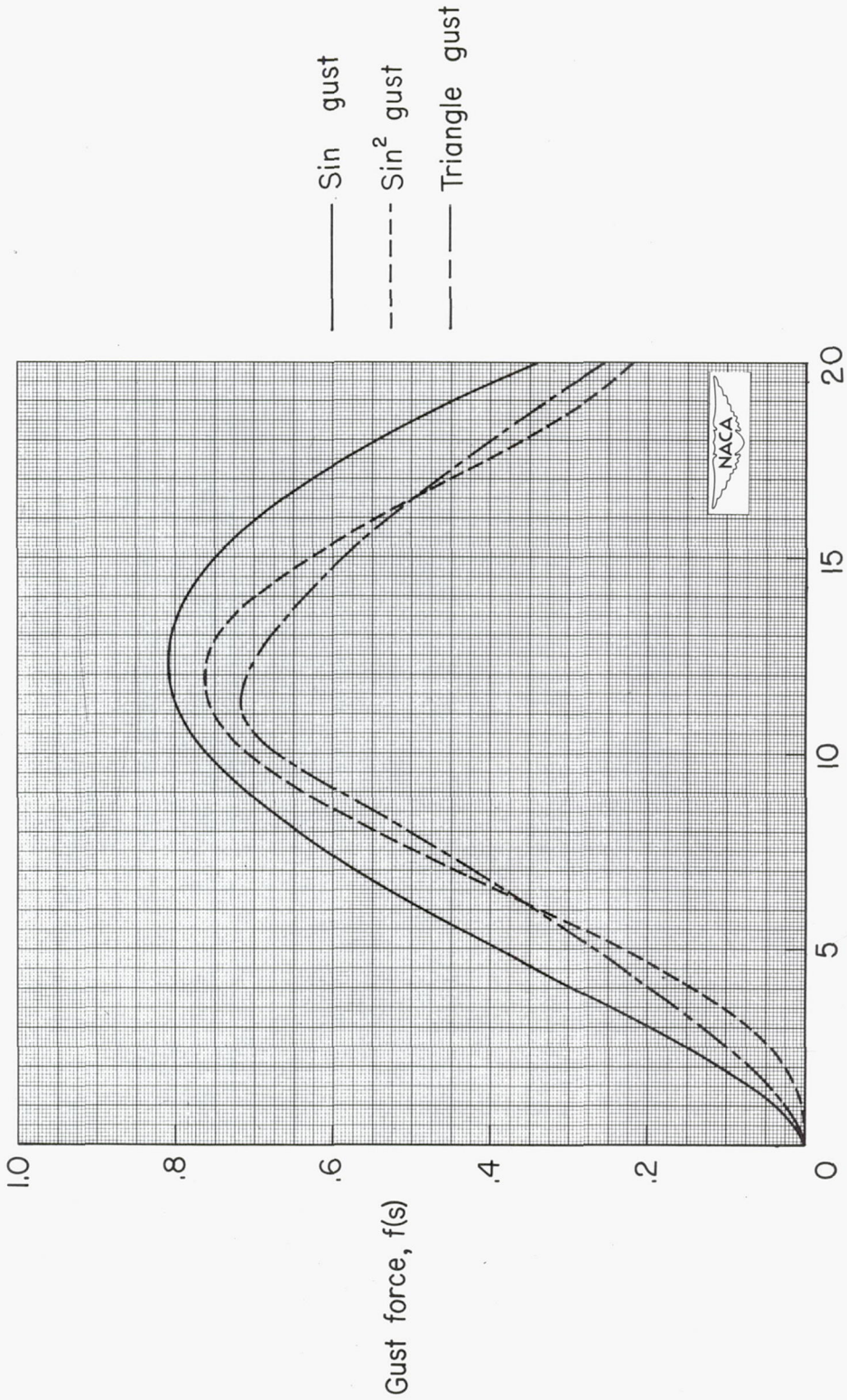



Figure 1.- Unsteady-lift functions (see equations (9) and (10)) where, for a sharp-edge gust, the gust force $f(s)=\psi(s)$.



