NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2771

THERMAL BUCKLING OF PLATES

By Myron L. Gossard, Paul Seide, and
William M. Roberts

Langley Aeronautical Laboratory
Langley Field, Va.

Washington
August 1952

1X

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2771

THERMAL BUCKLING OF PLATES

By Myron L. Gossard, Paul Seide, and
William M. Roberts

SUMMARY

An approximate method, based on large-deflection plate theory, for calculating the deflections of flat or initially imperfect plates subject to thermal buckling is outlined. The method is used to determine the deflections of a simply supported panel subjected to a tentlike temperature distribution over the plate surface. Experimental results for a particular panel are in good agreement with the theoretical results for the range of temperatures and deflections considered in the test.

INTRODUCTION

In supersonic aircraft, deflections of plate elements out of the plane of the plate may be caused by aerodynamic heating. Deflections may occur without thermal stresses appearing if the plate is unrestrained and if the temperature distribution is linear throughout the volume of the plate. If the plate is restrained or if the temperature distribution is nonlinear, however, thermal stresses are induced. Deflections of the plate occur at the beginning of heating if the temperature varies throughout the plate thickness but do not appear until a critical temperature is reached if the temperature is constant throughout the plate thickness. This last type of distortion is an example of buckling of a flat plate by middle-surface forces that vary throughout the plate and is the subject of the analysis of the present paper.

Since buckling of aerodynamic surfaces may have an adverse effect on aircraft performance, the temperature distributions for which buckling will occur should be known. In many cases it may not be feasible to design a structure which will not buckle and for these cases the magnitude of the distortions should be known in order that their effect on aircraft performance may be estimated.

For initially flat plates, solution of the Von Kármán large-deflection equations for plate buckling, modified to take into account the effect of thermal stresses, is required for the determination of the buckle magnitude. A procedure for the approximate solution of these equations is outlined in the present paper and is illustrated by the determination of the buckle magnitude of a simply supported plate that is subjected to a tentlike temperature distribution. This temperature distribution was chosen because it was easily obtained experimentally. An approximate method for readily extending the results for initially flat plates to plates with initial imperfections is also presented. The effects of plasticity and of variations in material properties due to temperature are excluded from the analysis.

The over-all validity of the assumptions made in the present analysis is tested by experimentally determining the variation with temperature of the deflections of an initially imperfect plate subjected to a tentlike temperature distribution and comparing these experimental results with the theory.

SYMBOLS

a	half-plate length in x-direction
a_{mn}	coefficients in series expansion for plate deflection
b	half-plate width in y-direction
$c_1, c_2, \ldots c_i$	coefficients in stress function F_1
D	plate flexural stiffness, $\dfrac{Et^3}{12(1 - \mu^2)}$
D^*	transformed plate flexural stiffness, $\dfrac{E^* t^{*3}}{12(1 - \mu^2)}$
E	Young's modulus of plate material
E^*	transformed Young's modulus of plate material, $E\left(1 - \dfrac{{w_i}_c^2}{w_c^2}\right)$
F	stress function defining stress distribution in plate, $F = F_0 + F_1$
F_0	stress function defining thermal stresses in unbuckled plate

F_1	stress function defining stresses due to stretching of plate middle surface during bending
t	plate thickness
t^*	transformed plate thickness, $t\Big/\sqrt{1 + \frac{w_{i_c}}{w_c}}$
m,n,p,q	integers
T	temperature distribution in plate
T_0	temperature differential, difference between center and edge temperatures in a tentlike temperature distribution (see fig. 2)
$T_{0_{cr}}$	critical value of T_0
T_1	uniform edge temperature (see fig. 2)
V	potential energy of an initially flat buckled plate
w	plate deflection
w_i	initial deflection
w_{i_c}	initial deflection at plate center
w_c	center deflection
w_s	plate deflection shape given by small-deflection theory
x,y	coordinate axes (see fig. 1)
$\frac{b^2E\alpha T_0t}{\pi^2D}$	temperature-differential parameter
$\frac{b^2E\alpha T_{0_{cr}}t}{\pi^2D}$	critical-temperature-differential parameter
a/b	plate aspect ratio
α	coefficient of thermal expansion

α^* transformed coefficient of thermal expansion, $\dfrac{\alpha}{1 - \dfrac{{w_i}_c^2}{w_c^2}}$

γ_{xy} shear strain in plane of plate

ϵ_x, ϵ_y normal strains in plane of plate in x- and y-directions, respectively

μ Poisson's ratio of plate material

σ_x, σ_y normal stresses in plane of plate in x- and y-directions, respectively, positive for tension

τ_{xy} shear stress in plane of plate

$\sigma_{x_0}, \sigma_{y_0}, \tau_{xy_0}$ thermal stresses in unbuckled plate

∇^2 differential operator, $\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$

∇^4 differential operator, $\dfrac{\partial^4}{\partial x^4} + 2\dfrac{\partial^4}{\partial x^2 \partial y^2} + \dfrac{\partial^4}{\partial y^4}$

ANALYSIS

Statement of Problem

The studies of thermal buckling presented herein are made for the panel shown in figure 1. The panel is heated along the longitudinal center line by a uniform line source of heat and cooled along the edges by two uniform and equal line sinks of heat. This arrangement supplies temperatures in the plate which are constant through the thickness and distributed in a tentlike manner over the faces as shown in figure 2. All edges of the panel are restrained in a direction normal to the plane of the plate by simple rigid supports but are free to slide in the plane of the plate.

The investigation is made in steps corresponding to the stages through which an initially flat plate passes as temperature is gradually increased. First the thermal stresses at temperatures below the critical are determined; next the critical temperature is found; and then the behavior of the plate at temperatures above the critical is obtained. The effects of initial imperfections throughout the entire temperature range are considered in a concluding step.

The procedure is presented in the succeeding sections together with pertinent results of the mathematical study, the details of which are given in appendixes. Specific numerical results are given for a panel with aspect ratio a/b equal to 1.57 which is the aspect ratio of the panel described in the section entitled "Experimental Results."

Thermal Stresses in the Unbuckled Plate

Details of the calculation and experimental verification of the thermal stresses in the unbuckled panel are given in reference 1. The calculations of reference 1 employ the first-order approximation that on any cross section normal to the x-axis the stress σ_x is distributed as shown in figure 3. The thermal-stress function F_0 can then be expressed as the product of a known function of y and an arbitrary function of x. The principle of minimum complementary energy is then used to determine an ordinary linear fourth-order differential equation for the function of x. The resulting approximate expression for F_0 is

$$F_0 = \frac{1}{12} b^2 E\alpha T_0\left(1 - 3\frac{y^2}{b^2} + 2\frac{y^3}{b^3}\right)\left(B_1 \sinh R_1\frac{x}{a} \sin R_2\frac{x}{a} + B_2 \cosh R_1\frac{x}{a} \cos R_2\frac{x}{a} + 1\right) \qquad (b \leqq y \leqq 0) \qquad (1)$$

where B_1, B_2, R_1, and R_2 are defined in appendix A.

The stress distribution in the plate is given by

$$\sigma_{x_0} = \frac{\partial^2 F_0}{\partial y^2} = E\alpha T_0\left(\frac{y}{b} - \frac{1}{2}\right)\left(B_1 \sinh R_1\frac{x}{a} \sin R_2\frac{x}{a} + B_2 \cosh R_1\frac{x}{a} \cos R_2\frac{x}{a} + 1\right) \qquad (2a)$$

$$\sigma_{y_0} = \frac{\partial^2 F_0}{\partial x^2}$$

$$= \frac{1}{12} E\alpha T_0\left(1 - 3\frac{y^2}{b^2} + 2\frac{y^3}{b^3}\right)\left(D_1 \sinh R_1 \frac{x}{a} \sin R_2 \frac{x}{a} +\right.$$

$$\left. D_2 \cosh R_1 \frac{x}{a} \cos R_2 \frac{x}{a}\right) \tag{2b}$$

$$\tau_{xy_0} = -\frac{\partial^2 F_0}{\partial x\, \partial y}$$

$$= \frac{1}{2} E\alpha T_0\left(1 - \frac{y}{b}\right)\frac{y}{b}\left(D_3 \sinh R_1 \frac{x}{a} \cos R_2 \frac{x}{a} + D_4 \cosh R_1 \frac{x}{a} \sin R_2 \frac{x}{a}\right)$$

$$(b \leqq y \leqq 0) \tag{2c}$$

where D_1, D_2, D_3, and D_4 are defined in appendix A. The stresses in the region $-b \leqq y \leqq 0$ are identical with those given by equations (2).

The stresses thus are a function of the temperature differential T_0 for the tentlike temperature distribution and are independent of the edge temperature T_1.

Critical Temperature Differential

As the temperature differential T_0 of the heated plate increases, a value $T_{0_{cr}}$ is reached at which the plate buckles under the action of the induced thermal stresses. If only small deflections of the buckled plate are considered, the assumptions can be made that the middle surface of the plate does not stretch and hence that the stress distribution in the plate does not change after the onset of buckling. The stress distribution then is given by equations (1) and (2). The deflection of the buckled plate is governed by the differential equation (ref. 2)

$$D\nabla^4 w = t\left(\sigma_{x_0}\frac{\partial^2 w}{\partial x^2} + \sigma_{y_0}\frac{\partial^2 w}{\partial y^2} + 2\tau_{xy_0}\frac{\partial^2 w}{\partial x\,\partial y}\right) \tag{3}$$

and the critical temperature differential may be found by methods appropriate for the investigation of the stability of flat plates with internally varying stresses.

