NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 3458

UNSTABLE CONVECTION IN VERTICAL CHANNELS WITH HEATING FROM BELOW, INCLUDING EFFECTS OF HEAT SOURCES AND FRICTIONAL HEATING

By Simon Ostrach

Lewis Flight Propulsion Laboratory
Cleveland, Ohio

Washington
July 1955

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 3458

UNSTABLE CONVECTION IN VERTICAL CHANNELS WITH HEATING FROM BELOW, INCLUDING EFFECTS OF HEAT SOURCES AND FRICTIONAL HEATING

By Simon Ostrach

SUMMARY

The effects of heat sources and frictional heating on the laminar fully developed channel flow subject to a body force between two parallel plates oriented in the direction of the body force are analyzed. Solutions are obtained for combined forced- and natural-convection flows for the cases in which the wall temperature variations are linear and (1) the wall temperatures are specified, (2) the walls are both insulated, and (3) the net mass flow in the channel is zero. These solutions depend on the Rayleigh number which was previously found to be the factor determining the stability and type of flow for horizontal and vertical layers of fluids heated from below but without heat sources or frictional heating. Similar stability characteristics are displayed in the present problem, and the heat sources affect the flows only in a quantitative manner.

When the effects of frictional heating are considered, two distinct solutions are obtained for each set of parametric values. Solutions neglecting frictional heating correspond to a different exact solution for values of the Rayleigh number smaller and larger than the critical. The related approximate and exact solutions are essentially coincident when the frictional-heating parameter is small but differ for unit order values of this parameter. The approximate solutions (i.e., those neglecting frictional heating) are shown to be always invalid for Rayleigh numbers near critical.

INTRODUCTION

Considerable theoretical work (refs. 1 to 5) has recently been done on problems dealing with internal natural-convection flows because of the many new practical applications of this phenomenon. The heat-transfer results of reference 5 are experimentally verified in reference 6. All this work is concerned with flows in vertical enclosures.

One of the many interesting aspects of the natural-convection process, which was originally studied theoretically by Rayleigh in 1916 (ref. 7) for horizontal rather than vertical layers of fluid, is the instability of the flow associated with heating a fluid at rest from below. When a stationary fluid has some layer with a density greater than others lying below it, its equilibrium becomes unstable in the sense that even a small disturbance may result in a completely altered regime. Bénard (ref. 8) and others had previously shown experimentally that such a configuration leads to a cellular type of flow. Numerous theoretical and experimental investigations have been performed to substantiate, clarify, and extend the early findings, but these works were restricted to horizontal layers of fluid. These studies, which are briefly summarized in reference 9, were essentially of the stability type in which eigen relations of the physical and cell-shape parameters were determined for neutral stability, and the details of the flow were usually not investigated further.

3573

The results of this work on horizontal layers indicate that the flow regimes depend on a dimensionless parameter sometimes referred to as the Rayleigh number, which is the product of the Prandtl and Grashof numbers. It has been demonstrated experimentally that there are essentially two different laminar types of flow: (1) a columnar type (ref. 10), occurring after a relatively low value of the Rayleigh number is attained, and (2) the cellular type (observed first), occurring for larger values of the Rayleigh number (approximately 1700 for two horizontal parallel plates).

In reference 5, it was pointed out that heating from below might be expected to affect the stability of vertical as well as horizontal layers despite the differences between the two configurations. As a matter of fact, heating from below in vertical enclosures is often actually encountered; a discussion of such a configuration in an atomic power unit is presented in reference 11. Configurations of this type can also be found in many other fields besides that of atomic power; for example, this phenomenon can also be found in conjunction with the cooling of turbine blades by natural convection. (The writer is unaware of any previous presentation of this connection.) Accordingly, the problem treated in reference 5 (namely, the fully developed flow with temperature increasing linearly upward in a vertical channel) was reconsidered in reference 12, with the modification that the heating was from below; that is, the temperature was specified to decrease linearly upward.

In reference 12, it was found that the stability characteristics and the critical values of the Rayleigh number for the vertical fluid layers were similar to those for horizontal layers. Representative velocity and temperature distributions for various sets of boundary conditions were determined from explicit expressions in reference 12 neglecting frictional heating. It was shown that, in certain ranges of the parametric values, the flow and heat transfer associated with heating from below are appreciably different from the corresponding quantities for the identical configuration in which heating is not from below.

The analysis of reference 12 is extended herein to include the effects of uniform internal heat sources in the fluid and of frictional heating. The importance of frictional heating in the natural-convection phenomenon was first discussed in reference 1 and later in references 5 and 13. The solutions obtained with frictional heating are exact, so that the assumption of the negligibility of frictional heating in reference 12 can be checked.

ANALYSIS

Basic Equations

The fully developed laminar flow of viscous fluids subject to a body force between two plane vertical parallel surfaces open at both ends (see fig. 1) is considered herein, as in references 1 and 5. (More generally, the surfaces could be taken to be oriented in the direction parallel to any generating body force.) In contradistinction to the work in references 1 and 5, the fluid here is heated from below, so that the axial (vertical) temperature gradient is specified as negative. It is further assumed that the physical properties of the fluids are constants, except that the essential influences of density changes on the flow are taken into account insofar as they modify the effects of the body forces. This last assumption is usually employed in problems of this nature, and its justification is discussed in references 1 and 14.

Under the conditions stated (which are essentially those for Poiseuille flows together with the specific thermal conditions), the temperature can be expressed as the sum of a linear function of the vertical coordinate and arbitrary function of the horizontal coordinate:

$$T^* = AX + T(Y) \tag{1}$$

Further, the continuity equation is identically satisfied, and the momentum and energy equations in dimensionless form are

$$u'' + \tau = -CK \tag{2}$$

$$\tau'' + \mathrm{Ra}\ u + (u')^2 + \alpha K = 0 \tag{3}$$

(All symbols are defined in the appendix.) The primes denote differentiation with respect to the dimensionless horizontal coordinate y. The terms in equation (2)(from left to right, respectively) denote the viscous, buoyancy, and axial-pressure forces, and the terms in equation (3) denote the conduction, convection, frictional heating, and heat sources. The details of the development of these equations are essentially presented in reference 5, the only modification in the present work being that the longitudinal (vertical) temperature gradient $\partial T^{*}/\partial X = A$ is taken as negative. Note that the Rayleigh and Grashof numbers are modified herein in that the characteristic temperature is Ad rather than a temperature difference as is conventional.

3573

The constant C which appears as a parameter in the problem merely specifies the temperature level (see eq. (2)) and is defined by (see ref. 5)

$$C = \frac{1}{\mu}\sqrt{\frac{Pr\ d^3}{c_p(-A)K}}\left(\frac{dP}{dX} + \rho_{w_0} f_X\right) \tag{4}$$

This parameter must in some way be related to the physics of the problem. From equation (4) it can be seen that C could be determined from the longitudinal pressure gradient; that is, C is essentially connected with the end conditions to which the channel is subject. The parameter C can also be related to the end conditions by the mass flow in the channel, which remains invariant over the entire channel length. Such a relation will be developed subsequently. The solutions of equations (2) and (3), together with appropriate boundary conditions, will, through the parameter C, apply to both natural-convection and combined natural- and forced-convection flows. The forced-convection pressure gradient merely alters the magnitude of C.

The system of equations (2) and (3) can, by eliminating τ, be written as a single fourth-order nonlinear ordinary differential equation:

$$u^{iv} - Ra\ u - (u')^2 - \alpha K = 0 \tag{5}$$

Equations of the same form as in reference 5 are obtained by letting

$$v = \frac{1}{64}(Ra\ u + \alpha K) \tag{6}$$

$$\eta = 2y - 1 \tag{7}$$

$$R = \frac{Ra}{16} \tag{8}$$

where equation (7) is merely for convenience. Equation (5) becomes, for $v(\eta)$,

$$v^{iv} - Rv - \frac{(v')^2}{R} = 0 \tag{9}$$

Boundary Conditions

3573

The present problem will be considered for two different sets of thermal-boundary conditions at the channel surfaces, namely, (1) that the wall temperatures are specified and (2) that both walls are insulated. Another case, which was essentially a combination of the above conditions, was treated in reference 12; but, since no basically new results were indicated, this case is not discussed herein.

For both sets of thermal conditions, the no-slip condition of viscous fluids must be satisfied; that is,

$$v(-1) = v(1) = \lambda \tag{10}$$

where

$$\lambda = \frac{\alpha K}{64} \tag{11}$$

Specified side-wall temperatures. - For the first set of thermal conditions the side-wall temperatures are specified. The temperature varies linearly along the walls (equal slopes on both walls), but the walls may each be at different temperatures. Thus, equation (1) implies uniform heat flux across each surface. When equations (2), (6), and (7) are used, this condition can be written as

$$v''(-1) = J\sqrt{R} \tag{12}$$

$$v''(1) = nJ\sqrt{R} \tag{13}$$

where

$$J = -\frac{Ra^{1/2}CK}{64} \tag{14}$$

and

$$n = 1 + \frac{\theta_{w_1}}{C(-A)d} \tag{15}$$

where n is essentially a measure of the difference between the two wall temperatures.

Both side walls insulated. - In order to simulate more closely the conditions used in the study of the horizontal layers where the side walls were taken to be insulated surfaces of symmetry in the fluid, the second set of thermal conditions chosen herein is that both side walls are insulated; that is,

$$\left.\begin{aligned} (\partial T^*/\partial Y)_{w_0,w_1} &= 0 \\ \text{or} \qquad & \\ v'''(-1) = v'''(1) &= 0 \end{aligned}\right\} \tag{16}$$

3573

Solutions

The nonlinear term in equation (9) is associated with frictional heating. If the effects of frictional heating are neglected, the problem then consists in solving the equation

$$v_0^{iv} - Rv_0 = 0 \tag{17}$$

together with the proper boundary conditions. For no internal heat sources ($\alpha = 0$), this problem is solved in reference 12 for three sets of boundary conditions. The solutions of equation (17), identified by the subscript zero, are hereinafter referred to as the approximate solutions. The effects of a uniform distribution of heat sources ($\alpha \neq 0$) are determined herein.

In the present paper the effects of frictional heating are determined from a direct solution of the complete equation (5), by means of a forward integration technique on a high-speed computing machine (Card-Programmed Calculator).[1]

The effects of frictional heating are clearly discernible by comparing the solutions of the complete and the approximate equations (eqs. (5) and (17), respectively). The solution of equation (17) also represents an extension (including heat sources) of the work in reference 12. The general solution of equation (17) is of the form

$$v_0 = C_1 \cos R^{1/4}\eta + C_2 \sin R^{1/4}\eta + C_3 \cosh R^{1/4}\eta + C_4 \sinh R^{1/4}\eta$$

[1]The effects of frictional heating could also be determined by an iteration procedure as described in reference 5, however, this procedure will not be used herein.

For each given set of boundary conditions, the solutions are presented in the succeeding sections in terms of v_0, and the corresponding temperature differences τ can be obtained by use of equation (2).

Specified side-wall temperatures. - The solution of equation (17) subject to the boundary conditions (eqs. (10), (12), and (13)) is

$$v_0 = \lambda v_{00} + \frac{J(n - 1)}{2} v_{01} + \frac{J(n + 1)}{2} v_{02} \tag{18}$$

where

$$v_{00} = \frac{1}{2}\left(\frac{\cosh R^{1/4}\eta}{\cosh R^{1/4}} + \frac{\cos R^{1/4}\eta}{\cos R^{1/4}}\right) \tag{19}$$

$$v_{01} = \frac{1}{2}\left(\frac{\sinh R^{1/4}\eta}{\sinh R^{1/4}} - \frac{\sin R^{1/4}\eta}{\sin R^{1/4}}\right) \tag{20}$$

$$v_{02} = \frac{1}{2}\left(\frac{\cosh R^{1/4}\eta}{\cosh R^{1/4}} - \frac{\cos R^{1/4}\eta}{\cos R^{1/4}}\right) \tag{21}$$

The solution given by equation (18) is a function of the parameter C. In order to relate this solution to a given physical problem, C can either be determined from the channel pressure gradient through equation (4) or from the dimensionless mass flow through the expression

$$M = \int_0^1 u_0 \, dy \tag{22}$$

Equation (22) is particularly convenient in the case of no net mass flow through the channel, for then C can be directly related to the parameters α, n, and Ra.

Examination of the solution neglecting frictional heating (eqs. (18) to (21)) shows that there are critical values of the Rayleigh number Ra for which the solutions become infinite. These values, namely, $Ra_k = (k\pi)^4$, where k denotes integers, obtained herein considering heat sources are identical with those given in reference 12 for no heat sources. It is significant that these critical values of the Rayleigh number when $k = 1$ and 2 ($Ra_1 = 97.41$ and $Ra_2 = 1558.55$) correlate closely with the critical values found for horizontal layers heated from below (see ref. 9). Further discussion of all the solutions and their physical simplications is given in the next section.