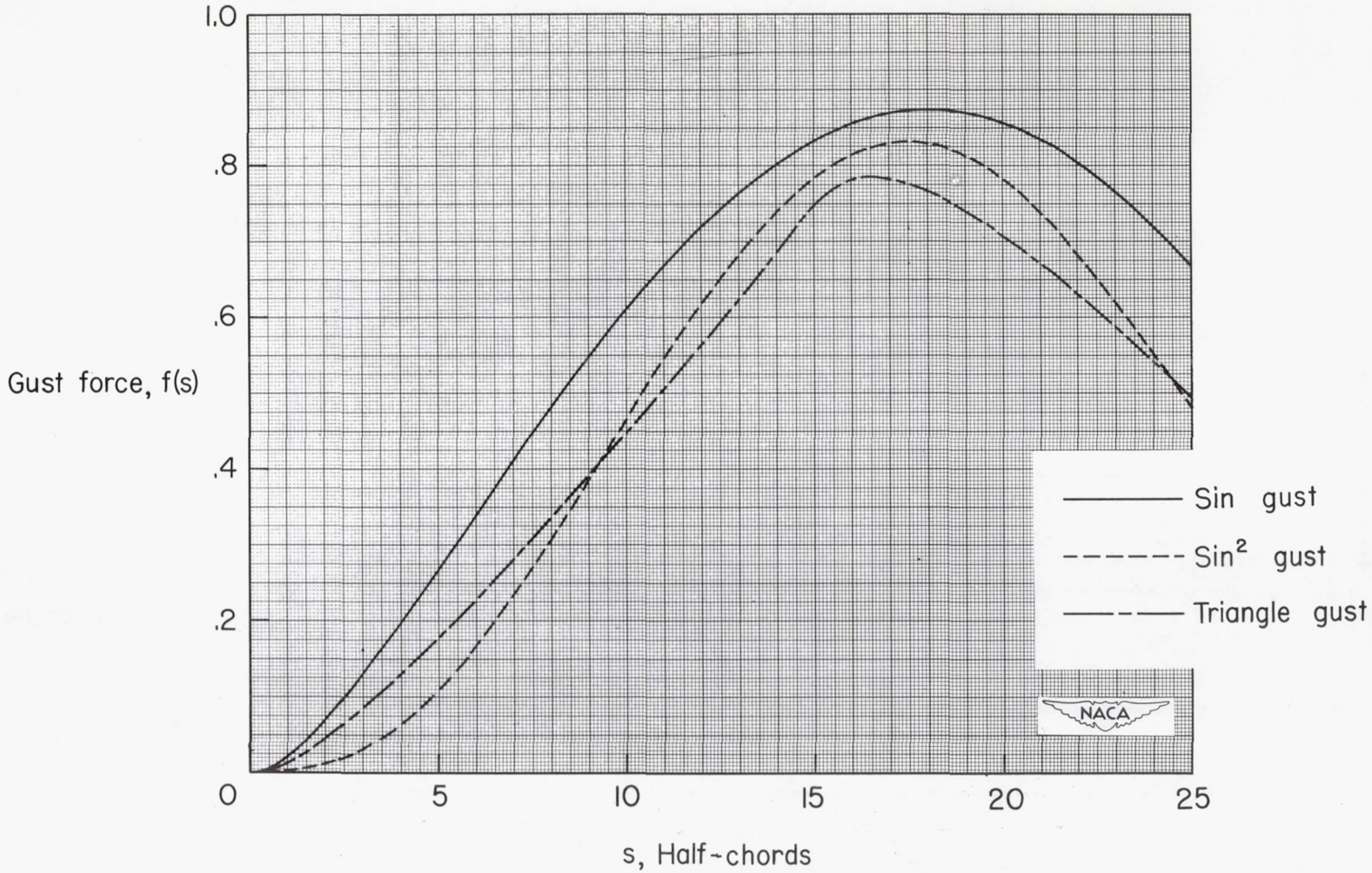

(a) H=2.5 chords.

Figure 2.- Value of the gust-force integral $f(s)=\int_0^s \frac{u'}{U}\Psi(s-\sigma)d\sigma$ for three gust shapes.



