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CORRELATIONS INVOLVING PRESSURE FLUCTUATIONS IN HOMOGENEOUS TURBULENCE

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SUMMARY

It is shown that the correlation of fluctuating static pressure (in an incompressible and homogeneous turbulence) with any fluctuating quantity in the flow field can be expressed in terms of the correlation of the same quantity with two or more components of the velocity.

The correlations of pressure with itself and of pressure with two velocity components are investigated in detail for the case of isotropic turbulence. These correlations can be expressed in terms of correlations involving two velocity components at a point and two velocity components at another point. A postulated relation between the fourth-order and second-order correlations is investigated. This relation is satisfied, for example, if the joint probability density of the four components of velocity is Gaussian. The consequences of this relation are compared with the measurements of the fourth-order correlations.

The root-mean-square pressure and pressure gradients are computed from second-order correlation for a range of turbulence Reynolds numbers. Since the pressure gradient is related to diffusion of marked particles from a source, the computed pressure-gradient level is compared with that calculated from a set of diffusion measurements.

The triple correlation equation and plausible hypotheses relating higher order correlations with second-order correlation are examined for the possibility of getting a determinant set of equations for isotropic turbulence.

INTRODUCTION

Although in isotropic turbulence the dynamical (correlation) equation does not contain any pressure correlations, the correlation of pressure with itself is of interest in turbulent diffusion and in the study of sound generation by turbulence (for very low Mach numbers of turbulence velocities when there is little effect of sound on the turbulent field, that is, only a slight amount of energy is drained away from the turbulent energy in the form of sound waves). The static pressure fluctuations are also intimately connected with the onset of cavitation in turbulent flow of liquids.

The pressure-gradient fluctuation, in connection with diffusion in isotropic turbulence, was first considered by Taylor (ref. 1). He postulated that $\frac{1}{\rho^2}\overline{\left(\frac{\partial p}{\partial x_1}\right)^2}$ is of the order $\overline{u_1^2}\,\overline{\left(\frac{\partial u_1}{\partial x_1}\right)^2}$ independent of Reynolds number of turbulence[1]. Heisenberg (ref. 2) later derived, from detailed spectral considerations, an expression for $\overline{\left(\partial p/\partial x_1\right)^2}$ as follows:

$$(1/\rho_2)\overline{\left(\partial p/\partial x_1\right)^2} \approx \frac{\overline{u_1^2}\,\overline{\left(\partial u_1/\partial x_1\right)^2}}{(0.13)^2 R_\lambda} \quad \text{where} \quad R_\lambda = u^2\Big/\nu\sqrt{\overline{\left(\partial u_1/\partial x_1\right)^2}}$$

is the Reynolds number of turbulence and the symbols are defined in the symbol list.

Obukhov (ref. 3) derived an expression for the correlation of pressure with itself for distances between two points whose spacing is within Kolmogoroff's "inertial subrange." Batchelor (ref. 4) and Limber (ref. 5) have considered the pressure-pressure and pressure-and-two-velocity-component correlations in isotropic turbulence. Chandrasekhar (ref. 6) has considered these correlations for the isotropic turbulence in magnetohydrodynamics. All these correlations can be expressed in terms of the correlations involving two components of velocity at one point and two components at another point. These investigators have from the very beginning assumed a simple relation between the fourth-order and the second-order correlations. In order to compare the primary result with experiment it is necessary to have the results in terms of the fourth-order correlation.[2]

Next to the simplest case of istropic turbulence is homogeneous axisymmetric turbulence, in which all statistical properties are symmetric about a particular axis, instead of being spherically symmetric as in the case of isotropic turbulence. This introduces one new element: The energies in different components of the velocity are not the same and there is a transfer of energy from one component of velocity to another.

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[1] In the present paper an overbar indicates mean value or statistical average and an underbar indicates a vector quantity.

[2] The main results were derived before the publication of references 4 to 6, but publication was delayed until the completion of the experiments reported here.

This transfer is essentially due to the correlation between velocity and pressure or pressure gradient (ref. 7). This is in addition to the transfer of energy from big eddies to small eddies; both of these transfers are expressible in terms of the nonlinear terms of the equations of motion and these can be expressed in terms of the triple-order correlations. Axisymmetric turbulence is the only case of homogeneous turbulence that has been theoretically studied and even this has been largely limited to general tensor forms for the correlations and equations governing these correlations. The physics and solution of the problem remain virtually untouched. In the general case of homogeneous turbulence the pressure-velocity correlation is undoubtedly of interest.

Before some special correlations are taken up, correlations involving static pressure in general will be considered and it will be shown that these correlations can be expressed in terms of correlations involving more than two velocity components. This is convenient from the theoretical point of view, since it may then be possible to express higher-order correlations in terms of second-order correlations on the basis of some plausible hypotheses, such as those used by other investigators. It may be also noted that Heisenberg's expression for the spectral transfer term (ref. 2) is a hypothesis for the triple correlation in terms of the second-order correlations. The fact that correlations involving static pressure can be expressed in terms of correlations involving more than two components of velocity is also convenient from an experimental point of view, since at present there exists no technique for the measurement of static pressure fluctuations. On the other hand, the standard hot-wire technique can be extended to correlations involving more than two velocity components.

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SYMBOLS

A_1,A_2	dimensional constants
a,b,c,d	arbitrary unit vectors
a_i,b_i	direction cosines of two arbitrary directions at x
C	constant

C_1 absolute constant (eq. (28))

c velocity of light

c_i, d_i direction cosines of two arbitrary directions at $\underline{x}$

$\underline{E}$ electric-field strength

e output of hot-wire

e_1, e_2 output of hot-wires set at $\underline{x}$ and $\underline{x}'$, respectively

$f(r), g(r)$ correlations used by Von Kármán and Howarth

$\underline{H}$ magnetic-field strength

$\underline{h} = (\mu/4\pi\rho)\underline{H}$

$\underline{j}$ current in conducting fluid in motion

K universal constant

$L = \nu^{3/4}\epsilon^{1/4}$

M mesh spacing

n integer

$P = (1/\rho)(p - \overline{p}) + (1/2)\left(|\underline{h}|^2 - \overline{|\underline{h}|^2}\right)$

P' value of P at $\underline{x}'$

p static pressure fluctuations

q' any quantity at another point $\underline{x}'$

R scalar correlation

$R_{1,2,3}$ correlations

R_λ Reynolds number of turbulence

r distance between two points $\underline{x}$ and $\underline{x}'$, $(\xi_i \xi_i)^{1/2}$

$S_{1\ldots n}$ set of random variables

t time

U velocity in x-direction

u velocity component

u_i components of instantaneous fluctuating velocity, u_1, u_2, u_3 or u, v, w

$V(\underline{y})$ volume

v velocity in direction perpendicular to hot-wire

v' root-mean-square velocity in y-direction

w

x_i coordinates of point x

$x' = \underline{x} + \underline{\xi}$

y variable of integration

$\Delta = \dfrac{\partial^2}{\partial x_i \partial x_i}$

δ Kronecker delta

ϵ energy dissipation per unit mass of fluid, $(1/2)\left(d\overline{u_i u_i}/dt\right)$

λ Eulerian microscale, $\left[\overline{u_1^2}\Big/\overline{\left(\dfrac{\partial u_1}{\partial x_1}\right)^2}\right]^{1/2}$, $\frac{1}{4}\pi\mu\sigma$

λ_η Lagrangian microscale

μ permeability

ν kinematic viscosity

ξ component of displacement vector

ρ density

ρ charge

σ conductivity

τ time difference

$\phi(S_1,S_2,S_3,S_4)$ joint probability density of $S_1 \ldots S_4$

$\psi\left(t_1,t_2,t_3,t_4\right)$ characteristic function of ϕ; t_1, t_2, t_3, and t_4 are independent variables

$(\underline{\ })$ vector

$\overline{(\ )}$ statistical average or mean

Subscripts:

e excess

i,j,k free indices characterizing general vector component; each can take value of 1, 2, or 3 corresponding to component in x-, y-, and z-directions, respectively

l,m,n directions (see fig. 1)

p,q,r,s free indices

p refers to pressure

GENERAL RELATIONS

Consider a statistically homogeneous and incompressible turbulence with no mean velocity (i.e., a "box turbulence"). The equations of motion for an arbitrary point $\underline{x}$ are for momentum

$$\left(\partial u_i/\partial t\right) + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\,\Delta u_i \tag{1}$$

and for continuity

$$\partial u_i/\partial x_i = 0 \tag{1a}$$

where the symbols have the usual meaning and repeated index means summation over the index. Taking the divergence of equation (1) and making use of equation (1a), the following well-known relation results:

$$\frac{1}{\rho}\frac{\partial^2 p}{\partial x_i\,\partial x_i} = -\frac{\partial^2 u_i u_j}{\partial x_i\,\partial x_j} \tag{2}$$

The fact that the pressure in an incompressible viscous fluid satisfies Poisson's equations means that it is not a primary variable, since the pressure can be expressed in terms of the velocity derivatives by using the well-known solution for Poisson's equation.[3]

Multiplying equation (2) by q', any quantity at another point $\underline{x}'$, and taking a statistical average,

$$\frac{1}{\rho}\overline{q'\frac{\partial^2 p}{\partial x_i\,\partial x_i}} = -\overline{q'\frac{\partial^2 u_i u_j}{\partial x_i\,\partial x_j}}$$

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[3]The solution of the equation $\Delta^n\phi = \Gamma(\underline{x})$ where $\Delta = \partial^2/\partial x_i\partial x_i$ is discussed in reference 8. In the following discussion the solutions for $n = 1$ (Poisson's equation) and $n = 2$, in three dimensions, are needed. Here ϕ and Γ are arbitrary functions of $\underline{x}$ and n is an integer.

Making use of the fact that the turbulence is statistically homogeneous and interchanging the order of differentiation and averaging,

$$\frac{1}{\rho}\frac{\partial^2 \overline{pq'}}{\partial\xi_i\,\partial\xi_i} = -\frac{\partial^2 \overline{u_i u_j q'}}{\partial\xi_i\,\partial\xi_j} \tag{3}$$

where $\overline{p(\underline{x})q'(\underline{x}')} = \overline{pq'}(\xi)$ and $\underline{x}' = \underline{x} + \underline{\xi}$. An overbar denotes statistical average. Equation (3) is just the Poisson equation for $\overline{pq'}$ and its solution (ref. 8) can be written immediately as

$$\frac{1}{\rho}\,\overline{pq'}(\xi) = \frac{1}{4\pi}\int\frac{\partial^2 \overline{u_i u_j q'}(\underline{y})}{\partial y_i\,\partial y_j}\,\frac{dV(\underline{y})}{|\underline{y} - \underline{\xi}|} \tag{4}$$

where $dV(\underline{y})$ is a volume element. Equation (3) in one form or another is the basis of the recent investigations reported in references 3 to 6 which have been mentioned earlier. Next particular cases of equation (3) are considered.

Case (1): $q' = u_k'$

When $q' = u_k'$,

$$\frac{1}{\rho}\,\overline{pu_k'} = \frac{1}{4\pi}\int\frac{\partial^2 \overline{u_i u_j u_k'}}{\partial y_i\,\partial y_j}\,\frac{dV(\underline{y})}{|\underline{y} - \underline{\xi}|} \tag{5}$$

The quantity $\overline{pu_k'}$ is zero in isotropic turbulence because it is a first-order isotropic tensor, but it is different from zero in the general case of homogeneous turbulence. Its role in the axisymmetric turbulence is discussed in reference 7. In axisymmetric turbulence $\overline{pu_k'}$ gives the transfer of energy from one component of velocity to another. As in isotropic turbulence there is transfer of energy from the large eddies to small eddies which is given directly by $\overline{u_i u_j u_k'}$. This transfer and the transfer of energy from one component to another can both be expressed in terms of $\overline{u_i u_j u_k'}$.

$$\text{Case (2):}\quad q' = \frac{1}{\rho}\frac{\partial^2 p'}{\partial x_i'\partial x_i'} = -\frac{\partial^2 u_k'u_l'}{\partial x_k'\partial x_l'}$$

Substituting $q' = \frac{1}{\rho}\frac{\partial^2 p'}{\partial x_i'\partial x_i'} = -\frac{\partial^2 u_k'u_l'}{\partial x_k'\partial x_l'}$ into equation (3), interchanging the order of differentiation, and averaging,

$$\frac{1}{\rho^2}\frac{\partial^4 \overline{pp'}}{\partial \xi_i\ \partial \xi_i\ \partial \xi_j\ \partial \xi_j} = \frac{\partial^4 \overline{u_i u_j u_k' u_l'}}{\partial \xi_i\ \partial \xi_j\ \partial \xi_k\ \partial \xi_l} \tag{6}$$

Making use of the fact that the elementary solution of the bi-Laplacian in three dimensions is $r/2$ where $r = \sqrt{\xi_i \xi_i}$ is the distance between the point of observation and the point of integration (ref. 8),

$$\frac{1}{\rho^2}\overline{pp'(\underline{\xi})} = -\frac{1}{8\pi}\int \frac{\partial^4 \overline{u_i u_j u_k' u_l'}}{\partial y_i\ \partial y_j\ \partial y_k\ \partial y_l}\left|\underline{y} - \underline{\xi}\right|\ dV(\underline{y}) \tag{7}$$

$$\text{Case (3):}\quad q' = u_k'u_l'$$

When $q' = u_k'u_l'$,

$$\frac{1}{\rho}\overline{pu_k'u_l'} = \frac{1}{4\pi}\int \frac{\partial^2 \overline{u_i u_j u_k' u_l'}}{\partial y_i\ \partial y_j}\ \frac{dV(\underline{y})}{\left|\underline{y} - \underline{\xi}\right|} \tag{8}$$

Case (4): Correlations Involving Pressure in an

Incompressible, Highly Conducting Fluid

The correlations involving pressure in an incompressible, highly conducting fluid have been investigated in reference 6. The current $\underline{j}$ in a conducting fluid in motion is given by

$$\underline{j} = \sigma(c\underline{E} + \mu\underline{u} \times \underline{H}) + (\rho_e \underline{u}/c)$$

where σ is the conductivity, $\underline{E}$ is the electric-field strength, $\underline{u}$ is velocity vector, $\underline{H}$ is the magnetic-field strength, μ is the permeability, c is the velocity of light, and ρ_e is the excess charge. The quantity $\rho_e \underline{U}/c$ represents the convection current. As $\sigma \longrightarrow \infty$ it may be assumed that

$$\underline{E} \approx -\mu \underline{u} \times \underline{H}/c$$

otherwise the current will become large. The energy in the electric field is of the order $|\underline{u}|^2/c^2$ of the energy in the magnetic field, therefore it can be negelected. In this approximation only the interaction between the two fields $\underline{u}$ and $\underline{H}$ needs to be considered, and the equations of motion are (ref. 6)

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}\left(u_i u_j - h_i h_j\right) = -\frac{1}{\rho}\frac{\partial}{\partial x_i}\left(p + \frac{1}{\rho}|\underline{h}|^2\right) + \nu \, \Delta u_i \tag{9}$$

$$\frac{\partial h_i}{\partial t} + \frac{\partial}{\partial x_k}\left(h_i h_j - u_i u_j\right) = \lambda \, \Delta h_i \tag{10}$$

where $\underline{h} = \left(\frac{\mu}{4\pi\rho}\right)\underline{H}$ and $\lambda = \frac{1}{4}\pi\mu\sigma$. Taking the divergence of equation (9) and making use of the condition of incompressibility (1a),

$$\frac{\partial^2 P}{\partial x_i \, \partial x_i} = -\frac{\partial^2\left(u_i u_j - h_i h_j\right)}{\partial x_i \, \partial x_j} \tag{11}$$

where

$$P = \frac{1}{\rho}\left(p - \overline{p}\right) + \frac{1}{2}\left(|\underline{h}|^2 - \overline{|\underline{h}|^2}\right)$$

Chandrasekhar (ref. 6) has considered three correlations $\overline{Pu_k'u_l'}$, $\overline{Ph_k'h_l'}$ and $\overline{PP'}$. It can be seen from equations (7) and (8) that these correlations can be expressed in terms of $\overline{u_i u_j u_k'u_l'}$, $\overline{h_i h_j h_k'h_l'}$, and $\overline{u_i u_j h_k'h_l'}$.

The quantities $\overline{Pu_k'u_l'}$ and $\overline{pu_k'u_l'}$ have similar tensor forms and they satisfy similar differential equations (equations (11) and (2)). Also $\overline{PP'}$ and $\overline{pp'}$ satisfy similar differential equations, so that it is not necessary to calculate separately all these expressions. When the expressions for $\overline{pp'}$ and $\overline{pu_k'u_l'}$ have been calculated the corresponding expressions for other correlations can be written at once. Furthermore, $\overline{pp'}$ and $\overline{pu_k'u_l'}$ are not independent. If $q' = p'$ in equation (3)

$$\frac{1}{\rho}\frac{\partial\overline{pp'}}{\partial\xi_i\,\partial\xi_i} = -\frac{\partial\overline{u_iu_jp'}}{\partial\xi_i\,\partial\xi_j}$$

In all these cases (except case (1)) the correlations involving pressure can be expressed in terms of correlations between two velocity components (or two magnetic-field components) at one point and similar quantities at another point. Some investigators have assumed that the joint probability density of these four quantities is Gaussian (refs. 4 and 9). If S_1, S_2, S_3, and S_4 are any such variables then it can be shown that

$$\overline{S_1S_2S_3S_4} = \overline{S_1S_3}\;\overline{S_2S_4} + \overline{S_1S_4}\;\overline{S_2S_3} + \overline{S_1S_2}\;\overline{S_3S_4} \tag{12}$$

and in particular

$$\overline{u_iu_ju_k'u_l'} = \overline{u_iu_k'}\;\overline{u_ju_l'} + \overline{u_iu_l'}\;\overline{u_ju_k'} + \overline{u_iu_j}\;\overline{u_k'u_l'} \tag{13}$$

$$\overline{h_ih_jh_k'h_l'} = \overline{h_ih_k'}\;\overline{h_jh_l'} + \overline{h_ih_l'}\;\overline{h_jh_k'} + \overline{h_ih_j}\;\overline{h_k'h_l'} \tag{14}$$

$$\overline{u_iu_jh_k'h_l'} = \overline{u_ih_k'}\;\overline{u_jh_l'} + \overline{u_ih_l'}\;\overline{u_jh_k'} + \overline{u_iu_j}\;\overline{h_k'h_l'} \tag{15}$$

(The opportunity has been taken to correct the expression corresponding to equation (15) in reference 6 where the last term $\overline{u_iu_j}\;\overline{h_k'h_l'} = \overline{u_i^2h_k^2}\delta_{ij}\delta_{kl}$ (no summation) is missing).

