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TECHNICAL NOTE 2900

THE CALCULATION OF PRESSURE ON SLENDER AIRPLANES IN SUBSONIC AND SUPERSONIC FLOW

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SUMMARY

Under the assumption that a wing, body, or wing-body combination is slender or flying at near sonic velocity, expressions are given which permit the calculation of pressure in the immediate vicinity of the configuration. The disturbance field, in both subsonic and supersonic flight, is shown to consist of two-dimensional disturbance fields extending laterally and a longitudinal field that depend on the streamwise growth of cross-sectional area. A discussion is al given of couplings, between lifting and thickness effects, that nec sarily arise as a result of the quadratic dependence of pressure on the induced velocity components.

INTRODUCTION

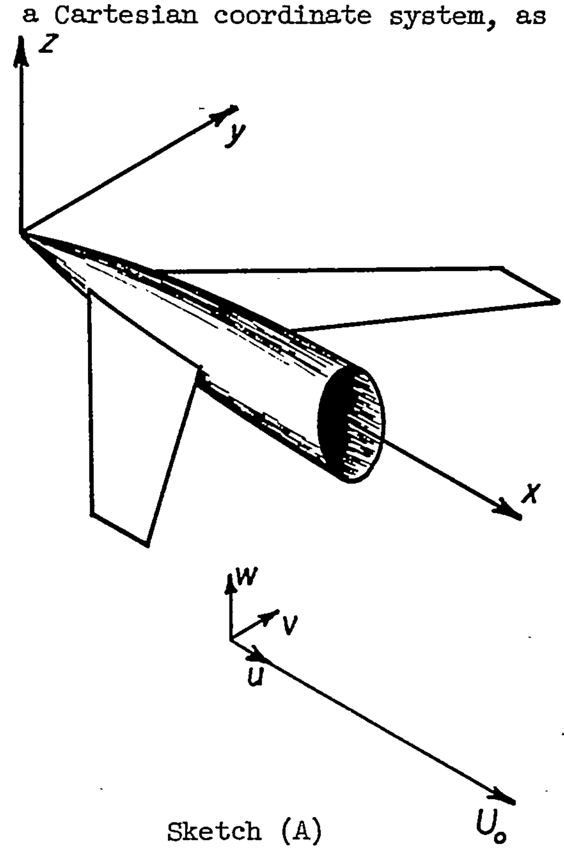
This paper is concerned with the prediction of pressure distribution on or in the immediate vicinity of a wing, body, or wing-body combination under conditions in which the geometric configuration is slender in the flight direction or is flying at near sonic velocity. The material to be presented is thus associated with the rather extensive group of results that belong to what is often referred to as slender-wing theory. The basic assumptions and methods can be found in publications by Munk, R. T. Jones, and Ward (refs. 1, 2, and 3)[1] and a discussion of the applicability of the methods to the prediction of loading on slender wings at sonic flight speeds has been given in reference 5. In reference 2, attention was directed toward the calculation of load distributions over wings in subsonic and supersonic flight and reference 3 was devoted essentially to the consideration of supersonic flight velocities. It is therefore of interest to investigate further the effects attributable to thickness on wings and wing-body combinations at both subsonic and supersonic flight speeds. Such investigations lead to valid approximations of interference effects and also indicate the way in which thickness and lifting effects can produce couplings in the calculations of pressures induced in the flow field.

[1]Reference should also be made to the recent extensions of slender-wing theory by Adams and Sears (ref. 4).

ANALYSIS

It is proposed to take the basic solutions of the linearized partial differential equations governing three-dimensional compressible flow and to obtain a simplification of the expressions by restricting attention to the induced field in the immediate vicinity of slender airplanes or missiles. These simplified expressions contain solutions used previously to study the forces and moments on lifting wings and bodies. In addition, however, they can be used to evaluate the first-order thickness effects on the pressure in the vicinity of the wing and body.

Consider, first, the construction of a weakly disturbed flow field. Let a uniform stream flow in the direction of the positive x axis of a Cartesian coordinate system, as in sketch (A). Immerse in the stream, which has a velocity U_o and a Mach number M_o, a slender wing-body shape the surface of which is inclined at a small angle to the free-stream direction. This angle of inclination must be small enough so that nearly everywhere in the fluid the magnitude of the perturbation velocity vector divided by the speed of the free stream is much less than one; that is,



Sketch (A)

$$\frac{\sqrt{u^2 + v^2 + w^2}}{U_o} << 1 \tag{1a}$$

Moreover, large supersonic Mach numbers are to be avoided and as a measure of this condition the inequality

$$\frac{M_o^2 \sqrt{u^2 + v^2 + w^2}}{U_o} << 1 \tag{1b}$$

is imposed.

Consider, next, the linearized partial differential equation governing weakly disturbed isentropic fluid flow. In terms of the perturbation velocity potential $\varphi(x,y,z)$, the lowest order approximation consistent with inequalities (1a) and (1b) is

$$(1-M_o^2)\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0 \tag{2}$$

where the subscripts denote partial differentiation with respect to the indicated variable.

Consider, finally, the expression for the pressure coefficient that is again consistent to the lowest order with inequalities (1a) and (1b). By expanding the pressure-velocity relation for steady isentropic flow and neglecting higher-order terms, one finds

$$C_p = \frac{p-p_o}{\frac{1}{2}\rho_o U_o^2} \approx -\frac{2u}{U_o} - \frac{(1-M_o^2)u^2 + v^2 + w^2}{U_o^2}$$

where p and ρ are pressure and density, respectively, and the subscript o refers to conditions in the free stream. It follows from inequalities (1a) and (1b) that pressure coefficient can be expressed in the form

$$C_p \approx -\frac{2u}{U_o} - \frac{v^2 + w^2}{U_o^2} \tag{3}$$

Equation (3) is the simplest general expression for pressure coefficient that is still entirely consistent with the assumptions basic to the development of equation (2).

Special solutions applying to problems of the class indicated can be obtained by appropriate simplification of general solutions to equation (2). Such a procedure will be discussed in the next section. The pressure coefficient is then determined by substituting these results into equation (3). The simplifications that can be made in evaluating the pressure on the surface of the airplane will also be discussed.

The Reduced Solutions

Subsonic.- As it applies to subsonic flow, equation (2) can be written in its normalized form as

$$\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0 \tag{4}$$

The analysis of equation (4) can be interpreted as applying to the condition $M_o = 0$ but one can extend the solutions throughout the subsonic Mach number range by applying the Prandtl-Glauert rule. It is important to stipulate, however, that the term "slender" will, unless otherwise indicated, have a dual interpretation - describing an aerodynamic configuration that is either much longer than it is wide or is flying close to the speed of sound.