One such method is the Rayleigh-Ritz energy method which employs the principle of minimum potential energy (see ref. 2). For the present problem a buckle pattern symmetrical about the center of the plate is chosen as

$$w = \sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty} a_{mn}\cos\frac{m\pi x}{2a}\cos\frac{n\pi y}{2b} \tag{4}$$

which, together with equations (2), is substituted into the potential-energy expression (ref. 2) to yield

$$V = \frac{D}{2}\int_{-b}^{b}\int_{-a}^{a}\left\{\left(\nabla^2 w\right)^2 - 2(1-\mu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\right]\right\}dx\,dy\ +$$

$$\frac{t}{2}\int_{-b}^{b}\int_{-a}^{a}\left[\sigma_{x_0}\left(\frac{\partial w}{\partial x}\right)^2 + \sigma_{y_0}\left(\frac{\partial w}{\partial y}\right)^2 + 2\tau_{xy_0}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]dx\,dy \tag{5}$$

Equation (5) is then minimized with respect to the unknown coefficients a_{mn}. This procedure leads to a set of simultaneous equations which constitute a characteristic-value problem, the solutions of which give sets of relative values of the coefficients a_{mn} and associated values of the critical temperature. The simultaneous equations are

$$\frac{1}{\dfrac{b^2 E\alpha T_{0_{cr}} t}{\pi^2 D}} K_{pq}a_{pq} + \sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty} K_{pqmn}a_{mn} = 0$$

$$(p = 1, 3, 5, \ldots;\ q = 1, 3, 5, \ldots) \tag{6}$$

Equations for the coefficients K_{pq} and K_{pqmn} are given in appendix A.

One method of solving equations (6) is a matrix iteration process which is described in reference 3. The lowest critical temperature coefficient thus obtained for a panel with aspect ratio of 1.57 is

$$\frac{b^2 E\alpha T_{0_{cr}} t}{\pi^2 D} = 5.39 \tag{7}$$

Since E appears in both the numerator and denominator (in D) of the critical-temperature-differential parameter, the critical temperature differential is independent of Young's modulus E.

The approximate buckle pattern associated with this critical temperature is given by the equation

$$w = \frac{w_c}{1.1767}\left(\cos\frac{\pi x}{2a}\cos\frac{\pi y}{2b} + 0.0365\cos\frac{\pi x}{2a}\cos\frac{3\pi y}{2b} + 0.1360\cos\frac{3\pi x}{2a}\cos\frac{\pi y}{2b} + 0.0042\cos\frac{3\pi x}{2a}\cos\frac{3\pi y}{2b}\right) \tag{8}$$

in which w_c is the deflection at the center of the plate and $\frac{a}{b} = 1.57$. A pictorial representation of equation (8) for one quarter of the plate is shown in figure 4.

Post-Buckling Behavior

In order that the post-buckling behavior of the heated plate may be obtained, stretching of the plate middle surface due to bending must be taken into account. Because of this stretching, the stresses in the plate change as the plate deflects and are determined by the equations

$$\nabla^4 F = -E\alpha\nabla^2 T + E\left[\left(\frac{\partial^2 w}{\partial x\, \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right] \tag{9}$$

and

$$\sigma_x = \frac{\partial^2 F}{\partial y^2} \tag{10a}$$

$$\sigma_y = \frac{\partial^2 F}{\partial x^2} \tag{10b}$$

$$\tau_{xy} = -\frac{\partial^2 F}{\partial x \, \partial y} \tag{10c}$$

in conjunction with the condition that the boundaries are stress free. The stress function is also related to the deflections by the equation of equilibrium

$$D\nabla^4 w = t\left(\frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 F}{\partial x \, \partial y}\frac{\partial^2 w}{\partial x \, \partial y}\right) \tag{11}$$

so that equations (9) and (11) must be solved simultaneously.

Equations (9) and (11) are the Von Kármán equations (ref. 2) for large deflections of a plate, modified for the effects of thermal expansion (see appendix B). Exact solutions of these equations are in general difficult to obtain, and approximate methods of solution must be used. A procedure using the Galerkin method (ref. 4) is used in the present paper.

The stress function F is obtained as the sum of two parts

$$F = F_0 + F_1 \tag{12}$$

The stress function F_0 is the thermal-stress function for the unbuckled plate which satisfies the equation

$$\nabla^4 F_0 = -E\alpha\nabla^2 T \tag{13}$$

and the boundary conditions on stresses. This solution is given by equation (1). The stress function F_1 is the solution (satisfying the boundary conditions on stresses) of the equation

$$\nabla^4 F_1 = E\left[\left(\frac{\partial^2 w}{\partial x \, \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right] \tag{14}$$

in which w is the buckle pattern given by equation (4). An approximate solution for F_1 is obtained by the Galerkin method as set forth in reference 4. The stress function F_1 is chosen as the sum of a series of functions for which the boundary stresses vanish, suggested in reference 5 as

$$F_1 = (x^2 - a^2)^2(y^2 - b^2)^2(c_1 + c_2x^2 + c_3y^2 + \ldots) \tag{15}$$

The coefficients $c_1, c_2, c_3, \ldots$ are found in terms of the coefficients a_{mn} by the equations

$$\int_{-a}^{a}\int_{-b}^{b} \frac{\partial F_1}{\partial c_i}\left\{\nabla^4 F_1 - E\left[\left(\frac{\partial^2 w}{\partial x\, \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]\right\} dx\, dy = 0$$

$$(i = 1, 2, 3, \ldots) \tag{16}$$

The resulting stress function F is substituted into equation (11). The Galerkin method may again be used to determine the values of the coefficients a_{mn} of the deflection function w. A set of simultaneous equations is obtained

$$\int_{-a}^{a}\int_{-b}^{b} \cos\frac{m\pi x}{2a} \cos\frac{n\pi y}{2b}\left(\frac{D}{t}\nabla^4 w - \frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + \right.$$

$$\left. 2\frac{\partial^2 F}{\partial x\, \partial y}\frac{\partial^2 w}{\partial x\, \partial y}\right) dx\, dy = 0 \qquad \left(\begin{array}{l} m = 1, 3, 5, \ldots; \\ n = 1, 3, 5. \ldots \end{array}\right) \tag{17}$$

which can be solved for values of the coefficients a_{mn}.

Equations (17) are nonlinear and their solution becomes difficult if many terms are retained in the deflection function. Experience has shown that very good results may be obtained if the shape of the deflected surface of the plate for large deflections is taken as the one existing at the critical temperature - that is, only the coefficient a_{11} is left arbitrary and the ratios a_{mn}/a_{11} are assumed to be those given by the small-deflection solution previously described. The Galerkin equation from which the coefficient a_{11} can be determined then becomes

$$\int_{-a}^{a}\int_{-b}^{b} w_S\left(\frac{D}{t}\nabla^4 w_S - \frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w_S}{\partial x^2} - \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w_S}{\partial y^2} + 2\frac{\partial^2 F}{\partial x\,\partial y}\frac{\partial^2 w_S}{\partial x\,\partial y}\right)dx\,dy = 0 \quad (18)$$

where

$$w_S = \sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty}\frac{a_{mn}}{a_{11}}\cos\frac{m\pi x}{2a}\cos\frac{n\pi y}{2b} \quad (19)$$

and the values of a_{mn}/a_{11} obtained from the small-deflection solution are also substituted into the stress function F. This procedure yields a relationship between the coefficient a_{11} and the temperature differential T_0. For a plate of aspect ratio a/b equal to 1.57, the relationship found in this manner is shown in appendix C to be

$$\frac{b^2 E\alpha T_0 t}{\pi^2 D} = 5.39 + 1.12(1 - \mu^2)\frac{w_c^2}{t^2} \quad (20)$$

where w_c is the plate-center deflection and is equal to $1.1767a_{11}$. It can be seen that the plate deflections are independent of Young's modulus E, since E appears in both the numerator and denominator of the temperature-differential parameter.

Effect of Initial Imperfections

Since actual plates are not usually flat, initial imperfections should be taken into account in the previously developed analysis. The analysis of the thermal buckling of initially imperfect plates involves the solution of the Von Kármán large-deflection equations for initially imperfect plates modified for the effects of nonuniform temperature distributions (appendix B):

$$\nabla^4 F = -E\alpha\nabla^2 T + E\left[\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w_i}{\partial x\,\partial y}\right)^2 + \frac{\partial^2 w_i}{\partial x^2}\frac{\partial^2 w_i}{\partial y^2}\right] \quad (21)$$

$$\frac{D}{t}\nabla^4(w - w_i) = \frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 F}{\partial x\,\partial y}\frac{\partial^2 w}{\partial x\,\partial y} \tag{22}$$

The solution of these equations could be effected in much the same manner as was done for the equations for flat plates. The outlined process, however, is tedious, and an approximate method for assessing the effect of initial imperfections would be more advantageous. Such a method is developed in appendix D.

In order to analyze an initially imperfect plate, a flat plate having the same aspect ratio is first analyzed, the quantities E, α, and t being left arbitrary. Then everywhere in the resulting expressions, E is replaced by $E\left(1 - \frac{{w_{i_c}}^2}{{w_c}^2}\right)$, α by $\frac{\alpha}{1 - \frac{{w_{i_c}}^2}{{w_c}^2}}$, and t by $\frac{t}{\sqrt{1 + \frac{w_{i_c}}{w_c}}}$. This relatively simple procedure yields the stresses and deflections of an initially imperfect plate.