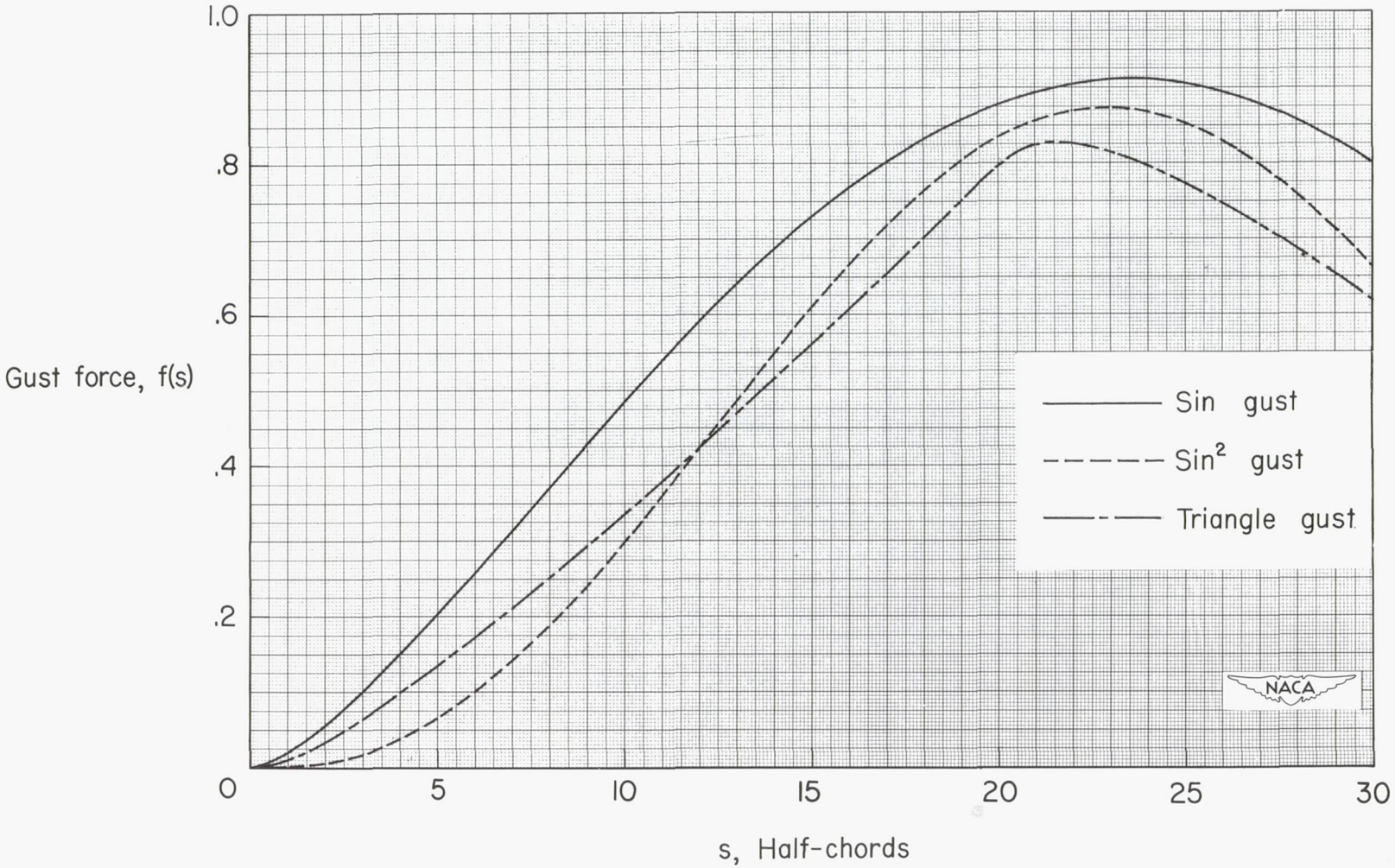

(b) H=5 chords.

Figure 2.- Continued.




(c) H= 7.5 chords.

Figure 2.- Continued.




(d) H=10 chords.