The hypothesis of Gaussian joint probability density makes all odd moments (e.g., triple correlations) zero. However, the above relations are only integral conditions on the joint probability density and the hypothesis of Gaussian probability density is strictly not necessary for their validity. Without making any such hypothesis about the probability density, equation (12) is taken as a plausible hypothesis.

The joint probability density $\phi(S_1,S_2,S_3,S_4)$ is the Fourier transform of the characteristic function $\psi(t_1,t_2,t_3,t_4)$

$$\psi(t_1,t_2,t_3,t_4) = \iiiint \phi(S_1,S_2,S_3,S_4)e^{iS_k t_k}\, dS_1\, dS_2\, dS_3\, dS_4$$

$$= e^{-\frac{1}{2}\left(\overline{S_k S_l} t_k t_l\right)}$$

where ϕ is Gaussian and ψ is a function of independent variables t_1 to t_4. The moments of all orders are determined from the behavior of the characteristic function near the origin. As has been pointed out by Chandrasekhar (ref. 6), if

$$\psi(t_1,t_2,t_3,t_4) = l^{-\frac{1}{2}\left(\overline{S_k S_l} t_k t_l\right)}\left(1 + C_{pqr}t_p t_q t_r\right)$$

is taken as the characteristic function, then there is the same relation between second- and fourth-order moment as for a Gaussian probability density, but the third-order moment will not, in general, be zero. In view of this it can be asserted that the hypothesis of equation (12) does not impose any restriction on the triple correlations.

Cases (2) and (3) will be investigated in detail for the simple case of isotropic turbulence and the results compared with experimental observations.

FOURTH-ORDER VELOCITY CORRELATIONS IN ISOTROPIC TURBULENCE

The fourth-order correlation $\overline{u_i u_j u_k' u_l'}$ enters prominently in the expressions for the correlations involving static pressure fluctuations. It is a fourth-order isotropic tensor and its form can be derived using some simple results of the invariant theory (see appendix). Thus:

$$\overline{u_i u_j u_k' u_l'} = \frac{R_{ll}^{ll}(r) + R_{nn}^{nn}(r) - 2R_{nn}^{ll}(r) - 4R_{nl}^{nl}}{r^4} \xi_i \xi_j \xi_k \xi_l + R_{nn}^{mm}(r)\delta_{ij}\delta_{kl} +$$

$$\frac{1}{2}\left[R_{nn}^{nn}(r) - R_{nn}^{mm}(r)\right]\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) + \frac{1}{r^2}\left[R_{nn}^{ll}(r) -\right.$$

$$\left.R_{nn}^{mm}(r)\right]\left(\xi_i \xi_j \delta_{kl} + \xi_k \xi_l \delta_{ij}\right) + \frac{1}{2r^2}\left[2R_{nl}^{nl}(r) - R_{nn}^{nn}(r) -\right.$$

$$\left.R_{nn}^{mm}(r)\right]\left(\xi_i \xi_k \delta_{lj} + \xi_j \xi_k \delta_{il} + \xi_j \xi_l \delta_{ki} + \xi_i \xi_l \delta_{kj}\right) \quad (16)$$

Since $\overline{u_i u_j u_k' u_l'}$ is not solenoidal, the continuity equation gives no relation between the five scalars defining $\overline{u_i u_j u_k' u_l'}$. However, it gives relations between derivatives of $\overline{u_i u_j u_k' u_l'}$ and other fourth-order correlations involving derivatives of velocity. Some representative correlations are shown in figure 1.

Under the assumption of equation (12),

$$\left.\begin{aligned} R_{ll}^{ll}(r) &= 2\left[R_l^l(r)\right]^2 + R_{ll}^{\ 2} \\ R_{nn}^{nn}(r) &= 2\left[R_n^n(r)\right]^2 + R_{nn}^{\ 2} \\ R_{nl}^{nl}(r) &= R_l^l(r)R_n^n(r) \\ R_{nn}^{ll} &= R_{ll}R_{nn} \\ R_{nn}^{mm} &= R_{nn}R_{mm} \end{aligned}\right\} \quad (17)$$

These relations and the expression for $\overline{u_i u_j u_k' u_l'}$ for the case $k = l = i$ were first given by Millionshtchikov (ref. 9). The second-order correlations $R_n^n(r)$ and $R_l^l(r)$ (essentially the Von Kármán-Howarth $g(r)$

and $f(r)$) are connected by the continuity equation of incompressible flow (ref. 10).

$$R_n^n(r) = R_l^l(r) + \frac{r}{2}\frac{\partial}{\partial r}R_l^l(r) \tag{18}$$

With the corresponding inverse relation,

$$R_l^l(r) = \frac{1}{r^2}\int_0^r R_n^n(y)y\,dy \tag{18a}$$

The fourth-order correlation $\overline{u_i u_j u_k u_l{}'}$ involving three velocity components at one point and one at another is not directly related to the pressure correlations that have been considered. It enters explicitly in the equation for the propagation of triple-order correlation which will be considered later. For isotropic turbulence, the correlation $\overline{u_i u_j u_k u_l{}'}$ has the following form (see the appendix):

$$\begin{aligned}\overline{u_i u_j u_k u_l{}'} = {} & \frac{1}{r^4}\left(R_{lll}^l + R_{nnn}^n - 3R_{lln}^n - 3R_{lnn}^n\right)\xi_i\xi_j\xi_k\xi_l + \\ & \frac{1}{3}R_{nnn}^n\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj}\right) + \\ & \frac{1}{r^2}\left(R_{lln}^n - \frac{1}{3}R_{nnn}^n\right)\left(\xi_i\xi_j\delta_{kl} + \xi_i\xi_k\delta_{jl} + \xi_j\xi_k\delta_{il}\right) + \\ & \frac{1}{r^2}\left(R_{lnn}^n - \frac{1}{3}R_{nnn}^n\right)\left(\xi_k\xi_l\delta_{ij} + \xi_l\xi_j\delta_{ik} + \xi_i\xi_l\delta_{jk}\right)\end{aligned} \tag{19}$$

The various correlations are shown in figure 1. Here $\overline{u_i u_j u_k u_l{}'}$ is a solenoidal tensor and the four correlations characterizing it are not independent. Taking the divergence of $\overline{u_i u_j u_k u_l{}'}$ with respect to the last index and equating it to zero,

$$\left.\begin{aligned}\frac{1}{r}\frac{\partial}{\partial r}\left(r^2R_{lnn}^l\right) &= -2R_{lln}^n + \frac{4}{3}R_{nnn}^n \\ \frac{1}{r}\frac{\partial}{\partial r}\left(r^2R_{lll}^l\right) &= 6R_{lln}^n\end{aligned}\right\} \tag{20}$$

Because of these two relations there are only two independent correlations characterizing the tensor $\overline{u_i u_j u_k u_l{}'}$. It is observed in passing that $\overline{u_i u_i u_k u_l{}'}$ is a solenoidal second-order tensor. This gives a relation between the four functions characterizing $\overline{u_i u_j u_k u_l{}'}$, which, however, is not independent of those written above and serves as a check on the algebraic errors. If the hypothesis of equation (12) is satisfied,

$$\left.\begin{aligned} R_{lll}^{l} &= 3R_{ll}R_{l}^{l} \\ R_{nnn}^{n} &= 3R_{nn}R_{n}^{n} \\ R_{lln}^{n} &= R_{ll}R_{n}^{n} \\ R_{lnn}^{n} &= R_{nn}R_{l}^{l} \end{aligned}\right\} \tag{21}$$

STATIC-PRESSURE CORRELATION IN ISOTROPIC TURBULENCE

For the isotropic case $\dfrac{\partial^4(\ )}{\partial\xi_i\ \partial\xi_i\ \partial\xi_j\ \partial\xi_j} = \dfrac{d^4[r(\ )]}{dr^4} + \dfrac{4}{r}\dfrac{d^3(\ )}{dr^3} = \dfrac{1}{r}\dfrac{d^4[r(\ )]}{dr^4}$ and the bi-Laplacian equation (6) simplifies to

$$\frac{1}{\rho^2 r}\frac{\partial^4 r\overline{pp'}(r)}{\partial r^4} = \frac{\partial^4 \overline{u_i u_j u_k{}' u_l{}'}}{\partial\xi_i\ \partial\xi_j\ \partial\xi_k\ \partial\xi_l} \tag{22}$$

After four repeated integrations, for any fixed value of t, corresponding to equation (7),

$$\frac{1}{\rho^2}\overline{pp'} = \frac{1}{6r}\int_r^\infty y(y - r)^3 \frac{\partial^4 \overline{u_i u_j u_k{}' u_l{}'}}{\partial y_i\ \partial y_j\ \partial y_k\ \partial y_l}\, dy \tag{23}$$

It was assumed that $\overline{pp'} \longrightarrow 0$ fast enough as $r \longrightarrow \infty$. Substituting equation (16) in equation (23) and carrying out some algebraic simplifications and integrations by parts,

$$\frac{1}{\rho^2}\overline{pp'}(r) = R_{ll}^{ll}(r) - R_{ll}{}^2 - 4\int_r^\infty\left[2R_{nl}^{nl} + R_{nn}^{ll} - R_{nn}^{nn} + \frac{r^2}{y^2}\left(R_{nn}^{nn} + R_{ll}^{ll} - 2R_{nn}^{ll} - 4R_{nl}^{nl}\right)\right]\frac{dy}{y} \tag{24}$$

The pressure gradient is of special interest since it is related to the diffusion of marked fluid particles from a fixed source (see the next section, "Pressure Gradient and Diffusion Measurements"). Now,

$$\overline{\left(\frac{\partial p}{\partial x_1}\right)^2} = \lim_{r \to 0} \overline{\frac{\partial p(x_1)}{\partial x_1}\,\frac{\partial p(x_1')}{\partial x_1'}}$$

where

$$x_1' = x_1 + r$$

Interchanging the order of integration and averaging,

$$\overline{\left(\frac{\partial p}{\partial x_1}\right)^2} = -\left[\frac{\partial^2\overline{pp'}(r)}{\partial r^2}\right]_{r=0}$$

Substituting for $\overline{pp'}$ from equation (24),

$$\frac{1}{\rho^2}\overline{\left(\frac{\partial p}{\partial x_1}\right)^2} = -\frac{d^2}{dr^2}\left[2R_{ll}^{ll}(r) - R_{nn}^{nn}(r)\right]_{r=0} + 8\int_0^\infty\left(R_{nn}^{nn} + R_{ll}^{ll} - 2R_{nn}^{ll} - 4R_{nl}^{nl}\right)\frac{dy}{y^3} \tag{25}$$

In order for the above integrals to converge it is necessary that the series expansion of $\left(R_{nn}^{nn} + R_{ll}^{ll} - 2R_{nn}^{ll} - 4R_{nl}^{nl}\right)$ at the origin begin with terms of order r^4; therefore $\frac{d^2}{dr^2}\left(R_{nn}^{nn} + R_{ll}^{ll} - 2R_{nn}^{ll} - 4R_{nl}^{nl}\right)_{r=0} = 0$. Use has been made of this fact in deriving expression (25). Under the assumption of equation (12), equations (24) and (25) become

$$\frac{1}{\rho^2}\,\overline{pp'}(r) = 2\int_r^{\infty}\left(y - \frac{r^2}{y}\right)\left[\frac{d}{dy}\,R_l^l(y)\right]^2 dy \tag{24a}$$

and

$$\frac{1}{\rho^2}\overline{\left(\frac{\partial p}{\partial x_1}\right)^2} = 4\int_0^{\infty}\frac{d}{dy}\left[R_l^l(y)\right]^2\frac{dy}{y} \tag{25a}$$

For vanishing turbulence Reynolds number, $R_\lambda \longrightarrow 0$ where $R_\lambda = \sqrt{\overline{u_1^2}}\,\lambda/\nu$ and

$$\lambda^2 = \overline{u_1^2}\Bigg/\overline{\left(\frac{\partial u_1}{\partial x_1}\right)^2}$$

The correlation $R_l^l(r)$ can be calculated neglecting the inertia terms of the equations of motion (ref. 11),

$$R_l^l(r) = \overline{u_1^2}\,\exp\left(-r^2/2\lambda^2\right)$$

Substituting this result in equations (24a) and (25a),

$$\overline{pp'}(r) = \rho^2\left[\overline{u_1^2}\right]^2 \exp\left(-r^2/\lambda^2\right) = R_l^{l^2}(r) \tag{26}$$

and

$$\overline{\left(\frac{\partial p}{\partial x_1}\right)^2} = \frac{2\rho^2\left(\overline{u_1^2}\right)^2}{\lambda^2} \tag{27}$$

On the basis of Kolmogoroff's hypothesis of local isotropy, for large Reynolds numbers and values of r with the "inertial subrange" (ref. 12)

$$\overline{(u - u')^2} = 2\left[\overline{u^2} - R_l^l(r)\right] = C_1(\nu\epsilon)^{1/2}\left(\frac{r}{L}\right)^{2/3} \tag{28}$$

where u is the velocity along the displacement vector, $\underline{\xi} = \underline{x}' - \underline{x}$, $r^2 = \xi_i \xi_i$, $\epsilon = \frac{1}{2}\frac{d\overline{u_i u_i}}{dt}$ is the energy dissipation per unit mass of the fluid, the length $L = \nu^{3/4}\epsilon^{1/4}$, and C_1 is an absolute constant. The quantities ϵ and λ are connected by Taylor's decay equation (ref. 1):

$$\frac{1}{2}\frac{d\overline{u_i u_i}}{dt} = -15\nu\overline{u_1^2}/\lambda^2$$

Hence for values of r within the inertial subrange

$$\frac{1}{2\rho^2}\left(\overline{p^2} - \overline{pp'}\right) = \int_0^r \left[\frac{d}{dy} R_l^l(y)\right]^2 y\, dy - r^2 \int_r^\infty \left[\frac{d}{dy} R_l^l(y)\right]^2 \frac{dy}{y} . \tag{29}$$

The integrals in equation (29) can be evaluated approximately by using equation (28). After some transformation the result is

$$\frac{1}{2}\,\rho^2\left(\overline{p^2} - \overline{pp'}\right) = \frac{C^2}{2}\,\epsilon\nu\left(\frac{r}{L}\right)^{4/3}$$

or

$$\frac{1}{\rho}\overline{(p - p')^2} = \left[\overline{(u - u')^2}\right]^2 \tag{30}$$

This relation was first given by Obukhov (ref. 3) and equations (24a) and (25a) were given by Batchelor (ref. 4).

PRESSURE GRADIENT AND DIFFUSION MEASUREMENTS

In reference (13) the "Lagrangian" microscale λ_η was related to the acceleration in isotropic turbulence without the neglect of viscosity. A brief résumé follows:

$$2/\lambda_\eta^{\,2} = \frac{1}{(v')^4}\left[\overline{\left(\frac{dv}{dt}\right)^2} - \left(\frac{dv'}{dt}\right)^2\right] \tag{31}$$

where v' is the root-mean-square velocity in the y-direction. Squaring and averaging the v-component equation of motion,

$$\overline{\left(\frac{dv}{dt}\right)^2} = \overline{\frac{1}{\rho^2}\left(\frac{\partial p}{\partial y}\right)^2} + \nu^2\overline{(\Delta v)^2} + \frac{2\nu}{\rho}\overline{\frac{\partial p}{\partial y}\Delta v} \tag{32}$$

The term $\overline{\left(\frac{\partial p}{\partial y}\Delta v\right)} = 0$ because of isotropy. All terms in the expansion of equation (32) can be expressed in terms of the derivatives of $R_l^l(r)$ by making use of the relations given by Von Kármán and Howarth (ref. 10). Thus $\overline{(\Delta v)^2} = \frac{35}{3}\left[\frac{\partial^4 R_l^l(r)}{\partial r^4}\right]_{r=0}$. Townsend (ref. 14) has made measurements of $\left[\frac{\partial^4 R_l^l(r)}{\partial r^4}\right]_{r=0}$ and his result is

$$\overline{(\Delta v)^2} = \frac{35}{3}\left[\frac{\partial^4}{\partial r^4}R_l^l(r)\right]_{r=0} = \frac{35}{3}\,\frac{(v')^2}{\lambda^4}\left(\frac{30}{7} + 0.2R_\lambda\right) \tag{33}$$

For the isotropic turbulence $\frac{d(v')^2}{dt} = -10\nu\,\frac{(v')^2}{\lambda^2}$ (the "decay equation" formulated by Taylor in ref. 1). Thus

$$\frac{\lambda^2}{(v')^4}\left(\frac{dv'}{dt}\right)^2 = 25/R_\lambda^{\,2} \tag{34}$$

Substituting equations (32), (33), and (34) in equation (31),[4]

$$\frac{\rho^2(v')^4}{\lambda^2\overline{\left(\frac{\partial p}{\partial y}\right)^2}} = \left(2\,\frac{\lambda^2}{\lambda_\eta{}^2} - \frac{25}{R_\lambda{}^2} - \frac{2.3}{R_\lambda}\right)^{-1} \tag{35}$$

Heisenberg (ref. 2) has calculated an expression for pressure gradient using the hypothesis that various Fourier components of the velocity field are statistically independent. As far as the fourth-order correlations are concerned the consequences of this hypothesis can be shown to be equivalent to equation (12) (see ref. 4). His result is

$$\frac{\rho^2(v')^4}{\lambda^2\overline{\left(\frac{\partial p}{\partial y}\right)^2}} = \frac{3K}{78.5}\,R_\lambda \tag{36}$$

Instead of Heisenberg's original extrapolation formula for the stationary velocity spectrum, Chandrasekhar's solution (ref. 16) was used to calculate equation (36). Here K is a "universal" constant which has to be determined from experimental data. Heisenberg found from turbulence decay that $K = 0.85$. Lee (ref. 17) gives its value as 0.13 based on

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[4]This is the first step toward the establishment of a possible relation between Lagrangian and Eulerian correlations. If nondecaying isotropic turbulence is considered then the Lagrangian correlation

$$\overline{u_1(t)u_1(t+\tau)} = R_1(\tau) = \overline{u_1{}^2} + (-1)^n\overline{\left(\frac{d^n u_1}{dt^n}\right)^2}\,\frac{\tau^{2n}}{(2n)!}$$

Each term of this series can be expressed in terms of the Eulerian correlations by using equations of motion. The first nonconstant term involves pressure or fourth-order velocity correlations. The next term will involve still higher order correlations. It may be possible to relate higher order correlations with second-order correlations by making a further hypothesis about the joint probability density of the velocities at two points. The first step of this computation has just been carried out. The next step will involve many more calculations. This is probably not the best method of attack of the problem of relating the Lagrangian and the Eulerian correlations. Frenkiel has related Lagrangian and Eulerian microscales under some assumptions (ref. 15).

skewness of the probability density of $\left(\frac{\partial u_1}{\partial x_1}\right)$ as measured by Townsend (ref. 14). Proudman (ref. 18) has calculated double and triple correlations from Chandrasekhar's solution of Heisenberg's self-preserving spectrum. He has estimated its value as 0.45 from comparison of theoretical correlations with measured correlation. Heisenberg's results are only approximate and the value of K depends on the range of spectrum (or correlation) which is made to fit the experimental data. The value of K = 0.45 has been taken as a compromise value.