For the problem of thermal buckling of a flat, simply supported plate of aspect ratio 1.57, subjected to a tentlike temperature distribution, the relationship between the temperature differential T_0 and the center deflection of the plate w_c has been found to be

$$\frac{b^2 E\alpha T_0 t}{\pi^2 D} = 5.39 + 1.12(1 - \mu^2)\frac{{w_c}^2}{t^2} \tag{23}$$

When α and t (note that t also appears in D) are replaced by $\frac{\alpha}{1 - \frac{{w_{i_c}}^2}{{w_c}^2}}$ and $\frac{t}{\sqrt{1 + \frac{w_{i_c}}{w_c}}}$ (no substitution need be made for E since the relationship is independent of E) the resulting equation may be written as

$$\frac{b^2 E\alpha T_0 t}{\pi^2 D} = 5.39\left(1 - \frac{w_{i_c}}{w_c}\right) + 1.12(1 - \mu^2)\frac{{w_c}^2 - {w_{i_c}}^2}{t^2} \tag{24}$$

which is the relationship between the temperature differential and the plate-center deflection for an initially imperfect plate.

This method of analysis may be expected to be fairly accurate for plates for which the initial deflected shape is similar to the shape of the first buckling mode of the corresponding flat plate. If the first buckling mode does not predominate in the initial deflected shape, no recourse appears possible except to solve equations (21) and (22) for the large deflections of an initially deflected plate.

EXPERIMENTAL RESULTS

In order to check the applicability of the theoretical analysis to actual plates, the deflections of a simply supported rectangular plate subjected to a tentlike temperature distribution were obtained experimentally and were compared with the theoretical results. The panel was tested under conditions which gave virtually complete boundary freedom in the plane of the plate and simple support.

A plate (see fig. 5) having an over-all length of 36 inches, an over-all width of 24 inches, and a thickness of 0.25 inch was used; this plate is similar to the panel of reference 1. The length and width between simple supports were 35.25 inches and 22.50 inches, respectively, which correspond to the plate aspect ratio of 1.57 used in the numerical calculations previously reported. The coefficient of thermal expansion α and Poisson's ratio μ for the 75S-T6 aluminum-alloy plate material were $0.127 \times 10^{-4} \frac{\text{in.}}{\text{in.-}^{\circ}\text{F}}$ and 0.33, respectively, for the range of temperatures of the test. The plate was initially imperfect and had a center deflection of 0.045 inch.

The arrangement and operation of the heat source and sinks are the same as for the panel of reference 1. Deflections of the plate along the longitudinal and transverse center lines of the sheet were measured by dial gages. These deflections are the difference between the total deflections and the initial deflections of the plate.

Figure 6 shows the experimental and theoretical plate-center deflections $w_c - w_{i_c}$ plotted as a function of the temperature differential T_0. The theoretical curve is given by equation (24) with μ, α, t, b, and w_{i_c} replaced by their respective values. With

$$\alpha = 0.127 \times 10^{-4} \frac{\text{in.}}{\text{in.-}^{\circ}\text{F}}$$

$$t = 0.25 \text{ in.}$$

$$b = 11.25 \text{ in.}$$

$$w_{1_c} = 0.045 \text{ in.}$$

and

$$\mu = 0.33$$

equation (24) becomes

$$T_0 = 194.1\left(1 - \frac{0.045}{w_c}\right) + 573.6\left[w_c^2 - (0.045)^2\right] \tag{25}$$

where T_0 is in degrees Fahrenheit and w_c is in inches. Figure 6 shows that good agreement exists between theory and experiment for the range of temperatures and deflections considered in the test.

Also shown in figure 6 is the theoretical variation of center deflection with temperature for an initially flat plate ($w_{1_c} = 0$). The additional deflection caused by initial imperfections of the plate can be seen to be appreciable.

The deflections measured along the longitudinal and transverse center lines of the plate for various values of the temperature differential T_0 are shown in figure 7. The theoretical deflections calculated for the same values of T_0 are also shown in the figure. Although the measured and calculated initial deflection shapes ($T_0 = 0^\circ$ F) are not in very good agreement, the measured and calculated deflection shapes tend to come into closer agreement as the temperature differential becomes larger and thus tend to fulfill the assumption of the theoretical analysis that the plate deflections have the same shape as the shape of the first mode of buckling of the corresponding flat plate.

CONCLUDING REMARKS

Good agreement exists between experimentally determined center deflections of an initially imperfect panel subjected to a tentlike temperature distribution and the center deflections calculated from an approximate solution of the large-deflection equations for an initially imperfect plate. Other cases of thermal bending or buckling of plates can very likely be calculated with good accuracy by the methods developed in the present paper if the temperature distribution and initial imperfections of the plate are known.

Langley Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Field, Va., May 12, 1952

APPENDIX A

EQUATIONS FOR CRITICAL TEMPERATURE AND BUCKLE PATTERN

The equations for the critical temperature and buckle pattern for symmetrical buckling of the panel subjected to the particular temperature distribution considered herein are as follows (eqs. (6)):

$$\frac{1}{\dfrac{b^2 E\alpha T_{0_{cr}} t}{\pi^2 D}} K_{pq} a_{pq} + \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} K_{pqmn} a_{mn} = 0$$

$$\begin{aligned}(p &= 1, 3, 5, \ldots; \\ q &= 1, 3, 5, \ldots)\end{aligned} \qquad \text{(A1)}$$

The coefficients K_{pq} and K_{pqmn} are defined as follows:

$$K_{pq} = \left[\frac{1}{4}\left(p^2 \frac{b}{a} + q^2 \frac{a}{b}\right)\right]^2 \qquad \text{(A2)}$$

$$K_{pqmn} = p\left[mA_{nq}\left(B_1 D_{mp} + B_2 E_{mp} + F_{mp}\right) + nB_{nq}\left(D_3 G_{mp} + D_4 H_{mp}\right)\right] + q\left[nC_{nq}\left(D_1 I_{mp} + D_2 J_{mp}\right) + mB_{qn}\left(D_3 G_{pm} + D_4 H_{pm}\right)\right] \qquad \text{(A3)}$$

The constants in equation (A3) that apply also in the formulas for direct thermal stress (eqs. (1) and (2)) are given by

$$B_1 = \frac{k_1 \sinh R_1 \cos R_2 - k_2 \cosh R_1 \sin R_2}{k_1 \sin R_2 \cos R_2 + k_2 \sinh R_1 \cosh R_2} \qquad \text{(A4a)}$$

$$B_2 = -\frac{k_1 \cosh R_1 \sin R_2 - k_2 \sinh R_1 \cos R_2}{k_1 \sin R_2 \cos R_2 + k_2 \sinh R_1 \cosh R_1} \qquad \text{(A4b)}$$

$$D_1 = B_1\left(k_1^2 - k_2^2\right) - 2B_2k_1k_2 \tag{A4c}$$

$$D_2 = B_2\left(k_1^2 - k_2^2\right) + 2B_1k_1k_2 \tag{A4d}$$

$$D_3 = B_1k_2 + B_2k_1 \tag{A4e}$$

$$D_4 = B_1k_1 - B_2k_2 \tag{A4f}$$

where

$$R_1 = k_1 \frac{a}{b}$$

$$R_2 = k_2 \frac{a}{b}$$

$$k_1 = \sqrt[4]{\frac{105}{13}}\sqrt{1 + \sqrt{\frac{21}{65}}}$$

$$k_2 = \sqrt[4]{\frac{105}{13}}\sqrt{1 - \sqrt{\frac{21}{65}}}$$

The other coefficients are

$$\left.\begin{aligned}
A_{nq} &= \int_0^b \left(\frac{y}{b} - \frac{1}{2}\right)\cos\frac{n\pi y}{2b}\cos\frac{q\pi y}{2b}\,d\frac{y}{b} \\
A_{nq} &= -\frac{1}{\left(\frac{n - q}{2}\pi\right)^2} \quad \text{if} \quad \left|\frac{n - q}{2}\right| \quad \text{is odd} \\
A_{nq} &= -\frac{1}{\left(\frac{n + q}{2}\pi\right)^2} \quad \text{if} \quad \left|\frac{n - q}{2}\right| \quad \text{is even or zero}
\end{aligned}\right\} \tag{A5a}$$

$$\left.\begin{aligned}
B_{nq} &= \frac{1}{2}\frac{a}{b}\int_0^b \frac{y}{b}\left(1 - \frac{y}{b}\right)\sin\frac{n\pi y}{2b}\cos\frac{q\pi y}{2b}\,d\frac{y}{b} \\
B_{nq} &= \frac{a/b}{\left(\frac{n - q}{2}\pi\right)^3} \quad \text{if} \quad \left|\frac{n - q}{2}\right| \quad \text{is odd} \\
B_{nq} &= \frac{a/b}{\left(\frac{n + q}{2}\pi\right)^3} \quad \text{if} \quad \left|\frac{n - q}{2}\right| \quad \text{is even or zero}
\end{aligned}\right\} \tag{A5b}$$

$$\left.\begin{aligned}
C_{nq} &= \frac{1}{12}\frac{a^2}{b^2}\int_0^b \left[1 - 3\left(\frac{y}{b}\right)^2 + 2\left(\frac{y}{b}\right)^3\right]\sin\frac{n\pi y}{2b}\sin\frac{q\pi y}{2b}\,d\frac{y}{b} \\
C_{nq} &= \frac{a^2}{b^2}\frac{1}{\left(\frac{n - q}{2}\pi\right)^4} \quad \text{if} \quad \left|\frac{n - q}{2}\right| \quad \text{is odd} \\
C_{nq} &= -\frac{a^2}{b^2}\frac{1}{\left(\frac{n + q}{2}\pi\right)^4} \quad \text{if} \quad \left|\frac{n - q}{2}\right| \quad \text{is even} \\
C_{nq} &= \frac{1}{48}\frac{a^2}{b^2} - \frac{a^2}{b^2}\frac{1}{\left(\frac{n + q}{2}\pi\right)^4} \quad \text{if} \quad \left|\frac{n - q}{2}\right| \quad \text{is zero}
\end{aligned}\right\} \tag{A5c}$$