A well-known solution to equation (4), resulting from an application of Green's theorem, is given by the expression

$$\varphi(x,y,z) = -\frac{1}{4\pi}\iint_S \left(\frac{\partial\varphi}{\partial n'} - \varphi\frac{\partial}{\partial n'}\right)\frac{dS_1}{\sqrt{(x-x_1)^2 + r^2}} \tag{5}$$

where dS_1 is the element of surface area on the airplane or its vortex wake, r equals $\sqrt{(y - y_1)^2 + (z - z_1)^2}$, and $\partial/\partial n'$ is the derivative normal to the surface S. When this solution is applied to boundary-value problems for slender configurations it can be simplified considerably. One method for bringing this about is suggested by studying the variation of $[r/(x - x_1)]^2$ over the area S. Consistent with the assumptions made, one has $[r/(x - x_1)]^2 << 1$ over almost all of the airplane surface and vortex wake provided the point x,y,z is on or in the vicinity of these surfaces. This implies the approximation

$$\sqrt{(x - x_1)^2 + r^2} \approx |x - x_1| \tag{6}$$

The singularity at $x = x_1$, which thus appears in the integrand of equation (5) for the limiting case $r = 0$, produces a divergent integral. An indication of the manner in which this difficulty can be avoided is obtained through consideration of the single integral

$$\int_a^b \frac{g(x_1)dx_1}{\sqrt{(x-x_1)^2 + r^2}}$$

If $g(x_1)$ is differentiable within the region of integration, it is easy to show that this expression can be written in the form

$$\frac{\partial}{\partial x}\int_a^b g(x_1)\sinh^{-1}\frac{x-x_1}{r}dx_1 = \frac{\partial}{\partial x}\int_a^b g(x_1)\frac{x-x_1}{|x-x_1|}\ln\frac{|x-x_1| + \sqrt{(x-x_1)^2+r^2}}{r}dx_1$$

and the approximation for small values of r then becomes

$$\frac{\partial}{\partial x}\int_a^b g(x_1)\frac{x-x_1}{|x-x_1|}\ln\frac{2|x-x_1|}{r}dx_1$$

Equation (5) is to be written subsequently with a logarithmic kernel in the integrand and a derivative operator outside the integrals. An estimate of the order of accuracy involved in using the approximate form of the integrand follows from the evaluation of the two expressions

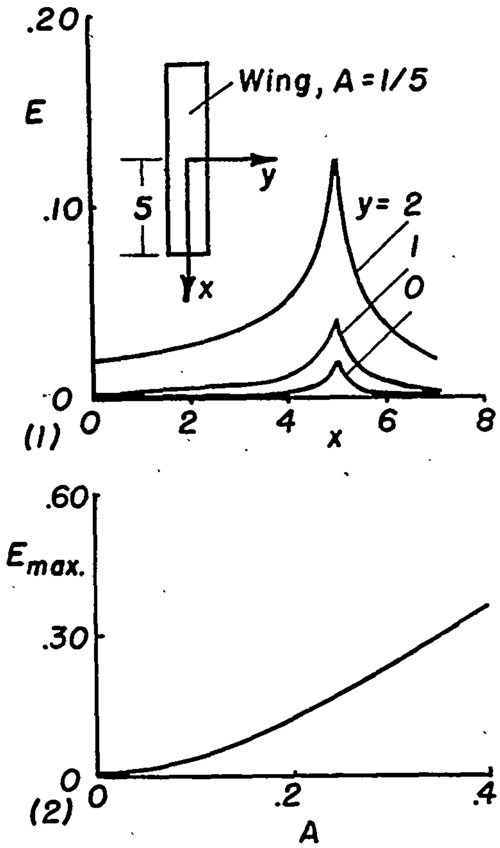
$$I_o = \int\int_{S_o} \ln \frac{|x-x_1|+\sqrt{(x-x_1)^2+(y-y_1)^2}}{|y-y_1|} \, dy_1 dx_1$$

and

$$I_1 = \int\int_{S_o} \frac{x-x_1}{|x-x_1|} \ln \frac{2|x-x_1|}{|y-y_1|} \, dy_1 dx_1$$

Sketch (B1) shows the variation of the ratio $E = (I_o - I_1)/I_o$ for a rectangle wing of area S_o and aspect ratio $A = 1/5$ as the point x,y covers the portion of the xy plane on and within one semispan of the wing.

The maximum value of E occurs when the point x,y lies along the trailing edge (or, by symmetry, along the leading edge) and it is significant to notice that the value of E decreases as the point moves from this location in either x direction and increases as it moves from the x axis in either y direction. This illustrates the necessity of restricting the approximate solutions to slender configurations and, further, to portions of the flow field in the vicinity of the longitudinal axis of these airplane shapes. Sketch (B2) shows how the maximum value of E within one semispan of the wing decreases with decreasing aspect ratio.



Sketches (B1) and (B2)

Under the restrictions that have been imposed, it is justifiable to introduce simplifications in the form of the derivative $\partial/\partial n'$ and the differential area dS_1 appearing in equation (5). The operator $\partial/\partial n'$ can be expressed as

$$\frac{n_1 \partial}{\partial x} + \frac{n_2 \partial}{\partial y} + \frac{n_3 \partial}{\partial z}$$

where n_1, n_2 and n_3 are the direction cosines between a normal to the surface S and the x,y, and z axes, respectively; the differential area dS_1 can be expressed as

$$\frac{ds_1 dx_1}{\sqrt{1 - n_1^2}}$$

where ds_1 is a differential length along the surface in a yz plane. If the airplane is slender, n_1 is small, and can be neglected, relative to either unity or $\sqrt{n_2^2 + n_3^2}$.

Combining the two simplifications discussed above, one can approximate equation (5) by the expression

$$\varphi(x,y,z) = -\frac{1}{4\pi}\frac{\partial}{\partial x}\int_0^\infty dx_1 \int_s ds_1\left(\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial}{\partial n}\right)\frac{x-x_1}{|x-x_1|}\, ln\, \frac{2|x-x_1|}{r} \tag{7}$$

where $\partial/\partial n$ represents $n_2\partial/\partial y + n_3\partial/\partial z$, the normal derivative to a section in a yz plane, s is the curve bounding this section, and l is the total length of the airplane. Equation (7) reduces to

$$\varphi(x,y,z) = \frac{1}{4\pi}\frac{\partial}{\partial x}\left(\int_0^x dx_1 - \int_x^\infty dx_1\right)\left[\int_s \left(\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial}{\partial n}\right) ln\ r\ ds_1\right] -$$

$$\frac{1}{4\pi}\frac{\partial}{\partial x}\int_0^\infty dx_1 \frac{x-x_1}{|x-x_1|}\ ln\ 2|x-x_1| \int_s \frac{\partial\varphi}{\partial n}\, ds_1$$

Since

$$\frac{1}{U_o}\int_s \frac{\partial\varphi}{\partial n}\, ds_1 = \frac{dS(x)}{dx} = S'(x) \tag{8}$$

where S(x) is the cross-sectional area of the specified shape in a yz plane, this expression becomes

$$\varphi(x,y,z) = \frac{1}{2\pi}\int_s \left(\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial}{\partial n}\right) ln\ r\ ds_1 - \frac{U_o}{4\pi}\frac{\partial}{\partial x}\int_0^l S'(x_1)\frac{x-x_1}{|x-x_1|}\, ln 2|x_1-x_1|dx_1$$

It is apparent that the first integral on the right-hand side of this equation is a solution to Laplace's equation in two (the y and z) dimensions. This portion of the solution will be represented by the symbol $\varphi_2(x;y,z)$. The x dimension, which does not appear explicitly, enters as a parameter when the solution is adapted to particular boundary values. Hence, by means of the usual Prandtl-Glauert transformation $x \longrightarrow x$, $y \longrightarrow y\sqrt{1-M_o^2}$, $z \longrightarrow z\sqrt{1-M_o^2}$, together with equation (8) and the relation

$$\frac{\partial}{\partial x}\int_0^l S'(x_1)\,\frac{x-x_1}{|x-x_1|}\,dx_1 = 2S'(x)$$

the final form of the solution is

$$\varphi(x,y,z) = \varphi_2(x;y,z) - \frac{U_o}{4\pi}\frac{\partial}{\partial x}\int_0^l S'(x_1)\,\frac{x-x_1}{|x-x_1|}\,ln\,\frac{2|x-x_1|}{\sqrt{1-M_o^2}}\,dx_1 \qquad (9)$$

The physical significance of the last expression can be interpreted easily. The three-dimensional perturbation velocity field induced by wing-body shapes that are slender or flying at near sonic speeds is approximated in the vicinity of such shapes by a field that satisfies the two-dimensional Laplace equation and the boundary conditions in transverse planes plus a longitudinal field which depends on the streamwise rate of change of cross-sectional area and is independent of y and z.