Both side walls insulated. - The solution of the boundary-value problem specified by equations (17), (10), and (16) is

$$v_0 = \frac{\lambda\left(\sin R^{1/4}\cosh R^{1/4}\eta - \sinh R^{1/4}\cos R^{1/4}\eta\right)}{\sin R^{1/4}\cosh R^{1/4} - \sinh R^{1/4}\cos R^{1/4}} \tag{23}$$

The critical values of Ra for equation (23) are those which satisfy $\tan R^{1/4} = \tanh R^{1/4}$ or $Ra_k \cong \left[\left(2k + \frac{1}{2}\right)\pi\right]^4$. For $k = 1$, $Ra_1 \cong 3803.22$. 3573

Note that the boundary-value problem as stated for this case (both walls insulated) by equations (5), (6), (10), and (16) defines τ to within an arbitrary constant. Therefore, in order that τ vanish at the wall $(y = 0)$, which it must by its definition, C must be so specified as to accomplish this. Hence, for this case, τ is determined from (see eq. (2)) $\tau = u''(0) - u''$.

Solutions for special case simulating an enclosed channel. - In order to simulate a completely enclosed channel, that is, one in which the ends are closed, the net mass flow as given by equation (22) can be specified as zero. For the case of linear side-wall temperatures, no net mass flow will be obtained if

$$\frac{C(n + 1)}{2} = \frac{\alpha}{4R^{1/2}} \frac{\tanh R^{1/4} + \tan R^{1/4} - 2R^{1/4}}{\tanh R^{1/4} - \tan R^{1/4}} \tag{24}$$

For the case wherein both walls are insulated, the condition of no net mass flow can be obtained only if there is no heat due to heat sources $(\alpha = 0)$. The situation for $\alpha = 0$ is as described in reference 12; that is, no flow except for certain specific values of Ra (those satisfying $\cos Ra^{1/4}\cosh Ra^{1/4} = 1$ or $Ra_k \cong \left[\left(k + \frac{1}{2}\right)\pi\right]^4$ as given in ref. 12).

MECHANICAL ANALOGY

Equation (17), obtained from the mathematical formulation of the problem treated herein, is identical with that describing the vibrations of uniform beams or of rotating shafts (see ref. 15). The case wherein both walls are insulated is analogous to the vibration problem if the ends of the shaft are fixed with no shear. The critical values of the Rayleigh number correspond to the critical or whirling speeds of the shaft.

If the shaft has initial deflections or bending moments at the ends, the rotating shaft becomes dynamically unstable and the solutions become infinite at the critical speeds. This situation has its counterpart in the present problem wherein the side-wall temperatures are specified.

RESULTS AND DISCUSSION

The discussion of the results is divided into two main sections according to the thermal boundary conditions. The main sections are further subdivided by first neglecting the frictional heating effects and then including them in order to demonstrate clearly the effects of heat sources and of frictional heating.

The primary effect of heating from below, namely, the instability of the flow, is demonstrated by comparing the present results with those for the stable case $(A > 0)$ reported in reference 5. In this regard, it should be noted that, since CK, αK, CA, and K/A are all independent of A, the problem treated herein (for $A < 0$) differs from that in reference 5 (for $A > 0$) only in the sign of the convection term. Therefore, in order to compare the two cases properly, the boundary conditions must be identical in the two problems. Choosing C with opposite signs in the two problems makes them identical. Since $C = -1$ for the present calculations, the corresponding cases for $A > 0$ are computed with the use of reference 5 for $C = 1$.

The results are all presented in dimensionless form, with the physical quantities related to the dimensionless ones by

$$U = \sqrt{\frac{c_p(-A)d}{Pr\ K}}\, u \tag{25}$$

$$\theta = \frac{(-A)d}{K}\, \tau \tag{26}$$

$$Y = yd \tag{27}$$

Linear Side-Wall Temperatures

<u>No frictional heating</u>. - In order to investigate closely the flow and heat transfer, including the effects of heat sources for values of Ra different from the critical, calculations of the velocity and temperature distributions were made from the approximate solutions for $\alpha = 10$, $K = 10$, and the two side walls either at the same temperature $(n = 1)$ or at different temperatures $(n = 0$, for convenience) and

$Ra = 10$, 81, 100, 1000, and 1600, so that the critical values of the Rayleigh number are straddled. The calculations of reference 12 for no heat sources ($\alpha = 0$) were also extended herein over the larger range of Rayleigh number values. All these computations are presented in figures 2 and 3 for the symmetric case ($n = 1$) and in figures 4 and 5 for the asymmetric case ($n = 0$) and are labeled with u_0 and τ_0 to denote that they are the solutions obtained neglecting frictional heating.

From the solutions (eqs. (18) to (21)), it can be seen that the only critical value of Ra for $n = 1$ in the range $0 < Ra \leq 1600$ (for which the calculations were made) is $Ra = \pi^4 = 97.41$. Comparison of the velocity profiles (fig. 2) for each fixed α and both walls at the same temperature shows a different flow direction on either side of this critical point, the distributions remaining essentially the same shape except for boundary-layer effects as Ra is further increased to 1600. This behavior can be clearly observed in figure 6, where the velocity extrema are presented as functions of Ra for $\alpha \neq 0$. The temperature distributions (see fig. 3) for $\alpha = 0$ are of opposite curvature on either side of $Ra = 97.41$; but for $\alpha = 10$ there is no such change, but merely a gradual decrease of the maximum values followed by an increase for large Rayleigh numbers.

For the walls at different temperatures ($n = 0$), the two critical values of Ra in the range up to 1600 are $\pi^4 = 97.41$ and $(2\pi)^4 = 1558.55$. For this case, it can be seen from figures 4(a) and (b) that again the velocity changes direction across $Ra = 97.41$ for $\alpha = 0$ and $\alpha = 10$. For values of $Ra > 100$, the flow is no longer entirely in one direction but is in opposite directions in adjacent sections of the channel (see fig. 4(c)). Beyond the second critical point, $Ra = (2\pi)^4 = 1558.55$, the flow directions in the adjacent sections are reversed. The temperature distributions (fig. 5) are also appreciably altered in passing the critical values of the Rayleigh number.

The effects of heat sources in the fluid can be determined by comparing the curves for $\alpha = 0$ with those for $\alpha = 10$ in figures 2 to 5. For $Ra = 10$ (figs. 2(a) and 4(a)), the addition of the heat ($\alpha = 10$) for these computations) causes the flow direction to be reversed essentially (with different magnitudes). Near the first critical point, the heat sources greatly reduce the velocity magnitudes. The addition of heat sources, of course, alters the temperature distributions and, hence, alters the heat transfer greatly.

In order to study the effect of heating from below, calculations made for the stable case ($A > 0$) from reference 5 are superposed on figures 2 to 5. The boundary conditions are specified to be identical for both problems, so that the two cases correspond except for the heating from below. Comparison of the velocity and temperature profiles shows that the heating from below changes the flow and heat-transfer directions and magnitudes in some ranges of Ra values. In all cases, greater velocities result with heating from below. Since, as has been previously discussed, the heat sources can also change the flow and heat-transfer magnitudes and directions from the $\alpha = 0$ case, the effect of the heat sources in this comparison is merely to shift some of the ranges of Ra values for which the stable and unstable flows and heat transfers are unidirectional.

Representative velocity and temperature profiles for the case simulating a fully enclosed region (i.e., zero net mass flow) are presented in figures 7 and 8 for Rayleigh numbers of 10, 100, and 1600 and for heat-source parameters of 0 and 10. For small Ra (see figs. 7(a) and 8(a)), the velocity profile with no heat sources ($\alpha = 0$) is antisymmetric and the temperature profile is essentially linear. For heat sources in the fluid ($\alpha = 10$), the velocity profile is symmetric about the channel axis and the temperature profile shows the effects of the increased convection by its almost parabolic shape. For large Rayleigh numbers (figs. 7(c) and 8(c)), both the velocity and temperature profiles for both $\alpha = 0$ and $\alpha = 10$ are antisymmetric, but for each different α the directions are opposite. Further, since $Ra = 1558.55$ is the first critical value due to heating from below for this case, the direction change from that for small Ra is also clearly evident. Note that the parameter n for no net mass flow and fixed C changes with α, and the appropriate values are indicated in the figures.

<u>Frictional heating</u>. - In reference 5, it was pointed out that, in general, the effects of frictional heating could be important if the ratio K/Ra is of unit order or larger. Therefore, in order to see how well the solutions neglecting frictional heating approximate the exact solutions (eq. (5)), the complete equation including the effects of frictional heating was solved for several cases by means of a Card-Programmed Calculator. A forward integration technique similar to that described in appendix B of reference 16 was employed. The results of these calculations, denoted by u and τ with no subscripts but with superscripts to indicate the exact solutions, are presented in tables I and II and also in figures 2(a) and (d), 3(a) and (d), 4(a) to (d), and 5(a) to (d). The significance of the superscripts will be discussed subsequently.

The approximate solutions deviate somewhat from the exact solutions for $K/Ra \approx 1$ and essentially coincide with them for $K/Ra \ll 1$. However, near the critical values of Ra where the approximate solutions become infinite, there are, of course, very great discrepancies between the two solutions. Therefore, solutions were computed from the exact equations for values of Ra at and on either side of the critical. These results are presented in figures 9 and 10 for the case of unequal wall temperatures $(n = 0)$. It should be recalled that in reference 1 it was first pointed out that solutions of the exact equations (taking into account the effects of frictional heating) were not unique. In fact, two distinct solutions were found for each given set of parameters up to a limiting set in the only problems which have been solved exactly (see refs. 1 and 5). Beyond the certain limiting set of parametric values, no exact solutions could be found. The conjecture is that this corresponds to a choking condition.

The two sets of exact solutions for several Rayleigh numbers (see table II) at and near the first critical value of Ra, shown in figures 9 and 10 for $\alpha = 0$ and $n = 0$, are denoted by $u^{(1)}$ and $u^{(2)}$. The superscripts are used merely to identify a given exact solution, and the (1) is used to denote that solution with the algebraically smaller initial slope. Computations from the corresponding approximate solutions are included in figures 9 and 10. For the case of equal wall temperatures $(n = 1)$, the velocity extrema of both exact solutions are superimposed on figure 6. It can be seen from figures 6, 9, and 10 that the flows computed from the approximate solutions correspond to different exact solutions on either side of the first critical Ra. 3573

Beyond the first critical Ra, separate consideration must be given to the symmetric case $(n = 1)$ and the asymmetric case $(n = 0)$. For $n = 1$, recall that $Ra = \pi^4$ is the only critical Rayleigh number (in the range computed, i.e., from $Ra = 10$ to 1600) indicated by the approximate solutions. Thus, for $97.41 < Ra < 7890.13$ (the latter being the second critical point $(3\pi)^4$ for $n = 1$), the approximate solutions indicate flows of the same general character. However, the exact solutions of the superscript (1) type for $n = 1$ appear to be discontinuous at approximately $Ra = 1579$ (see fig. 6). The change in flow type indicated thereby for the larger (than 97.41) Ra may imply that the frictional heating delays (in a Rayleigh number sense) the instability. Note that the approximate solution always corresponds to that exact solution with the lower velocities.

Exact (see table II) and approximate solutions near the second critical value of Ra $((2\pi)^4)$ for $\alpha = 0$ and $n = 0$ are shown in figures 11 and 12. The two exact solutions are shown only for $Ra = 1758.55$. Although the flow in adjacent sections of the channel is in opposite directions, the qualitative effects are essentially the same as those near the first critical Ra. No unusual behavior of the exact solutions for $n = 0$ was found, but not as many of these solutions for $n = 0$ were obtained as for $n = 1$.

Near the critical points there are appreciable quantitative and qualitative differences between the exact and approximate solutions, so that the exact solutions or higher-order approximations should be used there for reasonable accuracy. Therefore, it appears that near the critical Ra the effects of frictional heating are very important.

3573

It has been pointed out (figs. 6 and 9 to 12) that on either side of the critical points the solutions neglecting frictional heating correspond to different exact solutions but always those with the smaller velocity extremum. The question then arises: Since the exact solutions do not display a discontinuity (with Ra) at the values indicated by the approximate solutions, is the actual flow really of a different type on either side of the critical Ra as is indicated by the linearized solutions? Although a more general study of the present configuration would be necessary (perhaps relaxing the condition of fully developed flow) to answer this question conclusively, it seems reasonable that the actual flow does in fact change character in accord with the linearized solutions and the exact solutions for $n = 1$. The reason for this statement is that the approximate solutions are approximate only in that frictional heating is neglected. In starting a flow, the frictional heating effects are not of primary importance until velocities of appreciable magnitude are encountered. Therefore, it is felt that the larger velocity flows, indicated by one of the two exact solutions, result from a regenerative action of the frictional heating (see ref. 5) and that for a given set of conditions the flow first established would be that with the smaller velocities. Also, the changes in the character of the flow on either side of the critical Ra in the experiments with unstable horizontal fluid layers seem to lend further support to this contention.

Near the critical points themselves, however, the velocities are always large; and, therefore, the frictional heating may play an important role in the transition from one type of flow to the other.

The velocity and temperature distributions for the case of no net mass flow in the channel will be affected by frictional heating in exactly the same manner as was discussed above.