$$D_{mp} = \int_0^a \sinh R_1 \frac{x}{a} \sin R_2 \frac{x}{a} \sin \frac{m\pi x}{2a} \sin \frac{p\pi x}{2a} \, d\frac{x}{a}$$

$$= (-1)^{\frac{m-p}{2}+1} \Big[(C_1 + C_2 + C_3 + C_4)\sinh R_1 \cos R_2 - (C_5 + C_6 + C_7 + C_8)\cosh R_1 \sin R_2\Big] \tag{A5d}$$

$$E_{mp} = \int_0^a \cosh R_1 \frac{x}{a} \cos R_2 \frac{x}{a} \sin \frac{m\pi x}{2a} \sin \frac{p\pi x}{2a} \, d\frac{x}{a}$$

$$= (-1)^{\frac{m-p}{2}} \Big[(C_1 + C_2 + C_3 + C_4)\cosh R_1 \sin R_2 + (C_5 + C_6 + C_7 + C_8)\sinh R_1 \cos R_2\Big] \tag{A5e}$$

$$\left.\begin{aligned} F_{mp} &= \int_0^a \sin \frac{m\pi x}{2a} \sin \frac{p\pi x}{2a} \, d\frac{x}{a} \\ F_{mp} &= 0 \quad \text{if} \quad \left|\frac{m-p}{2}\right| \neq 0 \\ F_{mp} &= \frac{1}{2} \quad \text{if} \quad \left|\frac{m-p}{2}\right| = 0 \end{aligned}\right\} \tag{A5f}$$

$$G_{mp} = \int_0^a \sinh R_1 \frac{x}{a} \cos R_2 \frac{x}{a} \cos \frac{m\pi x}{2a} \sin \frac{p\pi x}{2a} \, d\frac{x}{a}$$

$$= (-1)^{\frac{m-p}{2}} \Big[(C_1 - C_2 + C_3 - C_4)\sinh R_1 \cos R_2 - (C_5 - C_6 + C_7 - C_8)\cosh R_1 \sin R_2\Big] \tag{A5g}$$

$$H_{mp} = \int_0^a \cosh R_1 \frac{x}{a} \sin R_2 \frac{x}{a} \cos \frac{m\pi x}{2a} \cos \frac{p\pi x}{2a}\, d\frac{x}{a}$$

$$= (-1)^{\frac{m-p}{2}} \Big[(C_1 - C_2 + C_3 - C_4)\cosh R_1 \sin R_2 + (C_5 - C_6 + C_7 - C_8)\sinh R_1 \cos R_2\Big] \tag{A5h}$$

$$I_{mp} = \int_0^a \sinh R_1 \frac{x}{a} \sin R_2 \frac{x}{a} \cos \frac{m\pi x}{2a} \cos \frac{p\pi x}{2a}\, d\frac{x}{a}$$

$$= (-1)^{\frac{m-p}{2}} \Big[(C_1 + C_2 - C_3 - C_4)\sinh R_1 \cos R_2 - (C_5 + C_6 - C_7 - C_8)\cosh R_1 \sin R_2\Big] \tag{A5i}$$

$$J_{mp} = \int_0^a \cosh R_1 \frac{x}{a} \cos R_2 \frac{x}{a} \cos \frac{m\pi x}{2a} \cos \frac{p\pi x}{2a}\, d\frac{x}{a}$$

$$= (-1)^{\frac{m-p}{2}+1} \Big[(C_1 + C_2 - C_3 - C_4)\cosh R_1 \sin R_2 + (C_5 + C_6 - C_7 - C_8)\sinh R_1 \cos R_2\Big] \tag{A5j}$$

where

$$C_1 = \frac{1}{4} \frac{R_2 + \frac{m+p}{2}\pi}{R_1^2 + \left(R_2 + \frac{m+p}{2}\pi\right)^2}$$

$$C_2 = \frac{1}{4} \frac{R_2 - \frac{m + p}{2} \pi}{R_1^2 + \left(R_2 - \frac{m + p}{2} \pi\right)^2}$$

$$C_3 = \frac{1}{4} \frac{R_2 + \frac{m - p}{2} \pi}{R_1^2 + \left(R_2 + \frac{m - p}{2} \pi\right)^2}$$

$$C_4 = \frac{1}{4} \frac{R_2 - \frac{m - p}{2} \pi}{R_1^2 + \left(R_2 - \frac{m - p}{2} \pi\right)^2}$$

$$C_5 = \frac{1}{4} \frac{R_1}{R_1^2 + \left(R_2 + \frac{m + p}{2} \pi\right)^2}$$

$$C_6 = \frac{1}{4} \frac{R_1}{R_1^2 + \left(R_2 - \frac{m + p}{2} \pi\right)^2}$$

$$C_7 = \frac{1}{4} \frac{R_1}{R_1^2 + \left(R_2 + \frac{m - p}{2} \pi\right)^2}$$

$$C_8 = \frac{1}{4} \frac{R_1}{R_1^2 + \left(R_2 - \frac{m - p}{2} \pi\right)^2}$$

If only the terms a_{11}, a_{13}, a_{31}, and a_{33} are retained in the deflection function (eq. (4)), equations (A1) may be written in matrix form, for a plate having an aspect ratio of 1.57, as

$$\begin{bmatrix} 15.73 & 22.52 & 14.88 & -5.96 \\ 0.504 & 1.426 & 0.871 & 0.377 \\ 1.35 & 3.54 & 7.42 & 7.69 \\ -0.0735 & 0.208 & 1.043 & 0.437 \end{bmatrix} \begin{Bmatrix} a_{11} \\ a_{13} \\ a_{31} \\ a_{33} \end{Bmatrix} = \frac{100}{\dfrac{b^2 E\alpha T_{0_{cr}} t}{\pi^2 D}} \begin{Bmatrix} a_{11} \\ a_{13} \\ a_{31} \\ a_{33} \end{Bmatrix} \tag{A6}$$

The solution for the largest value of $100\Big/\dfrac{b^2 E\alpha T_{0_{cr}} t}{\pi^2 D}$, and hence for the smallest value of $\dfrac{b^2 E\alpha T_{0_{cr}} t}{\pi^2 D}$, is obtained from matrix iteration (ref. 3) of equation (A6). The critical-temperature parameter so obtained is

$$\frac{b^2 E\alpha T_{0_{cr}} t}{\pi^2 D} = 5.39 \tag{A7}$$

The relative values of the four coefficients retained in the deflection function are found to be

$$\begin{Bmatrix} a_{11} \\ a_{13} \\ a_{31} \\ a_{33} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0.0365 \\ 0.1360 \\ 0.0042 \end{Bmatrix} \tag{A8}$$

in which case the deflection function may be written as

$$w = a_{11}\left(\cos \frac{\pi x}{2a} \cos \frac{\pi y}{2b} + 0.0365 \cos \frac{\pi x}{2a} \cos \frac{3\pi y}{2b} + 0.1360 \cos \frac{3\pi x}{2a} \cos \frac{\pi y}{2b} + 0.0042 \cos \frac{3\pi x}{2a} \cos \frac{3\pi y}{2b}\right) \tag{A9}$$

The deflection at the center of the plate is $1.1767a_{11}$. Let this quantity be denoted by w_c; then

$$w = \frac{w_c}{1.1767}\left(\cos\frac{\pi x}{2a}\cos\frac{\pi y}{2b} + 0.0365\cos\frac{\pi x}{2a}\cos\frac{3\pi y}{2b} + \right.$$

$$\left. 0.1360\cos\frac{3\pi x}{2a}\cos\frac{\pi y}{2b} + 0.0042\cos\frac{3\pi x}{2a}\cos\frac{3\pi y}{2b}\right) \qquad \text{(A10)}$$

which is the form of the deflection function presented as equation (8).

The choice of the particular coefficients a_{11}, a_{13}, a_{31}, and a_{33} depends on the fact that these are the most important coefficients in the series for the deflection w (eq. (4)). The following table shows the convergence of the critical-temperature parameter as more terms are taken in the deflection function:

Terms retained	$\frac{b^2E\alpha T_{0_{cr}}t}{\pi^2D}$
a_{11}	6.35
a_{11}, a_{31}	5.65
a_{11}, a_{31}, a_{13}	5.40
a_{11}, a_{31}, a_{13}, a_{33}	5.39

The retention of terms other than the four chosen has a negligible effect on the critical temperature and on the buckle pattern.