Supersonic.- In the case of supersonic flow, the normalized form of equation (2) becomes

$$\varphi_{xx} - \varphi_{yy} - \varphi_{zz} = 0 \qquad (10)$$

An analysis based on equation (10) applies specifically to the condition $M_o = \sqrt{2}$ but these results can be extended throughout the supersonic Mach number range by applying the Prandtl-Glauert rule. Volterra's solution to equation (10), (see, e.g., ref. 6, p. 190) which is analogous to the subsonic form given in equation (5), is expressible as

$$\varphi(x,y,z) = -\frac{1}{2\pi}\frac{\partial}{\partial x}\iint_T\left(\frac{\partial\varphi}{\partial\nu} - \varphi\frac{\partial}{\partial\nu}\right)ln\,\frac{|x-x_1| + \sqrt{(x-x_1)^2 - r^2}}{r}\,dS_1 \qquad (11)$$

where, as in the subsonic case, dS_1 is an element of surface area on the airplane or its vortex sheet and r equals $\sqrt{(y - y_1)^2 + (z - z_1)^2}$. In distinction to the subsonic solutions, the area of integration is now the portion of the airplane and its vortex wake within the forecone from the point $x,y,z,$ and $\partial/\partial\nu$ is the derivative along the conormal[2] rather than the normal.

If the application of equation (11) is limited to slender configurations, the approximation, similar to expression (6) for the subsonic case,

$$\sqrt{(x-x_1)^2-r^2} \approx |x-x_1|$$

is implied. Furthermore, if the conormal and differential area are expressed in terms of the direction cosines and n_1 is again neglected relative to unity or $\sqrt{n_2^2 + n_3^2}$, equation (11) reduces to the form

$$\varphi = \frac{1}{2\pi}\frac{\partial}{\partial x}\int_0^x dx_1\int_s\left(\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial}{\partial n}\right) \ln r\, ds_1 - \frac{1}{2\pi}\frac{\partial}{\partial x}\int_0^x \ln 2|x-x_1|dx_1\int_s \frac{\partial\varphi}{\partial n}\, ds_1 \tag{12}$$

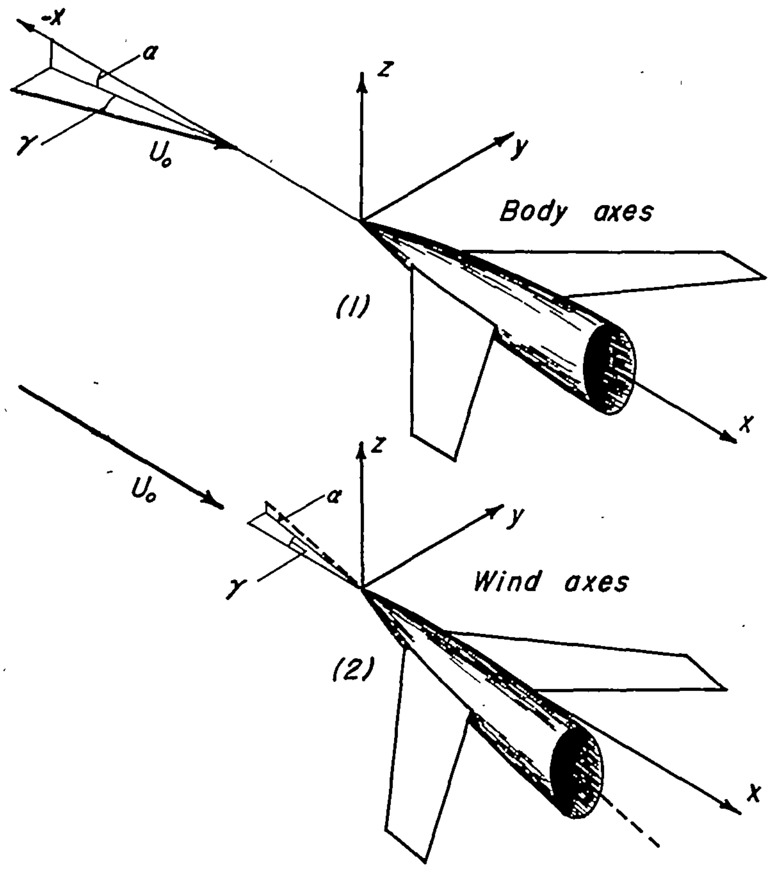
This differs from the corresponding expression for subsonic flow only by a factor of two and by the extent of the x_1 integration. The latter is carried only to x in the supersonic case since the original integration area τ included only points in the forecone from x,y,z. It is obvious, however, that these two differences are compensating in the first integral term so the final expression for the perturbation potential, appropriately modified by the Prandtl-Glauert rule, becomes

$$\varphi(x,y,z) = \varphi_2(x;y,z) - \frac{U_0}{2\pi}\frac{\partial}{\partial x}\int_0^x S'(x_1)\ \ln\ \frac{2(x-x_1)}{\sqrt{M_0^2-1}}\, dx_1 \tag{13}$$

The Reference Coordinate Systems

Equation (2) was developed specifically for the case in which the undisturbed stream at infinity is parallel to the x axis. A coordinate system so orientated is usually referred to as the wind axes. (See sketch (C).) When the configuration is tilted with respect to the

[2] The conormal is the vector that results from changing the sign of the x component of the normal.



Sketch (C)

free-stream vector, however, it is often easier to study the boundary-value problem with the x axis placed along the center line of the fuselage. Such a coordinate system is usually referred to as the body axes.

Obviously the wind and body axes differ significantly only by rotations about the y and z axes. When M_o is zero, equation (2) is invariant to such a rotation, but for values of M_o greater than zero this is no longer true. However, when M_o is greater than zero, equation (2) represents the governing differential equation only to a certain order, and, if the magnitude of the rotation is similar to that of the parameters by which the equation is ordered, it is, in this sense, still invariant to rotations about all three axes for both subsonic and supersonic Mach numbers. Thus equation (2) is to the lowest order the governing partial differential equation for both wind and body axes, provided the airplane is slender and the angles of attack and sideslip are small.

Although the partial differential equation is invariant with respect to a small rotation of the coordinate system, the boundary conditions and expression for the pressure coefficient in terms of the perturbation velocities are not. The following will contain a discussion of the boundary conditions and the pressure and loading coefficients with reference to a body axes system.