Both Side Walls Insulated

No frictional heating. - The effect of heat sources in the fluid for no frictional heating and with both side walls insulated can be studied by comparing equation (23) with the solution for $\alpha = 0$ as given in reference 12. For example, in reference 12 it was shown that for no heat sources flow does not ensue except for values of Ra that satisfy the equation $\cos \mathrm{Ra}^{1/4} \cosh \mathrm{Ra}^{1/4} = 1$. However, when $\alpha \neq 0$, the present results indicate that the flow and heat transfer are much the same as for the linear surface temperature cases, except that the critical values of Ra are higher (the first is 3803.22). A representative set of profiles for $\alpha = 10$, $K = 10$, and $\mathrm{Ra} = 10$, 7^4, and 8^4, with both walls insulated, is shown in figures 13 and 14. Again, the different regimes on opposite sides of the critical Ra are evident.

Frictional heating. - Solutions including the effects of frictional heating obtained by means of the high-speed computing machine for $Ra = 7^4$ and 8^4 are presented in figures 13 and 14. The initial values of the exact solutions are presented in table III. These solutions coincide with those computed neglecting frictional heating, as is expected, since the ratio $K/Ra \ll 1$. At the critical Rayleigh number (3803.22), no solution could be obtained with frictional heating. This insolvability may be the result of the "choking condition" previously mentioned and discussed in reference 1.

3573

CONCLUDING REMARKS

An analysis of the flow subject to a body force between two parallel plane surfaces oriented parallel to the body force direction with heating from below and including the effects of heat sources and frictional heating shows that the basic physical characteristics are similar to flows in horizontal fluid layers with heating from below. The prime characteristics in this respect are the existence of critical Rayleigh numbers and the difference in the flow and heat transfer on opposite sides of critical Rayleigh number.

Detailed velocity and temperature distributions were computed for linearly varying wall temperatures when the wall temperatures were specified or when both walls were insulated, and also for the special case of no net mass flow in the channel. The critical Rayleigh numbers are the same for corresponding flows with and without heat sources for the specified wall temperatures but are different when both walls are insulated. Comparison of two identical configurations, in one of which, however, the fluid was heated from below, indicated that the flow and heat transfer can be appreciably affected by heating from below.

The effects of frictional heating were once again found to be important when K/Ra is of unit order and are also important in the vicinity of the critical Ra. Two distinct solutions were obtained for each set of parametric values when frictional heating is considered. The solutions obtained neglecting frictional heating correspond to one of the two sets of exact solutions except near the critical regions. The approximate solutions on one side of a critical region correspond to one of the two distinct solution types obtained with frictional heating but on the other side of the critical region correspond to the other type exact solution. Hence, the approximate solutions indicate completely altered regimes on opposite sides of the critical Rayleigh number, analogous to the situation experimentally found for horizontal layers heated from below. However, close to the critical region the approximate solutions are invalid; and, hence, the question as to which of the two exact solutions properly describes the flow in this region can be answered perhaps by a more general analysis.

Lewis Flight Propulsion Laboratory
National Advisory Committee for Aeronautics
Cleveland, Ohio, April 29, 1955

3573

APPENDIX - SYMBOLS

The following symbols are used in this report:

A	longitudinal temperature gradient, $\partial T^*/\partial X$
C	parameter defined by eq. (4)
C_i	constants, $i = 1, 2, 3, 4$
c_p	specific heat at constant pressure
d	distance between parallel plates
f_X	negative of X-component of body force per unit mass
Gr	modified Grashof number, $\beta f_X(-A)d^4/\nu^2$
J	constant, $-Ra^{1/2}CK/64$
K	frictional-heating parameter, $\mathrm{PrGr}\,\dfrac{\beta f_X d}{c_p} = \dfrac{\rho^2\beta^2 f_X^2(-A)d^5}{\varkappa\mu}$
k	integers
M	dimensionless mass flow
n	wall-temperature parameter, $1 + \dfrac{\theta_{w_1}}{C(-A)d}$
P	pressure
Pr	Prandtl number, $c_p\mu/\varkappa$
Q	heat due to heat sources
R	constant, Ra/16
Ra	modified Rayleigh number, $\dfrac{1}{\gamma}$ PrGr
T,T^*	temperature
U	velocity
u	dimensionless velocity
v	dimensionless velocity defined by eq. (6)

3573

X longitudinal or axial (vertical) coordinate

Y transverse (horizontal) coordinate

y dimensionless transverse coordinate, Y/d

α dimensionless heat-source parameter, $Qd/(-A)\varkappa$

β volumetric expansion coefficient, $\rho\left[\frac{\partial(1/\rho)}{\partial T}\right]_p$

γ ratio of specific heats

η dimensionless coordinate, $2y - 1$

θ temperature difference, $T - T_{w_0}$ or $\frac{(-A)d}{K}\tau$

$\varkappa$ coefficient of thermal conductivity

λ constant, $\alpha K/64$

μ absolute viscosity coefficient

ν kinematic viscosity coefficient

ρ density

τ dimensionless temperature difference, $K\theta/(-A)d$

Subscripts:

e extremum value

w_0 conditions at $y = 0$

w_1 conditions at $y = 1$

REFERENCES

1. Ostrach, Simon: Laminar Natural-Convection Flow and Heat Transfer of Fluids with and without Heat Sources in Channels with Constant Wall Temperatures. NACA TN 2863, 1952.

2. Lighthill, M. J.: Theoretical Considerations on Free Convection in Tubes. F.M.1758, British A.R.C., July 21, 1952.

3. Woodrow, J.: Free Convection in Heat Generating Fluid (Laminar Flow). A.E.R.E. E/R 1267, Atomic Energy Res. Est. (British), Oct. 1953.

4. Wordsworth, D. V.: Laminar Free Convection Between Heat Producing Vertical Plates in a Liquid. A.E.R.E. E/R 1270, Atomic Energy Res. Est. (British), Oct. 1953.

5. Ostrach, Simon: Combined Natural- and Forced-Convection Laminar Flow and Heat Transfer of Fluids with and without Heat Sources in Channels with Linearly Varying Wall Temperatures. NACA TN 3141, 1954.

6. Lietzke, A. F.: Theoretical and Experimental Investigation of Heat Transfer by Laminar Natural Convection Between Parallel Plates. NACA TN 3328, 1954.

7. Rayleigh, Lord: On Convection Currents in a Horizontal Layer of Fluid, when the Higher Temperature is on the Under Side. Phil. Mag. and Jour. Sci., ser. 6, vol. 32, no. 192, Dec. 1916, pp. 529-547.

8. Bénard, H.: Tourbillons cellulaires dans une nappe liquide. Revue générale des sci. pures et appliquées, t. 11, 1900, pp. 1261-1271; pp. 1309-1328.

9. Bosworth, R. C. L.: Heat Transfer Phenomena. John Wiley & Sons, Inc., 1952, pp. 104-107.

10. Chandra, Krishna: Instability of Fluids Heated from Below. Proc. Roy. Soc. (London), ser. A, vol. 164, no. A917, Jan. 21, 1938, pp. 231-242.

11. Timo, D. P.: Free Convection in Narrow Vertical Sodium Annuli. KAPL-1082, Knolls Atomic Power Lab., General Electric Co., Mar. 5, 1954. (Contract No. W-31-109 Eng-52.)

12. Ostrach, Simon: On the Flow, Heat Transfer, and Stability of Viscous Fluids Subject to Body Forces and Heated from Below in Vertical Channels. 50 Jahre Grenzschichtforschung, Eine Festschrift in Originalbeiträgen. Verlag Friedr. Vieweg & Sohn, Braunschweig (Germany), 1955.

13. Ostrach, Simon: New Aspects of Natural-Convection Heat Transfer. Trans. A.S.M.E., vol. 75, no. 7, Oct. 1953, pp. 1287-1290.

14. Sutton, O. G.: On the Stability of a Fluid Heated from Below. Proc. Roy. Soc. (London), ser. A, vol. 204, no. 1078, Dec. 22, 1950, pp. 297-309.

15. von Kármán, Theodore, and Biot, Maurice A.: Mathematical Methods in Engineering. McGraw-Hill Book Co., Inc., 1940, pp. 286-290.

16. Ostrach, Simon: An Analysis of Laminar Free-Convection Flow and Heat Transfer About a Flat Plate Parallel to the Direction of the Generating Body Force. NACA Rep. 1111, 1953. (Supersedes NACA TN 2635.)

TABLE I. - EXACT SOLUTIONS FOR WALLS AT SAME TEMPERATURE

[n = 1, K = 10, C = -1.]

(a) Ra = 10, α = 0

y	$u^{(1)}$	$u^{(1)'}$	$u^{(1)''}$	$u^{(1)'''}$
0	0	-5.205	10.00	-0.2362
.10	-.4704	-4.202	10.09	1.742
.20	-.8399	-3.183	10.31	2.452
.30	-1.106	-2.140	10.54	2.189
.40	-1.267	-1.076	10.72	1.262
.50	-1.321	.0001010	10.79	-.002926
.60	-1.267	1.077	10.72	-1.267
.70	-1.106	2.141	10.54	-2.194
.80	-.8399	3.183	10.30	-2.457
.90	-.4704	4.202	10.08	-1.748
1.00	$-.1986 \times 10^{-4}$	5.204	9.997	.2300

(b) Ra = 10, α = 10

y	$u^{(1)}$	$u^{(1)'}$	$u^{(1)''}$	$u^{(1)'''}$
0	0	-0.8240	10.00	-50.06
.10	-.04033	-.05760	5.494	-40.07
.20	-.02488	.3081	1.986	-30.10
.30	.01126	.3729	-.5240	-20.09
.40	.04300	.2367	-2.032	-10.05
.50	.05526	.00004873	-2.534	-.001519
.60	.04301	-.2366	-2.032	10.05
.70	.01128	-.3728	-.5246	20.09
.80	-.02486	-.3082	1.985	30.10
.90	-.04032	.05746	5.493	40.06
1.00	$-.1445 \times 10^{-4}$	.8237	9.998	50.06

(c) Ra = 100, α = 0

y	$u^{(2)}$	$u^{(2)'}$	$u^{(2)''}$	$u^{(2)'''}$
0	0	15.34	10.00	-227.2
.10	1.547	15.25	-11.27	-195.6
.20	2.985	13.22	-28.72	-152.0
.30	4.140	9.666	-41.48	-102.6
.40	4.884	5.090	-49.18	-51.42
.50	5.141	$.3334 \times 10^{-3}$	-51.75	.008463
.60	4.885	-5.089	-49.18	51.44
.70	4.140	-9.665	-41.47	102.7
.80	2.986	-13.22	-28.71	152.0
.90	1.548	-15.25	-11.26	195.6
1.00	$.6819 \times 10^{-3}$	-15.34	10.01	227.3

(d) Ra = 1600, α = 0

y	$u^{(1)}$	$u^{(1)'}$	$u^{(1)''}$	$u^{(1)'''}$
0	0	-0.8000	10.00	-31.05
.10	-.03526	.04072	6.745	-34.98
.20	-.003506	.5321	3.017	-38.72
.30	.05839	.6431	-.7247	-34.44
.40	.1138	.4182	-3.538	-20.33
.50	.1354	-.004572	-4.574	.1771
.60	.1130	-.4256	-3.505	20.61
.70	.05699	-.6461	-.6719	34.54
.80	-.004927	-.5297	3.068	38.59
.90	-.03622	-.03424	6.773	34.65
1.00	$-.2323 \times 10^{-3}$	.8073	9.988	30.63

(e) Ra = 1600, α = 10

y	$u^{(1)}$	$u^{(1)'}$	$u^{(1)''}$	$u^{(1)'''}$
0	0	-0.9800	10.00	-39.61
.10	-.05429	-.1666	6.342	-35.00
.20	-.04503	.2944	2.900	-33.56
.30	-.006514	.4236	-.2216	-27.84
.40	.03053	.2800	-2.449	-15.71
.50	.04432	-.01711	-3.219	.6815
.60	.02733	-.3076	-2.321	16.81
.70	-.01168	-.4341	-.01617	28.25
.80	-.05018	-.2837	3.104	33.11
.90	-.05745	.1942	6.464	33.87
1.00	$.2669 \times 10^{-5}$	1.013	9.991	38.22

3573

TABLE II. - EXACT SOLUTIONS FOR UNEQUAL WALL TEMPERATURES

[n = 0, K = 10, C = -1.]