PRESSURE-VELOCITY CORRELATIONS $\overline{pu_k{}'u_l{}'}$

Equation (8) relates the correlation $\overline{pu_k{}'u_l{}'}$ with the fourth-order correlations

$$\frac{1}{\rho}\frac{\partial^2\overline{pu_k{}'u_l{}'}}{\partial\xi_i\,\partial\xi_j} = -\frac{\partial^2\overline{u_iu_ju_k{}'u_l{}'}}{\partial\xi_i\,\partial\xi_j} \tag{37}$$

Here $\overline{pu_k{}'u_l{}'}$ is a second-order isotropic tensor; therefore it has the same form as, for example, the tensor $\overline{u_ku_l{}'}$:

$$\begin{aligned}\overline{pu_k{}'u_l{}'} &= \frac{R_p^{ll}(r) - R_p^{nn}(r)}{r^2}\,\xi_k\xi_l + R_p^{nn}(r)\delta_{kl} \\ &= p_1(r)\frac{\xi_k\xi_l}{r^2} + p_2(r)\delta_{kl}\end{aligned} \tag{38}$$

The correlations $R_p^{ll}(r)$ and $R_p^{nn}(r)$ are shown in figure 1. Substituting the expression for $\overline{u_iu_ju_k{}'u_l{}'}$ (eq. (16)) in the right-hand side of equation (37),

$$-\frac{\partial^2 \overline{u_i u_j u_k' u_l'}}{\partial \xi_i \, \partial \xi_j} = -\left[\frac{\partial^2}{\partial r^2}\left(R_{ll}^{ll} - R_{nn}^{ll}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(R_{nn}^{nn} + 4R_{ll}^{ll} - 6R_{nn}^{ll} - \right.\right.$$

$$\left.\left. 12R_{nl}^{nl} + R_{nn}^{mm}\right) + \frac{2}{r^2}\left(5R_{nn}^{nn} + R_{ll}^{ll} - 5R_{nn}^{ll} - 12R_{nl}^{nl} - R_{nn}^{mm}\right)\right]\frac{\xi_k \xi_l}{r^2} -$$

$$\left[\frac{1}{2r}\frac{\partial^2}{\partial r^2} r\left(2R_{nl}^{nl} + 3R_{nn}^{mm} - R_{nn}^{nn}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(2R_{nn}^{ll} + 4R_{nl}^{nl} - \right.\right.$$

$$\left.\left. R_{nn}^{nn} - R_{nn}^{mm}\right) + \frac{4}{r^2}\left(R_{nn}^{ll} + 2R_{nl}^{nl} - R_{nn}^{nn}\right)\right]\delta_{kl}$$

$$= -\psi_1(r)\frac{\xi_k \xi_l}{r^2} - \psi_2(r)\delta_{kl} \tag{39}$$

where ψ_1 and ψ_2 are functions of r which are defined by the above equation. Substituting equations (38) and (39) in equation (37),

$$\frac{1}{r}\frac{d}{dr}\left[\frac{1}{r^4}\frac{d}{dr}\left(p_1 r^3\right)\right]\xi_k \xi_l + \left(\frac{2p_1}{r^2} + \Delta p_2\right)\delta_{kl} = -\psi_1 \frac{\xi_k \xi_l}{r^2} - \psi_2 \delta_{kl}$$

Equating coefficients of the above equation,

$$\frac{d}{dr}\left[\frac{1}{r^4}\frac{d}{dr}\left(p_1 r^3\right)\right] = -\frac{1}{r}\psi_1(r)$$

After integrating twice,

$$p_1 r^3 = -\frac{1}{5}\int_r^\infty \left(y^5 - r^5\right)\psi_1 \frac{dy}{y} \tag{40}$$

and

$$\Delta p_2 = \frac{1}{r}\frac{\partial^2 r p_2}{\partial r^2} = -\psi_2 - \frac{2p_1}{r^2}$$

After integrating twice,

$$p_2 = \int_r^\infty \left(\psi_2 + \frac{2p_1}{y^2}\right)\left(y - \frac{r^2}{y}\right) dy$$

However, it is more convenient to calculate $p_1 + 3p_2$, the correlation between static and dynamic pressure. It follows directly from equations (37) and (39) that

$$\Delta\left(p_1 + 3p_2\right) = \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(p_1 + 3p_2\right)\right] = -\left(\psi_1 + 3\psi_2\right)$$

After integrating twice,

$$p_1 + 3p_2 = \int_r^\infty \left(\psi_1 + 3\psi_2\right)\left(y - \frac{y^2}{r}\right) dy \tag{41}$$

Substituting for ψ_1 in equation (40) and integrating by parts,

$$p_1(r) = -R_{ll}^{ll} + R_{nn}^{ll} - \frac{1}{5r^3}\int_0^r \left(7R_{nn}^{nn} + 2R_{ll}^{ll} + 12R_{nl}^{nl} - 5R_{nn}^{mm} - 4R_{nn}^{ll}\right)y^2\, dy - \frac{12r^5}{5}\int_r^\infty \left(R_{nn}^{nn} + R_{ll}^{ll} - 2R_{nn}^{ll} + 4R_{nl}^{nl}\right)\frac{dy}{y^3} \tag{42}$$

and from equation (41),

$$p_1 + 3p_2 = \frac{1}{\rho}\overline{pu_k'u_k'}$$

$$= -R_{ll}^{ll} - 2R_{nn}^{ll} + 3\left(\overline{u_1^2}\right)^2 - 2\int_r^\infty \left(R_{nn}^{nn} - R_{nn}^{ll} - R_{ll}^{ll} + R_{nn}^{mm}\right)\frac{dy}{y} \tag{43}$$

From equation (38) and equations (42) and (43),

$$R_p^{nn} = -R_{nn}^{ll} - \left(\overline{u_1^2}\right)^2 - \frac{1}{15r^3}\int_0^r \left(7R_{nn}^{nn} + 2R_{ll}^{ll} + 12R_{nl}^{nl} - 5R_{nn}^{mm} + 4R_{nn}^{ll}\right)y^2\,dy - \frac{4r^5}{5}\int_r^\infty \left(R_{nn}^{nn} + R_{ll}^{ll} - 2R_{nn}^{ll} - 4R_{nl}^{nl}\right)\frac{dy}{y^3} - \frac{2}{3}\int_r^\infty \left(R_{nn}^{nn} - R_{nn}^{ll} - R_{ll}^{ll} - R_{nn}^{mm}\right)\frac{dy}{y} \tag{44}$$

and

$$R_p^{ll} = -R_{ll}^{ll} + \left(\overline{u_1^2}\right)^2 + \frac{2}{15r^3}\int_0^r \left(7R_{nn}^{nn} + 2R_{ll}^{ll} + 12R_{nl}^{nl} - 5R_{nn}^{mm} - 4R_{nn}^{ll}\right)y^2\,dy + \frac{8}{5}\int_r^\infty \left(R_{nn}^{nn} + R_{ll}^{ll} - 2R_{nn}^{ll} - 4R_{nl}^{nl}\right)\frac{dy}{y^3} - \frac{2}{3}\int_r^\infty \left(R_{nn}^{nn} - R_{nn}^{ll} - R_{ll}^{ll} - R_{nn}^{mm}\right)\frac{dy}{y} \tag{45}$$

Using the hypothesis of equation (12), equations (43), (44), and (45) become

$$\frac{1}{\rho}\overline{pu_k'u_k'} = R_p^{ll} + 2R_p^{nn} = -\int_r^\infty \left[\frac{d}{dy} R_l^l(y)\right]^2 y\,dy \tag{43a}$$

$$R_p^{nn} = -\int_0^r \left(\frac{7}{30}\frac{y^4}{r^3}\right)\left(\frac{d}{dy} R_l^l\right)^2 dy - \int_r^\infty \left(\frac{2r^2}{y} + \frac{2y}{3}\right)\left(\frac{d}{dy} R_l^l\right)^2 dy \tag{44a}$$

$$R_p^{ll} = \int_0^r \left(\frac{14y^4}{30r^3}\right)\left(\frac{d}{dy} R_l^l\right)^2 dy + \int_r^\infty \left(\frac{4r^2}{5y} - \frac{2y}{3}\right)\left(\frac{d}{dy} R_l^l\right)^2 dy \tag{45a}$$

Comparing equations (24a) and (43a) it is seen that

$$\frac{1}{\rho^2}\,\overline{p^2} = -2\overline{pu_k u_k} \tag{46}$$

This result is a consequence of the equations of motion and of equation (12).

As in the case of pressure-pressure correlation, an explicit relation for pressure-velocity correlations for the limiting case of small and large Reynolds numbers can be derived (see eq. (26) and the related text). For $R_\lambda \longrightarrow 0$,

$$R_l^l(r) = \overline{u_1{}^2}\, \exp\left(-r^2/2\lambda^2\right)$$

Substituting this in equation (43a),

$$\frac{1}{\rho}\,\overline{pu_k{}'u_k{}'} = -\frac{\left(\overline{u_1{}^2}\right)^2}{2}\left(1 + \frac{r^2}{\lambda^2}\right) \exp\left(-r^2/\lambda^2\right) \tag{47}$$

For $R_\lambda \longrightarrow \infty$, and, on the basis of Kolmogoroff's hypothesis of local similarity, for values of r within the inertial subrange (see eqs. (28), (29), and (30) and the related text)

$$\overline{(u - u')^2} = 2\left[-R_l^l(r) + \overline{u^2}\right] = c(\nu\epsilon)^{1/2}\left(\frac{r}{L}\right)^{2/3} \tag{48}$$

Substituting this result in equation (43a),

$$\frac{1}{\rho}\left(\overline{pu_k u_k} - \overline{pu_k{}'u_k{}'}\right) = \frac{1}{12}\, c^2 \nu\epsilon\left(\frac{r}{L}\right)^{2/3} \tag{49}$$

or using equation (48)

$$\frac{1}{\rho}\left(\overline{pu_k u_k} - \overline{pu_k{}'u_k{}'}\right) = 12\left[\overline{(u - u')^2}\right]^2 \tag{50}$$

or using equation (30)

$$\frac{1}{\rho}\left(\overline{pu_k u_k} - \overline{pu_k' u_k'}\right) = \frac{1}{6\rho^2}\left(\overline{p^2} - \overline{pp'}\right) \tag{51}$$

Equations (43a), (44a), and (45a) and (49), (50), and (51) were first given by Limber (ref. 5).

TRIPLE-CORRELATION EQUATION IN ISOTROPIC TURBULENCE

Consider the equation of motion at the point $\underline{x}$ and time t

$$\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\,\Delta u_i$$

Multiplying this equation by $u_k' u_l'$ and taking the statistical average,

$$\overline{u_k' u_l' \frac{\partial u_i}{\partial t}} + \frac{\partial \overline{u_i u_j u_k' u_l'}}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p u_k' u_l'}}{\partial x_i} + \nu\,\Delta\overline{u_i u_k' u_l'} \tag{52}$$

In order to express $\overline{u_k' u_l' \frac{\partial u_i}{\partial t}}$ in terms of the triple correlation $\overline{u_k' u_l' u_i'}$, it is necessary to proceed as follows: Consider the equations of motion at $\underline{x}'$ and t:

$$\frac{\partial u_k'}{\partial t} + \frac{\partial u_k' u_s'}{\partial x_s'} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_k'} + \nu\,\Delta' u_k'$$

$$\frac{\partial u_l'}{\partial t} + \frac{\partial u_l' u_s'}{\partial x_s'} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_l'} + \nu\,\Delta' u_l'$$

Multiply the first equation by u_l' and the second equation by u_k' and add the two equations:

$$\frac{\partial u_k' u_l'}{\partial t} + \frac{\partial u_k' u_l' u_s'}{\partial x_s'} = -\frac{1}{\rho}\left(u_l' \frac{\partial p'}{\partial x_k'} + u_k' \frac{\partial p'}{\partial x_l'}\right) + \nu\left(u_l' \Delta' u_k' + u_k' \Delta' u_l'\right)$$

Multiplying the last equation by u_i and taking the statistical average,

$$\overline{u_i \frac{\partial u_k' u_l'}{\partial t}} + \frac{\partial \overline{u_k' u_l' u_s' u_i}}{\partial x_s'} = -\frac{1}{\rho}\overline{u_i\left(u_l' \frac{\partial p'}{\partial x_k'} + u_k' \frac{\partial p'}{\partial x_l'}\right)} + \nu\overline{\left(u_l' \Delta' u_k' + u_k' \Delta' u_l'\right)u_i} \tag{53}$$

It is not possible to express $\overline{u_l' \Delta' u_k' u_i}$ in terms of $\overline{u_l' u_k' u_i}$ or to express $\overline{u_l' \frac{\partial p'}{\partial x_k'} u_i}$ in terms of $\overline{u_l' p' u_i}$ without introducing new correlations. The attempt to express $\overline{u_k' u_l' \frac{\partial u_i}{\partial t}}$ in terms of $\overline{u_k' u_l' u_i}$ has resulted in introducing two new correlations, each of which is solenoidal third-order isotropic tensor characterized by a single scalar. Adding equations (52) and (53) gives the equation for the propagation of $\overline{u_i u_k' u_l'}$

$$\frac{\partial \overline{u_i u_k' u_l'}}{\partial t} + \frac{\partial \overline{u_i u_j u_k' u_l'}}{\partial x_j} + \frac{\partial \overline{u_i u_k' u_l' u_s'}}{\partial x_s'} = -\frac{1}{\rho}\frac{\partial \overline{p u_k' u_l'}}{\partial x_i} - \frac{1}{\rho}\overline{\left(u_l' \frac{\partial p'}{\partial x_k'} + u_k' \frac{\partial p'}{\partial x_l'}\right)u_i} + \nu \Delta \overline{u_i u_k' u_l'} + \nu\overline{\left(u_l' \Delta' u_k' + u_k' \Delta' u_l'\right)u_i} \tag{54}$$

Millionshtchikov has derived an equation relating third- and fourth-order correlations (ref. 9). His result is

$$\frac{\partial \overline{u_k'u_l'u_i}}{\partial t} + \frac{\partial \overline{u_iu_ju_k'u_l'}}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{pu_k'u_l'}}{\partial x_i} + \nu\,\Delta\overline{u_iu_k'u_l'}$$

The first term is wrong; it should be replaced by $\overline{u_k'u_l'\frac{\partial u_l'}{\partial t}}$. Millionshtchikov used this equation together with the hypothesis of equation (12) to derive an expression for the second-order correlation which is valid when the inertia terms are small. As a first approximation he calculated the double correlation from the Kármán-Howarth equation by neglecting the triple correlation, which is valid as $R_\lambda \longrightarrow 0$. As a second approximation he used his equation for the triple correlation and the hypothesis of equation (12) to express the triple correlation in terms of double correlation. In his calculation he neglected the correlation $\overline{pu_k'u_l'}$; however, this omission can be rectified since $\overline{pu_k'u_l'}$ can be expressed in terms of the second-order correlation by using equations of motion and the hypothesis of equation (12). The error in his equation for the triple correlation is more serious. When this is corrected a rather complicated equation for the triple correlation results (eq. (53) above).

The purpose in deriving an equation for the triple correlation was to express the triple correlation in terms of correlations which can be expressed in terms of double correlation by use of a plausible hypothesis. If this were possible there would be (with the Kármán-Howarth equation) two equations for two unknowns, that is, the two scalar functions characterizing double and triple correlation tensors. This has been only partly successful, however, since $\overline{u_iu_ju_k'u_l'}$, $\overline{u_iu_ju_k'u_l'}$, and $\overline{pu_k'u_l'}$ can only be expressed in terms of double correlations by using the hypothesis of equation (12) and the equation of motion. The two remaining correlations appearing in the equation for the third-order correlation cannot be expressed in terms of known correlations without making further assumptions. Some of these difficulties can be overcome if correlations can be considered that involve quantities at three different points and later make two of the points coincide.

Previously it was believed that higher-order correlation equations could not be investigated because of pressure-velocity correlations entering in these equations. Most of the pressure-velocity correlations can be expressed in terms of correlations involving velocity components at two points, and others, in terms of velocity correlations at more than two points. Consider, for example, the correlation $\overline{pu_iu_j{}'}$ which is related to $\overline{\frac{\partial p}{\partial x_k} u_iu_j{}'}$. The latter quantity enters in the triple-correlation equation. Now, $\overline{pu_iu_j{}'} = \lim_{x''\to x} \overline{pu_i{}''u_j{}'}$ and it is seen from equation (3) that

$$\frac{\partial^2 \overline{p(\underline{x})u_i(\underline{x}'')u_j(\underline{x}')}}{\partial x_i\ \partial x_i} = -\rho \frac{\partial^2 \overline{u_k(\underline{x})u_l(\underline{x})u_i(\underline{x}'')u_j(\underline{x}')}}{\partial x_k\ \partial x_l}$$

so that by using three point correlations, it is possible to express all the pressure-velocity correlations entering in the triple-correlation equation in terms of velocity correlation. It may be possible to relate $\overline{u_ku_lu_i{}'u_j{}''}$ with double correlations $\overline{u_ku_i{}'}$, $\overline{u_i{}'u_j{}''}$, $\overline{u_ku_j{}''}$, and so forth, using, for instance, the hypothesis of equation (12).