The Boundary Conditions

The boundary conditions require that the gradient of the total velocity potential evaluated infinitely far from the aircraft be consistent with a uniform free stream there (the direction of which depends on the orientation of the coordinate system) and when evaluated normal to and on the surface of the airplane itself be zero. Let $\Phi(x,y,z)$ denote total velocity potential, $\varphi(x,y,z)$ perturbation velocity potential, and refer the analysis to body axes in a free stream. If the orientation of the free-stream velocity vector to the system of axes is fixed by the angles α and γ as shown in sketch (C), one can write

$$\Phi(x,y,z) = U_0(x \cos \alpha \cos \gamma + y \sin \gamma + z \cos \gamma \sin \alpha) + \varphi(x,y,z)$$

such that on the aircraft surface

$$U_0(n_1 \cos \alpha \cos \gamma + n_2 \sin \gamma + n_3 \sin \alpha \cos \gamma) + n_1\varphi_x + n_2\varphi_y + n_3\varphi_z = 0$$

$n_1, n_2,$ and n_3 again being the direction cosines of a normal to the airplane surface with respect to the $x, y,$ and z axes, respectively. By the assumptions basic to the present theory, the latter equation reduces to

$$U_0(n_1 + n_2\, \gamma + n_3\, \alpha) + \frac{\partial}{\partial n}\varphi_2(x;y,z) = 0 \tag{14}$$

where, as before, n is the normal to the curve bounding a cross section in the yz plane.

Equation (14), which applies to arbitrary slender shapes, can be simplified for many specific problems. Consider now three types of configurations that lead to such simplifications: first, a surface, such as a wing, which deviates only slightly from a plane; second, a surface which forms a body of revolution; and, third, a surface which is a combination of the above two.

Planar problems.- Let $h(x,y)$ be the distance a surface deviates from the $z = 0$ plane, and b represent the wing span. (See sketch (D1).) Assume that $\alpha/(db/dx) << 1$ holds; then furthermore, if the inequality $(\partial h/\partial x)/(db/dx) << 1$ is satisfied, it is consistent with the previous approximations to neglect the y component of the normal along the wing surface and to project the velocity vector represented by the resulting vertical derivative to the upper or lower surface of the $z = 0$ plane. In this way equation (14) becomes

$$U_0(n_1+n_3\alpha) + n_3\left(\frac{\partial\varphi_2}{\partial z}\right)_{z=0} = 0$$

and, since $n_1/n_3 = -\partial h/\partial x$, the boundary conditions for planar problems[3] are expressed by the equation

$$\left(\frac{\partial\varphi_2}{\partial z}\right)_{z=0} = -U_0\alpha + U_0\frac{\partial h}{\partial x} \qquad (15)$$

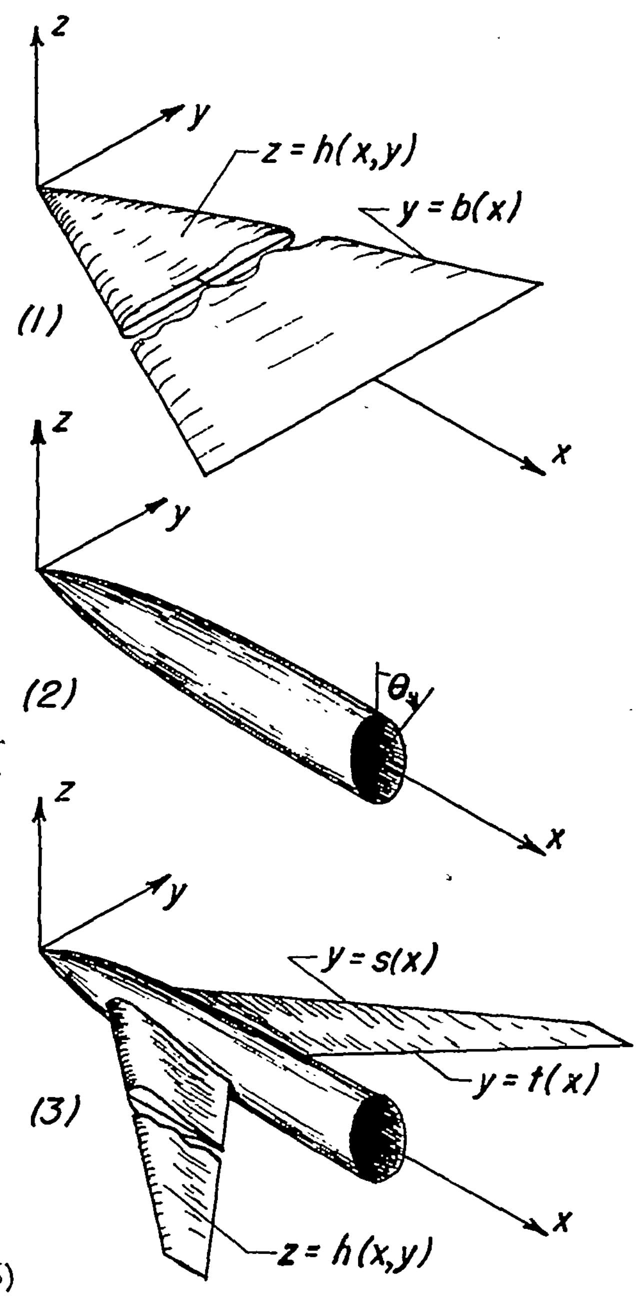
Body of revolution problems.- Let R be the radius of a body of revolution. (See sketch (D2).) If θ is measured from the z axis in the yz plane and γ is set equal to zero, the relations

$$n_1 = \frac{-dR/dx}{\sqrt{1+(dR/dx)^2}} \approx -\frac{dR}{dx},$$

$$n_3 = \frac{\cos\theta}{\sqrt{1+(dR/dx)^2}} \approx \cos\theta$$

together with equation (14), give for the normal derivative on the surface of the body

$$\left(\frac{\partial\varphi_2}{\partial n}\right)_B = U_0\frac{dR}{dx} - U_0\alpha\cos\theta \qquad (16)$$



Sketches (D1), (D2), and (D3)

[3]Certain planar problems, such as the cruciform wing, require more than one plane but the concepts are essentially the same as those presented here.

which is the simplified expression of the boundary conditions pertaining to bodies of revolution.

Interference problems.- Consider, finally, surfaces which are a combination of the above two as, for example, the one shown in sketch (D3). The rather obvious extension of the above concepts is to apply equation (15) over the winged portion of the configuration and equation (16) over the body. It is then necessary, however, to consider the relative magnitudes of the terms $\partial h/\partial x$, ds/dx, dt/dx, and dR/dx, since they appear in the solutions in various combinations. If the winged portion is to be treated as a planar problem, the magnitude of $\partial h/\partial x$ must be small enough to be neglected in comparison to the leading- and trailing-edge slopes, ds/dx and dt/dx. But this does not imply that $\partial h/\partial x$ can be neglected in comparison to dR/dx or that dR/dx can be neglected in comparison to either ds/dx or dt/dx. The latter approximations will not, in general, be made.