(a) Ra = 10, α = 0

y	u(2)	u(2)'	u(2)''	u(2)'''
0	0	-3.500	10.00	-9.160
.10	-.3015	-2.544	9.129	-8.402
.20	-.5116	-1.673	8.296	-8.368
.30	-.6389	-.8858	7.441	-8.783
.40	-.6917	-.1866	6.532	-9.422
.50	-.6793	.4182	5.555	-10.11
.60	-.6115	.9221	4.513	-10.71
.70	-.4985	1.319	3.419	-11.14
.80	-.3514	1.605	2.292	-11.35
.90	-.1813	1.777	1.157	-11.33
1.00	$.3222\times10^{-3}$	1.837	.03429	-11.09

(b) Ra = 10, α = 10

y	u(2)	u(2)'	u(2)''	u(2)'''
0	0	1.172	10.00	-63.56
.10	.1571	1.872	4.157	-53.24
.20	.3566	2.039	-.6361	-42.58
.30	.5507	1.780	-4.354	-31.75
.40	.7021	1.204	-6.985	-20.89
.50	.7846	.4195	-8.532	-10.06
.60	.7827	-.4660	-8.999	7.345
.70	.6916	-1.344	-8.384	11.57
.80	.5177	-2.107	-6.682	22.49
.90	.2778	-2.644	-3.885	33.47
1.00	$.3038\times10^{-4}$	-2.847	.009144	44.38

(c) Ra = 77.41, α = 0

y	$u^{(1)}$	$u^{(2)}$	$u^{(1)'}$	$u^{(2)'}$	$u^{(1)''}$	$u^{(2)''}$	$u^{(1)'''}$	$u^{(2)'''}$
0	0	0	-8.150	20.90	10.00	10.00	32.42	-297.1
.10	-.7594	2.092	-6.980	20.49	13.44	-17.24	35.18	-245.2
.20	-1.384	4.017	-5.464	17.64	16.79	-38.75	30.71	-184.3
.30	-1.842	5.560	-3.647	12.95	19.39	-54.08	20.23	-122.9
.40	-2.107	6.567	-1.630	7.027	20.70	-63.45	5.551	-65.17
.50	-2.166	6.944	.4405	.4475	20.43	-67.23	-11.05	-10.67
.60	-2.023	6.653	2.401	-6.239	18.51	-65.59	-27.15	43.61
.70	-1.695	5.710	4.092	-12.49	15.09	-58.40	-40.55	101.1
.80	-1.218	4.189	5.381	-17.72	10.54	-45.22	-49.62	163.1
.90	-.6353	2.221	6.179	-21.32	5.343	-25.71	-53.45	227.0
1.00	$.3330\times10^{-3}$	$.8638\times10^{-3}$	6.446	-22.65	.03330	-.02029	-51.88	285.0

(d) Ra = 97.41, α = 0

y	$u^{(1)}$	$u^{(2)}$	$u^{(1)'}$	$u^{(2)'}$	$u^{(1)''}$	$u^{(2)''}$	$u^{(1)'''}$	$u^{(2)'''}$
0	0	0	-13.45	10.83	10.00	10.00	73.30	-162.4
.10	-1.282	1.107	-12.06	11.04	17.97	-5.451	83.39	-144.8
.20	-2.384	2.160	-9.849	9.816	26.15	-18.64	77.55	-117.6
.30	-3.226	3.031	-6.876	7.417	32.99	-28.78	57.14	-84.54
.40	-3.741	3.616	-3.340	4.176	37.20	-35.44	25.67	-48.39
.50	-3.886	3.850	.4481	.4518	37.93	-38.42	-11.45	-11.06
.60	-3.655	3.702	4.122	-3.383	34.94	-37.65	-47.83	26.37
.70	-3.077	3.182	7.323	-6.955	28.59	-33.17	-77.46	63.01
.80	-2.216	2.332	9.760	-9.898	19.82	-25.12	-95.97	97.39
.90	-1.157	1.235	11.25	-11.87	9.843	-13.85	-101.3	127.0
1.00	$.1560\times10^{-4}$	$.2910\times10^{-3}$	11.73	-12.58	-.003217	$.7480\times10^{-3}$	-93.57	148.3

3573

TABLE II. - Continued. EXACT SOLUTIONS FOR UNEQUAL WALL TEMPERATURES

[n = 0, K = 10, C = -1.]

(e) Ra = 100, α = 0

y	$u^{(1)}$	$u^{(1)'}$	$u^{(1)''}$	$u^{(1)'''}$
0	0	9.877	10.00	-150.7
.10	1.013	10.14	-4.395	-135.4
.20	1.984	9.065	-16.77	-110.9
.30	2.789	6.883	-26.37	-80.33
.40	3.334	3.899	-32.73	-46.42
.50	3.554	.4525	-35.61	-11.12
.60	3.420	-3.105	-34.95	24.34
.70	2.941	-6.420	-30.78	58.81
.80	2.156	-9.149	-23.27	90.74
.90	1.141	-10.97	-12.79	117.7
1.00	$.1563 \times 10^{-3}$	-11.63	$-.2000 \times 10^{-3}$	136.5

(f) Ra = 117.41, α = 0

y	$u^{(1)}$	$u^{(2)}$	$u^{(1)'}$	$u^{(2)'}$	$u^{(1)''}$	$u^{(2)''}$	$u^{(1)'''}$	$u^{(2)'''}$
0	0	0	-22.65	5.400	10.00	10.00	129.2	-98.41
.10	-2.192	.5738	-20.93	5.916	24.95	.4285	164.3	-91.80
.20	-4.132	1.153	-17.60	5.520	41.69	-8.128	164.6	-78.27
.30	-5.657	1.652	-12.65	4.345	56.70	-15.04	130.1	-59.19
.40	-6.620	2.002	-6.427	2.582	66.74	-19.84	66.97	-36.31
.50	-6.921	2.156	.4536	.4578	69.56	-22.24	-11.89	-11.42
.60	-6.533	2.090	7.217	-1.781	64.40	-22.11	-89.65	13.81
.70	-5.507	1.805	13.10	-3.882	52.21	-19.52	-150.1	37.73
.80	-3.963	1.326	17.50	-5.608	35.34	-14.67	-182.1	58.59
.90	-2.067	.7023	20.11	-6.753	16.88	-7.962	-181.7	74.51
1.00	$-.4120 \times 10^{-3}$	$.8901 \times 10^{-4}$	20.93	-7.159	-.002140	.007880	-151.3	83.61

(g) Ra = 1000, α = 0

y	$u^{(1)}$	$u^{(1)'}$	$u^{(1)''}$	$u^{(1)'''}$
0	0	-2.040	10.00	8.015
.10	-.1528	-1.007	10.52	-.2448
.20	-.2017	.01490	9.608	-18.79
.30	-.1561	.8490	6.762	-37.35
.40	-.04417	1.317	2.419	-47.62
.50	.09160	1.319	-2.330	-45.07
.60	.2048	.8815	-6.165	-29.75
.70	.2581	.1536	-7.995	-5.961
.80	.2336	-.6327	-7.308	19.29
.90	.1379	-1.231	-4.340	38.47
1.00	$-.1136 \times 10^{-4}$	-1.455	-.01511	45.74

(h) Ra = 1000, α = 10

y	$u^{(1)}$	$u^{(1)'}$	$u^{(1)''}$	$u^{(1)'''}$
0	0	-2.412	10.00	2.010
.10	-.1906	-1.394	10.36	2.002
.20	-.2783	-.3673	9.927	-12.22
.30	-.2682	.5339	7.793	-30.29
.40	-.1815	1.137	4.045	-43.20
.50	-.05509	1.317	-.4729	-45.02
.60	.06701	1.057	-4.529	-34.06
.70	.1452	.4663	-6.941	-12.89
.80	.1560	-.2500	-6.955	12.78
.90	.09944	-.8405	-4.465	36.09
1.00	$-.1579 \times 10^{-3}$	-1.077	-.009459	51.35

3573

TABLE II. - Concluded. EXACT SOLUTIONS FOR UNEQUAL WALL TEMPERATURES

[n = 0, K = 10, C = -1.]

(i) Ra = 1358.55, α = 0

y	$u^{(1)}$	$u^{(1)'}$	$u^{(1)''}$	$u^{(1)'''}$
0	0	-4.660	10.00	109.4
.10	-.3982	-3.135	20.04	82.18
.20	-.6003	-.8246	25.00	12.18
.30	-.5592	1.598	22.09	-69.28
.40	-.3035	3.346	11.87	-129.2
.50	.06772	3.841	-2.237	-144.4
.60	.4174	2.934	-15.31	-109.2
.70	.6188	.9675	-22.79	-36.11
.80	.5991	-1.349	-22.11	49.30
.90	.3650	-3.188	-13.52	117.5
1.00	$-.7953\times10^{-4}$	-3.887	-.01161	144.4

(j) Ra = 1558.55, α = 0

y	$u^{(1)}$	$u^{(1)'}$	$u^{(1)''}$	$u^{(1)'''}$
0	0	-64.10	10.00	2081
.10	-6.005	-52.45	221.0	1959
.20	-9.854	-22.06	367.9	839.9
.30	-10.15	16.32	372.7	-755.5
.40	-6.837	47.44	228.8	-2002
.50	-1.309	59.33	3.164	-2345
.60	4.262	48.52	-209.8	-1789
.70	7.809	20.29	-336.3	-679.4
.80	8.096	-14.59	-339.6	616.7
.90	5.094	-43.41	-217.3	1779
1.00	$-.1463\times10^{-3}$	-54.84	-.008320	2452

(k) Ra = 1600, α = 0

y	$u^{(2)}$	$u^{(2)'}$	$u^{(2)''}$	$u^{(2)'''}$
0	0	18.88	10.00	-854.1
.10	1.800	15.79	-68.70	-673.5
.20	2.938	6.163	-117.1	-267.8
.30	2.944	-6.076	-119.5	220.5
.40	1.795	-16.18	-65.74	627.3
.50	-.08615	-20.20	-1.733	804.9
.60	-1.982	-16.44	74.70	670.5
.70	-3.155	-6.210	123.0	260.4
.80	-3.140	6.530	123.1	-259.0
.90	-1.934	16.78	75.13	-663.2
1.00	$-.7300\times10^{-5}$	20.64	-.01997	-785.5

(l) Ra = 1600, α = 10

y	$u^{(2)}$	$u^{(2)'}$	$u^{(2)''}$	$u^{(2)'''}$
0	0	18.81	10.00	-866.8
.10	1.791	15.68	-69.51	-677.1
.20	2.913	5.959	-117.9	-264.3
.30	2.896	-6.334	-119.7	228.2
.40	1.721	-16.42	-75.12	635.7
.50	-.1793	-20.34	-.3720	810.6
.60	-2.082	-16.42	76.38	670.8
.70	-3.244	-6.030	124.4	255.2
.80	-3.204	6.821	123.8	-266.5
.90	-1.967	17.11	75.23	-667.8
1.00	$-.1390\times10^{-4}$	20.97	$-.9340\times10^{-3}$	-781.5

(m) Ra = 1758.55, α = 0

y	$u^{(1)}$	$u^{(2)}$	$u^{(1)'}$	$u^{(2)'}$	$u^{(1)''}$	$u^{(2)''}$	$u^{(1)'''}$	$u^{(2)'''}$
0	0	0	3.210	163.9	10.00	10.00	-200.7	-9719
.10	.3381	14.95	3.233	121.7	-9.026	-792.1	-169.8	-6090
.20	.5909	22.32	1.604	18.91	-22.14	-1196	-85.07	-2029
.30	.6309	18.05	-.8536	-104.3	-25.15	-1200	26.06	2012
.40	.4286	2.146	-3.062	-206.8	-17.41	-775.1	122.9	6559
.50	.05856	-21.13	-4.085	-244.3	-2.286	91.46	168.7	10378
.60	-.3334	-43.32	-3.481	-181.8	13.99	1145	145.1	9578
.70	-.5902	-54.42	-1.476	-29.47	24.71	1777	61.66	2186
.80	-.6084	-48.51	1.124	143.5	25.45	1535	-47.51	-6518
.90	-.3809	-27.78	3.266	256.9	15.87	690.6	-137.1	-8983
1.00	$-.6685\times10^{-4}$	$.2300\times10^{-4}$	4.084	287.0	-.06536	-.005290	-170.4	-3740

TABLE III. - INITIAL VALUES OF EXACT SOLUTIONS FOR BOTH WALLS INSULATED

[$\alpha K = 0.01$]

(a) Ra = 2401

$u^{(1)}(0)$	$u^{(2)}(0)$	$u^{(1)'}(0)$	$u^{(2)'}(0)$	$u^{(1)''}(0)$	$u^{(2)''}(0)$	$u^{(1)'''}(0)$	$u^{(2)'''}(0)$
0	0	-0.000035	166.0	-0.0004495	-3172	0	0

(b) Ra = 4096

$u^{(1)}(0)$	$u^{(2)}(0)$	$u^{(1)'}(0)$	$u^{(2)'}(0)$	$u^{(1)''}(0)$	$u^{(2)''}(0)$	$u^{(1)'''}(0)$	$u^{(2)'''}(0)$
0	0	-50	-0.00028	387.1	0.002085	0	0