APPENDIX B

MODIFICATION OF VON KÁRMÁN LARGE-DEFLECTION EQUATIONS

FOR EFFECTS OF THERMAL EXPANSION

The condition of compatibility of strains for a deflected plate with initial imperfections is represented by the following equation (see refs. 2 and 6):

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_x}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x\, \partial y} = \left(\frac{\partial^2 w}{\partial x\, \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w_i}{\partial x\, \partial y}\right)^2 + \frac{\partial^2 w_i}{\partial x^2}\frac{\partial^2 w_i}{\partial y^2} \qquad \text{(B1)}$$

When heat is applied to a plate, strains ϵ_x, ϵ_y, and γ_{xy} are caused by thermal expansion of the material as well as by stresses that may arise from various sources. The strains are given by

$$\left.\begin{aligned} \epsilon_x &= \frac{1}{E}(\sigma_x - \mu\sigma_y) + \alpha T \\ \epsilon_y &= \frac{1}{E}(\sigma_y - \mu\sigma_x) + \alpha T \\ \gamma_{xy} &= \frac{2(1 + \mu)}{E}\tau_{xy} \end{aligned}\right\} \qquad \text{(B2)}$$

With the introduction of a stress function F defined by

$$\left.\begin{aligned} \sigma_x &= \frac{\partial^2 F}{\partial y^2} \\ \sigma_y &= \frac{\partial^2 F}{\partial x^2} \\ \tau_{xy} &= -\frac{\partial^2 F}{\partial x\, \partial y} \end{aligned}\right\} \qquad \text{(B3)}$$

the strains are given by

$$\left.\begin{aligned}
\epsilon_x &= \frac{1}{E}\left(\frac{\partial^2 F}{\partial y^2} - \mu\,\frac{\partial^2 F}{\partial x^2}\right) + \alpha T \\
\epsilon_y &= \frac{1}{E}\left(\frac{\partial^2 F}{\partial x^2} - \mu\,\frac{\partial^2 F}{\partial y^2}\right) + \alpha T \\
\gamma_{xy} &= -\,\frac{2(1 + \mu)}{E}\,\frac{\partial^2 F}{\partial x\,\partial y}
\end{aligned}\right\} \tag{B4}$$

Substitution of equations (B4) into equation (B1) gives the compatibility condition in the form

$$\nabla^4 F = -E\alpha\nabla^2 T + E\left[\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\,\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w_i}{\partial x\,\partial y}\right)^2 + \frac{\partial^2 w_i}{\partial x^2}\,\frac{\partial^2 w_i}{\partial y^2}\right] \tag{B5}$$

This is the first of the Von Kármán equations, modified for effects of thermal expansion. The stresses derived from the stress function F satisfy equilibrium in the plane of the plate because F, as defined by equations (B3), identically satisfies the equilibrium equations

$$\left.\begin{aligned}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} &= 0 \\
\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} &= 0
\end{aligned}\right\} \tag{B6}$$

The second Von Kármán equation, the equation of equilibrium of forces normal to the plate middle surface, remains unchanged as

$$D\nabla^4(w - w_i) = t\left(\frac{\partial^2 F}{\partial y^2}\,\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 F}{\partial x^2}\,\frac{\partial^2 w}{\partial y^2} - 2\,\frac{\partial^2 F}{\partial x\,\partial y}\,\frac{\partial^2 w}{\partial x\,\partial y}\right) \tag{B7}$$

APPENDIX C

APPROXIMATE SOLUTION OF THE VON KÁRMÁN LARGE-DEFLECTION EQUATIONS

Determination of the Stress Distribution in the Plate

The first Von Kármán equation for flat plates relates the stress distribution in the plate to the plate deflection w as follows:

$$\nabla^4 F = -E\alpha\nabla^2 T + E\left[\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right] \tag{C1}$$

The stress function F can be separated into two parts, the primary stress function F_0, which satisfies the equation

$$\nabla^4 F_0 = -E\alpha\nabla^2 T \tag{C2}$$

and is given by equation (1) for the present problem, and a stress function F_1, which satisfies the equation

$$\nabla^4 F_1 = E\left[\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right] \tag{C3}$$

and the given condition of stress-free edges. For the buckle pattern given by equation (4), equation (C3) becomes

$$\nabla^4 F_1 = E\sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty}\sum_{p=1,3,5}^{\infty}\sum_{q=1,3,5}^{\infty}\frac{m\pi}{2a}\frac{q\pi}{2b}a_{mn}a_{pq}\left(\frac{p\pi}{2a}\frac{n\pi}{2b}\sin\frac{m\pi x}{2a}\sin\frac{p\pi x}{2a}\sin\frac{n\pi y}{2b}\sin\frac{q\pi y}{2b} - \frac{n\pi}{2a}\frac{q\pi}{2b}\cos\frac{m\pi x}{2a}\cos\frac{p\pi x}{2a}\cos\frac{n\pi y}{2b}\cos\frac{q\pi y}{2b}\right) \tag{C4}$$

Equation (C4) may be solved approximately by the Galerkin method. A solution for F_1 is taken as

$$F_1 = \sum_{i=1}^{\infty} c_i f_i \tag{C5}$$

Each function f_i is a function of x and y that yields stresses which satisfy the given boundary conditions. The arbitrary coefficients c_i are then obtained from the equations

$$\sum_{i=1}^{\infty} c_i \int_{-a}^{a} \int_{-b}^{b} f_s \nabla^4 f_i \, dx \, dy = \int_{-a}^{a} \int_{-b}^{b} f_s G(x,y) \, dx \, dy \qquad (s = 1, 2, 3, \ldots) \tag{C6}$$

where $G(x,y)$ denotes the terms on the right side of equation (C4).

For the present problem the boundary conditions to be satisfied are those of stress-free edges of the plate. Furthermore, the stress distribution is symmetrical about the x- and y-axes. A series for F_1 that satisfies these conditions is suggested in reference 5 as

$$F_1 = (x^2 - a^2)^2 (y^2 - b^2)^2 (c_1 + c_2 x^2 + c_3 y^2 + \ldots) \tag{C7}$$

In the present solution only the first three terms of the series are considered.

The substitution of equation (C7) into equations (C6) yields the following equations for the determination of the coefficients c_1, c_2, and c_3:

$$7\left(1 + \frac{4}{7}\frac{a^2}{b^2} + \frac{a^4}{b^4}\right)a^2 b^6 c_1 + \left(1 + \frac{7}{11}\frac{a^4}{b^4}\right)a^4 b^6 c_2 + \left(\frac{7}{11} + \frac{a^4}{b^4}\right)a^2 b^8 c_3 = -\frac{11025}{32768} E\left[\frac{16}{5}\left(a_{11}{}^2 + a_{33}{}^2\right) + \right.$$
$$\frac{656}{45}\left(a_{13}{}^2 + a_{31}{}^2\right) + \left(\frac{12}{5} + \frac{567}{\pi^4}\right)\left(a_{13} + a_{31}\right)a_{11} + \left(\frac{108}{5} + \frac{63}{\pi^4}\right)\left(a_{13} + a_{31}\right)a_{33} - \frac{162}{\pi^4} a_{11} a_{33} +$$
$$\left. \frac{1863}{\pi^4} a_{13} a_{31} + \ldots\right] \tag{C8a}$$

$$\left(1 + \frac{7}{11}\frac{a^4}{b^4}\right)a^2b^6c_1 + 3\left(1 + \frac{4}{33}\frac{a^2}{b^2} + \frac{7}{143}\frac{a^4}{b^4}\right)a^4b^6c_2 + \frac{1}{11}\left(1 + \frac{a^4}{b^4}\right)a^2b^8c_3 = -\frac{11025}{32768}E\left[\left(\frac{176}{35} - \frac{48}{\pi^2}\right)a_{11}{}^2 + \right.$$

$$\left(\frac{13616}{315} - \frac{432}{\pi^2}\right)a_{13}{}^2 + \left(\frac{272}{105} - \frac{16}{27\pi^2}\right)a_{31}{}^2 + \left(\frac{176}{35} - \frac{16}{3\pi^2}\right)a_{33}{}^2 + \left(\frac{12}{35} + \frac{1701}{\pi^4} - \frac{17010}{\pi^6}\right)a_{11}a_{13} +$$

$$\left(\frac{36}{5} - \frac{90}{\pi^2} + \frac{1701}{\pi^4} - \frac{34425}{2\pi^6}\right)a_{11}a_{31} - \left(\frac{486}{\pi^4} - \frac{6075}{2\pi^6}\right)a_{11}a_{33} + \left(\frac{5589}{\pi^4} - \frac{60750}{\pi^6}\right)a_{13}a_{31} + \left(\frac{324}{5} - \frac{810}{\pi^2} + \right.$$

$$\left.\left.\frac{189}{\pi^4} - \frac{3825}{2\pi^6}\right)a_{13}a_{33} + \left(\frac{108}{35} + \frac{189}{\pi^4} - \frac{210}{\pi^6}\right)a_{31}a_{33} + \ldots\right] \tag{C8b}$$

$$\left(\frac{7}{11} + \frac{a^4}{b^4}\right)a^2b^6c_1 + \frac{1}{11}\left(1 + \frac{a^4}{b^4}\right)a^4b^6c_2 + 3\left(\frac{7}{143} + \frac{4}{33}\frac{a^2}{b^2} + \frac{a^4}{b^4}\right)a^2b^8c_3 = -\frac{11025}{32768}E\left[\left(\frac{176}{35} - \frac{48}{\pi^2}\right)a_{11}{}^2 + \right.$$

$$\left(\frac{272}{105} - \frac{16}{27\pi^2}\right)a_{13}{}^2 + \left(\frac{13616}{315} - \frac{432}{\pi^2}\right)a_{31}{}^2 + \left(\frac{176}{35} - \frac{16}{3\pi^2}\right)a_{33}{}^2 + \left(\frac{36}{5} - \frac{90}{\pi^2} + \frac{1701}{\pi^4} - \frac{34425}{2\pi^6}\right)a_{11}a_{13} +$$

$$\left(\frac{12}{35} + \frac{1701}{\pi^4} - \frac{17010}{\pi^6}\right)a_{11}a_{31} - \left(\frac{486}{\pi^4} - \frac{6075}{2\pi^6}\right)a_{11}a_{33} + \left(\frac{5589}{\pi^4} - \frac{60750}{\pi^6}\right)a_{13}a_{31} +$$

$$\left.\left(\frac{108}{35} + \frac{189}{\pi^4} - \frac{210}{\pi^6}\right)a_{13}a_{33} + \left(\frac{324}{5} - \frac{810}{\pi^2} + \frac{189}{\pi^4} - \frac{3825}{2\pi^6}\right)a_{31}a_{33} + \ldots\right] \tag{C8c}$$