The Pressure Coefficient

The expression for the pressure coefficient given by equation (3) is written in terms of velocity components that are referred to the wind axes. Its re-expression in terms of velocities referred to the body axes is readily determined. For the orientation shown in sketch (C2), the equation becomes

$$C_p = -\frac{2}{U_o}\left(\varphi_x + \gamma\varphi_y + \alpha\varphi_z\right) - \frac{1}{U_o^2}\left(\varphi_y^2 + \varphi_z^2\right) \tag{17}$$

Equation (17) can be used, in general, to evaluate the pressure in a perturbation velocity field that is referred to the body axes. If the interest is limited to the pressure on the surface of the aircraft, however, certain simplifications can be made. For example, consider the configuration illustrated in sketch (D3) consisting of a swept-back wing mounted on a body of revolution. For simplicity, let $\gamma = 0$. Applying the boundary conditions given by equations (15) and (16), one can show that on the surface of the wing

$$C_p = -\left[\frac{2\varphi_x}{U_o} + \left(\frac{\varphi_y}{U_o}\right)^2\right]_{z=0} + \alpha^2 - \left(\frac{\partial h}{\partial x}\right)^2 \tag{18a}$$

and on the surface of the body

$$C_p = -\left[\frac{2\varphi_x}{U_o} + \frac{1}{U_o^2}\left(U_o\alpha\sin\theta - \frac{1}{R}\varphi_\theta\right)^2\right]_B + \alpha^2 - \left(\frac{dR}{dx}\right)^2 \tag{18b}$$

These solutions can be simplified further by considering the detailed nature of the perturbation velocity field induced by shapes such as that shown in sketch (D3). Thus, the results given by equations (9) and (13) can be expressed in the form

$$\varphi(x,y,z) = \varphi_2(x;y,z) + A(x) \tag{19}$$

where the expression for $A(x)$ depends on whether the speed is subsonic or supersonic. Further, for the particular configurations being considered, the expression for $\varphi_2(x;y,z)$ can be written in the general form

$$\varphi_2(x;y,z) = \alpha\varphi_a(t,s,R;y,z) + \frac{\partial h}{\partial x}\,\varphi_b(t,s,R;y,z) + \frac{dR}{dx}\,\varphi_c(t,s,R;y,z) \tag{20}$$

since the dependency on x can enter only through the boundary conditions which, in turn, are specified by the body radius $R(x)$, the wing thickness, $h(x,y)$, and the lateral distances from the center line to the trailing edge and leading edge, $t(x)$ and $s(x)$, respectively. The term $\alpha\varphi_a$ will be referred to as the potential due to angle of attack, since it vanishes when the angle of attack vanishes and increases linearly with increasing α; the term $(\partial h/\partial x)\varphi_b + (dR/dx)\varphi_c$ will be referred to as the potential due to thickness, since it exists when the angle of attack is zero, does not change with angle-of-attack change and vanishes when the thicknesses of the wing and body do not vary with x.

By breaking φ_2 down into its component parts as in equation (20), it has been ordered in that the magnitudes of the terms on the right-hand side of equation (20) are controlled by the coefficients of the φ's, and the derivatives of φ_a, φ_b, and φ_c with respect to s,t,R,y, and z can all be considered equal. Since α and $\partial h/\partial x$ are negligible relative to dt/dx, ds/dx, and dR/dx (as was pointed out in the discussion of the boundary conditions for interference problems), equations (18a) and (18b) can be written:

on the surface of the wing

$$C_p = -\left\{\frac{2}{U_o}\frac{\partial\varphi}{\partial x} + \frac{1}{U_o^{\,2}}\frac{dR}{dx}\left[2\alpha\frac{\partial\varphi_a}{\partial y}\frac{\partial\varphi_c}{\partial y} + 2\frac{\partial h}{\partial x}\frac{\partial\varphi_b}{\partial y}\frac{\partial\varphi_c}{\partial y} + \frac{dR}{dx}\left(\frac{\partial\varphi_c}{\partial y}\right)^2\right]\right\}_{z=0} \tag{21a}$$

and on the surface of the body

$$C_p = -\left\{\frac{2}{U_o}\frac{\partial\varphi}{\partial x} + \left(\frac{dR}{dx}\right)^2 + \frac{1}{U_o{}^2R^2}\frac{dR}{dx}\left[2\alpha\frac{\partial\varphi_a}{\partial\theta}\frac{\partial\varphi_c}{\partial\theta} + 2\frac{\partial h}{dx}\frac{\partial\varphi_b}{\partial\theta}\frac{\partial\varphi_c}{\partial\theta} + \frac{dR}{dx}\left(\frac{\partial\varphi_c}{\partial\theta}\right)^2 - 2\alpha U_oR\sin\theta\frac{\partial\varphi_c}{\partial\theta}\right]\right\}_B \tag{21b}$$

If the body is a cylinder so that its radius does not vary with x, the pressure coefficient reduces to

$$C_p = \left[-\frac{2}{U_o}\frac{\partial\varphi}{\partial x}\right]_B \tag{22}$$

Loading Coefficient

By definition the loading coefficient is

$$\frac{\Delta p}{q} = (C_p)_l - (C_p)_u \tag{23}$$

where the subscripts u and l refer to the upper and lower surfaces of the airplane, respectively. It is immediately apparent from an inspection of equations (23) and (19) that the loading is not affected by $A(x)$. Hence, the lift, pitching moment, rolling moment, and induced drag can all be expressed entirely in terms of $\varphi_2(x;y,z)$.

Consider again the type of airplane shapes represented in sketch (D3). The velocity potential φ_2 for such a class of configurations has been expressed in equation (20) as the sum of three potentials: one due to angle of attack, one due to the thickness of the wing, and one due to the thickness of the body. It is now useful to remark that φ_a has odd symmetry with reference to the $z = 0$ plane and φ_b and φ_c have even symmetry. Placing equations (21a) and (21b) into equation (23) and using these properties, one finds

$$\left(\frac{\Delta p}{q}\right)_{wing} = \frac{2\alpha}{U_o}\left[\Delta\frac{\partial\varphi_a}{\partial x} + \frac{1}{U_o}\frac{dR}{dx}\Delta\left(\frac{\partial\varphi_a}{\partial y}\frac{\partial\varphi_c}{\partial y}\right)\right]_{z=0} \tag{24a}$$

and

$$\left(\frac{\Delta p}{q}\right)_{body} = \frac{2\alpha}{U_o}\left[\Delta\frac{\partial\varphi_a}{\partial x} + \frac{1}{U_oR^2}\frac{dR}{dx}\Delta\left(\frac{\partial\varphi_a}{\partial\theta}\frac{\partial\varphi_c}{\partial\theta}\right)\right]_B \tag{24b}$$

where Δ indicates the difference between a quantity on vertically opposed points of the upper and lower surface of the airplane.