3573

CY-4

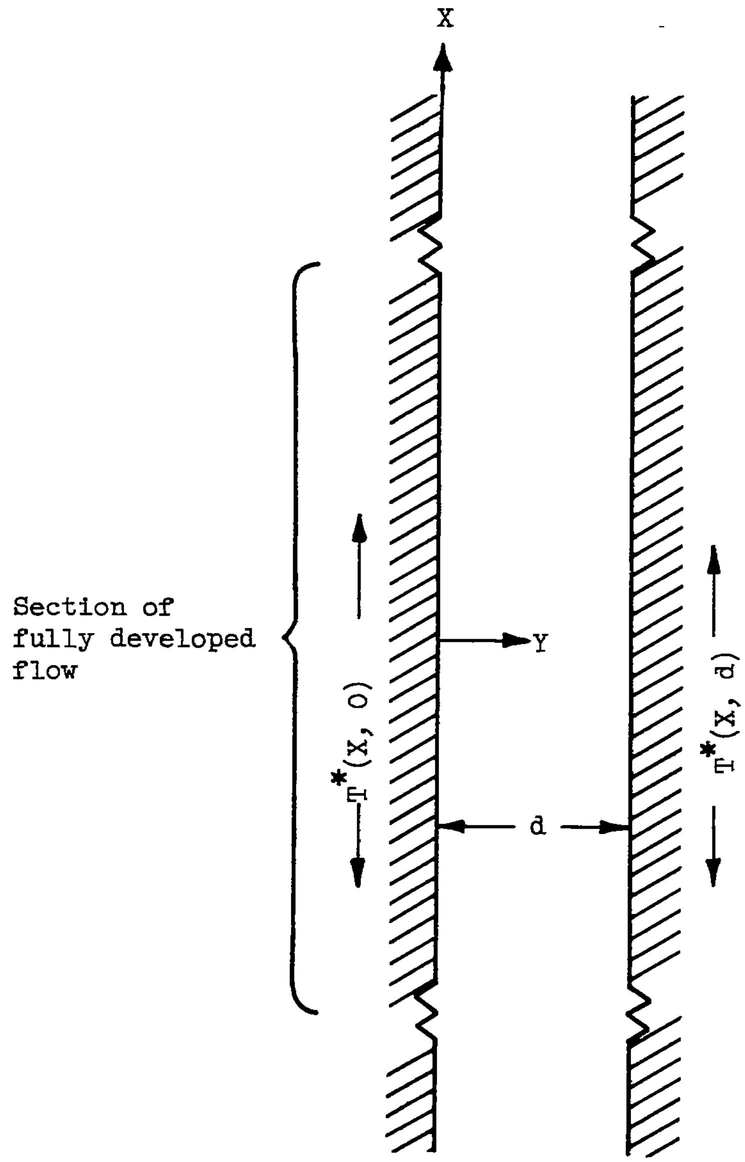


Figure 1. - Schematic sketch of configuration considered.

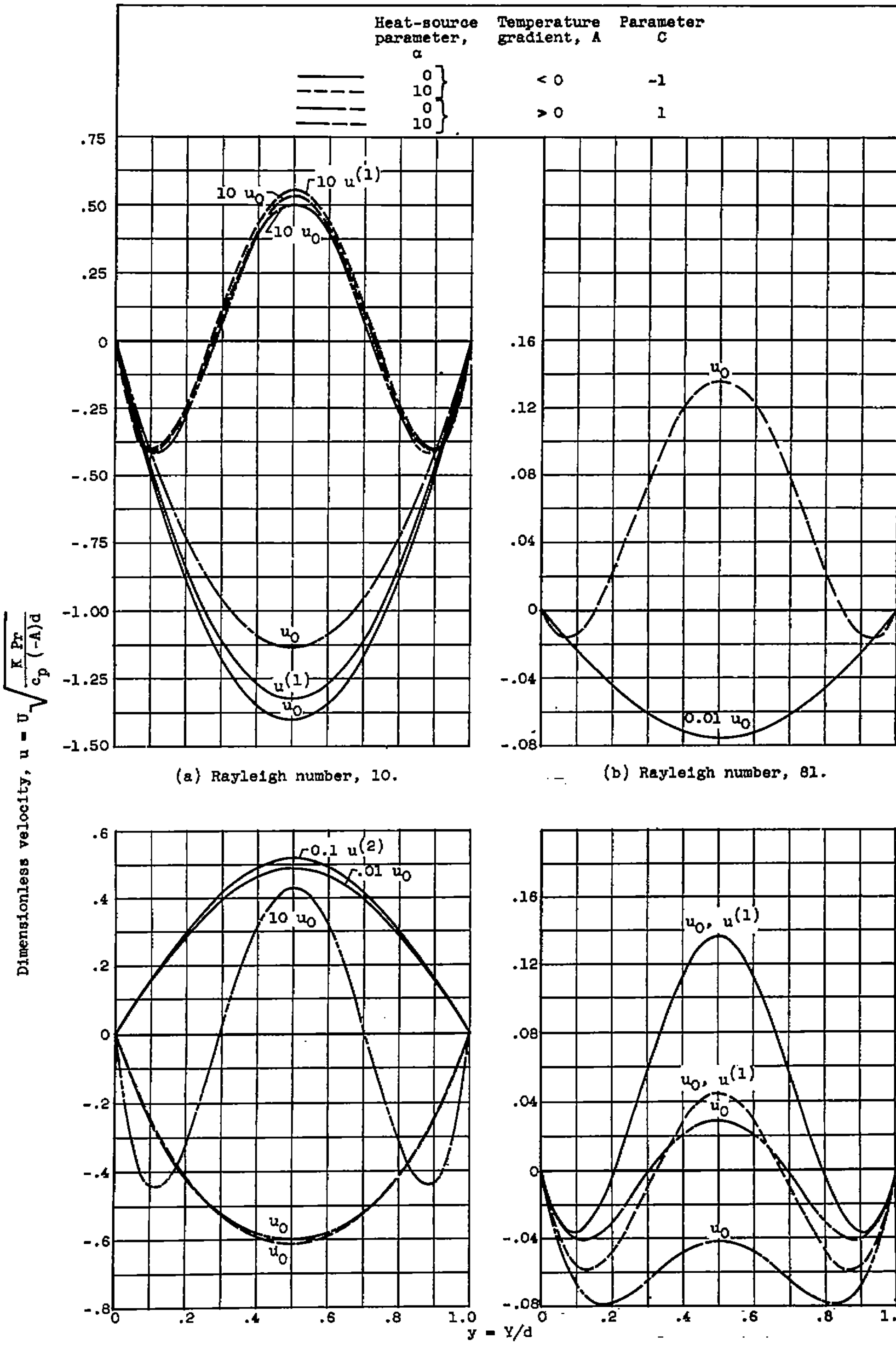


Figure 2. - Velocity distributions with and without heat sources for heating from below ($A < 0$) and not ($A > 0$). Identical wall temperatures ($n = 1$); $K = 10$.

3573

3573

CY-4 back

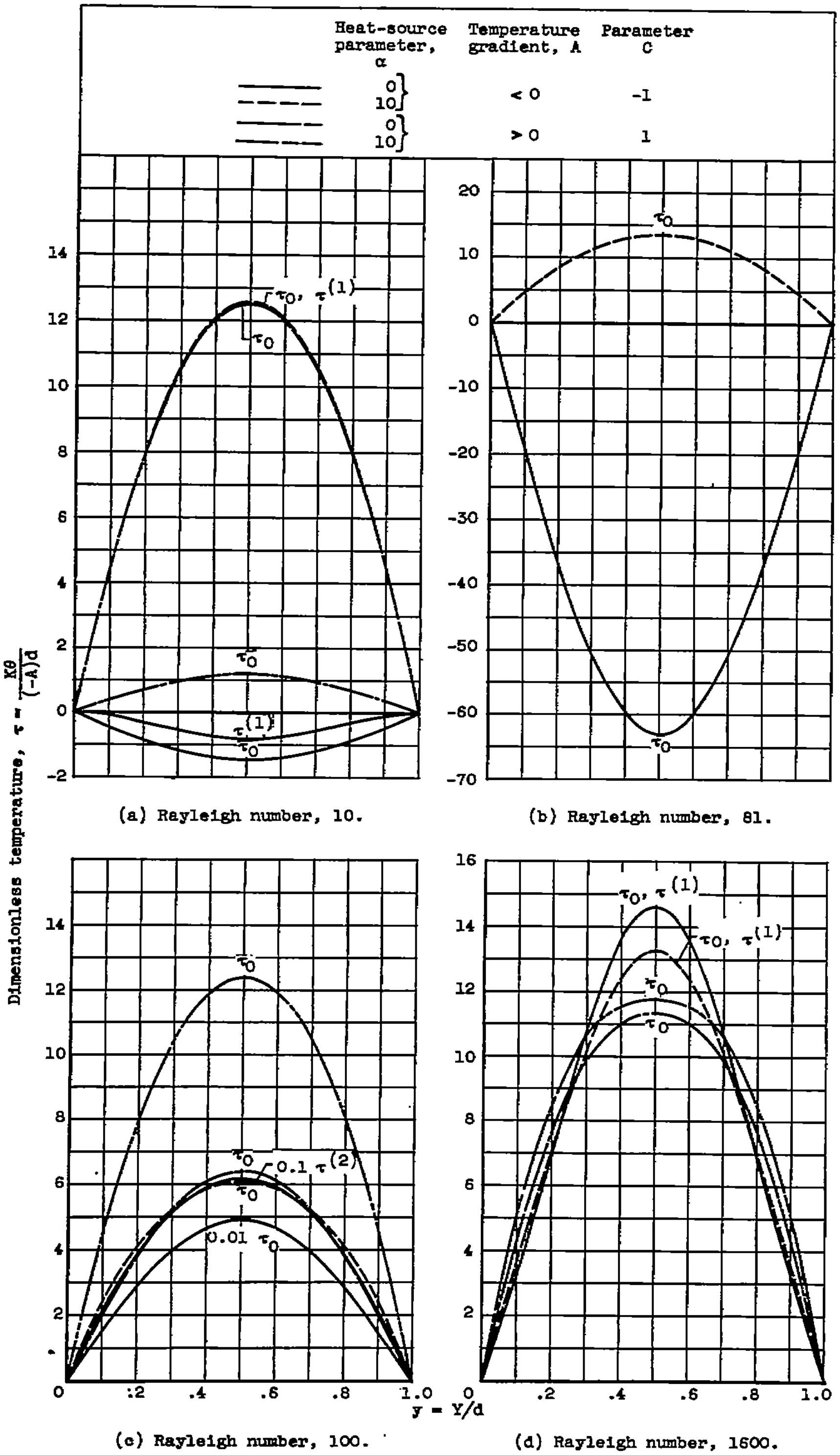


Figure 3. - Temperature distributions with and without heat sources for heating from below (A < 0) and not (A > 0). Identical wall temperatures (n = 1); K = 10.

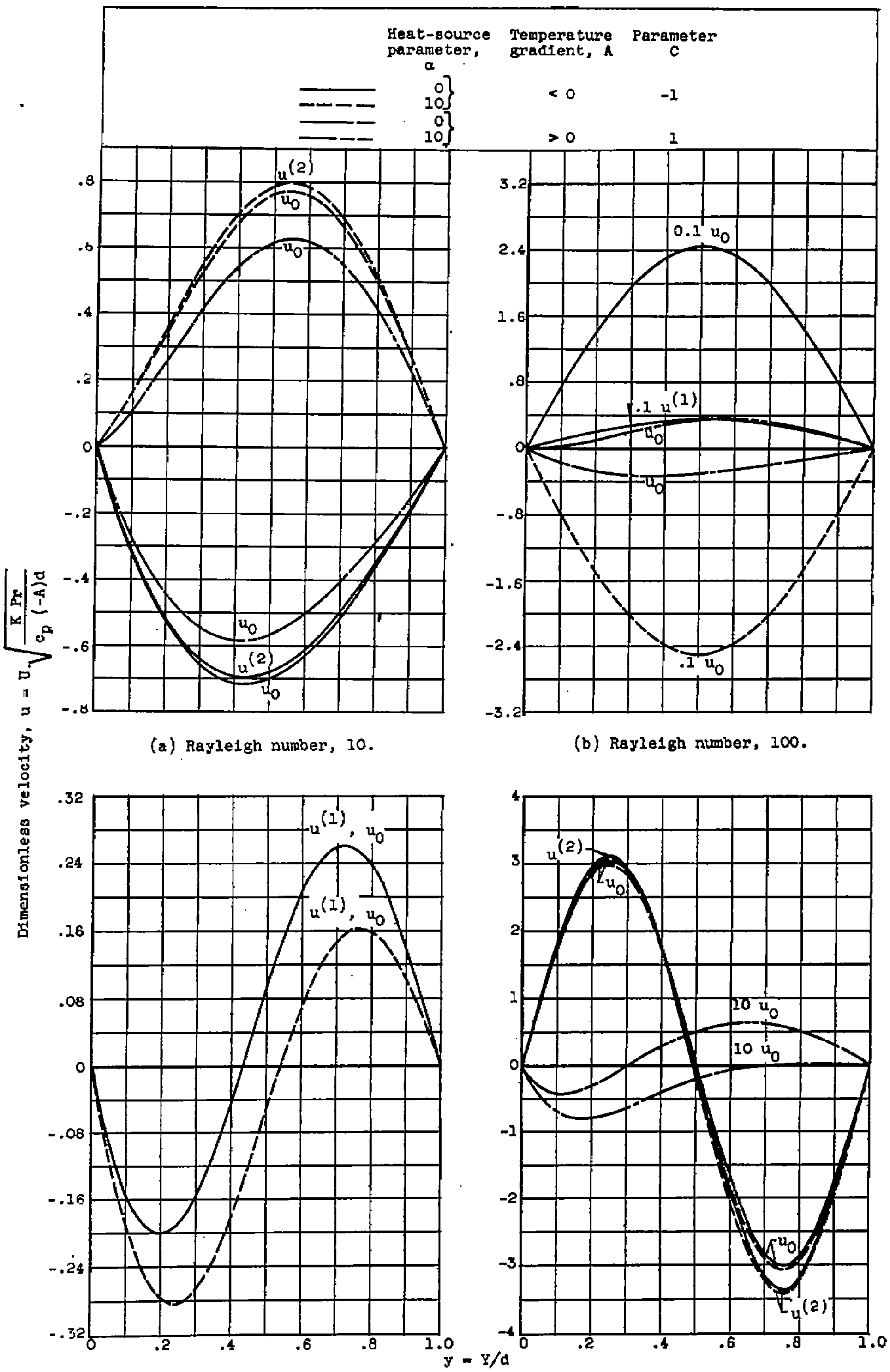


Figure 4. - Velocity distributions with and without heat sources for heating from below (A < 0) and not (A > 0). Unequal wall temperatures (n = 0); K = 10.