It is of interest to investigate what limitations and errors, if any, are introduced in the dynamics of turbulence by the use of the hypothesis of equation (12). If the fourth-order correlations could be simply related to the triple correlations it would be possible to investigate theoretically some of the consequences of the hypothesis of equation (12).

MEASUREMENT OF SECOND- AND FOURTH-ORDER CORRELATIONS

Equipment and Technique

The measurements of the second- and the fourth-order correlations were made in a 2- by 2-foot wind tunnel at 48 mesh lengths downstream from a 1-inch-square mesh grid made from 1/4-inch circular rods. The electronic equipment used is described in reference 19. The new equipment used here is an electronic squaring circuit which has been developed by Kovásznáy (ref. 20). The circuits consist of pairs of rectifiers with series resistors acting as a full-wave rectifier. All pairs are in parallel; however, each pair is biased more than the preceding, so that as the input voltage increases more and more of these pairs of rectifiers conduct. The bias voltage and the series resistors are chosen so that the total rectified current of all the conducting pairs is proportional to the square of the instantaneous voltage. The squaring circuit responds instantaneously within the limitations of the capacity effects in the diodes. Calibration with a harmonic signal showed that its response is good up to 70 kilocycles.

The schematic diagram of the equipment used in measuring the correlations is shown in figure 2. The preamplified hot-wire signal is fed to a push-pull power amplifier which drives the squaring circuit. The output of the squaring circuit is passed through a microammeter and a thermocouple. Because of the inertia of the moving coil of the ammeter it is sensitive only to the average current or mean square of the input voltage. The output of thermocouple, due to lag, is proportional to the average square of the current and hence it is proportional to the average fourth-power of the input voltage. This arrangement measures simultaneously the average square and average fourth power of the input voltage and this enables one to measure simultaneously the fourth-order and the corresponding second-order correlations.

The squaring circuit has hardly any measurable error. However, the thermocouple, because of radiation and other losses, is not a perfect squaring device, and the error depends on the probability density of the signal. The complete fourth-power circuit (the squaring circuit and the thermocouple) was calibrated with a "noise" generator which gives a signal having Gaussian probability density (see fig. 3). In view of the fact that in most of the turbulence measurements approximately Gaussian signals are dealt with, the above calibration was used to correct all the measured data. The error in the corrected results is expected to be within ±5 percent. If e_1 is the output of the hot-wire set at $\underline{x}$ and e_2 is the output of the hot-wire set at $\underline{x}'$ and furthermore if the two wires are set perpendicular to the mean flow, then

$$e_1 = a_1 u$$

$$e_2 = a_2 u'$$

where u is the fluctuating velocity at $\underline{x}$ in the direction of the mean flow and u' is the corresponding quantity at $\underline{x}'$; then

$$\frac{\overline{(e_1 + e_2)^2} - \overline{(e_1 - e_2)^2}}{4\sqrt{\overline{e_1^2}\ \overline{e_2^2}}} = \frac{\overline{e_1 e_2}}{\sqrt{\overline{e_1^2}\ \overline{e_2^2}}} = \frac{\overline{uu'}}{\sqrt{\overline{u^2}\ \overline{(u')^2}}}$$

$$\frac{\overline{(e_1 + e_2)^4} + \overline{(e_1 - e_2)^4} - 2\overline{e_1^4} - 2\overline{e_2^4}}{12\sqrt{\overline{e_1^4}\ \overline{e_2^4}}} = \frac{\overline{e_1^2\ e_2^2}}{\sqrt{\overline{e_1^4}\ \overline{e_2^4}}} = \frac{\overline{u^2\ (u')^2}}{\sqrt{\overline{u^4}\ \overline{(u')^4}}}$$

and

$$\frac{\overline{(e_1 + e_2)^4} - \overline{(e_1 - e_2)^4}}{8\left(\overline{e_1^4}\ \overline{e_2^4}\right)^{1/4}\left(\overline{e_1^4} + \overline{e_2^4}\right)^{1/2}} = \frac{\overline{e_1^3 e_2} + \overline{e_1 e_2^3}}{2\left(\overline{e_1^4}\ \overline{e_2^4}\right)^{1/4}\left(\overline{e_1^4} + \overline{e_2^4}\right)^{1/2}} = \frac{\overline{u(u')^3}}{\overline{u^2}}$$

if $\overline{u(u')^3} = \overline{u^3 u'}$ so that in order to measure the fourth-order correlations $\overline{u^2(u')^2}$ and $\overline{u^3u'}$ and the second-order correlation $\overline{uu'}$ it is only necessary to measure the average square and average fourth power of (a) the outputs of two hot-wires, (b) their sums, and (c) their difference.

The correlations $R_{ll}^{ll}(r)$ and $R_{lll}^{l}(r)$ were measured by moving one hot-wire with respect to the other in the direction of the mean flow.

The correlations $R_{nn}^{nn}(r)$ and $R_{nnn}^{n}(r)$ were measured by moving one hot-wire with respect to the other in a direction perpendicular to the mean flow.

The combination $\left(R_{ll}^{ll} + R_{nn}^{nn} + 2R_{ll}^{nn} + 4R_{nl}^{nl}\right)$ was measured by moving one hot-wire with respect to the other along a line making an angle of 45° with the mean motion (see fig. 8).

The combination $\left(R_{ll}^{ll} + R_{nn}^{nn} + 2R_{ll}^{nn} - 4R_{nl}^{nl}\right)$ was measured by using two wires perpendicular to each other and each inclined 45° to the mean flow. When one wire is on top of the other the combination of wires is essentially an x-type hot-wire which is sensitive of v-component of the fluctuating velocity.

The correlation R_{ll}^{nn} was measured by using one x-type hot-wire sensitive to v and a single wire sensitive to u and moving apart the two probes in the direction of v.

The correlation R_{nn}^{mm} was measured by using an x-type hot-wire sensitive to w and a single wire sensitive to u and moving the two probes apart in the direction of v.

The length of the hot-wire was quite small (the ratio of the microscale λ to the length being about 10), so that no appreciable length correction was necessary.

Results

The measurement of various correlations was made in isotropic turbulence at $R_\lambda = 60$. The correlations $R_l^l(r)$ and $R_n^n(r)$ were measured (see fig. 4) and they satisfy reasonably well the relation

$$R_l^l(r) - R_n^n(r) = -\frac{r}{2}\frac{\partial}{\partial r}R_l^l(r)$$

which arises from the continuity equation of the incompressible flow and the condition of isotropy (ref. 10). This served as a check on the isotropy of the turbulence.

The correlations $R_{ll}^{ll}(r)$, $R_{nn}^{nn}(r)$, and $R_{nn}^{mm}(r)$ and two independent combinations $\left(R_{ll}^{ll} + R_{nn}^{nn} + 2R_{nn}^{ll} + 4R_{nl}^{nl}\right)$ and $\left(R_{ll}^{ll} + R_{nn}^{nn} + 2R_{nn}^{ll} - 4R_{nl}^{nl}\right)$ were measured. The measurements are compared with the hypothesis of equation (12) in figures 5 to 9. For large displacement of the points the correlation between u_i^2 and $(u_j')^2$ disappears and $\overline{u_i^2(u_j')^2}$ tends to a constant value $\overline{u_i^2}\;\overline{(u_j')^2} = \left(\overline{u_1^2}\right)^2$ for isotropic turbulence. If the probability density of u_1 is assumed to be Gaussian then $\overline{u_1^4} = 3\left(\overline{u_1^2}\right)^2$ and therefore for large displacement of the points $\overline{u_i^2\left(u_j'\right)^2}\Big/\overline{u_1^4}$ tends to a constant value of 1/3. This is indicated by a dashed line in figures (5), (6), and (8). While the fourth-order correlation was measured the corresponding second-order correlation was also measured simultaneously. The correlation R_{ll}^{nn} (fig. 10) was also measured; this is not independent of the above five correlations. Since the hypothesis of equation (12) gives it a constant value it is of interest to see if the measured value deviates from this constant. Within the experimental accuracy, all the fourth-order correlations are close to those computed from corresponding second-order correlations using equation (12).

The correlations $R^{l}_{lll}(r)$ and $R^{n}_{nnn}(r)$ characterizing the quadruple correlation $\overline{u_i u_j u_k u_l{}'}$ were measured. These are compared with those computed from corresponding second-order correlations using equation (12) in figures 11 and 12. The hypothesis of equation (12) is satisfied for R^{l}_{lll} and R^{n}_{nnn} within experimental scatter.

CALCULATION OF ROOT-MEAN-SQUARE PRESSURE AND PRESSURE GRADIENT FROM VELOCITY CORRELATIONS AND DIFFUSION MEASUREMENTS

Let R_1, R_2, and R_3 denote the correlations R^{ll}_{ll} and R^{nn}_{nn} and the combination $\frac{1}{4}\left(R^{ll}_{ll} + R^{nn}_{nn} + 2R^{ll}_{nn} + 4R^{nl}_{nl}\right)$, respectively. This set of correlations, among other possible sets, suffices to determine the pressure correlation. In terms of the above notation equation (24) becomes

$$\frac{1}{\rho^2}\,\overline{pp'}(r) = R_1(r) - \left(\overline{u_1{}^2}\right)^2 + 2\int_r^\infty \left[\left(3R_2 + R_1 - 4R_3\right) - \frac{4r^2}{y^2}\left(R_1 + R_2 - 2R_3\right)\right]\frac{dy}{y}$$

where R_2 was measured for both positive and negative values of r (see fig. 6) and the point $r = 0$ was determined from the fact that it is an even function of r. The correlations R_1 and R_3 were measured for positive values of r and there is some uncertainty (0.025 inch) in the determination of the point $r = 0$ for these two correlations. This uncertainty, the experimental scatter, and the fact that small differences between relatively large quantities are required to compute the pressure correlation have made the results very uncertain.

Two widely different sets of curves were drawn for each measured correlation, taking into account the experimental scatter and the uncertainty in the determination of the point $r = 0$. Two pressure correlations were computed from these two different sets of fourth-order correlations. The result of these computations and those from the second-order correlation are compared in table I and figure (13).

It appears that for the computation of pressure correlation one should measure directly the differences $(3R_2 + R_1 - 4R_3)$ and $(R_1 + R_2 - 2R_3)$ instead of the individual correlations. This involves the difficult problem of measuring small correlation between relatively large quantities. In view of the fact that the hypothesis of equation (12) is approximately satisfied, the double correlation can be used to compute root-mean-square pressure and pressure gradient. However, one unfortunate circumstance must not be overlooked. It is possible that the hypothesis of equation (12) is approximately satisfied but the root-mean-square pressure and the pressure gradient computed from using this hypothesis are still in error because the root-mean-square pressure and pressure gradient depend on the differences of various fourth-order correlations.

Equations (24a) and (25a) express the desired quantities in terms of the longitudinal second-order correlation R_l^l. The correlations R_l^l and R_n^n are connected by equations (18) and (18a). It is necessary to express $\frac{\partial}{\partial r} R_l^l$ in terms of R_n^n. Since

$$\frac{\partial}{\partial r} R_n^n = \frac{1}{2r^2} \frac{\partial}{\partial r}\left(r^3 \frac{\partial}{\partial r} R_l^l\right)$$

therefore

$$\frac{\partial}{\partial r} R_l^l = \frac{2}{r^3} \int_0^r \frac{\partial}{\partial y} R_n^n(y) y^2 \, dy$$

Using this result, equations (24a) and (25a) become

$$\overline{p^2}\Big/\rho^2\left(\overline{u_1^2}\right)^2 = \frac{8}{\left(\overline{u_1^2}\right)^2} \int_0^\infty \left[\int_0^r \frac{\partial}{\partial y} R_n^n(y) y^2 \, dy\right]^2 \frac{dr}{r^5}$$

$$\overline{\left(\frac{\partial p}{\partial x_1}\right)^2}\Big/\rho^2\left(\overline{u_1^2}\right)^2 = \frac{16}{\left(\overline{u_1^2}\right)^2} \int_0^\infty \left[\int_0^r \frac{\partial}{\partial y} R_n^n(y) y^2 \, dy\right]^2 \frac{dr}{r^7}$$

Root-mean-square pressure and pressure gradient were computed from measured values of $R_n^n(r)$ for various Reynolds numbers. The results of the computation are shown in figures 14 and 15. The data for the two highest Reynolds numbers were taken from reference 21. The data in reference 21 are given in terms of grid Reynolds numbers. Turbulence Reynolds numbers R_λ and λ/M (where M is the mesh spacing) were estimated from grid Reynolds numbers by assuming that $U^2/\overline{u_1^2} \approx t^{-1}$ in the initial period where U is the mean velocity.

Equation (35) relates the Lagrangian microscale λ_η with the root-mean-square pressure gradients. Measurements of λ_η have been made by Simmons (reported in ref. 1) and Collis (ref. 22), and an extensive set of measurements is given in reference (13). The experimental data (table II) are used to compute the pressure gradient (eq. (35)) and the results are compared with Heisenberg's analysis (eq. (36)) in figure 15.

CONCLUDING REMARKS

It is of theoretical interest and experimentally convenient that correlations involving static pressure fluctuations can be expressed in terms of higher velocity correlations; the latter in turn can be related to second-order velocity correlations by using some plausible hypothesis. The fourth-order correlation, which enters prominently in the correlations $\overline{pp'}$ and $\overline{pu_k'u_l'}$, may be related to the second-order correlation by using the hypothesis of equation (12). Experiments lend support to this hypothesis. Since the differences of quadruple correlations are involved in the expression for $\overline{pp'}$, a slight deviation of the quadruple correlations from those computed using double correlations can lead to considerable error in $\overline{pp'}$. Further experimental improvement involves the difficult problem of the measurement of small correlation between relatively large quantities.

The experimental determination of root-mean-square pressure gradient from diffusion measurements is also not very accurate because of the inherent double differentiation of the experimental curves involved in this technique. The hypothesis of equation (12) relates the Lagrangian microscale to Eulerian microscale. It may be possible to relate the Lagrangian and Eulerian correlations for larger values of the independent variables by use of the equations of motion and higher-order correlations.

The dynamic equation relating quadruple correlations to triple correlations is worth further investigation, firstly, to see if one can possibly get a closed system of equations for the dynamic of isotropic turbulence by use of the hypothesis of equation (12), and, secondly, to see the limitations and errors introduced by the use of the above hypothesis. In this respect it is noted that experiments lend support to the hypothesis of equation (12) for $\overline{u_i u_j u_k' u_l'}$, the quadruple correlation involving two components at one point and two at another, and also for $\overline{u_i u_j u_k u_l'}$, the quadruple correlation involving three components at one point and one at another. If this were strictly true then the quantity $\overline{(u_1 - u_1')^4} \Big/ \left[\overline{(u_1 - u_1')^2}\right]^2$ would be constant and equal to a numerical value of 3 (the value for Gaussian joint distribution) independent of position, where u_1 and u_1' are velocity components at two separate points and perpendicular to the displacement vector. The quantity $\overline{(u_1 - u_1')^4} \Big/ \left[\overline{(u_1 - u_1')^2}\right]^2$ has been measured (fig. 16) and its value deviates from 3 for small displacement of the points and in this region it approximately equals $\overline{\left(\frac{\partial u_1}{\partial x_2}\right)^4} \Big/ \left[\overline{\left(\frac{\partial u_1}{\partial x_2}\right)^2}\right]^2$. The deviation of $\overline{(u - u')^4} \Big/ \left[\overline{(u - u')^2}\right]^2$ from the numerical value of 3 for small displacement of points shows that the hypothesis of equation (12) is not satisfied for small eddies; however, the maximum error is only about 20 percent. Since the differences of correlations are involved in the quantity $\overline{(u - u')^4} \Big/ \left[\overline{(u - u')^2}\right]^2$ small deviations in the values of $\overline{u_i u_j u_k' u_l'}$ and $\overline{u_i u_j u_k u_l'}$ from that computed from the hypothesis of equation (12) show up prominently in the values for $\overline{(u - u')^4} \Big/ \left[\overline{(u - u')^2}\right]^2$ while these small deviations are hardly noticeable in the values for $\overline{u_i u_j u_k' u_l'}$ and $\overline{u_i u_j u_k u_l'}$.

Batchelor (ref. 4) has discussed the correlation $\overline{u_i u_j u_k' u_l'}$ in connection with pressure fluctuations. He has presented Stewart's measurements of $\overline{(u - u')^4} \Big/ \left[\overline{(u - u')^2}\right]^2$ to show that the joint probability density of u_i, u_j, u_k' and u_l' is Gaussian. As far as the pressure fluctuations are concerned it is only necessary to assume that $\overline{u_i u_j u_k' u_l'}$ satisfies the hypothesis of equation (12) which is less restrictive than the hypothesis of Gaussian joint probability density of u_i, u_j, u_k', and u_l'. Even if $\overline{(u - u')^4} \Big/ \left[\overline{(u - u')^2}\right]^2$ is equal to a numerical value of 3 this does not prove that u_i, u_j, u_k', and u_l' are jointly Gaussian or that $\overline{u_i u_j u_k' u_l'}$ satisfies the hypothesis of equation (12) since $\overline{u_i u_j u_k u_l'}$ also enters in $\overline{(u - u')^4} \Big/ \left[\overline{(u - u')^2}\right]^2$.

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APPENDIX A

Consider the fourth-order correlation $\overline{u_i u_j u_k' u_l'}$ where u_i is the velocity at the point $\underline{x}$ and u_k' is the velocity at $\underline{x}' = \underline{x} + \underline{\xi}$. An outline of the procedure, based on invariant theory and essentially following Robertson (ref. 23), will be given for deriving the form for the isotropic tensor $\overline{u_i u_j u_k' u_l'}$. Consider the scalar correlation between the two velocity components in two arbitrary directions at $\underline{x}$ and two velocity components in two arbitrary directions at $\underline{x}'$. Let a_i and b_i be the direction cosines of the two arbitrary directions at $\underline{x}$ and c_i and d_i be the corresponding quantities at $\underline{x}'$. Then the scalar correlation R is

$$R(\underline{a},\underline{b},\underline{c},\underline{d};r) = \overline{u_i u_j u_k' u_l'}\ a_i b_i c_i d_i \tag{A1}$$

where $r = \left(\xi_i \xi_i\right)^{1/2}$ is the distance between two points $\underline{x}$ and $\underline{x}'$. The correlation $R(\underline{a},\underline{b},\underline{c},\underline{d};r)$ has the following special properties:

(1) It is invariant under an arbitrary translation or rotation, as a rigid body, of the configuration defined by the points $\underline{x}$ and $\underline{x}'$ and the unit vectors $\underline{a}$, $\underline{b}$, $\underline{c}$, and $\underline{d}$.