It is apparent from the last two equations that, in general, the angle-of-attack and thickness solutions have a coupling effect on the loading coefficient and therefore their contribution to the load distribution cannot be treated separately. It is also important to notice the two special cases in which the coupling effects vanish; namely, a body of revolution without wings, and an airplane with a cylindrical body between the foremost and rearmost extent of the wing. In the former case the term $\partial\varphi_c/\partial\theta$ is zero and in the latter dR/dx is zero. In both these cases the equation for the loading coefficient is

$$\frac{\Delta p}{q} = 2\alpha\Delta\frac{\partial\varphi_a}{\partial x} = \frac{2}{U_o}\Delta\frac{\partial\varphi_2}{\partial x} \tag{25}$$

The Total Lift

Total lift can be obtained, of course, by integrating the loading coefficient over the aircraft surface. A much simpler way of finding the lift, however, can be derived from a momentum balance. Thus, by momentum considerations it is possible to show that the vectorial force $\vec{F}$ on a body inside a control surface S is given by the surface integral

$$\vec{F} = -\int_S\int(p-p_o)d\vec{S} - \int_S\int\rho\left(\vec{V} - \vec{V}_o\right)\left[\vec{V}\cdot d\vec{S}\right]$$

where vector notation is used, the o subscript indicates free-stream conditions, p and ρ are the local static pressure and density, and $\vec{V}$ is the local velocity vector. Let the surface S be a cylinder of infinite radius and two yz planes closing the cylinder and located infinitely far ahead of and behind the airplane. Then the lift force is given to the lowest order by

$$L = F_z = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\rho w(U_o+u)\right]_{X=\infty} dy\, dz$$

which reduces to

$$L = -\rho_o U_o \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (w)_{x=\infty}\, dy\, dz$$

This can be simplified since $w = \partial\varphi/\partial z$ and $(\Delta\varphi)_{x=\infty}$ is the same as the jump in the potential evaluated at the airplane trailing edge. Thus the expression for lift becomes

$$L = \rho_o U_o \int_{\text{span}} (\Delta\varphi)_{T.E.}\, dy \tag{26}$$

Equation (26) applies to all slender shapes. In special cases represented by sketch (D3), thickness effects always have even symmetry with respect to the $z = 0$ plane, and it follows that the total lift and the vortex distribution in the wake of such configurations are affected only by the part of the potential due to angle of attack, even though the detailed load distribution depends upon both thickness and angle-of-attack solutions.

EXAMPLES

Pressure on a Triangular Wing With Elliptic Cross Section

It is of interest to calculate, by equation (13), the pressure on nonlifting wings of triangular plan form and elliptic cross section flying at supersonic speeds, since examples of this type have been solved without restriction to slender-wing theory. It is proposed, therefore, to study two cases given first by Squire (ref. 7) and then to compare the analytical results.

Let the wing be placed at zero angle of attack in a supersonic free stream of Mach number M_o. Consider first the thickness distribution for which the ordinate of the upper surface is

$$h(x,y) = \frac{t}{2mc_o}\sqrt{m^2x^2 - y^2} \tag{27}$$

where c_o is the root chord, t is wing thickness at $x = c_o$, and m is the tangent of the semiapex angle of the plan form. Since the flow is supersonic, it is unnecessary to consider closure.

Since attention is confined to symmetric nonlifting wings, the boundary values are planar and are expressed by equation (15) for $\alpha = 0$. Further, the solution is given in terms of these boundary values by equation (12) since $(\partial\varphi/\partial n)_{z=0}$ becomes $U_o(\partial h/\partial x)$ and $(\Delta\varphi)_{z=0}$ is zero by symmetry. Hence, if $\beta = \sqrt{M_o^2-1}$,

$$\frac{1}{U_o}\varphi(x,y,0) = \frac{1}{\pi}\int_{-mx}^{mx}\frac{\partial h}{\partial x}\,ln|y-y_1|\,dy_1 - \frac{1}{2\pi}\frac{\partial}{\partial x}\int_0^x S'(x_1)\,ln\,\frac{2(x-x_1)}{\beta}\,dx_1$$

It follows from equation (27) that the elliptic section in the plane $x = x_1$ has major and minor semiaxes equal to mx and $tx/2c_o$, respectively. The cross-sectional area and the surface slope are, therefore,

$$S(x) = \pi tmx_1^2/2c_o,\quad \partial h/\partial x = tmx/2c_o\sqrt{m^2x^2-y^2}$$

Since

$$\frac{1}{\pi}\int_{-mx}^{mx}\frac{tmx\,ln|y-y_1|\,dy_1}{2c_o\sqrt{m^2x^2-y_1^2}} = \begin{cases}\dfrac{mtx}{2c_o}\,ln\,\dfrac{mx}{2}; & |y|<mx\\[2ex] \dfrac{mtx}{2c_o}\,ln\,\dfrac{|y|+\sqrt{y^2-m^2x^2}}{2}; & |y|>mx\end{cases}$$

the expression for perturbation potential becomes

$$\frac{1}{U_o}\varphi(x,y,0) = \begin{cases}\dfrac{tmx}{2c_o}\left(1-ln\,\dfrac{4}{m\beta}\right); & |y|<mx\\[2ex] \dfrac{tmx}{2c_o}\left[1-ln\,\dfrac{4x}{\beta\left(|y|+\sqrt{y^2-m^2x^2}\right)}\right]; & |y|>mx\end{cases}$$

From equation (22), pressure coefficient in the plane of the wing is.

$$C_p = \begin{cases} \dfrac{tm}{c_o}\left(ln\dfrac{4}{m\beta} - 1\right); & |y| < mx \\ \dfrac{tm}{c_o}\left[\dfrac{y}{\sqrt{y^2-m^2x^2}} + ln\dfrac{4x}{\beta(|y|+\sqrt{y^2-m^2x^2})} - 1\right]; & |y| > mx \end{cases} \tag{28}$$

The pressure distribution on the wing is uniform and off the wing has a square-root singularity at the leading edges. Analysis not limited by the assumption of slenderness yields for pressure coefficient on the wing

$$C_p = \frac{tm}{c_o}\frac{K-E}{\sqrt{1-\beta^2m^2}}$$

where K and E are complete elliptic integrals with modulus $\sqrt{1-\beta^2m^2}$. Since for values of the modulus near one the asymptotic relations

$$K \approx ln\frac{4}{\beta m}, \qquad E \approx 1$$

apply, the pressure coefficient in slender-wing theory is seen to be a first-order approximation.

Squire has also considered the wing with ordinates given by

$$h(x,y) = \frac{tx}{2{c_o}^2m}\sqrt{m^2x^2-y^2} \tag{29}$$

The lateral section is again elliptic, with semimajor and semiminor axes equal to mx and $tx^2/2c_o^2$. Cross-sectional area and surface slope are, respectively,

$$S(x) = \frac{\pi mtx^3}{2{c_o}^2}, \qquad \frac{\partial h}{\partial x} = \frac{t}{2mc_o^2}\frac{2m^2x^2-y^2}{\sqrt{m^2x^2-y^2}}$$

By direct integration it can be shown that

$$C_p = \frac{mtx}{c_o^2}\left(3\ ln\ \frac{4}{\beta m} - 4\right) \tag{30}$$

Analysis not limited to slender wings yields the expression

$$C_p = \frac{mtx}{c_o^2(1-\beta^2m^2)}\ [(3-\beta^2m^2)K-(4-2\beta^2m^2)E]$$

and again the results are in agreement if higher-order terms in βm are neglected.

A study of generalized conical flow fields in linearized supersonic theory reveals that the linear pressure distribution in the above problem can also be obtained on a wing with thickness specified by the relation

$$h(x,y) = \frac{ky^2}{m^2} \cosh^{-1} \frac{mx}{y}$$

where k is a constant that can be related to the maximum thickness ratio (attained along the line $mx/y = 1.31$) of the wing. The cross-sectional area and surface slope are, respectively,

$$S(x) = \frac{\pi kmx^3}{3}, \qquad \frac{\partial h}{\partial x} = \frac{ky^2}{m\sqrt{m^2x^2-y^2}}$$

and pressure coefficient on the wing is

$$C_p = 2kmx\left(ln\ \frac{4}{\beta m} - 2\right)$$

The latter expression agrees to the first order in βm with the general linearized solution for such a wing presented in reference 8. Slender-wing theory thus retains the property of the more general linear theory in that a given pressure distribution does not necessarily yield a unique thickness distribution.