Figure 2.- Concluded.

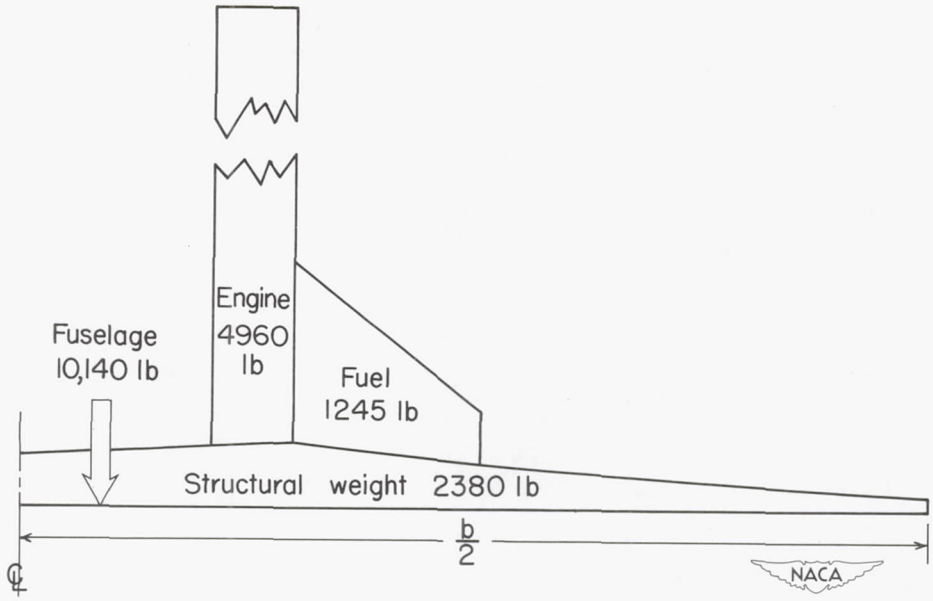



Figure 3.- Semispan weight distribution for the two-engine airplane of example A.

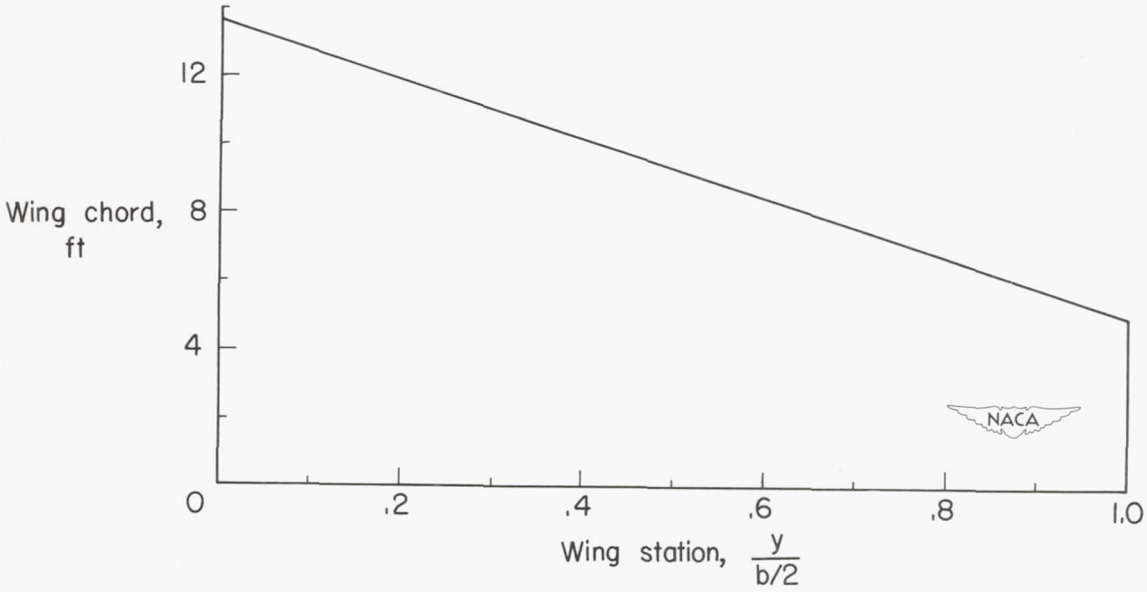



Figure 4.- Wing chord distribution for airplane of example A.