(2) Its value is unchanged by the reflection of the above configuration in any point. These two are the conditions of homogeneity and isotropy.

(3) It is homogeneous quadrilinear in the components of the four vectors $\underline{a}$, $\underline{b}$, $\underline{c}$, and $\underline{d}$.

The form of $R(\underline{a},\underline{b},\underline{c},\underline{d};r)$ has to be determined under the above three conditions. According to the invariant theory of rotation groups in three dimensions (ref. 23), any invariant function of any number of vectors $\underline{\xi}$, $\underline{a}$, $\underline{b}$, . . . can be expressed in terms of the fundamental invariants of the following types: (1) The scalar product $(\underline{\xi}\ \underline{b}) = \xi_i b_i$ of any two vectors including the scalar square $\xi_i \xi_i$, and (2) the determinants

$$\left[\underline{\xi}\ \underline{a}\ \underline{b}\right] = \begin{vmatrix} \xi_1 & a_1 & b_1 \\ \xi_2 & a_2 & b_2 \\ \xi_3 & a_3 & b_3 \end{vmatrix}$$

of any three vectors. In terms of geometrical notions the invariants associated with a set of vectors are their lengths $\xi_i \xi_i$, the angle $\xi_i a_i$ between any two vectors, and the volume $[\underline{\xi}\ \underline{a}\ \underline{b}]$ of the parallelepiped whose edges are any three given vectors. However, the volume of the parallelepiped changes sign on reflection, $[\underline{\xi}\ \underline{a}\ \underline{b}] = -[\underline{a}\ \underline{\xi}\ \underline{b}]$, hence these invariants do not appear in the expression for $R(\underline{a},\underline{b},\underline{c},\underline{d};r)$. Since $R(\underline{a},\underline{b},\underline{c},\underline{d};r)$ is a homogeneous quadrilinear in the components of the unit vectors $\underline{a}$, $\underline{b}$, $\underline{c}$, and $\underline{d}$, it must be a linear function of the form $(\underline{\xi}\ \underline{a})(\underline{\xi}\ \underline{b})(\underline{\xi}\ \underline{c})(\underline{\xi}\ \underline{d})$, the six forms of type $(\underline{\xi}\ \underline{a})(\underline{\xi}\ \underline{b})(\underline{c}\ \underline{d})$, and three of the type $(\underline{a}\ \underline{b})(\underline{c}\ \underline{d})$, with coefficients which are even functions of r. Thus,

$$
\begin{aligned}
R(\underline{a},\underline{b},\underline{c},\underline{d};r) = {} & R_1(r)(\underline{\xi}\ \underline{a})(\underline{\xi}\ \underline{b})(\underline{\xi}\ \underline{c})(\underline{\xi}\ \underline{d}) + R_2(r)(\underline{\xi}\ \underline{a})(\underline{\xi}\ \underline{b})(\underline{c}\ \underline{d}) + \\
& R_3(r)(\underline{\xi}\ \underline{c})(\underline{\xi}\ \underline{d})(\underline{a}\ \underline{b}) + R_4(r)(\underline{\xi}\ \underline{d})(\underline{\xi}\ \underline{b})(\underline{a}\ \underline{c}) + \\
& R_5(r)(\underline{\xi}\ \underline{a})(\underline{\xi}\ \underline{c})(\underline{b}\ \underline{d}) + R_6(r)(\underline{\xi}\ \underline{b})(\underline{\xi}\ \underline{c})(\underline{a}\ \underline{d}) + \\
& R_7(r)(\underline{\xi}\ \underline{a})(\underline{\xi}\ \underline{d})(\underline{c}\ \underline{b}) + R_8(r)(\underline{a}\ \underline{b})(\underline{c}\ \underline{d}) + \\
& R_9(r)(\underline{a}\ \underline{c})(\underline{b}\ \underline{d}) + R_{10}(r)(\underline{a}\ \underline{d})(\underline{b}\ \underline{c})
\end{aligned}
$$

$$
\begin{aligned}
R(\underline{a},\underline{b},\underline{c},\underline{d};r) = {} & R_1(r)\xi_i\xi_j\xi_k\xi_l a_i b_j c_k d_l + R_2(r)\xi_i\xi_j\delta_{kl} a_i b_j c_k d_l + \\
& R_3(r)\xi_k\xi_l\delta_{ij} a_i b_j c_k d_l + R_4(r)\xi_l\xi_j\delta_{ik} a_i b_j c_k d_l + \\
& R_5(r)\xi_i\xi_k\delta_{jl} a_i b_j c_k d_l + R_6(r)\xi_j\xi_k\delta_{il} a_i b_j c_k d_l + \\
& R_7(r)\xi_i\xi_l\delta_{jk} a_i b_j c_k d_l + R_8(r)\delta_{ij}\delta_{kl} a_i b_j c_k d_l + \\
& R_9(r)\delta_{ik}\delta_{jl} a_i b_j c_k d_l + R_{10}(r)\delta_{il}\delta_{kj} a_i b_j c_k d_l
\end{aligned} \qquad (A2)
$$

Equating equations (A1) and (A2) and making use of the fact that this equality is true for arbitrary unit vectors $\underline{a}$, $\underline{b}$, $\underline{c}$, and $\underline{d}$,

$$\overline{u_i u_j u_k' u_l'} = R_1(r)\xi_i\xi_j\xi_k\xi_l + R_2(r)\xi_i\xi_j\delta_{kl} + R_3(r)\xi_k\xi_l\delta_{ij} +$$
$$R_4(r)\xi_l\xi_j\delta_{ik} + R_5(r)\xi_i\xi_k\delta_{jl} + R_6(r)\xi_j\xi_k\delta_{il} +$$
$$R_7(r)\xi_i\xi_l\delta_{jl} + R_8(r)\delta_{ij}\delta_{kl} + R_9(r)\delta_{ik}\delta_{jl} + R_{10}(r)\delta_{il}\delta_{kj} \tag{A3}$$

This is the general form of the correlation involving four velocity components, either two velocity components at one point and two at another, or three velocity components at one point and one at another. Use is made of the fact that for the correlation $\overline{u_i u_j u_k' u_l'}$ the indexes i and j can be interchanged, k and l can be interchanged, and ij can be interchanged with kl. Using these symmetry conditions it is found that

$$\overline{u_i u_j u_k' u_l'} = R_1\xi_i\xi_j\xi_k\xi_l + \frac{1}{2}\left(R_2 + R_3\right)\left(\xi_i\xi_j\delta_{kl} + \xi_k\xi_l\delta_{ij}\right) +$$
$$\frac{1}{4}\left(R_4 + R_5 + R_6 + R_7\right)\left(\xi_l\xi_j\delta_{ik} + \xi_i\xi_k\delta_{jl} + \xi_j\xi_k\delta_{il} +\right.$$
$$\left.\xi_i\xi_l\delta_{jl}\right) + R_8\delta_{ij}\delta_{kl} + \frac{1}{2}\left(R_9 + R_{10}\right)\left(\delta_{il}\delta_{kj} + \delta_{ik}\delta_{jl}\right)$$

The five scalars characterizing $\overline{u_i u_j u_k' u_l'}$ can be expressed in terms of five special correlations. When this is done equation (16) is obtained.

As mentioned earlier, equation (3) gives the form for the general fourth-order correlation, so that $\overline{u_i u_j u_k u_l'}$, the correlation involving three velocity components at one point and one at another point, has the form

$$\overline{u_i u_j u_k u_l'} = S_1(r)\xi_i\xi_j\xi_k\xi_l + S_2(r)\xi_i\xi_j\delta_{kl} + S_3(r)\xi_k\xi_l\delta_{ij} + S_4(r)\xi_l\xi_j\delta_{ik} +$$
$$S_5(r)\xi_i\xi_k\delta_{jl} + S_6(r)\xi_j\xi_k\delta_{il} + S_7(r)\xi_i\xi_l\delta_{jk} + S_8(r)\delta_{ij}\delta_{kl} +$$
$$S_9(r)\delta_{ik}\delta_{jl} + S_{10}(r)\delta_{il}\delta_{kj}$$

Making use of the fact that indexes i, j, and k can be interchanged,

$$\overline{u_i u_j u_k u_l{}'} = S_1(r)\xi_i\xi_j\xi_k\xi_l + \frac{1}{3}\left[S_2(r) + S_5(r) + S_6(r)\right]\left(\xi_i\xi_j\delta_{kl} + \xi_i\xi_k\delta_{jl} + \xi_j\xi_k\delta_{il}\right) + \frac{1}{3}\left[S_3(r) + S_4(r) + S_7(r)\right]\left(\xi_k\xi_l\delta_{ij} + \xi_l\xi_j\delta_{ik} + \xi_i\xi_l\delta_{jk}\right) + \frac{1}{3}\left[S_8(r) + S_9(r) + S_{10}(r)\right]\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj}\right)$$

The four scalars defining $\overline{u_i u_j u_k u_l{}'}$ can be expressed in terms of four special correlations. When this is done equation (19) is obtained.

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TABLE I

PRESSURE CORRELATIONS COMPUTED FROM FOURTH- AND SECOND-ORDER CORRELATIONS

	$\overline{p^2}/\rho^2\left(\overline{u_1^2}\right)^2$	$\overline{\left(\frac{\partial p}{\partial x_1}\right)^2}/\rho^2\left(\overline{u_1^2}\right)^2$ (1)	$\lambda_p = \left[2\overline{p}_2 / \overline{\left(\frac{\partial p}{\partial x_1}\right)^2}\right]^{1/2}$ (2)
Two extreme values computed from fourth-order correlation	3.3 .45	85 11	0.28 .20
Computed from second-order correlation	0.50	12	0.20

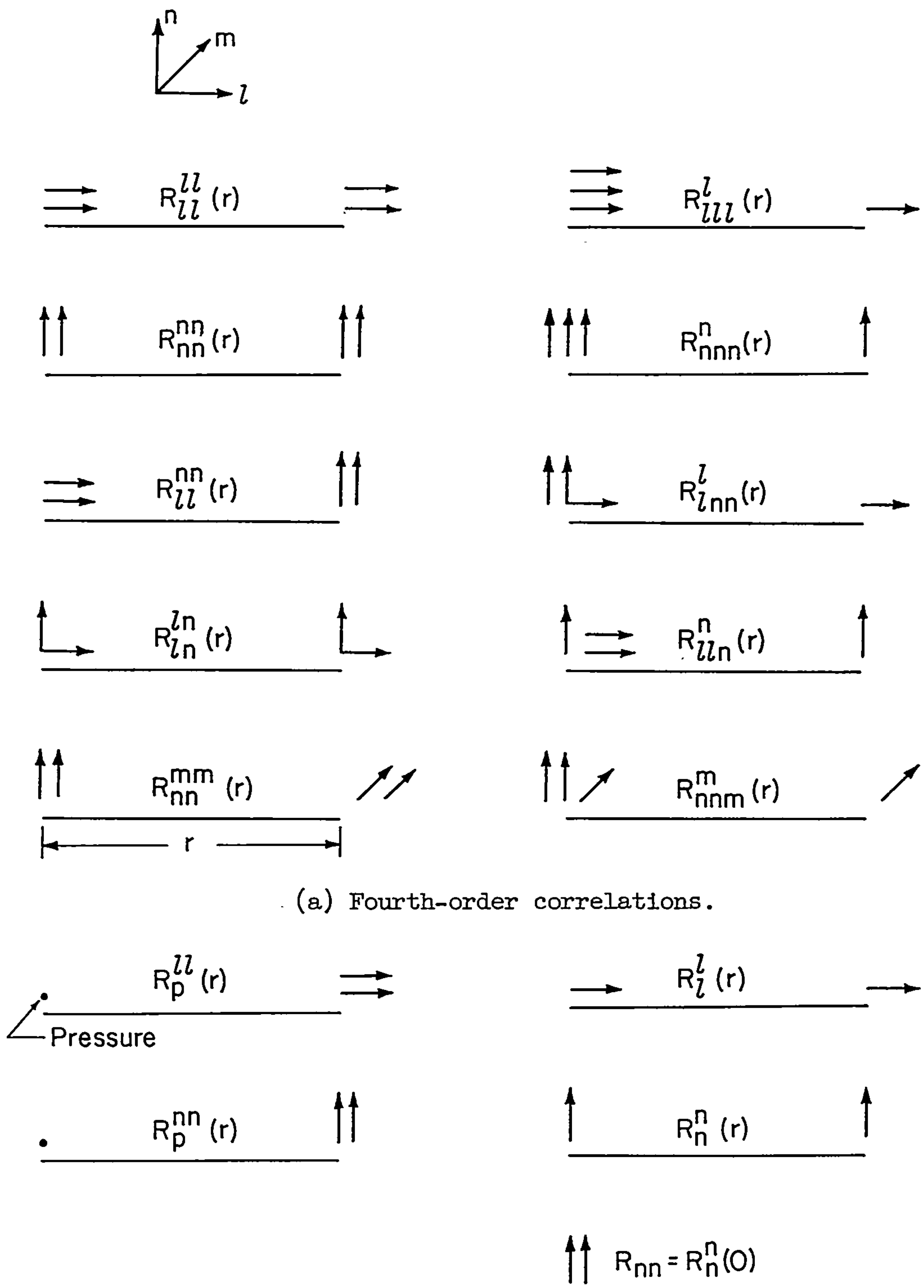
[1]Dimension of values is approximately $(\text{in.})^{-2}$

[2]Dimension of values is approximately in.

TABLE II

EXPERIMENTAL DATA

Investigator and date	R_λ	λ, in.	λ_η, in.	$\frac{2\lambda^2}{\lambda_\eta^2}$	$\frac{25}{R_\lambda^2} + \frac{2.3}{R_\lambda}$	$\frac{\lambda^2\overline{\left(\frac{\partial p}{\partial y}\right)^2}}{\rho^2\overline{(v')^4}}$	$\sqrt{\frac{\rho^2\overline{(v')^4}}{\lambda^2\overline{\left(\frac{\partial p}{\partial y}\right)^2}}}$
Simmons 1935	37.0	0.16	0.08	8.00	0.08	7.92	0.35
Collis 1948	23.0	.14	.20	1.00	.15	.85	1.08
Collis 1948	38.0	.14	.20	1.00	.09	.91	1.05
Reference (13), 1951	29.0	.17	.11	4.78	.11	4.67	.46
Do.---------	32.0	.12	.12	2.00	.10	1.90	.73
Do.---------	36.0	.41	.36	2.60	.08	2.52	.63
Do.---------	42.5	.23	.235	1.92	.07	1.9	.74
Do.---------	43.5	.165	.245	.910	.07	.84	1.09
Do.---------	49.0	.35	.22	5.04	.06	4.98	.45
Do.---------	61.0	.23	.34	.914	.04	.87	1.06
Do.---------	67.0	.34	.22	4.76	.04	4.72	.46
Do.---------	74.0	.19	.27	.990	.04	.95	1.02



(a) Fourth-order correlations.

(b) Pressure-velocity correlations.

(c) Second-order correlations.

Figure 1.- Correlation diagrams.

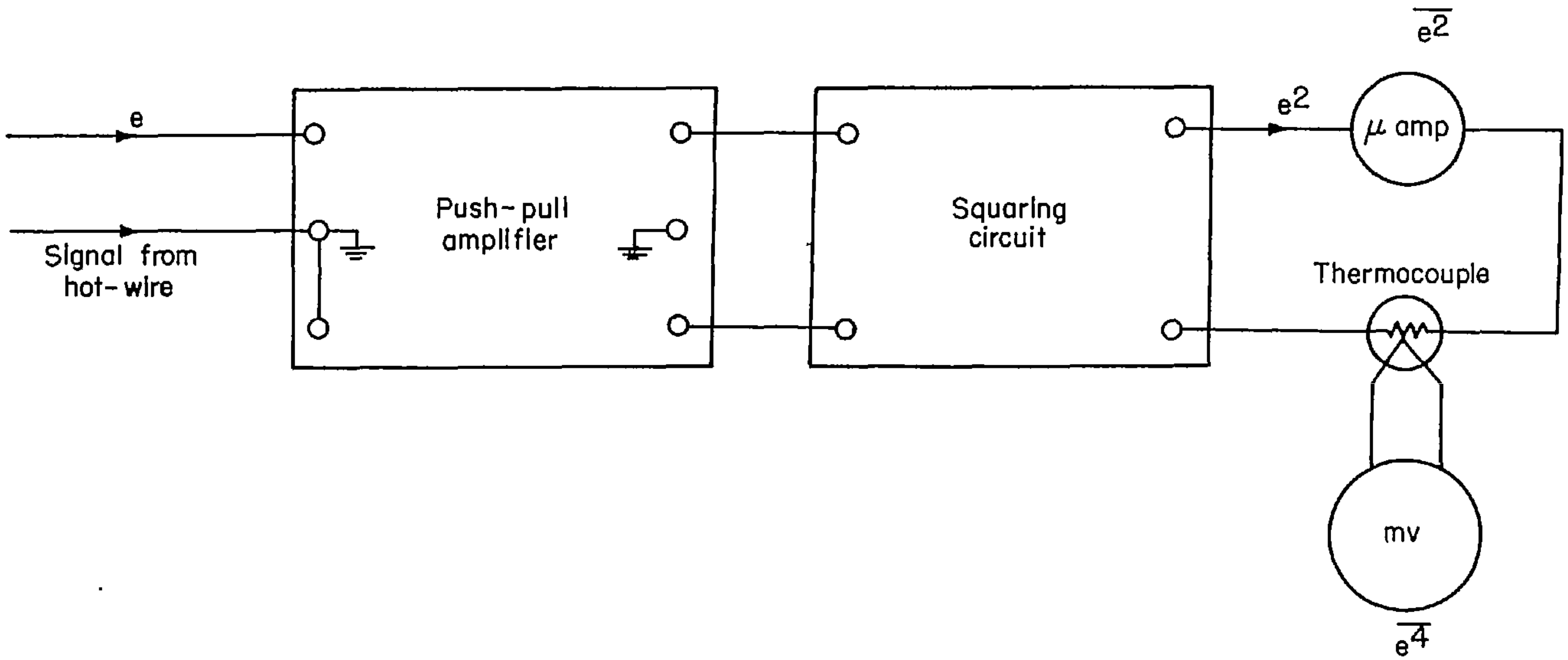


Figure 2.- Schematic diagram of measuring equipment.