Supersonic Drag of Wings at Zero Incidence

The general expression for the supersonic drag of a slender aerodynamic shape has been derived by Ward (ref. 3) through the use of momentum methods. It is also possible to obtain these results by direct integration of the product of pressure and surface slope over the specified surface; the analysis, however, requires rather careful attention to orders of integration, when planar problems are involved. Consider, for example, the drag of a wing at zero incidence and with a specified thickness distribution $z = \pm h(x,y)$. The drag of the wing is expressible in the form

$$D = D_e + 2q \int_W \int C_{p_u} \frac{\partial h_u}{\partial x} \, dx \, dy \tag{31}$$

where the first term includes possible contributions to the drag that result from a finite leading-edge radius of curvature. From reference 9, this drag per unit of span is, in slender wing theory,

$$\frac{dD_e}{dy} = \pi q r_n \left(\frac{ds}{dx}\right)^2 \tag{32}$$

where r_n is the radius of curvature normal to the wing leading edge and s is the local semispan. If the ordinate of the wing, in the vicinity of the leading edge, is

$$z_u = f(s,y)\sqrt{s - y}$$

equation (32) becomes

$$\frac{dD_e}{dy} = \pi q \frac{f^2(s,s)}{2} \left(\frac{ds}{dx}\right)^2$$

From equation (13) the potential of the wing, evaluated in the plane of the wing is

$$\varphi(x,y,0) = \frac{U_o}{\pi} \int_{-s}^{s} \frac{\partial z_u(x,y_1)}{\partial x} \ln|y-y_1| \, dy_1 - \frac{U_o}{2\pi} S'(x) \ln \frac{2}{\beta} - \frac{U_o}{2\pi} \int_0^x S''(x_1) \ln (x-x_1) \, dx_1 \tag{33}$$

and, since pressure coefficient in the planar case is directly proportional to the streamwise gradient of φ, the contribution of each of the terms on the right-hand side of equation (33) can be calculated separately in equation (31). The second and third terms offer no difficulty but simplification of the expression resulting from the first term necessitates an inversion of order of integration and, if the leading edge has a finite radius of curvature, such an inversion cannot be carried out in the conventional manner. However, a method by means of which such an inversion can be carried out is presented in reference 10. Thus, set

$$I_1 = \int_0^s dy \,⨍_0^s dy_1 \frac{\partial z_u(x,y)}{\partial x} \frac{\partial^2 z_u(x,y_1)}{\partial x^2} \ln |y-y_1|$$

and

$$I_2 = ⨍_0^s dy_1 \int_0^s dy \frac{\partial z_u(x,y)}{\partial x} \frac{\partial^2 z_u(x,y_1)}{\partial x^2} \ln |y-y_1|$$

where $⨍$ refers to a finite-part integral[4] and the notation $\int dy \int dy_1$ signifies that the y_1 integration must be performed first. Then if

$$z_u(x,y) = f(s,y)\sqrt{s - y}$$

it can be shown that

$$I_1 - I_2 = \frac{\pi^2}{4}\left(\frac{ds}{dx}\right)^s f^2(s,s)$$

Detailed analysis reveals that the residual term (i.e., the value of $I_1 - I_2$) yields a drag component that is equal in magnitude but opposite in sign to D_e.

The final expression for the drag of the wing is then

$$\frac{D}{q} = -\frac{2}{\pi}\int_{-s(l)}^{s(l)} \left.\frac{\partial h_u(x,y)}{\partial x}\right]_{x=l} dy \int_{-s(l)}^{s(l)} \left.\frac{\partial h_u(x,y_1)}{\partial x}\right]_{x=l} \ln|y-y_1| \, dy_1 + \frac{S'(l)^2}{2\pi} \ln \frac{2}{\beta} +$$

$$\frac{1}{\pi} S'(l) \int_0^l S''(x_1) \ln (l-x_1) \, dx_1 - \frac{1}{2\pi}\int_0^l S''(x) dx \int_0^l S''(x_1) \ln|x-x_1| \, dx_1 \tag{34}$$

where l is the over-all length of the wing.

[4]For a definition of the finite-part integration technique as used here, see reference 10.

As a particular example, consider the wing-like surface of triangular plan form (ref. 7) which results from a combination of the surfaces specified in equations (27) and (29) and has ordinates given by the expression

$$h_u(x,y) = \frac{2t}{mc_o^2}(c_o-x)\sqrt{m^2x^2-y^2} \tag{35}$$

This wing has rounded leading edges and a finite trailing-edge angle, and from equation (34) its drag coefficient based on wing area is found to be

$$C_D = -2m\pi\left(\frac{t}{c_o}\right)^2\left(1 + \ln\frac{\beta m}{4}\right) \tag{36}$$

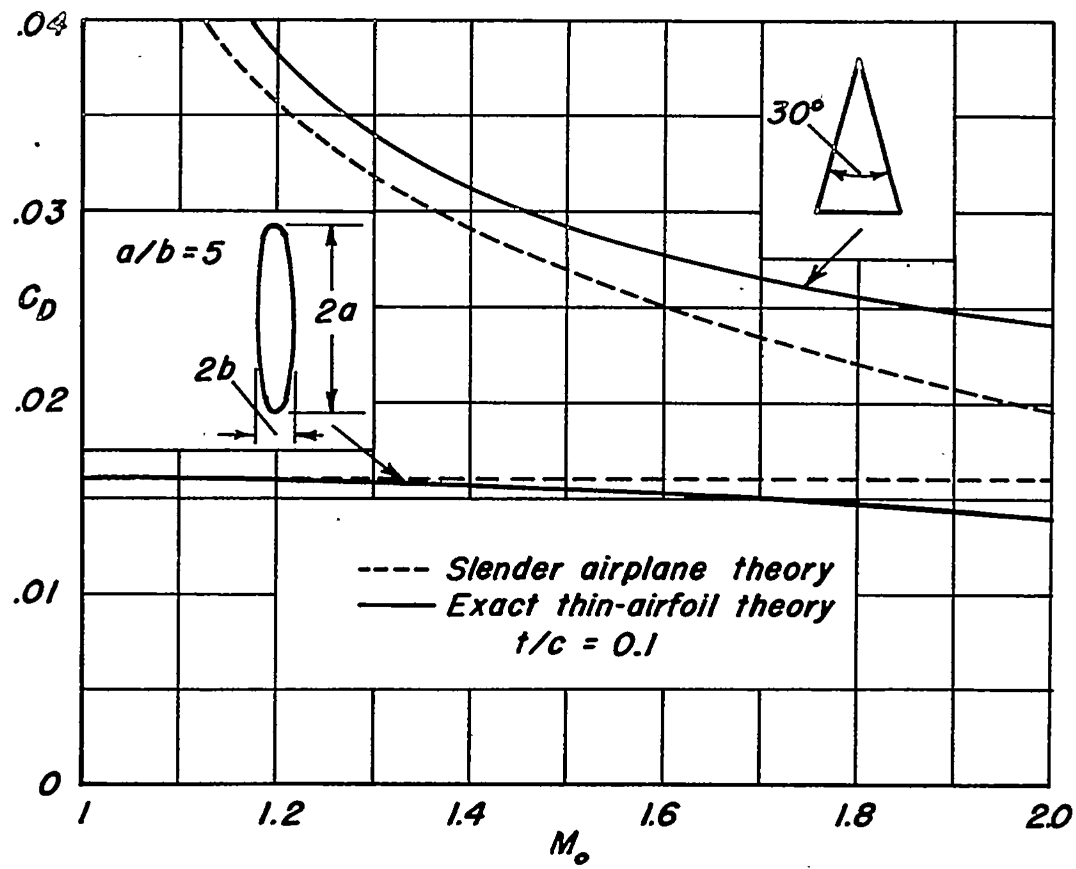
It is apparent from equation (34) that wing drag varies with Mach number so long as the streamwise gradient of area is finite at the rear of the wing; conversely, there is no dependence on Mach number when the gradient of area vanishes there. For example, a wing with an elliptic plan form and biconvex sections satisfies the latter condition, and its drag coefficient based on wing area is