The assumption is now made that the buckled shape of the plate does not change as the deflections increase, in which case relative values of the coefficients a_{mn} for large deflections are the same as those at the critical temperature. For the plate of aspect ratio a/b equal to 1.57, these relative values are given by equation (A8). Only the coefficients a_{11}, a_{13}, a_{31}, and a_{33} are considered. The solutions of equations (C8) are then

$$\left.\begin{aligned} c_1 &= -0.01337 \frac{E{a_{11}}^2}{b^8} \\ c_2 &= 0.00685 \frac{E{a_{11}}^2}{b^{10}} \\ c_3 &= 0.00389 \frac{E{a_{11}}^2}{b^{10}} \end{aligned}\right\} \tag{C9}$$

The stress function for stresses due to stretching of the middle surface of a plate with aspect ratio 1.57 is therefore given approximately by

$$F_1 = E\left(\frac{w_c}{1.1767}\right)^2\left[\frac{x^2}{b^2} - (1.57)^2\right]^2\left(\frac{y^2}{b^2} - 1\right)^2\left(-0.01337 + 0.00685 \frac{x^2}{b^2} + 0.00389 \frac{y^2}{b^2}\right) \qquad \left(-1.57 \leqq \frac{x}{b} \leqq 1.57;\ -1 \leqq \frac{y}{b} \leqq 1\right) \tag{C10}$$

where w_c is the plate-center deflection and is equal to $1.1767a_{11}$. The stress function F is given by the sum of equations (1) and (C10).

Determination of Relationship Between Temperature and Center Deflection

With the stress distribution in the plate known as a function of the plate-center deflection w_c, the second Von Kármán large-deflection equation for flat plates can be solved to determine the relationship between temperature and deflection for temperatures above the critical. Equation (B7) can be satisfied only on an average over the plate when the buckle pattern for temperatures above critical is assumed to have the same shape as at the critical temperature. Use of the Galerkin method (ref. 4) for this averaging process gives the following equation governing the relation between temperature and center deflection:

$$\int_{-a}^{a}\int_{-b}^{b} w_S\left(\frac{D}{t}\nabla^4 w_S - \frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w_S}{\partial x^2} - \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w_S}{\partial y^2} + 2\frac{\partial^2 F}{\partial x\,\partial y}\frac{\partial^2 w_S}{\partial x\,\partial y}\right)dx\,dy = 0 \qquad \text{(C11)}$$

where

$$w_S = \sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty}\frac{a_{mn}}{a_{11}}\cos\frac{m\pi x}{2a}\cos\frac{n\pi y}{2b} \qquad \text{(C12)}$$

and the values of a_{mn}/a_{11} are given by the small-deflection solution of appendix A.

For the plate of aspect ratio a/b equal to 1.57,

$$w_S = \cos\frac{\pi x}{2a}\cos\frac{\pi y}{2b} + 0.0365\cos\frac{\pi x}{2a}\cos\frac{3\pi y}{2b} + 0.1360\cos\frac{3\pi x}{2a}\cos\frac{\pi y}{2b} +$$
$$0.0042\cos\frac{3\pi x}{2a}\cos\frac{3\pi y}{2b} \qquad \text{(C13)}$$

and the stress function F is given by the sum of equations (1) and (C10). Equation (C11) then yields the following relationship between the temperature T_0 and the center deflection w_c:

$$\frac{b^2 E\alpha T_0 t}{\pi^2 D} = 5.39 + 1.12(1 - \mu^2)\frac{{w_c}^2}{t^2} \qquad \text{(C14)}$$

The deflection at any other point of the plate with stress-free edges and of aspect ratio a/b equal to 1.57 is given by

$$\frac{w}{t} = 0.723\sqrt{\frac{\frac{b^2 E\alpha T_0 t}{\pi^2 D} - 5.39}{1.12(1 - \mu^2)}}\, w_S \tag{C15}$$

where w_S is given by equation (C13).

APPENDIX D

CORRECTION FOR INITIAL IMPERFECTIONS

The large-deflection equations for bending of an initially imperfect plate subjected to thermal stresses are derived in appendix B as

$$\nabla^4 F = -E\alpha\nabla^2 T + E\left[\left(\frac{\partial^2 w}{\partial x\ \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w_i}{\partial x\ \partial y}\right)^2 + \frac{\partial^2 w_i}{\partial x^2}\frac{\partial^2 w_i}{\partial y^2}\right] \tag{D1a}$$

$$\frac{D}{t}\nabla^4(w - w_i) = \frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 F}{\partial x\ \partial y}\frac{\partial^2 w}{\partial x\ \partial y} \tag{D1b}$$

The assumption is made that the bending deflections of the plate are merely a magnification of the initial deflections w_i; that is, the deflections w have the same shape as the initial deflections w_i. This assumption may be written as

$$\frac{w_i}{w} = \frac{w_{i_c}}{w_c} \tag{D2}$$

The substitution of equation (D2) into equations (D1) yields

$$\nabla^4 F = -E\alpha\nabla^2 T + E\left(1 - \frac{{w_{i_c}}^2}{{w_c}^2}\right)\left[\left(\frac{\partial^2 w}{\partial x\ \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right] \tag{D3a}$$

$$\frac{D}{t}\left(1 - \frac{w_{i_c}}{w_c}\right)\nabla^4 w = \frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 F}{\partial x\ \partial y}\frac{\partial^2 w}{\partial x\ \partial y} \tag{D3b}$$

If the following substitutions are made:

$$\left.\begin{aligned} E^* &= E\left(1 - \frac{{w_{i_c}}^2}{{w_c}^2}\right) \\ \alpha^* &= \frac{\alpha}{1 - \frac{{w_{i_c}}^2}{{w_c}^2}} \\ t^* &= \frac{t}{\sqrt{1 + \frac{w_{i_c}}{w_c}}} \end{aligned}\right\} \tag{D4}$$

equations (D3) become

$$\nabla^4 F = -E^*\alpha^*\nabla^2 T + E^*\left[\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right] \tag{D5a}$$

$$\frac{D^*}{t^*}\nabla^4 w = \frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 F}{\partial x\,\partial y}\frac{\partial^2 w}{\partial x\,\partial y} \tag{D5b}$$

Equations (D5) are identical with the large-deflection equations for buckling of a flat plate with Young's modulus E^*, coefficient of thermal expansion α^*, and thickness t^*. If the initial deflections w_i are assumed to satisfy the same homogeneous boundary conditions as would be satisfied by the deflections of an initially flat plate, the solution of equations (D5) is identical with the solution for the deflections of a flat plate having the same aspect ratio but with E replaced by E^*, α by α^*, and t by t^*.

As an example of this method of correction, consider the relationship between the temperature and the center deflection for the plate of aspect ratio 1.57

$$\frac{b^2 E\alpha T_0 t}{\pi^2 D} = 5.39 + 1.12(1 - \mu^2)\frac{{w_c}^2}{t^2} \tag{D6}$$

To find the corresponding relationship for a plate with aspect ratio 1.57 but having initial deflections of the same shape as the initial buckling deflections of the flat plate, α is replaced by α^* or $\dfrac{\alpha}{1 - \dfrac{{w_{i_c}}^2}{{w_c}^2}}$ and t (note that t also appears in D) by t^* or $t\bigg/\sqrt{1 + \dfrac{w_{i_c}}{w_c}}$. ($E$ need not be replaced by E^* since the relationship is independent of E.) Equation (D6) then becomes

$$\frac{b^2 E\alpha T_0 t}{\pi^2 D} = 5.39\left(1 - \frac{w_{i_c}}{w_c}\right) + 1.12\left(1 - \mu^2\right)\frac{{w_c}^2 - {w_{i_c}}^2}{t^2} \tag{D7}$$

In order that the validity of the foregoing method of analyzing plates with initial deflections be checked, the method was used to calculate curves of center deflection plotted against average edge compressive stress for the problem considered in reference 6, the bending under edge compressive stress of simply supported square plates with initial imperfections. These curves were in excellent agreement with the numerical results of reference 6 which are obtained from an approximate but accurate solution of the Von Kármán large-deflection equations for initially imperfect plates. The agreement was found to exist for all cases in which the initial imperfection was a half-sine-wave deflection in both directions.

REFERENCES

1. Heldenfels, Richard R., and Roberts, William M.: Experimental and Theoretical Determination of Thermal Stresses in a Flat Plate. NACA TN 2769, 1952.

2. Timoshenko, S.: Theory of Elastic Stability. McGraw-Hill Book Co., Inc., 1936, pp. 305-323.

3. Von Kármán, Theodore, and Biot, Maurice A.: Mathematical Methods in Engineering. First ed., McGraw-Hill Book Co., Inc., 1940, pp. 196-203.

4. Sokolnikoff, I. S.: Mathematical Theory of Elasticity. First ed., McGraw-Hill Book Co., Inc., 1946, pp. 313-316.

5. Timoshenko, S., and Goodier, J. N.: Theory of Elasticity. Second ed., McGraw-Hill Book Co., Inc., 1951, p. 169.

6. Hu, Pai C., Lundquist, Eugene E., and Batdorf, S. B.: Effect of Small Deviations From Flatness on Effective Width and Buckling of Plates in Compression. NACA TN 1124, 1946.