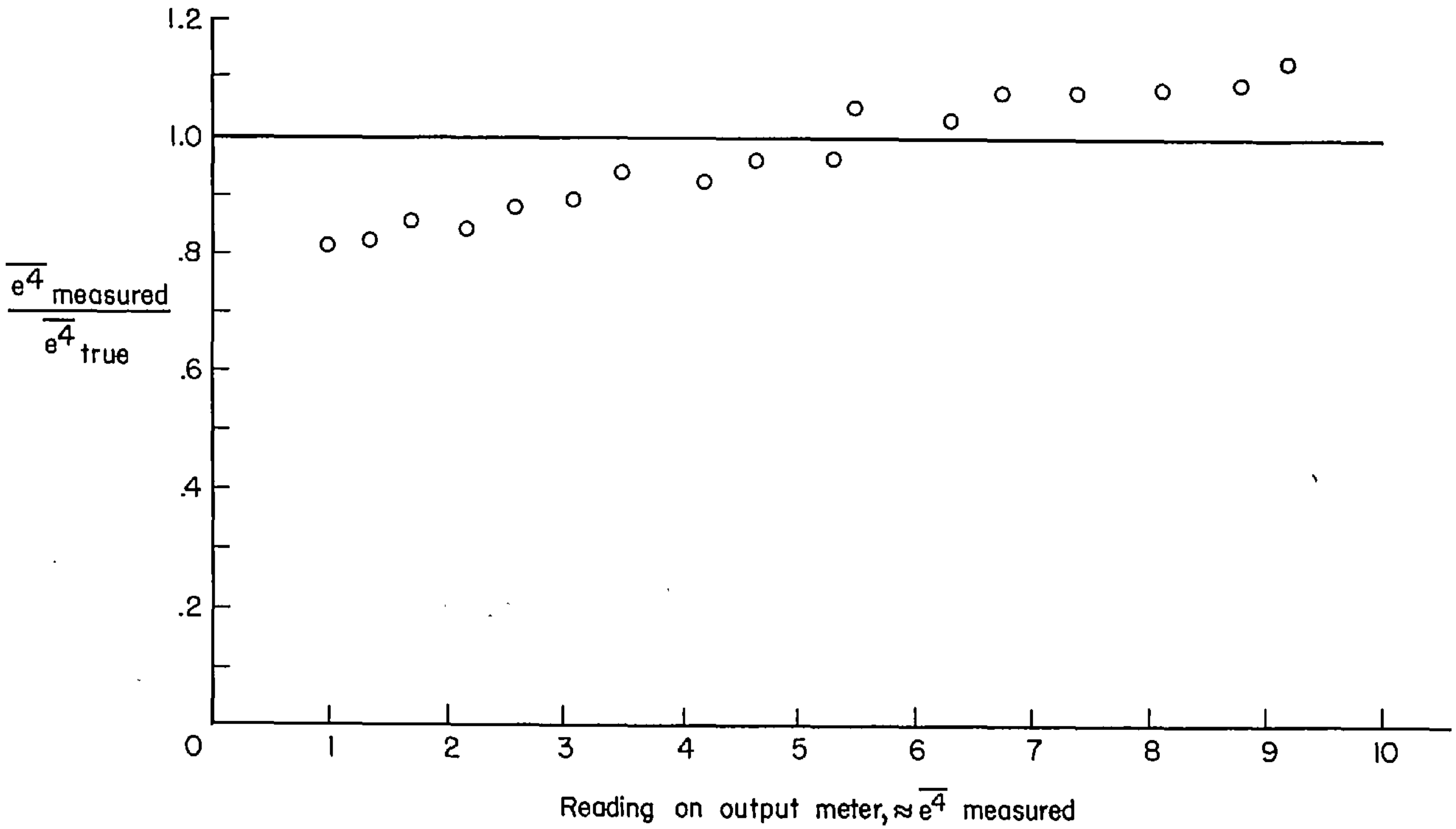


Figure 3.- Calibration of fourth-power circuit with signal having gaussian probability density.

8X

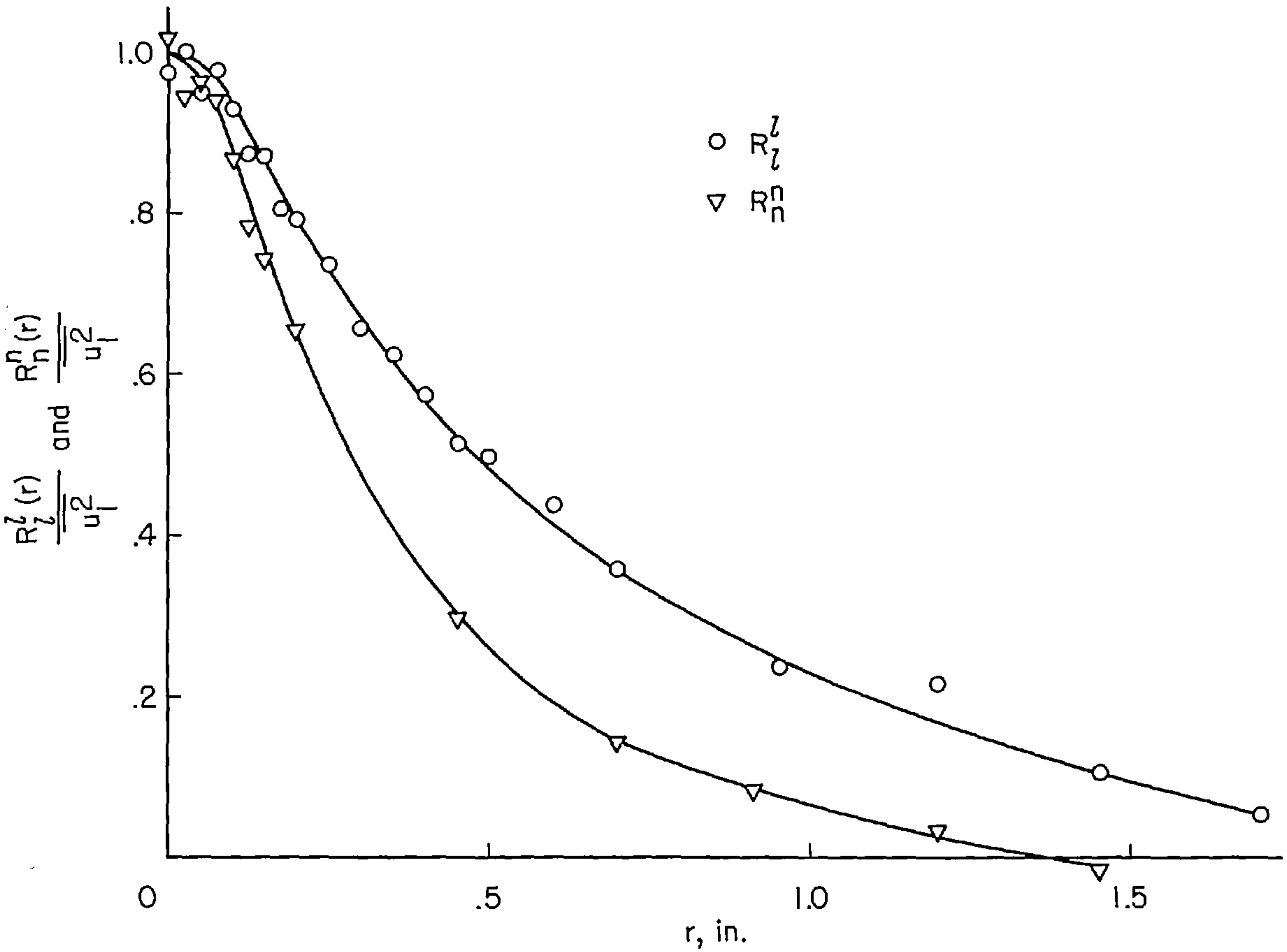


Figure 4.- Correlations $R_l^l(r)$ and $R_n^n(r)$.

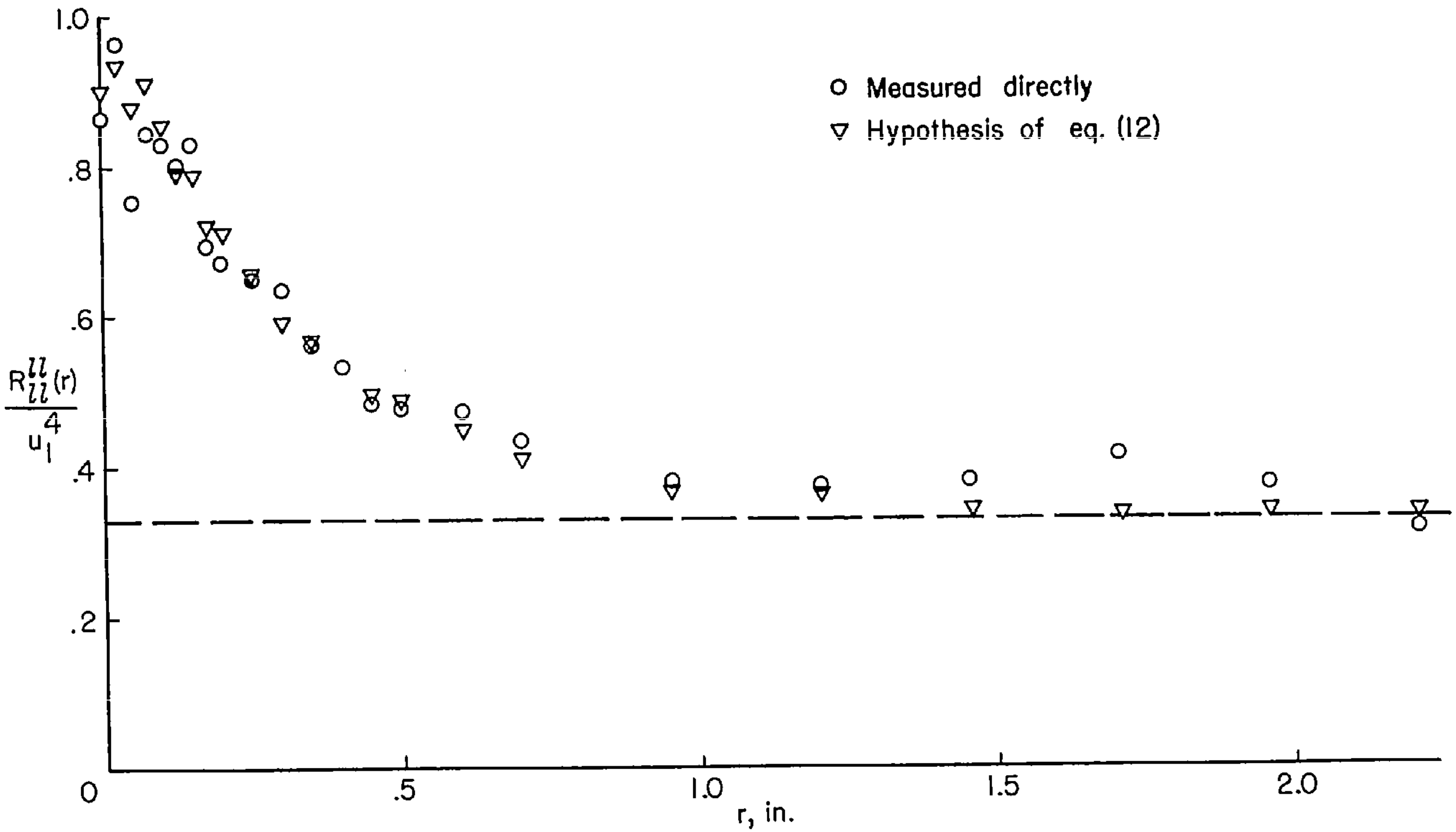


Figure 5.- Correlation $R_{ll}^{ll}(r)$.

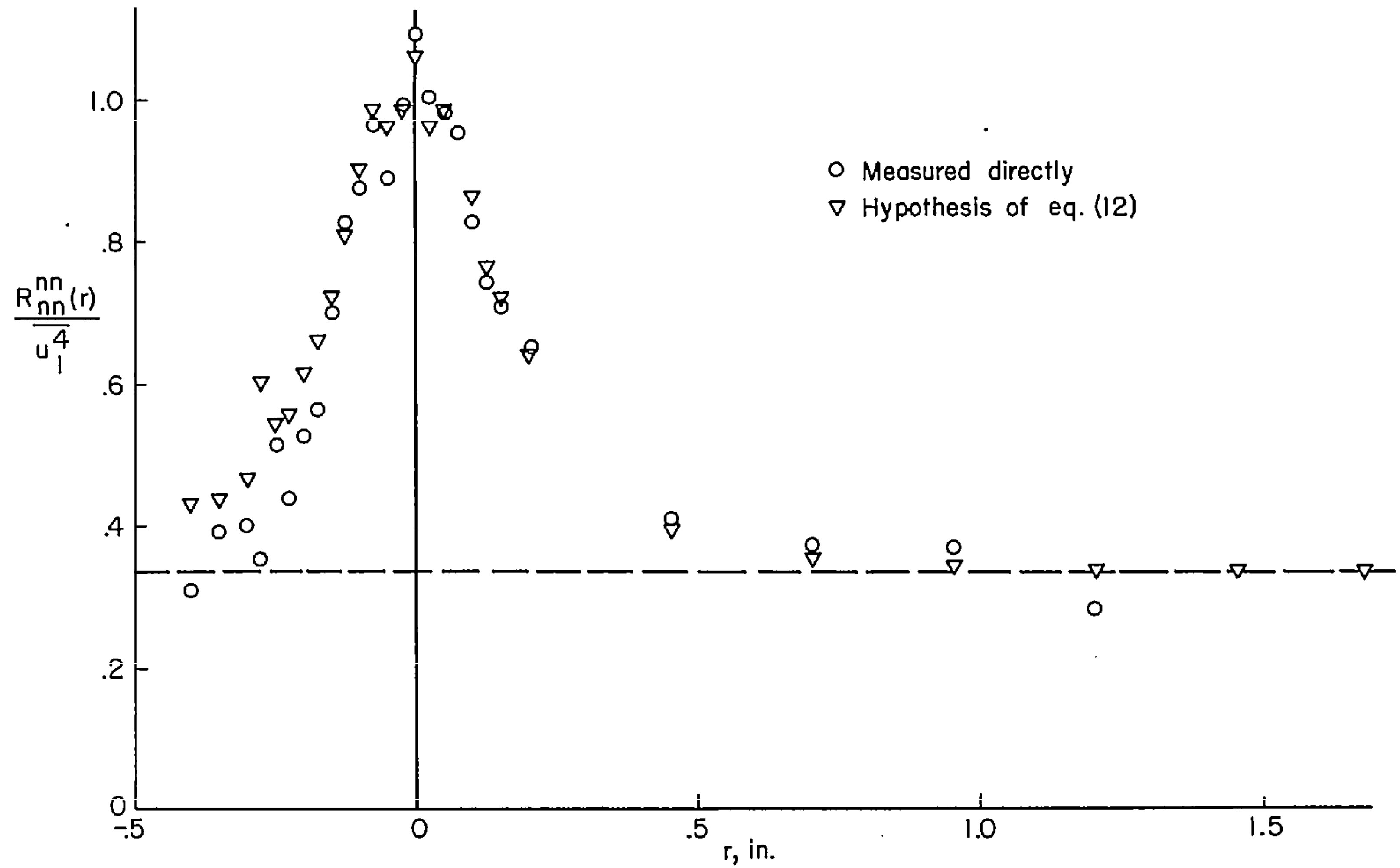


Figure 6.- Correlation $R_{nn}^{nn}(r)$.

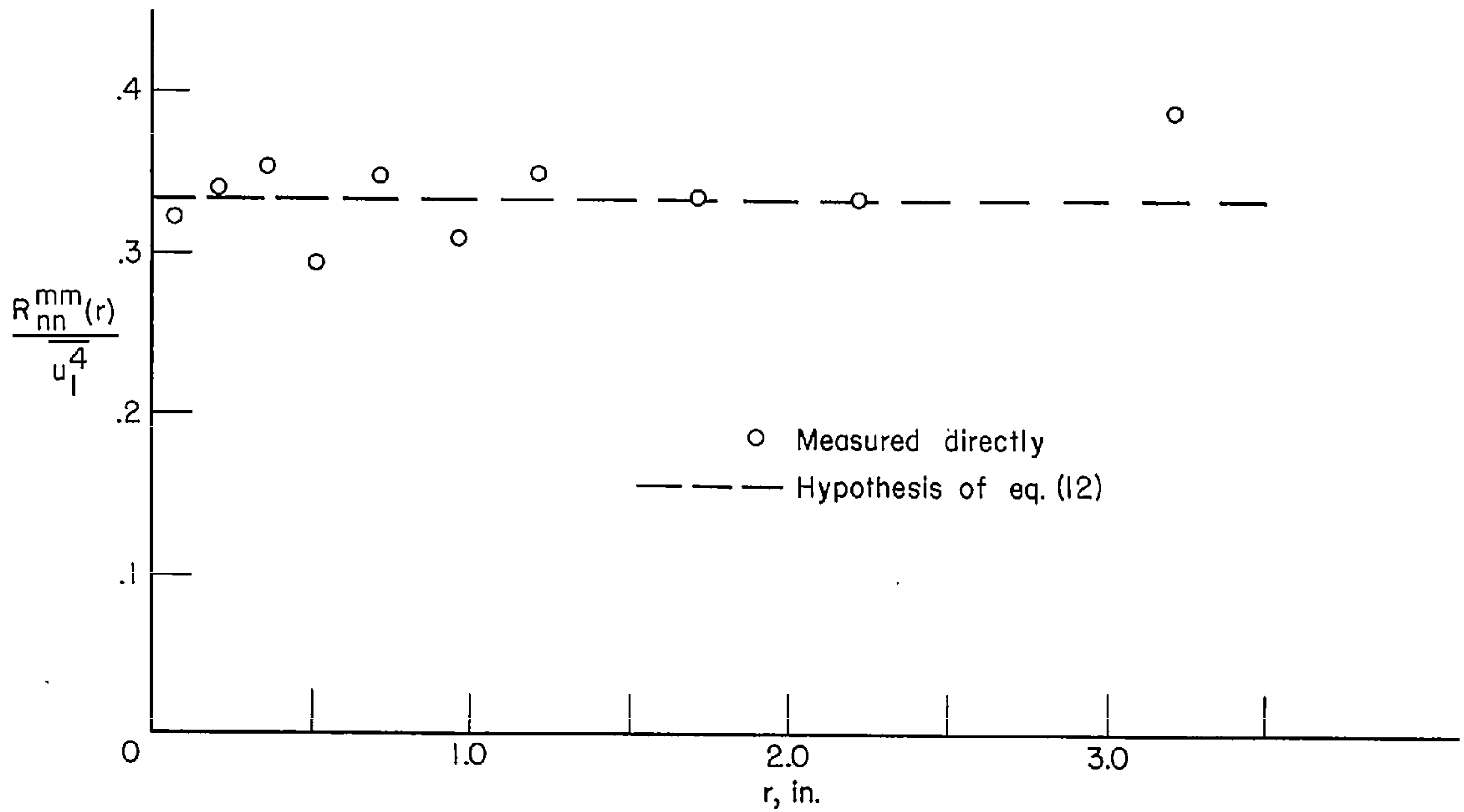


Figure 7.- Correlation $R_{nn}^{mm}(r)$.