3573

3573

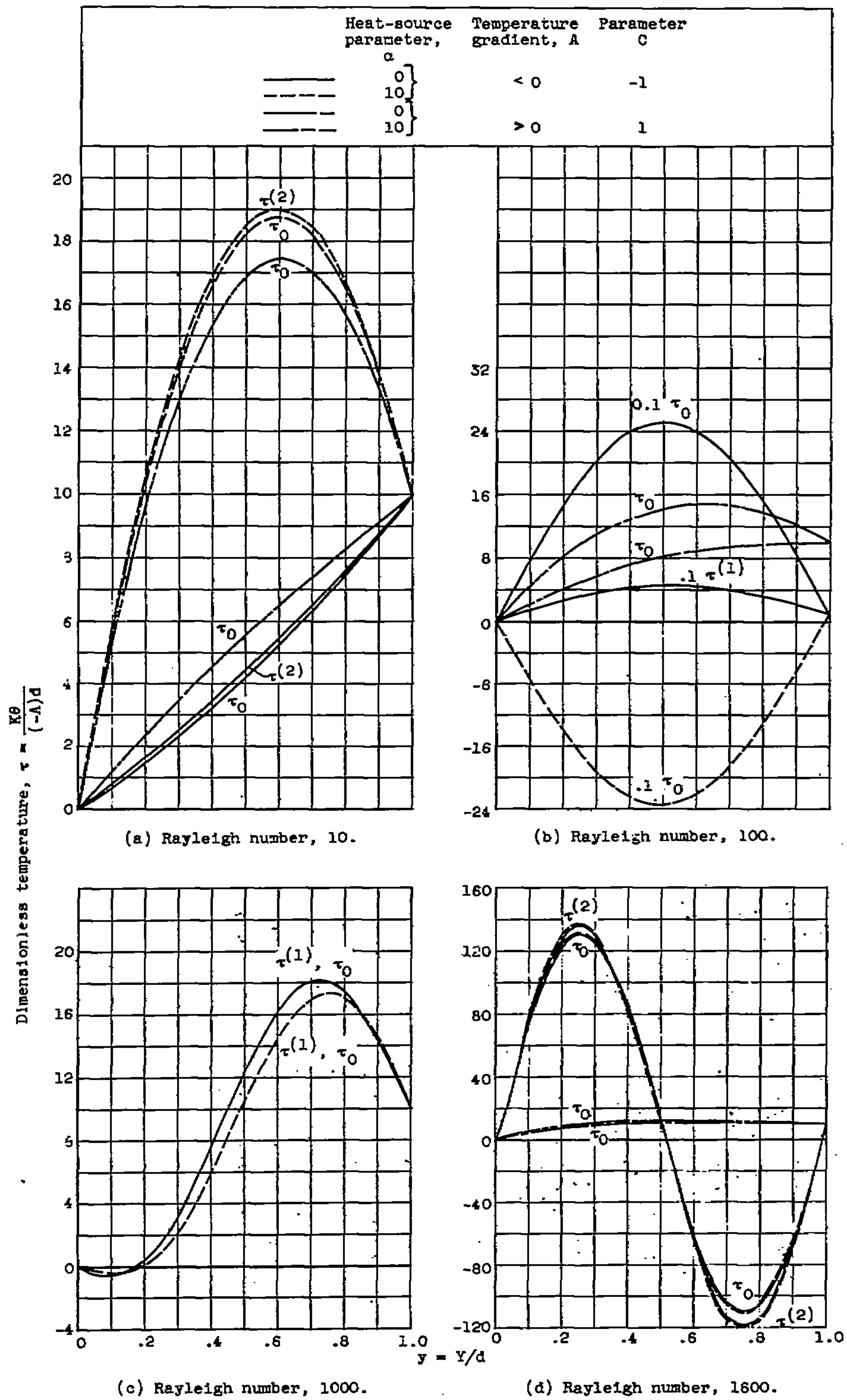


Figure 5. - Temperature distribution with and without heat sources for heating from below (A < 0) and not (A > 0). Unequal wall temperatures (n = 0); K = 10.

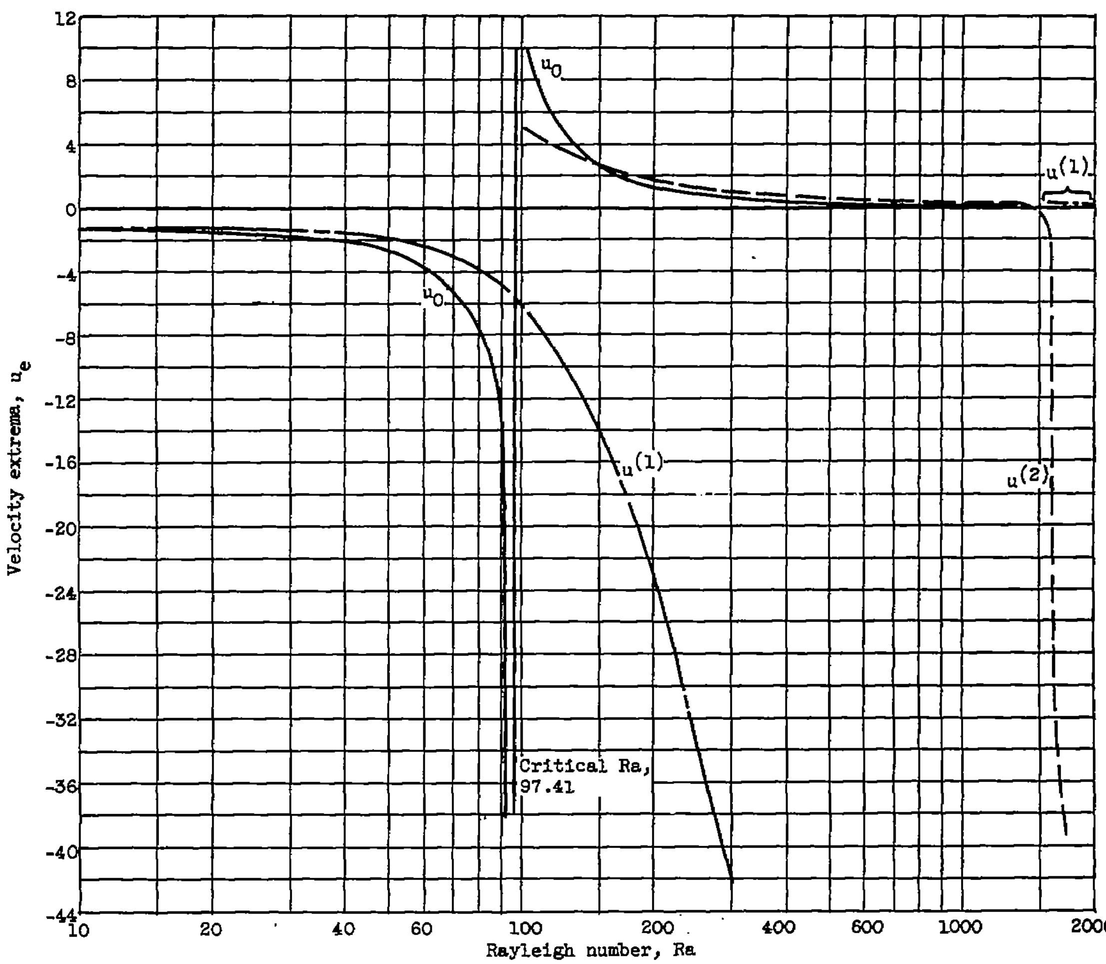
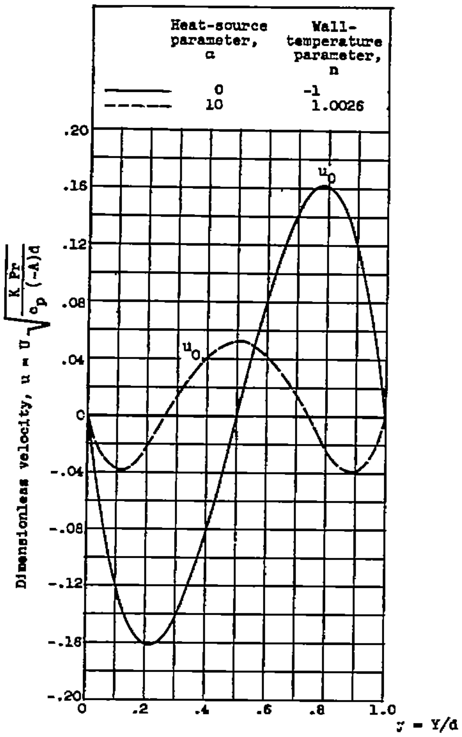


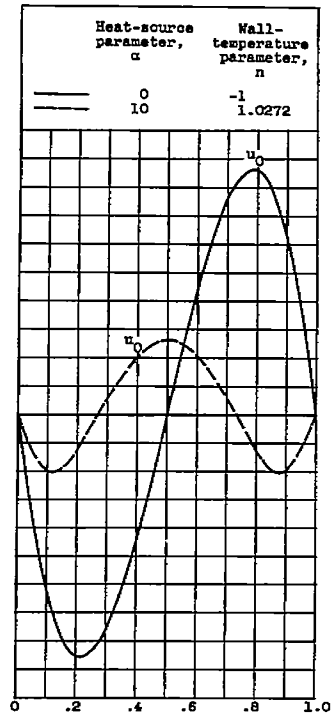
Figure 6. - Velocity extrema as functions of Rayleigh number. Identical wall temperatures ($n = 1$); $K = 10$; $\alpha = 0$; $C = -1$.

3573

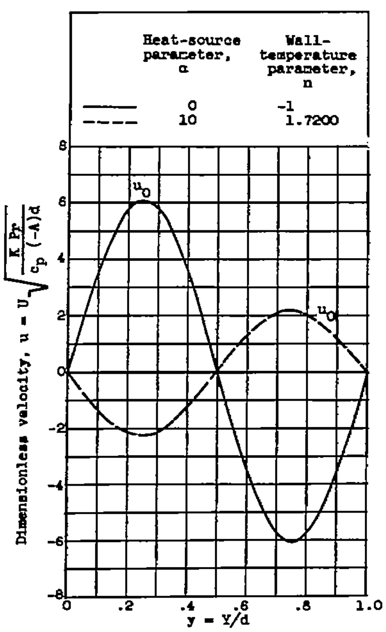
3573



(a) Rayleigh number, 10.



(b) Rayleigh number, 100.



(c) Rayleigh number, 1600.

Figure 7. - Velocity distributions with and without heat sources for zero net mass flow. K = 10; C = -1.