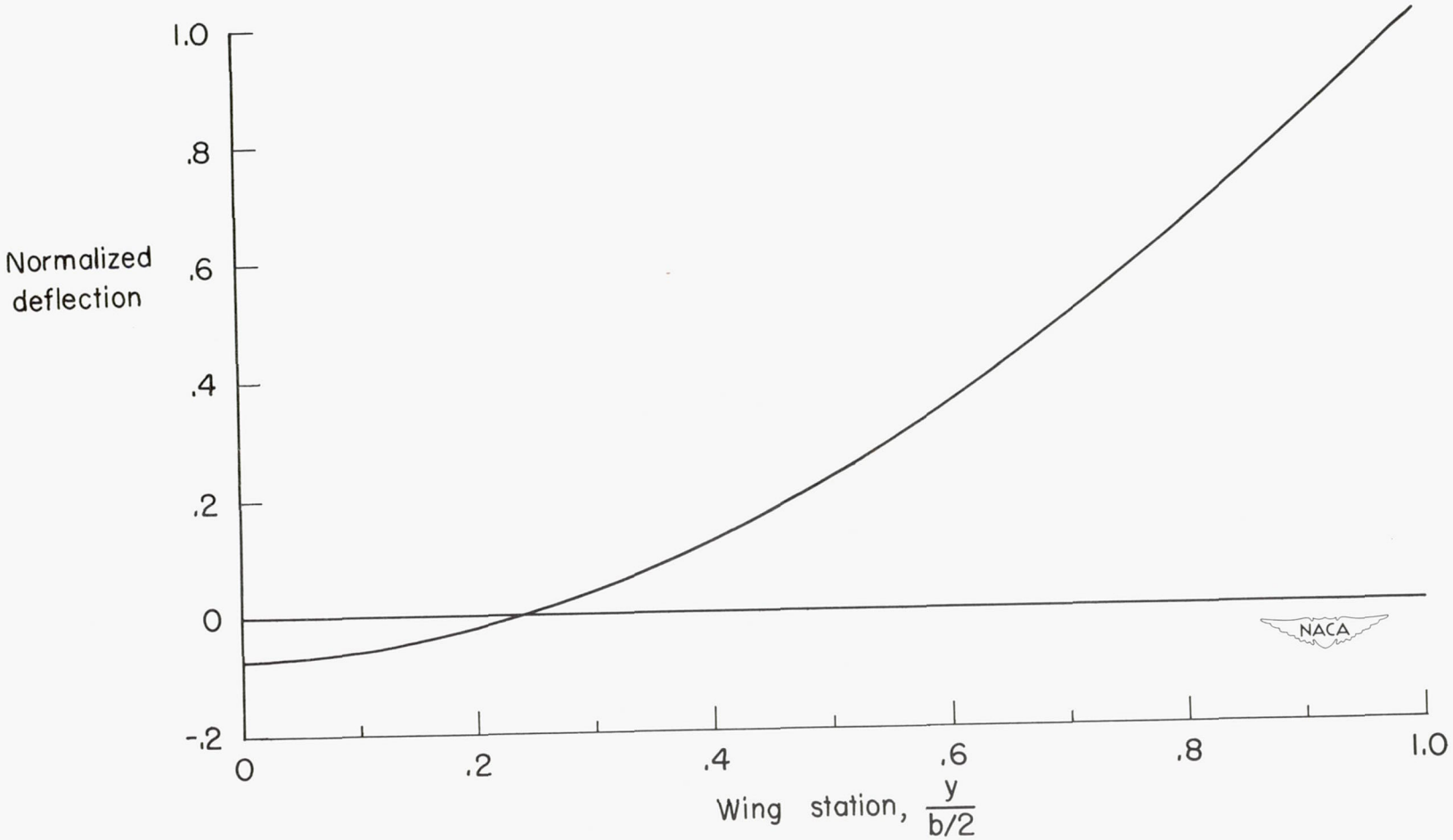



Figure 5.- First-symmetrical-bending-mode deflection curve of example airplane A. $\omega_1 = 20.9$ radians per second.