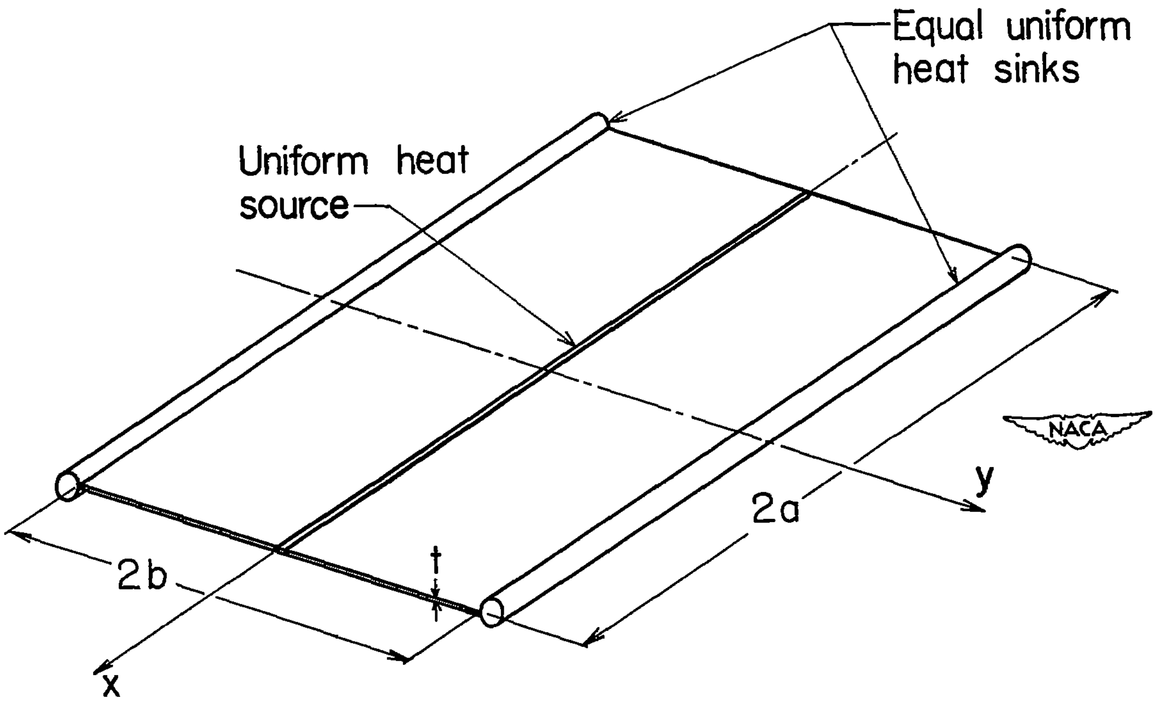


Figure 1.- Thermal-buckling problem treated in present paper.

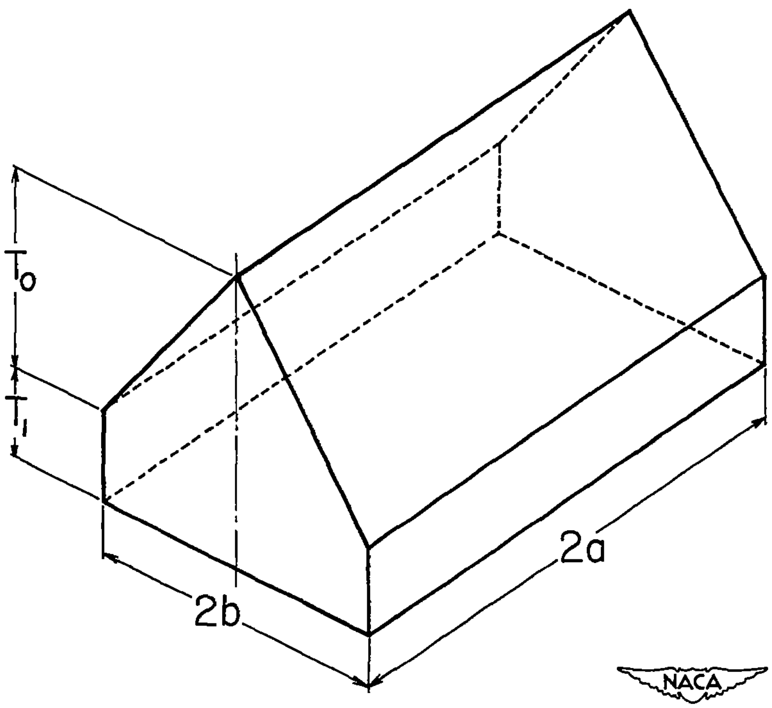


Figure 2.- Tentlike temperature distribution.

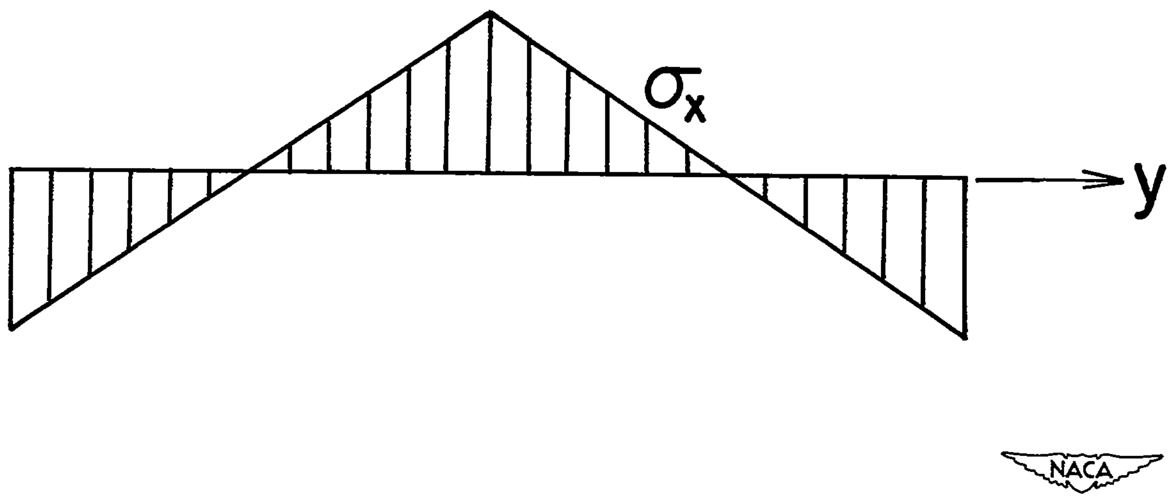


Figure 3.- Assumed variation of primary normal stress σ_x.

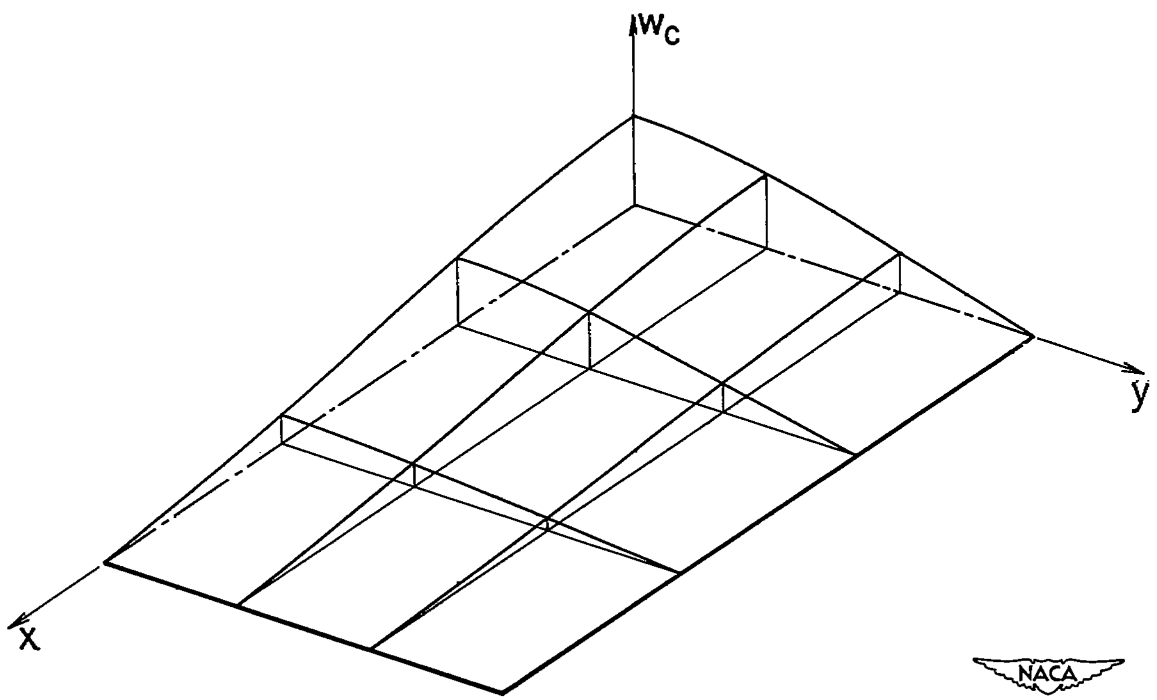


Figure 4.- Small-deflection buckle pattern in one quadrant of a plate of aspect ratio 1.57.