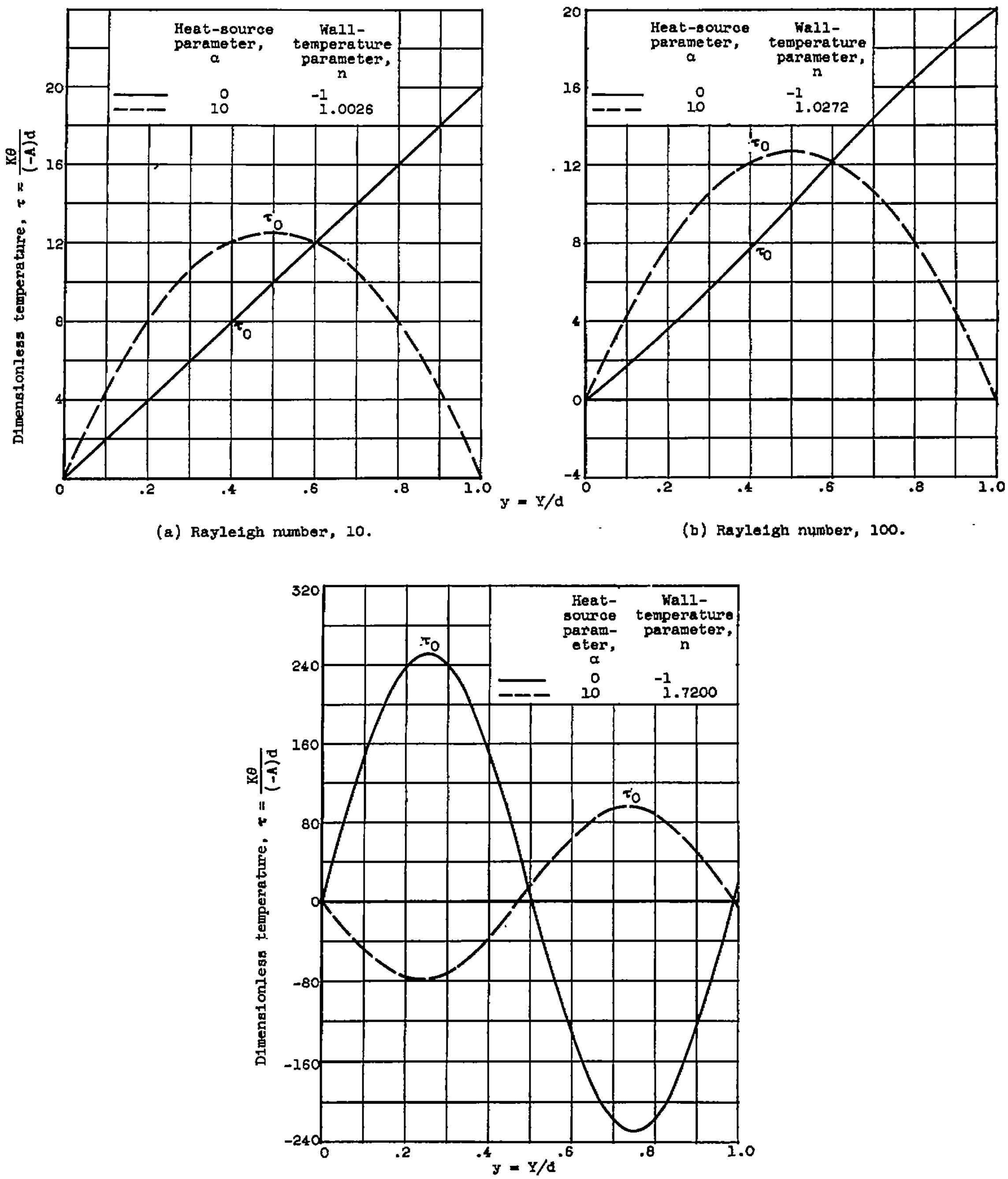


Figure 8. - Temperature distributions with and without heat sources for zero net mass flow. K = 10; C = -1.

3573

3573

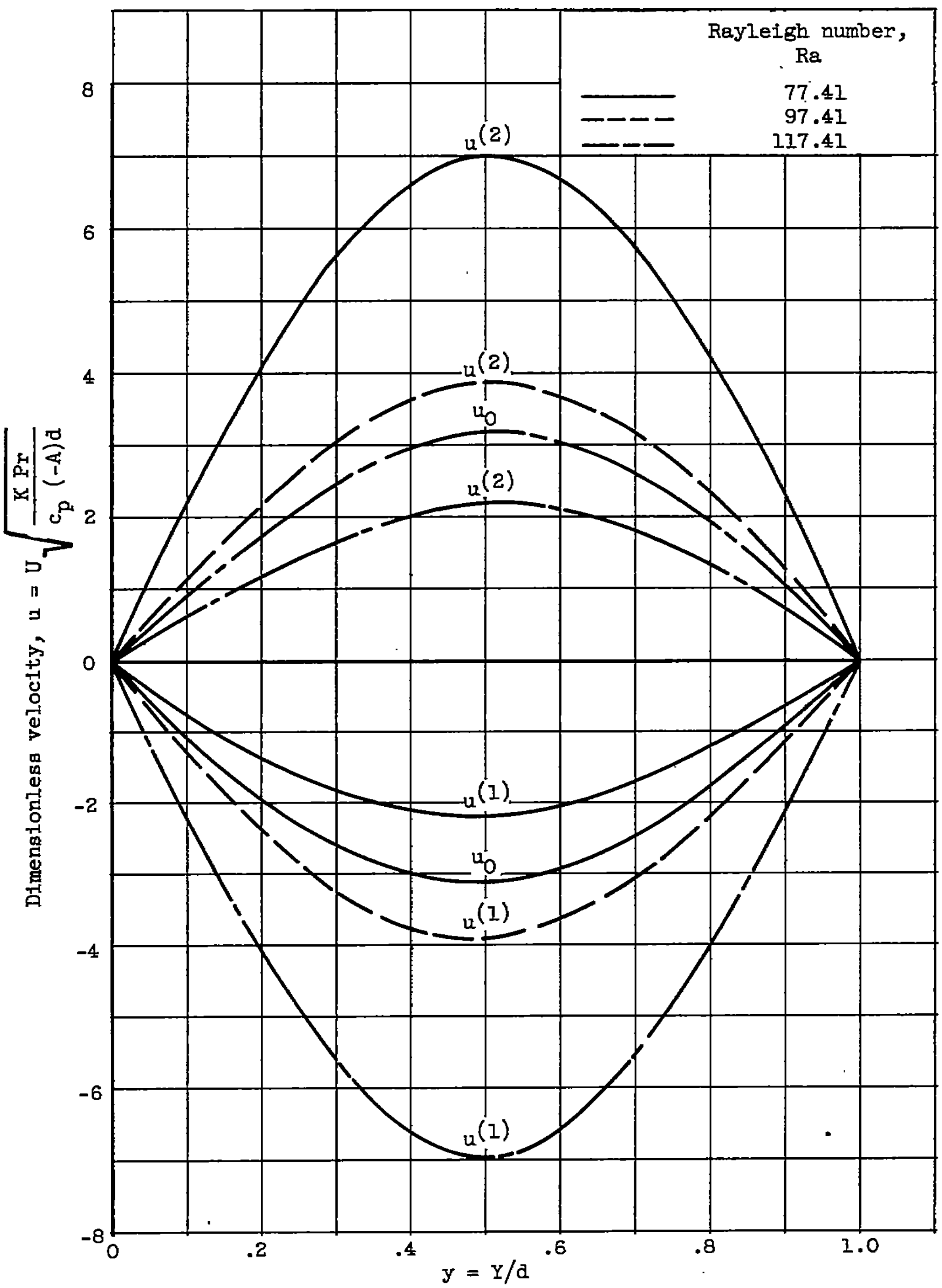


Figure 9. - Velocity distributions for Rayleigh numbers at and on both sides of first critical Rayleigh number. Unequal wall temperatures (n = 0); K = 10; α = 0; C = -1.

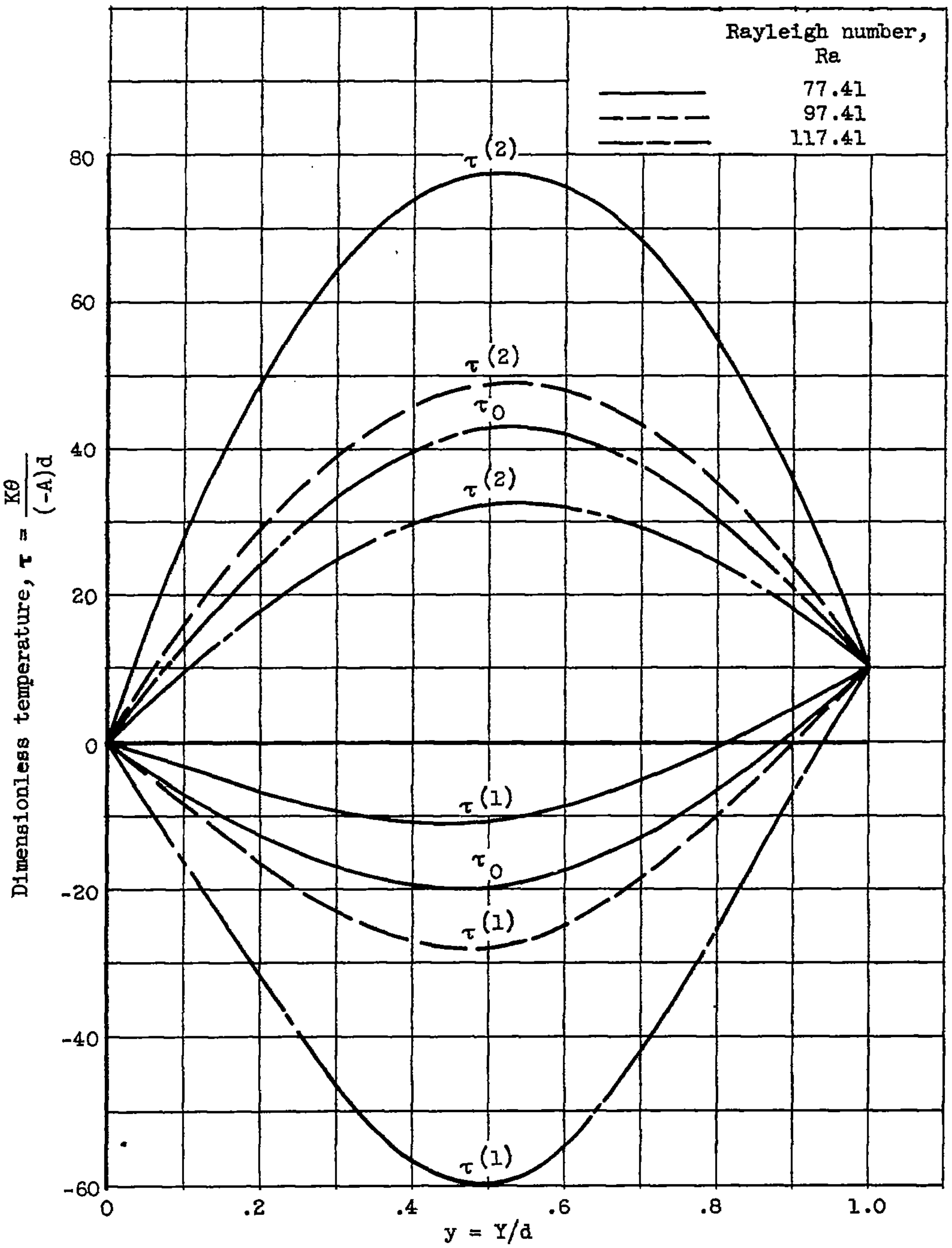


Figure 10. - Temperature distributions for Rayleigh numbers at and on both sides of first critical Rayleigh number. Unequal wall temperatures ($n = 0$); $K = 10$; $\alpha = 0$; $C = -1$.

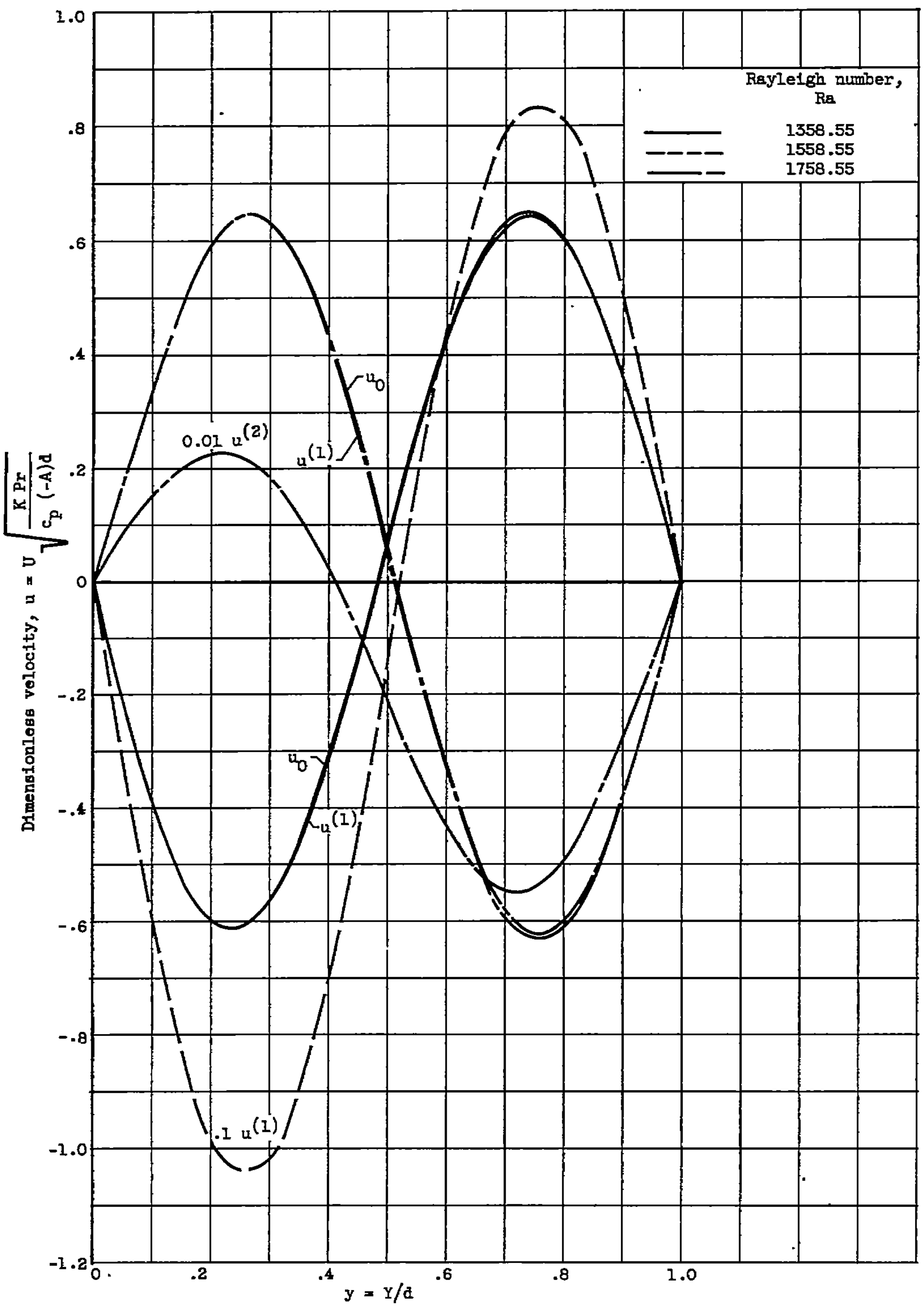


Figure 11. - Velocity distributions for Rayleigh numbers at and on both sides of second critical Rayleigh number. Unequal wall temperatures ($n = 0$); $K = 10$; $\alpha = 0$; $C = -1$.