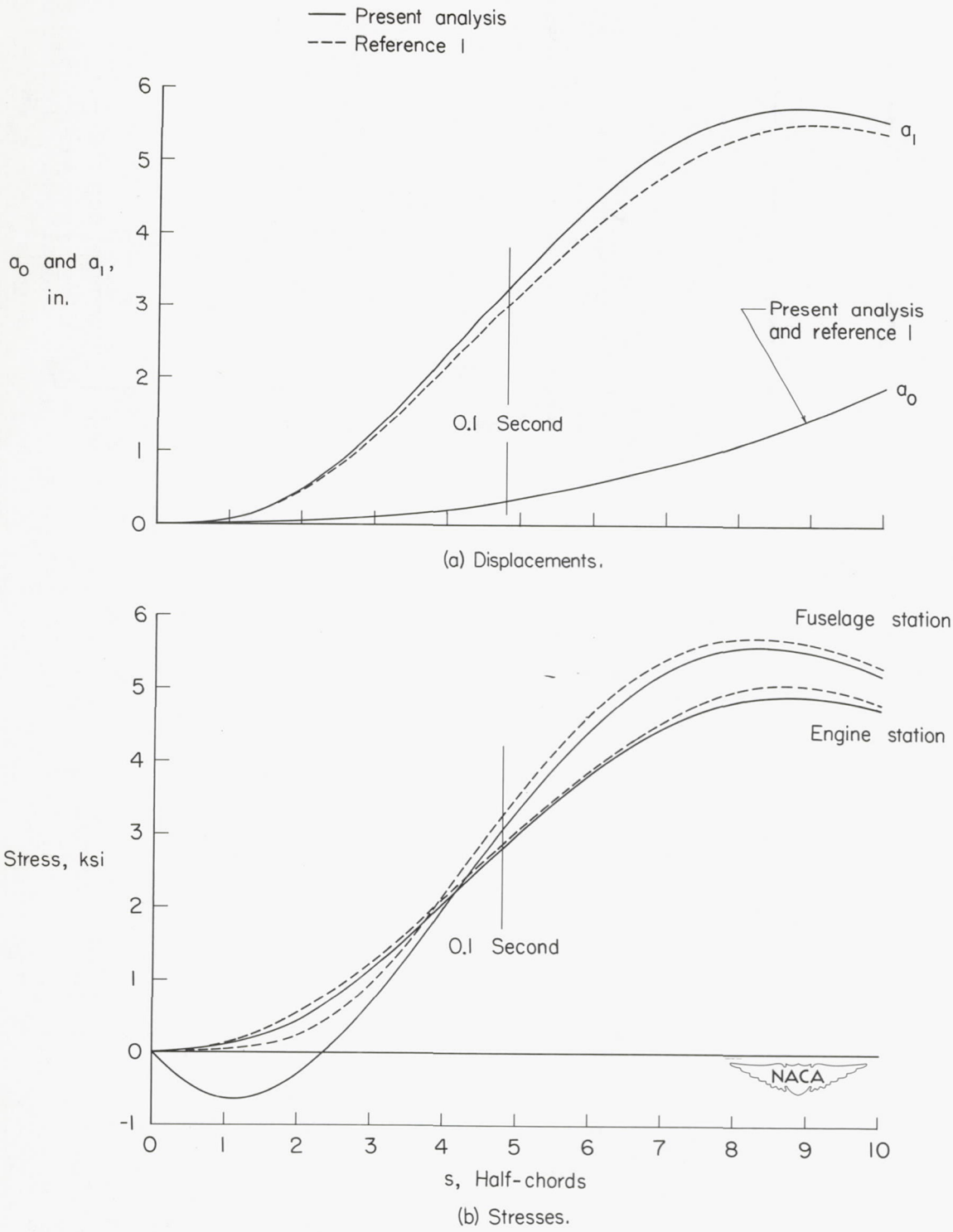



Figure 6.- Response of example airplane A to a 10-foot-per-second sharp-edge gust. V=210 miles per hour.