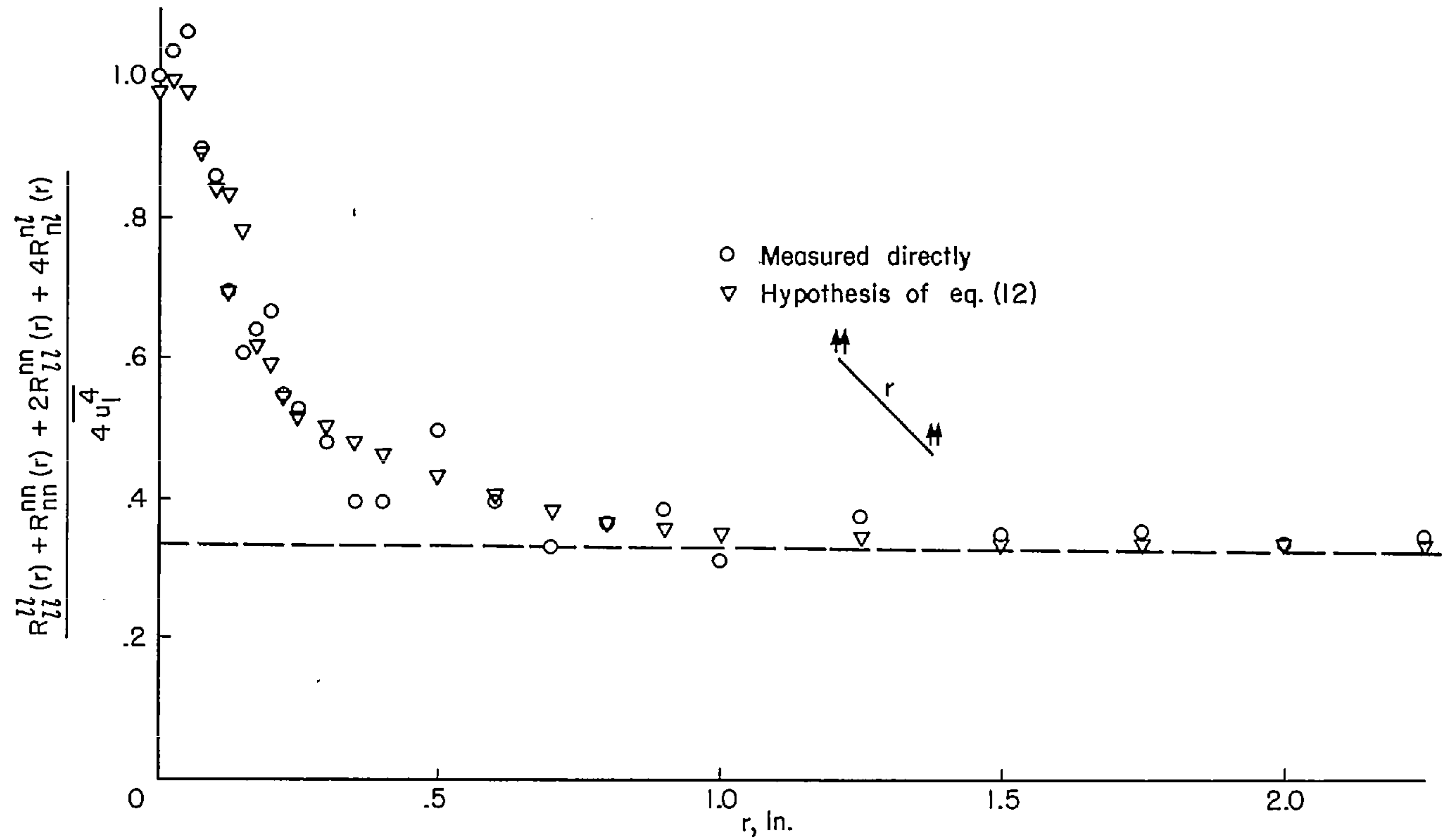


Figure 8.- Correlation $\frac{R_{ll}^{ll}(r) + R_{nn}^{nn}(r) + 2R_{ll}^{nn}(r) + 4R_{nl}^{nl}(r)}{4\overline{u_l^4}}$.

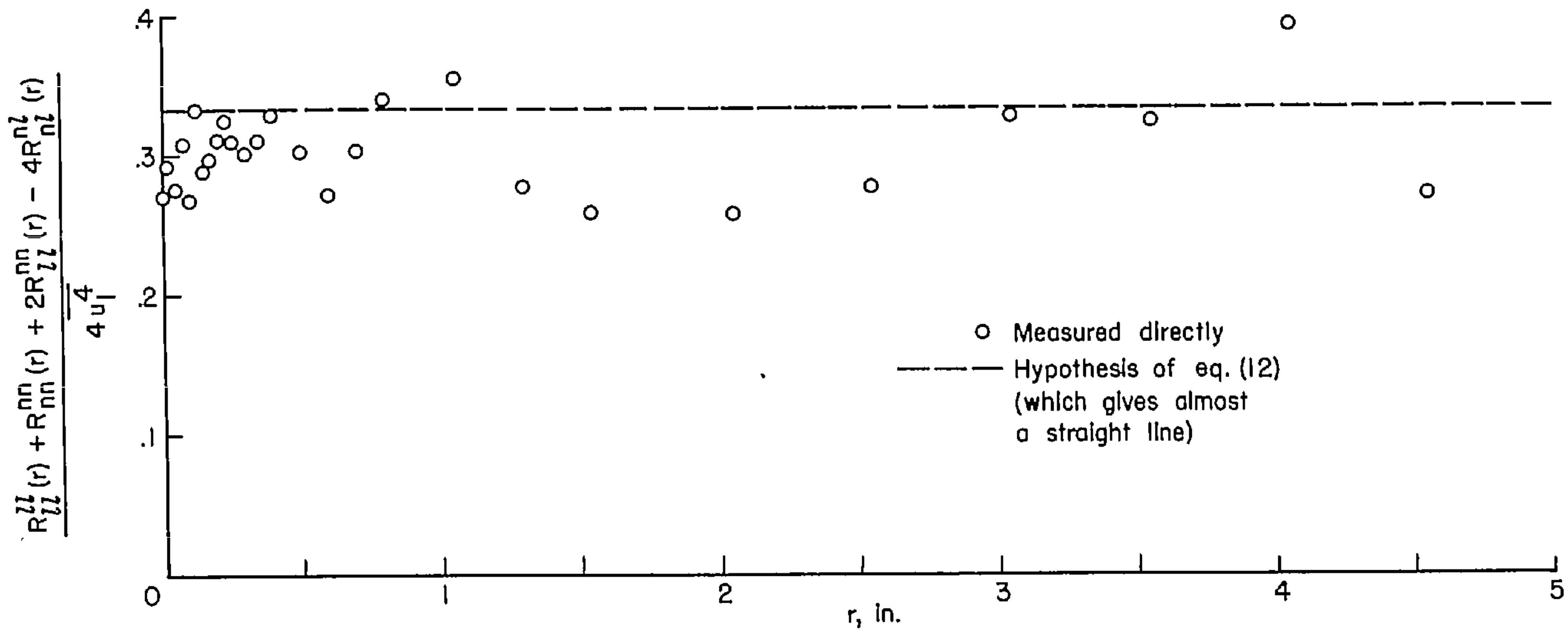


Figure 9.- Correlation $\dfrac{R_{ll}^{ll}(r) + R_{nn}^{nn}(r) + 2R_{ll}^{nn}(r) - 4R_{nl}^{nl}(r)}{4\overline{u_l^4}}$.

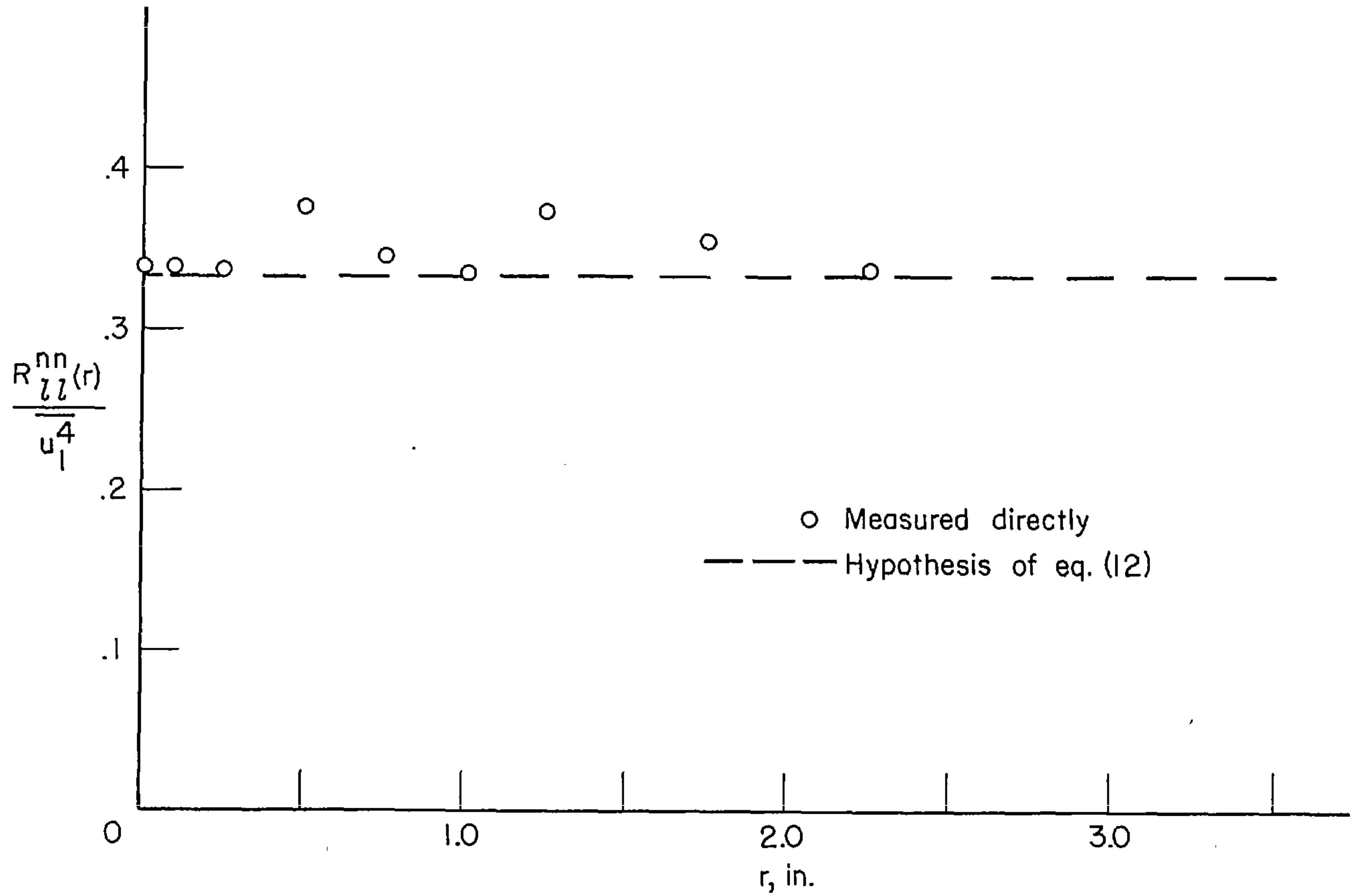


Figure 10.- Correlation $R_{ll}^{nn}(r)$.

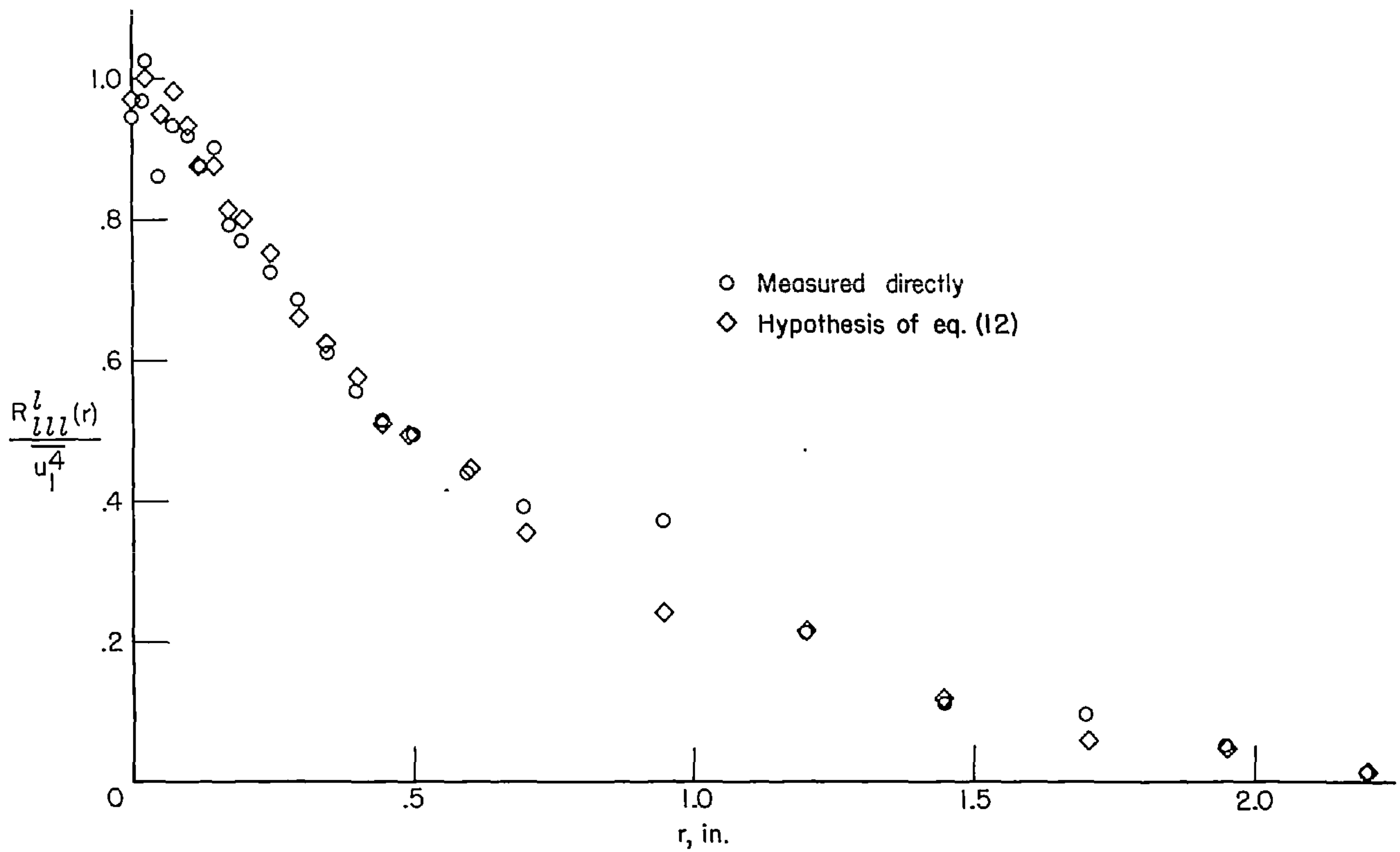


Figure 11.- Correlation $R^{l}_{lll}(r)$.