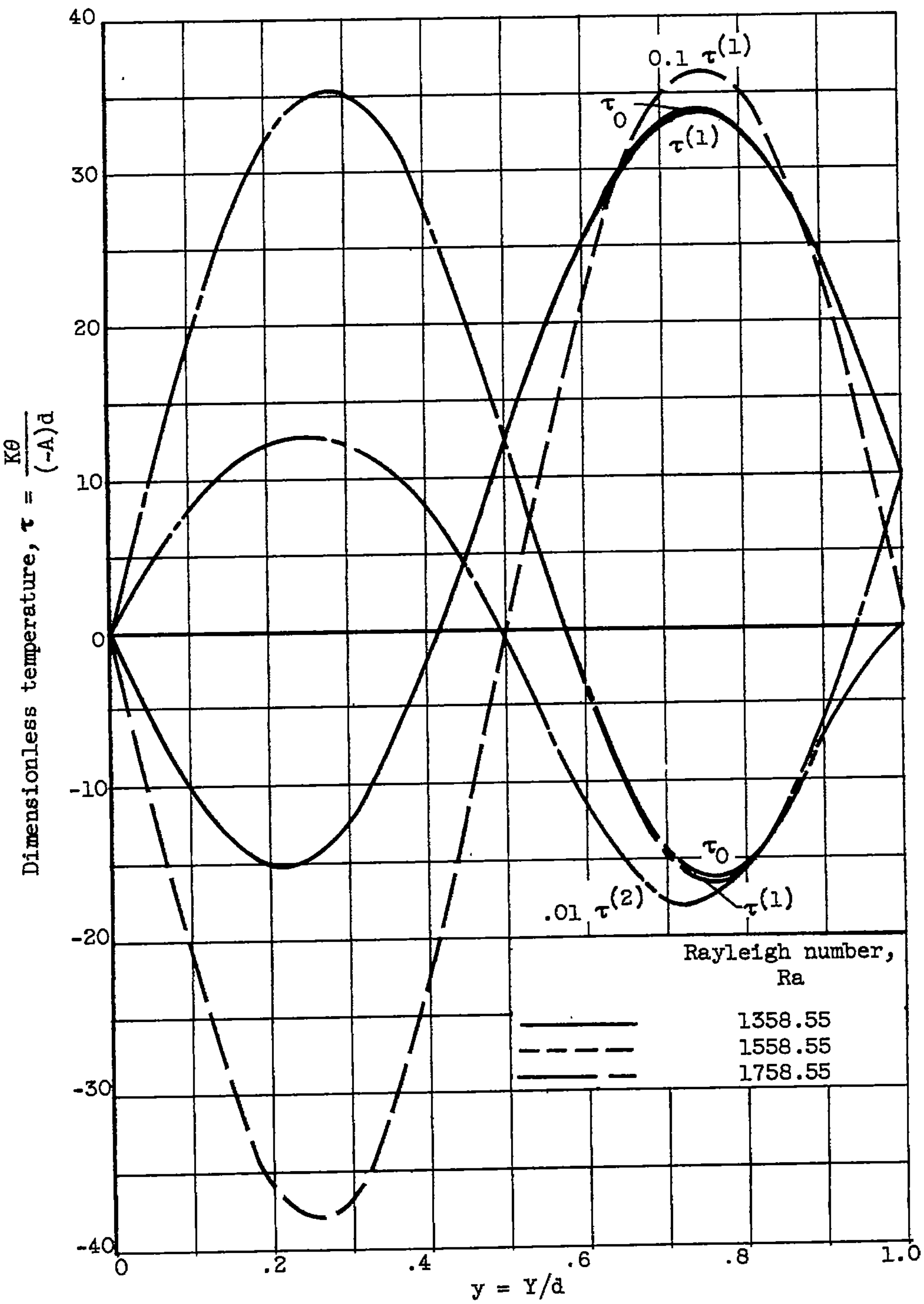


Figure 12. - Temperature distributions for Rayleigh numbers at and on both sides of second critical Rayleigh number. Unequal wall temperatures ($n = 0$); $K = 10$; $\alpha = 0$; $C = -1$.

3573

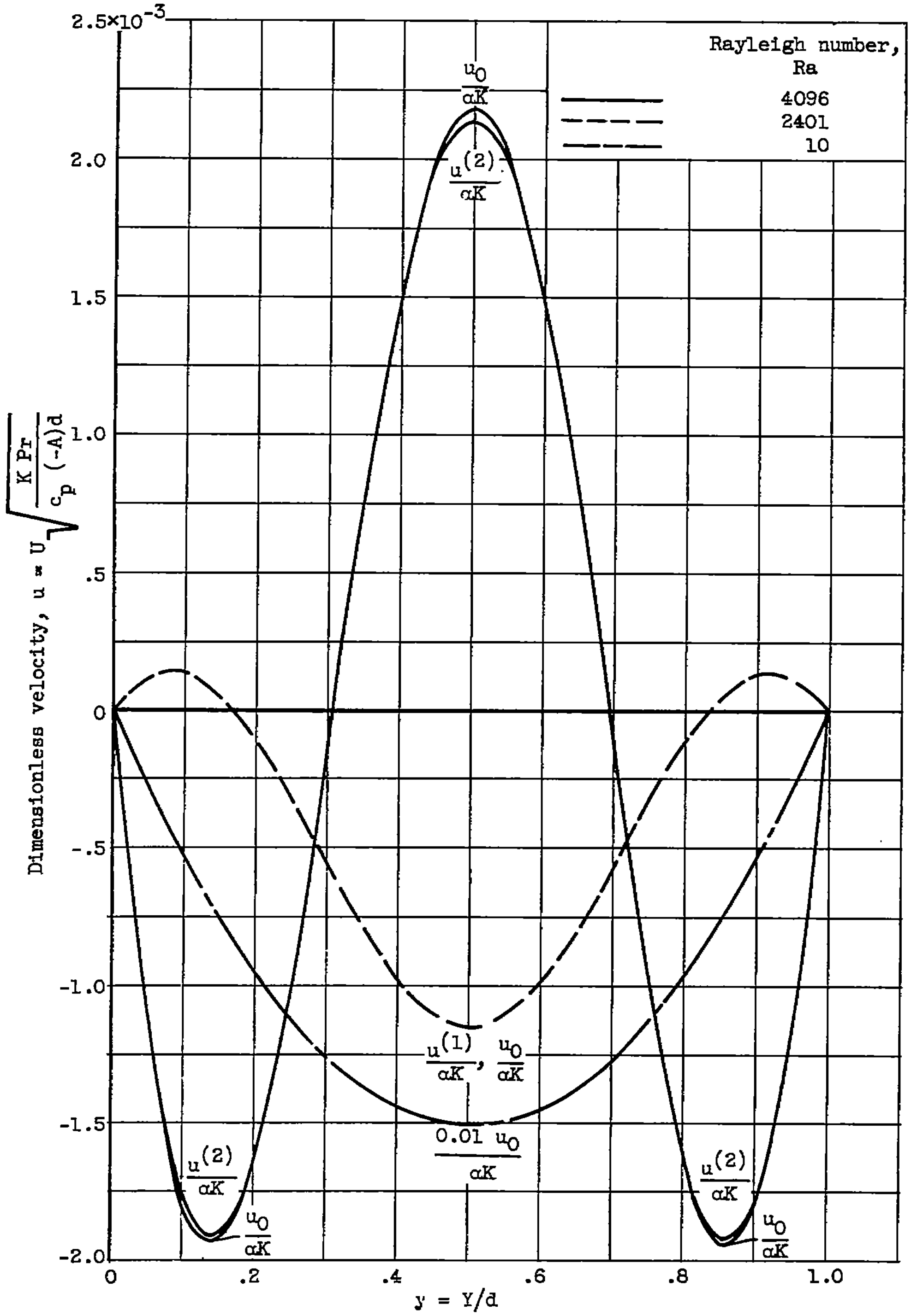


Figure 13. - Velocity distributions for Rayleigh numbers on both sides of first critical (3803.22) for both walls insulated. $K = 10$; $\alpha \neq 0$.

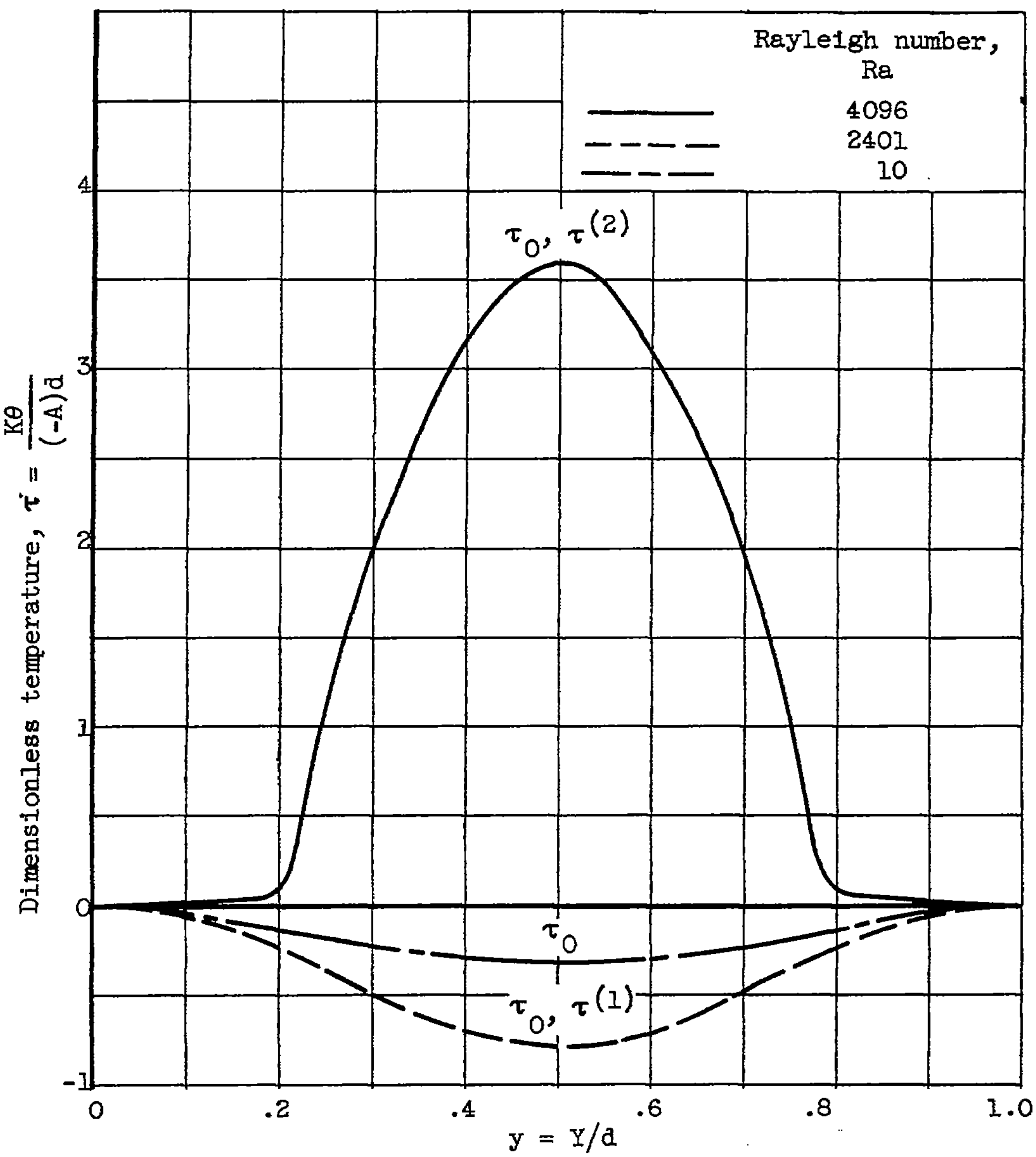


Figure 14. - Temperature distributions for Rayleigh numbers on both sides of first critical Rayleigh number (3803.22) for both walls insulated. $K = 10$; $\alpha = 0.001$.