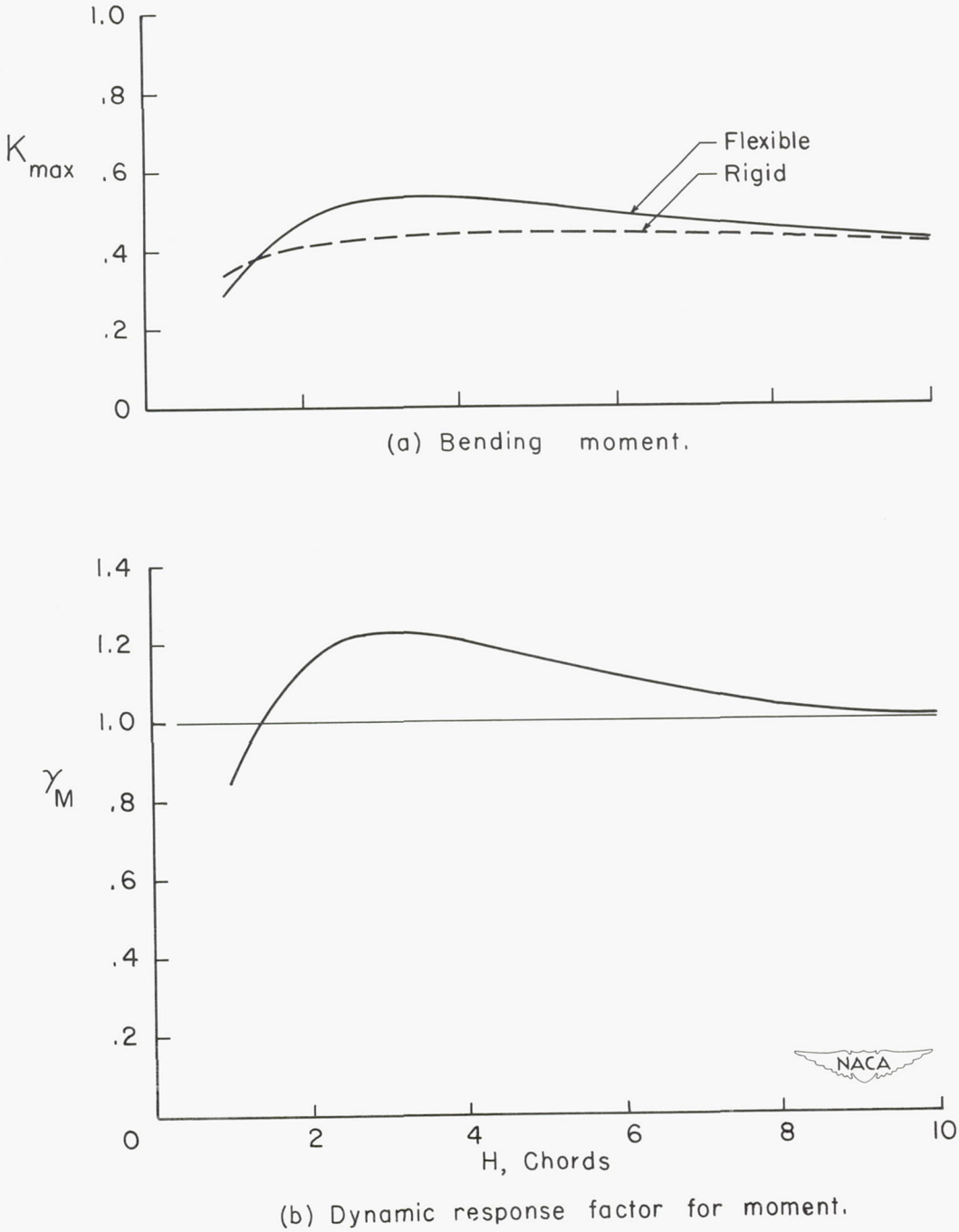



Figure 7.- Bending moment and dynamic response factor for airplane of example B due to flight through sine gusts.