$$C_D = 2\,\frac{b}{a}\left(\frac{t}{a}\right)^2 \tag{37}$$

where t is total maximum thickness, a is the semiaxis of the elliptic plan form in the stream direction, and b is the semiaxis measured normal to the stream direction.

A comparison between the values of C_D given by slender airplane theory for the Squire wing (eq. (36)) and the elliptic lens (eq. (37)) and the exact thin-airfoil-theory values[5] for the same wings is shown in sketch (E).

[5] The exact values were given, respectively, by Squire in reference 7 and by R. T. Jones in a paper entitled "Theoretical Determination of the Minimum Drag of Airfoils at Supersonic Speeds" and presented at the July 1952 meeting of the Institute of Aeronautical Sciences.



Sketch (E)

Thickness Distributions Having Minimal Drag
in Supersonic Flight

The expression for the perturbation velocity potential given in equation (13) provides a ready way of expressing analytically certain criteria that have been given by R. T. Jones (ref. 11) for wings possessing minimum drag characteristics. An essential feature of Jones' results involves the use of combined flow fields which are obtained by superimposing the disturbance fields in forward and reversed motions. So long as the governing equations of flow are linear, it is possible to establish reciprocity relations between the induced fields of arbitrarily situated sources and doublets in combined flow fields. Conditions for minimum drag under imposed restrictions are then expressed in terms of the pressure induced in the superimposed fields. For example, it is found that if the thickness distribution for a symmetrical nonlifting wing yields a specified volume, then drag is a minimum if the thickness is distributed in such a way that the pressure gradient in the combined field remains constant over the plan form of the wing. The application of this condition to a slender wing in supersonic flight is simple, since from equation (13) the perturbation potential in the two directions of flow can be written explicitly. In the plane of the wing, the forward flow yields

$$\varphi_f(x,y,0) = \frac{1}{2\pi}\int_{-b}^{b}\Delta\frac{\partial\varphi_f}{\partial z}\, l n\,|y-y_1|\,dy_1 - \frac{U_o}{2\pi}\frac{\partial}{\partial x}\int_o^x S'(x_1)\; l n\,\frac{2(x-x_1)}{\beta}\,dx_1$$

and the reverse flow yields

$$\varphi_r(x,y,0) = \frac{1}{2\pi}\int_{-b}^{b}\Delta\frac{\partial\varphi_r}{\partial z}\, l n\,|y-y_1|\,dy_1 + \frac{U_o}{2\pi}\frac{\partial}{\partial x}\int_l^x S'(x_1)\; l n\,\frac{2(x_1-x)}{\beta}\,dx_1$$

where l is the streamwise length of the wing and b is local semi-span. Since

$$\frac{\partial\varphi_f}{\partial n} = -\frac{\partial\varphi_r}{\partial n}$$

the perturbation potential φ_Σ in the combined field is

$$\varphi_\Sigma(x,y,0) = \frac{-U_o}{2\pi}\frac{\partial}{\partial x}\int_o^l S'(x_1)\; l n\;\frac{2|x-x_1|}{\beta}\,dx_1$$

and it follows directly that if $S'(0) = S'(l) = 0$, pressure coefficient in the combined field is

$$C_{p_\Sigma} = \frac{1}{\pi}\int_o^l \frac{S''(x_1)}{x-x_1}\,dx_1 \qquad (38)$$

In the case of thickness distribution with given volume, C_{p_Σ} is a linear function in x and equation (38) is precisely the same integral equation that arises in the determination of thickness distribution with given volume for a slender body of revolution in supersonic flight (refs. 12, 13, and 14). The same chordwise distribution of area therefore exists for wings and bodies of revolution under the given conditions.

Ames Aeronautical Laboratory
National Advisory Committee for Aeronautics
Moffett Field, Calif., Nov. 28, 1952.

REFERENCES

1. Munk, Max M.: The Aerodynamic Forces on Airship Hulls. NACA Rep. 184, 1924.

2. Jones, Robert T.: Properties of Low-Aspect-Ratio Pointed Wings at Speeds Below and Above the Speed of Sound. NACA Rep. 835, 1946.

3. Ward, G. N.: Supersonic Flow Past Slender Pointed Bodies. Quart. Jour. Mech. and Appl. Math., vol. II, pt. 1, 1949.

4. Adams, Mac C., and Sears, William Rees: On an Extension of Slender-Wing Theory. Jour. Aero. Sci., vol. 19, no. 6, June 1952, pp. 424-425.

5. Heaslet, Max. A., Lomax, Harvard, and Spreiter, John R.: Linearized Compressible-Flow Theory for Sonic Flight Speeds. NACA Rep. 956, 1950. (Supersedes NACA TN 1824)

6. Bateman, Harry: Partial Differential Equations of Mathematical Physics. Dover Pub., N. Y., 1944.

7. Squire, H. B.: Theory of the Flow Over a Particular Wing in a Supersonic Stream. R.A.E. Aero Rep. No. 2184, (British), 1947.

8. Lomax, Harvard, and Heaslet, Max. A.: Generalized Conical-Flow Fields in Supersonic Wing Theory. NACA TN 2497, 1951.

9. Jones, Robert T.: Leading-Edge Singularities in Thin-Airfoil Theory. Jour. Aero. Sci., vol. 17, no. 5, pp. 307-310, May 1950.

10. Lomax, Harvard, Heaslet, Max. A., and Fuller, Franklyn B.: Integrals and Integral Equations in Linearized Wing Theory. NACA Rep. 1054, 1951. (Supersedes NACA TN 2252)

11. Jones, Robert T.: The Minimum Drag of Thin Wings in Frictionless Flow. Jour. Aero. Sci., vol. 18, no. 2, Feb. 1951, pp. 75-81.

12. Sears, William R.: On Projectiles of Minimum Wave Drag. Quart. Appl. Math., vol. IV, no. 4, 1947.

13. Haack, W.: Geschossformen kleinsten Wellenwiderstandes. Lilienthal-Gesellschaft für Luftfahrtforschung, Bericht 139, Oct. 9-10, 1941, pp. 14-28.

14. Fuller, Franklyn B., and Briggs, Benjamin R.: Minimum Wave Drag of Bodies of Revolution With a Cylindrical Center Section. NACA TN 2535, 1951.

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