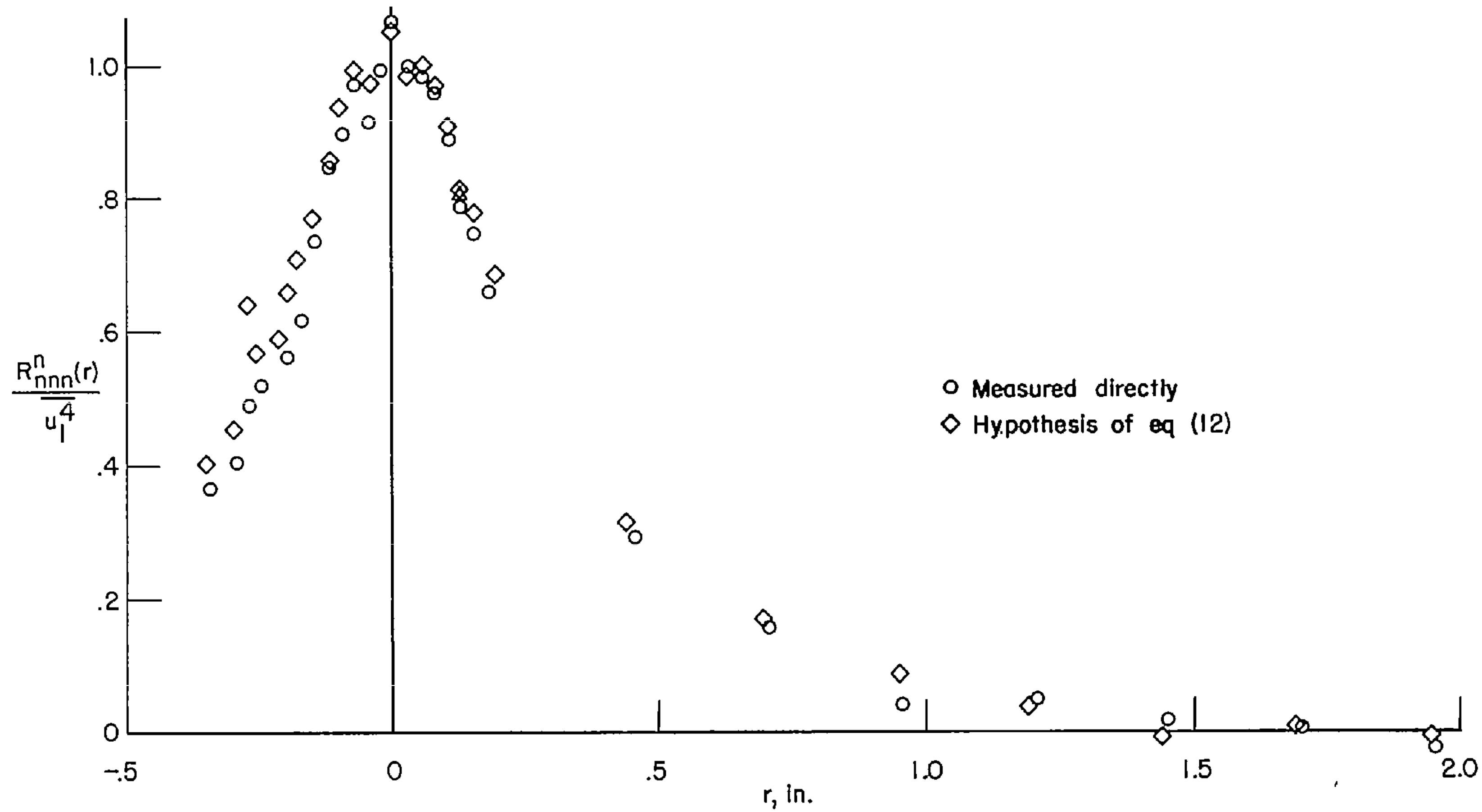


Figure 12.- Correlation $R^n_{nnn}(r)$.

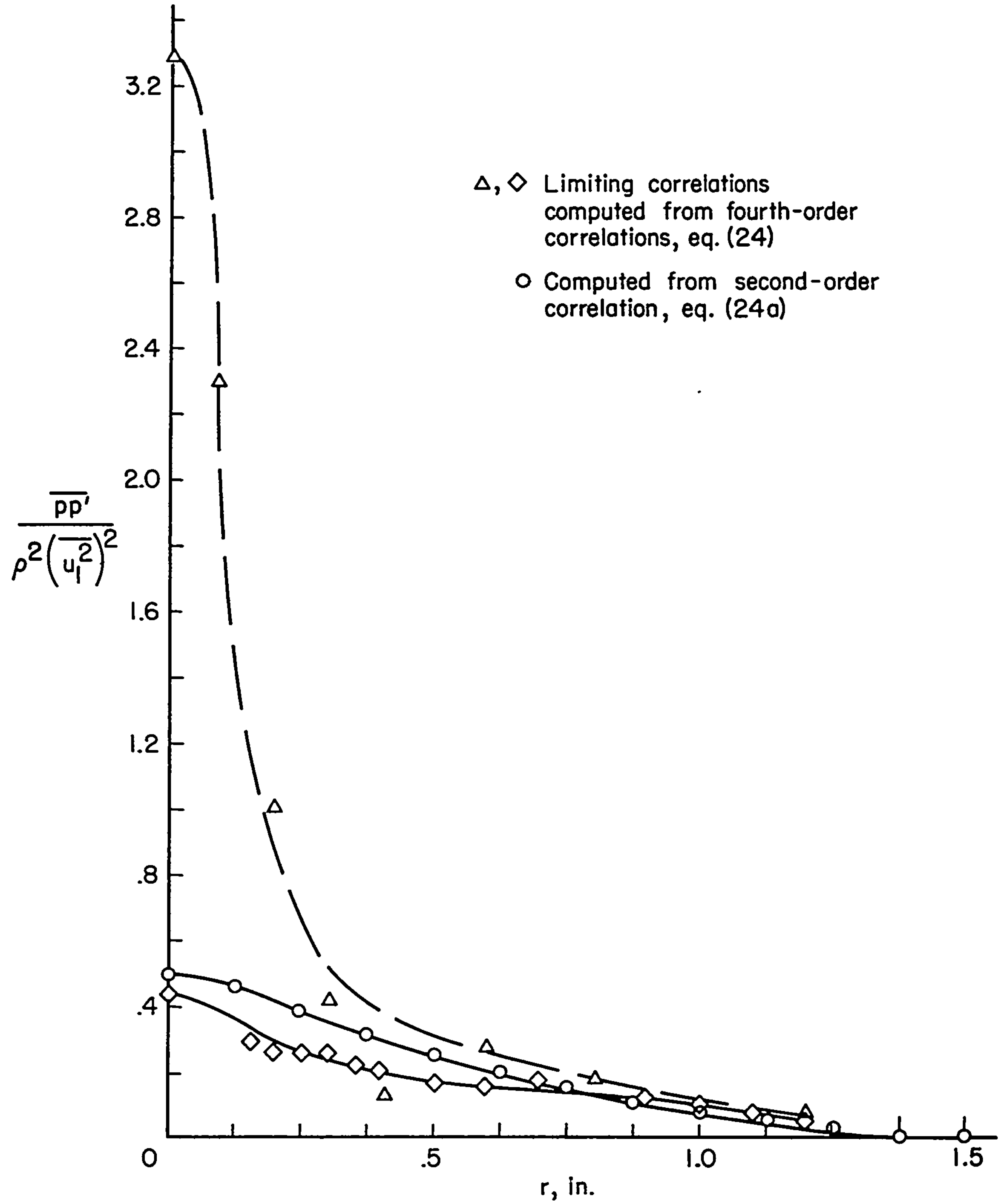


Figure 13.- Correlation $\overline{pp'(r)}$.

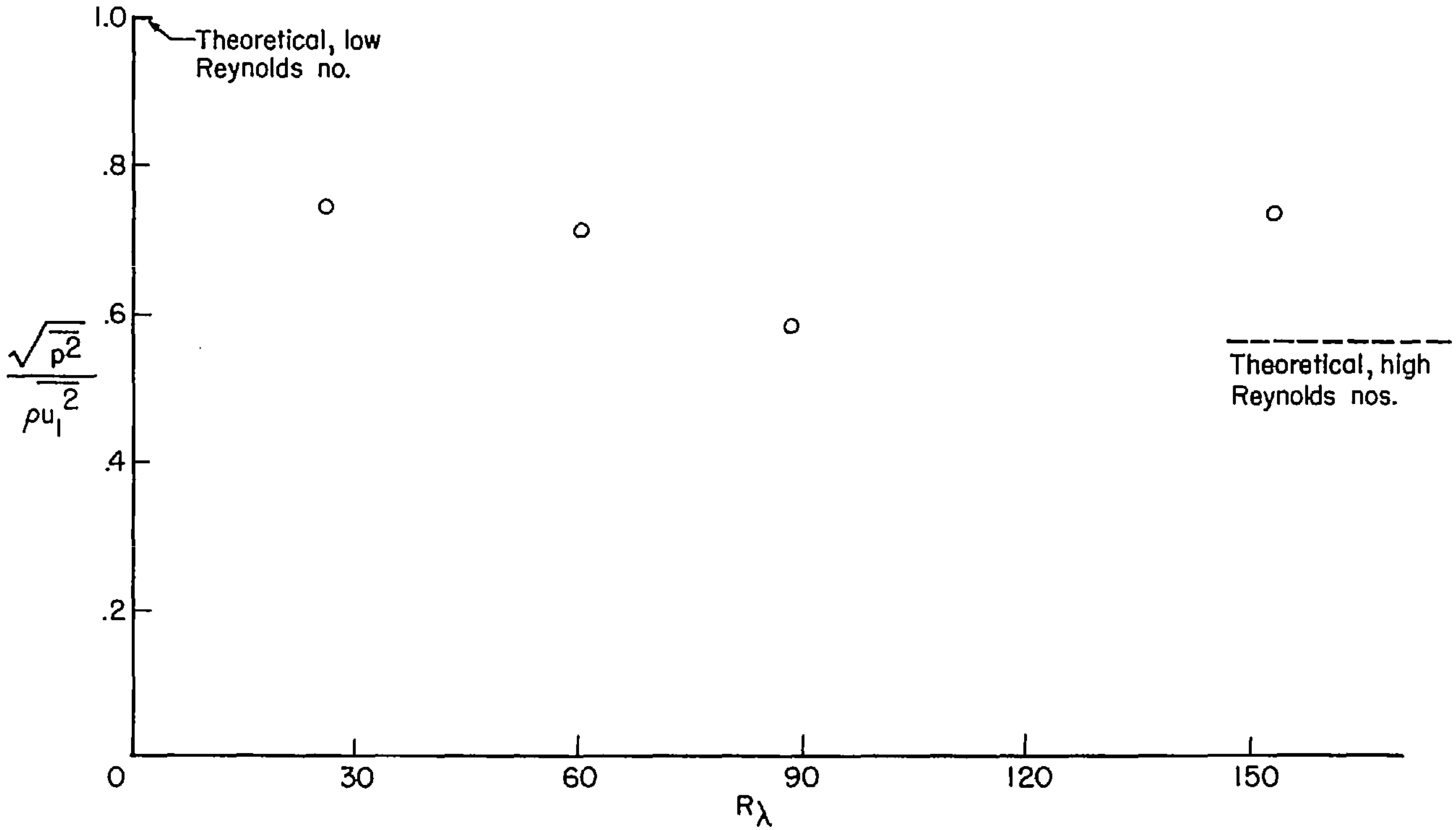


Figure 14.- Variation of $\sqrt{\overline{p^2}}/\rho\overline{u_1^2}$ with Reynolds number.

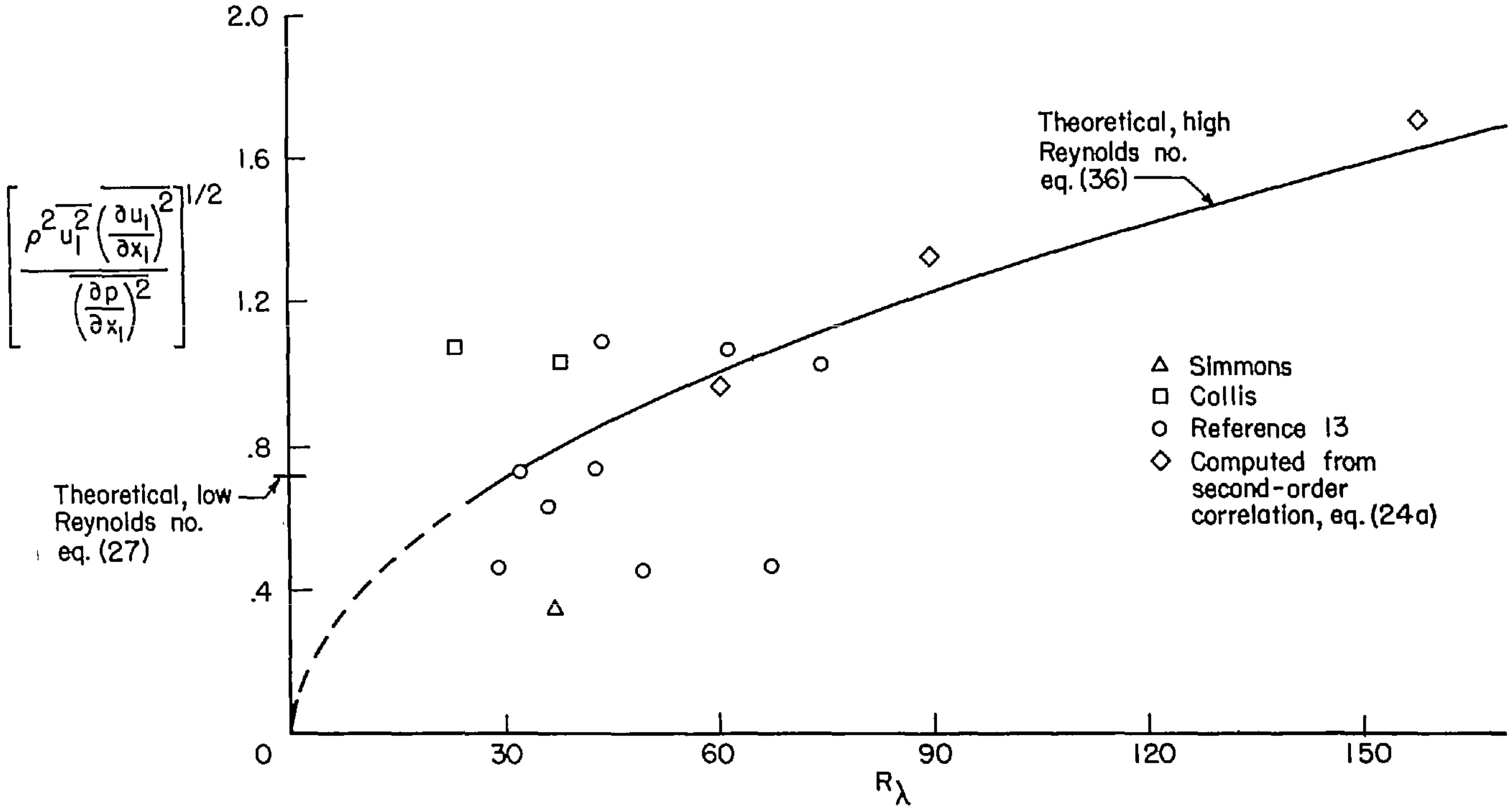


Figure 15.- Variation of $\left[\rho^2 \overline{u_1^2} \overline{\left(\frac{\partial u_1}{\partial x_1}\right)^2} \Big/ \overline{\left(\frac{\partial p}{\partial x_1}\right)^2}\right]^{1/2}$ with Reynolds number.

$$\lambda^2 = \overline{u_1^2} \Big/ \overline{\left(\frac{\partial u_1}{\partial x_1}\right)^2}.$$

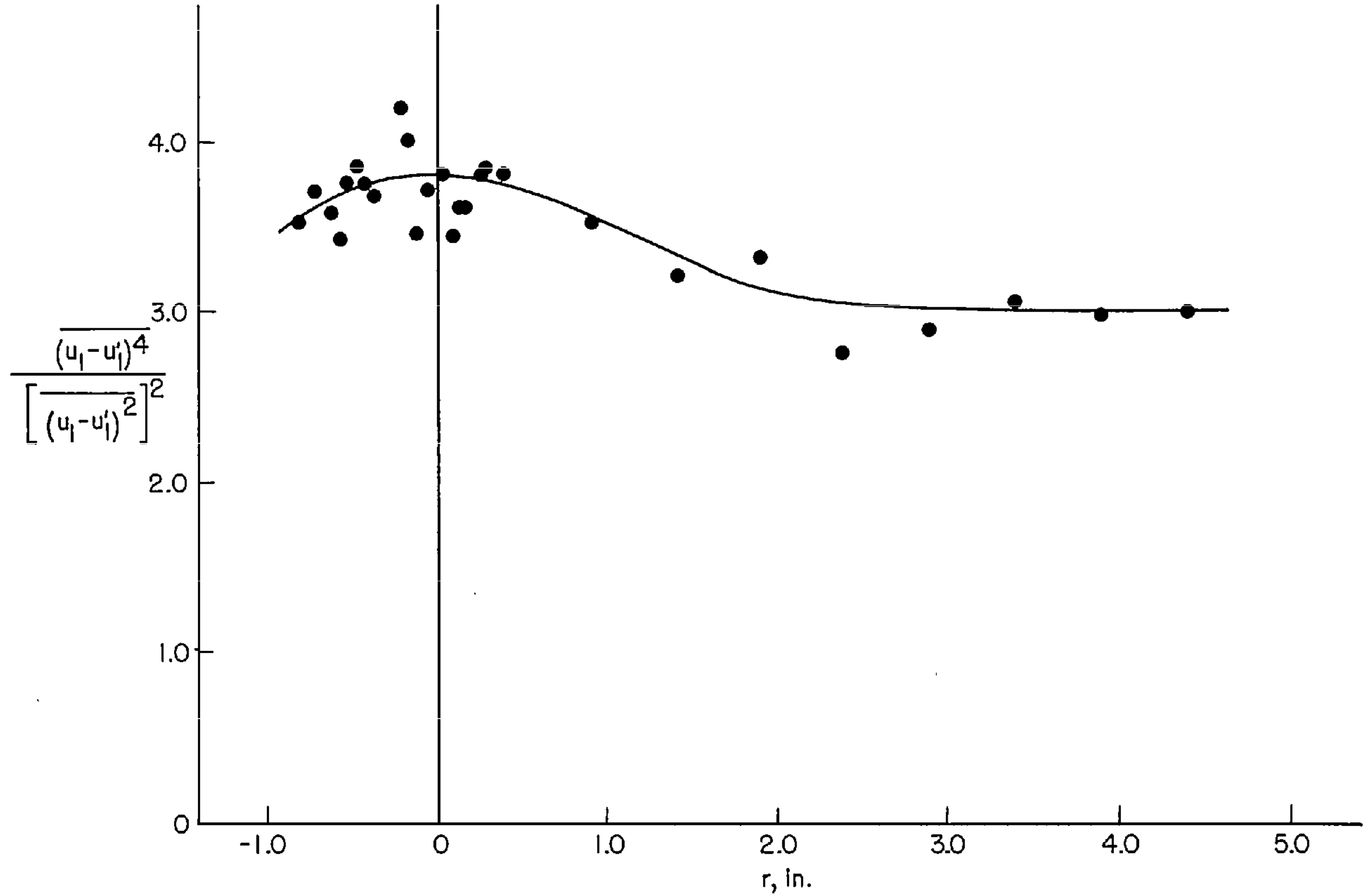


Figure 16.- Variation of $\overline{(u_1 - u_1')^4} \Big/ \left[\overline{(u_1 - u_1')^2}\right]^2$ with distance.

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