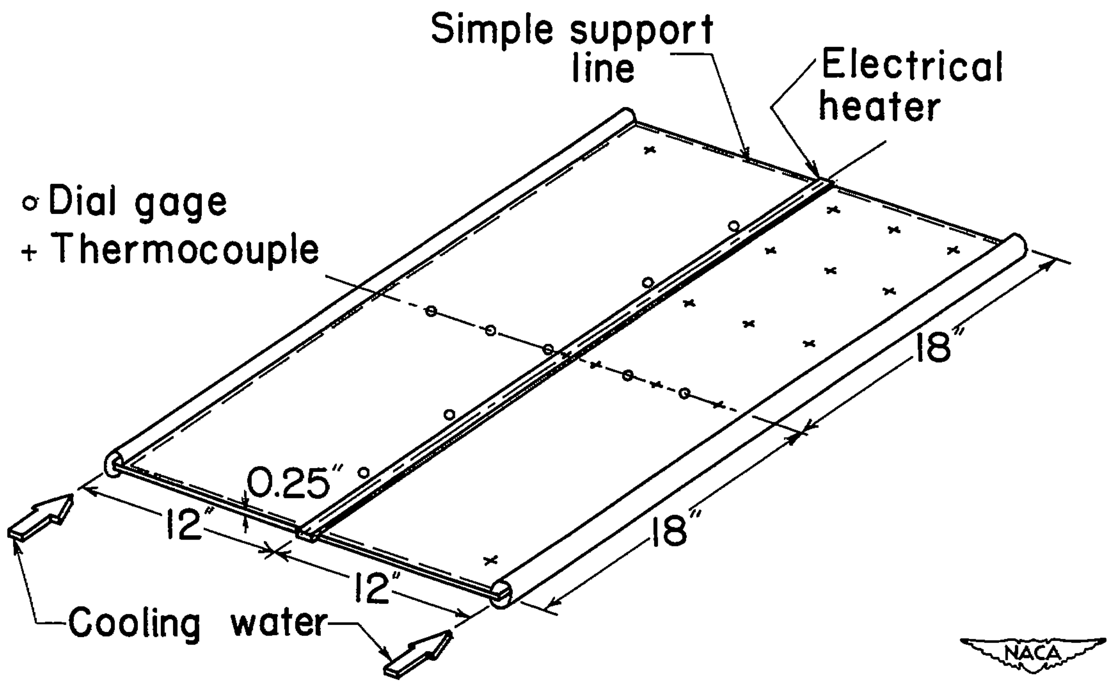


Figure 5.- Location of dial gages and thermocouples on test panel.

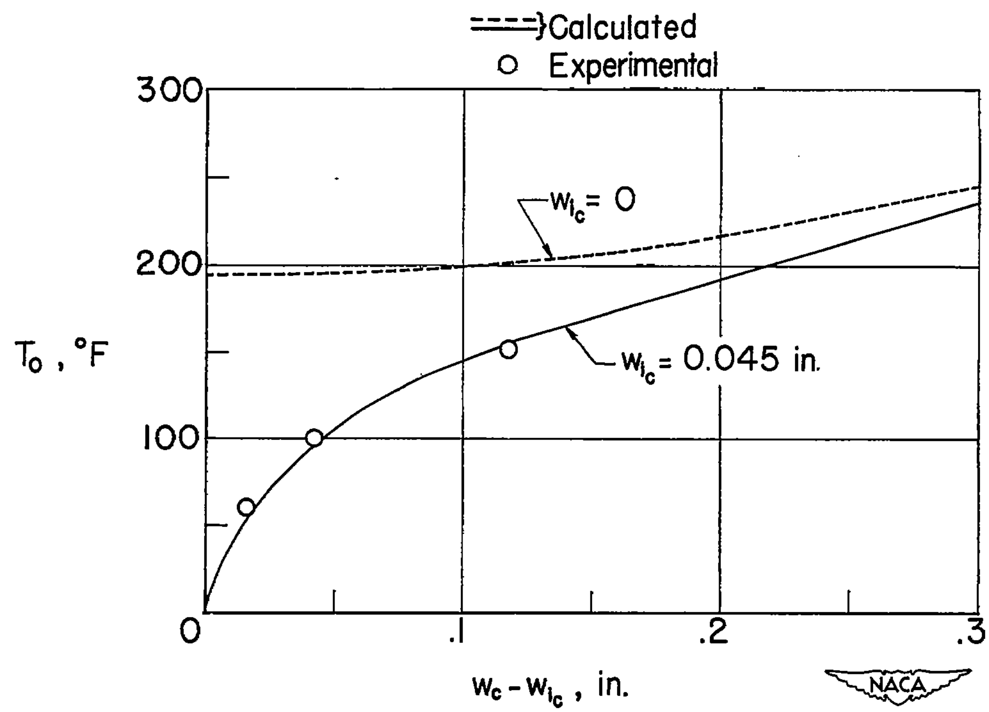
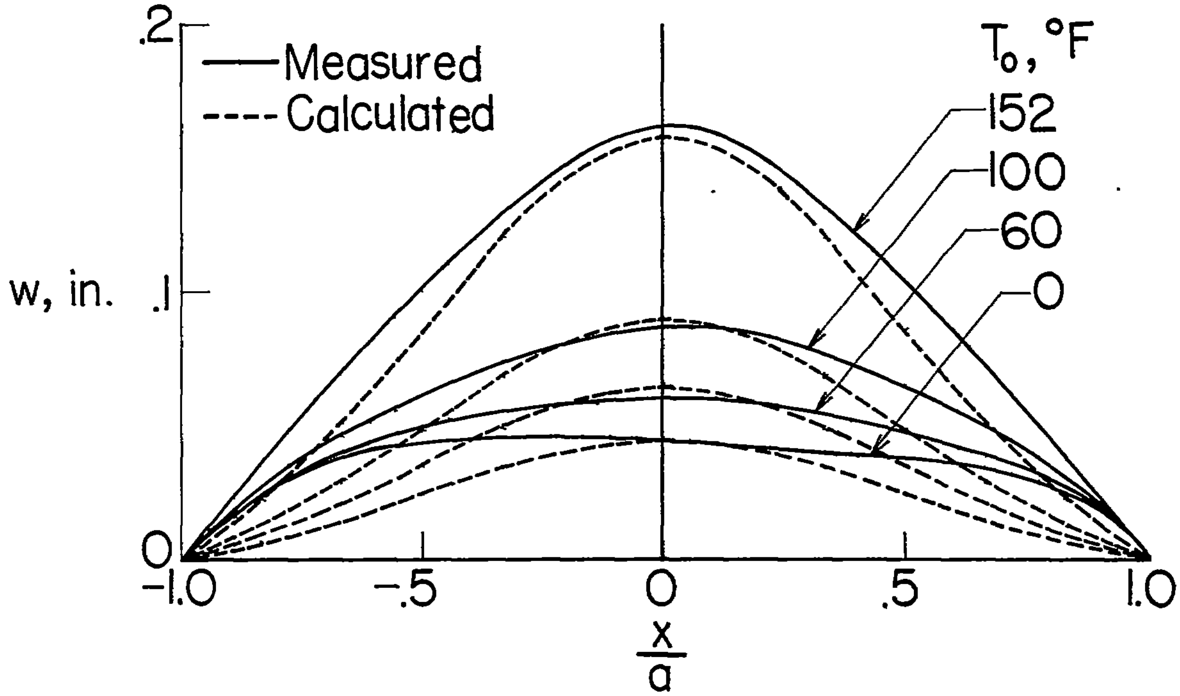
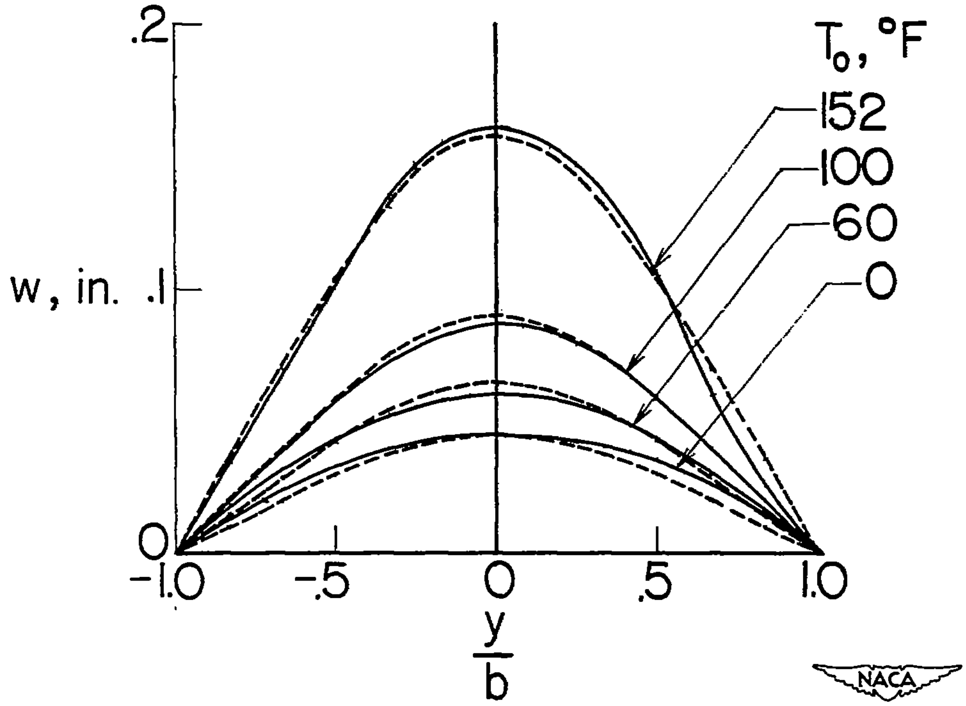


Figure 6.- Comparison of calculated and experimental deflections at plate center.



(a) Longitudinal center line.



(b) Transverse center line.

Figure 7.- Growth of deflections with temperature.

NACA-Langley - 5-7-53 - 150