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INFLUENCE OF LARGE AMPLITUDES ON FLEXURAL MOTIONS OF ELASTIC PLATES

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SUMMARY

Starting with the fundamental equations of the general three-dimensional nonlinear theory of elasticity for the case of small elongations and shears but moderately large rotations, a set of plate equations of motion is derived for an isotropic material obeying linear Hooke's law. Certain assumptions as to plate displacements are made at the outset. The resulting nonlinear plate theory of motion, valid for large deflections, is discussed in the light of the three-dimensional theory and other nonlinear plate theories, in particular, the static Von Kármán plate theory.

In order to investigate the influence of large rotations in a dynamic problem, the nonlinear equations are solved for the case of propagation of straight-crested waves and the wave velocities are computed for various values of the parameters involved. Certain similarities to free vibrations of a mass on a nonlinear spring, governed by Duffing's equation, are established.

In addition, a variant of plate equations is derived which permits tracing back the origin of certain terms appearing in other nonlinear plate theories, thus revealing more clearly the underlying assumptions of those theories.

INTRODUCTION

A variety of important problems of structural strength and stability of plates, arising in modern aircraft and missile construction, cannot be adequately analyzed on the basis of the classical theory because the plate deflections experienced are not small in comparison with the plate thickness. Several theories have been suggested to take into account the influence of large displacements and large rotations, the most widely known being the static theory of Von Kármán. It is a common characteristic of these theories that the equations of equilibrium were obtained by considering the free-body diagram of a deformed plate element and by equating to zero all the forces and moments. This procedure does not guarantee a rigorous consistency relative to the order of the terms retained, does not permit a clear identification of the order of the terms involved, and precludes the possibility of occurrence of plate stresses typical of a nonlinear theory which vanish in the linearized case. Moreover, since the relationship between, for example, the Von Kármán theory and the general three-dimensional nonlinear theory of elasticity has been established only as far as the strain and displacement components are concerned, a more complete study of this relationship is needed. In view of these shortcomings an attempt has been made to derive a large-deflection plate theory of motion, starting with the general equations of the three-dimensional nonlinear theory of elasticity.

The idea of deriving approximate theories of plates and other bodies, one or two of whose dimensions are small as compared with the third, from the general three-dimensional equations has been, in recent times, repeatedly applied with success within the class of linear theories, as for example, in references 1 and 2. It is demonstrated herein that the same procedure, offering analogous advantages, may be used in the case of a nonlinear theory.

In a first section of the present report, the main equations of the general theory are written down in the formulation given by Biot in references 3 to 5. These are followed by plate equations deduced from the general equations by assuming at the start the same displacements as in the Von Kármán plate theory and performing appropriate integrations in the expressions to obtain the potential and kinetic energies used in the formulation of Hamilton's principle.

The resulting stress equations of motion appear to be different from those obtained by Von Kármán and by Love, in that inertia terms have been added. However, the equations presented herein are reducible to Von Kármán's equations in the absence of external moments, tangential forces, and compressional inertia forces. The present theory is also reducible to the classical plate theory for the case where the rotations are taken to be small as compared with the strains and the rotatory inertia is neglected.

All the nonlinear terms of the present theory appear to be of the same type as in the general three-dimensional theory and may be designated, using Biot's terminology, as "curvature" terms and as "buoyancy" terms. A "torsion" term, occurring in the three-dimensional theory, is absent in the plate theory. The plate stresses entering the equations are defined automatically during the process.

The plate stress-displacement relations are also derived by starting with the corresponding relations of the three-dimensional theory and performing appropriate integrations and omitting certain terms. They appear to be precisely the same as those given in the literature which, however, were derived quite differently.

The present theory consists, in essence, just as in the linear case, of the equations of motion, the stress-displacement relations, and boundary conditions. The role of the latter, however, is greatly reduced in the nonlinear case. While, in the linear case, the boundary conditions are understood to be those quantities, in number and combinations, which have to be specified at the boundary in order to insure a unique solution, the analogous conditions presently ascertain merely the nonviolation of the principle of the conservation of energy.

As an application of the equations of motion of plates with large deflections, the propagation of harmonic, straight-crested waves in an infinite plate has been studied. The inclusion, in the theory, of the effect of large rotations introduces a coupling between the classical compressional and flexural waves, and a redefinition of the normal plate stress permits exhibiting the nature of the coupling term more clearly. If, for this purpose, the rotation is considered independent of the transverse displacement, the operator matrix of the set of two equations is shown to be symmetric, just as it should be in any linear theory.

The two coupled nonlinear wave equations are solved simultaneously and certain similarities to Duffing's equation and his method of solution are noted in the process. The biquadratic velocity equation describes the propagation of two modes, one essentially compressional, the other flexural. It contains the effects of transverse inertia, rotatory inertia, longitudinal inertia, flexural stiffness, and compressional stiffness. The influence of each of these effects on the wave velocity is discussed in detail.

Within the range of validity of the present theory, that is, for flexural wave velocities which are small as compared with the velocity of compressional waves, the influence of rotatory and longitudinal inertia is shown to be negligible. It is remarkable that in this case the two coupled, nonlinear equations can be solved exactly. For a given wave length, the amplitude has the same effect on the wave length as the plate thickness divided by the square root of three.

In order to gain a deeper insight into the assumptions underlying the plate equations derived by Love (ref. 6) and the bar equations derived by Eringen (ref. 7), a variant of plate equations was derived, in the light of the three-dimensional theory, in the last section of this report. The same plate displacements were assumed, but more complete expressions for the components of strain were employed.

Shortly after the manuscript of this report had been completed, Reissner, in reference 8, noted that Kirchhoff (ref. 9) had been the first to analyze motions of plates with large deflections. The present equations coincide, in essence, with those given by Kirchhoff, who, however, did not make an attempt to solve any special problems. The first to consider equilibrium of plates with large deflections appears to be Clebsch (ref. 10). Both Clebsch and Kirchhoff used different procedures and arguments in deriving their theories, as compared with the ones presently employed. A detailed analysis of these differences would require a separate study and is beyond the scope of the present report. A critical review of the work of Clebsch and Kirchhoff is found in reference 11.

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SYMBOLS

A,B	wave amplitudes
c	phase velocity
c_p	phase velocity defined by equation (46)
D	plate stiffness
E	Young's modulus
e_{xx}, e_{yz}, ω_x e_{yy}, e_{zx}, ω_y e_{zz}, e_{xy}, ω_z	quantities defined by equations (3)
F_x, F_y m_x, m_y, q $N_n^*, M_n^*, N_{ns}^*, M_{ns}^*$ Q_n	external plate forces and moments defined by equations (18)
f_x, f_y, f_z	components of traction (external force per unit original area acting on plate surface) in x-, y-, and z-directions, respectively
G	shear modulus; $G \equiv \mu$
h	plate thickness

K	kinetic potential
L	wave length
M_{31}^*, M_{32}^*	plate stresses defined by equations (68)
N_1, N_2, N_3, N_{12}, M_1, M_2, M_{12}	plate stresses defined by equations (12)
n, s	coordinates perpendicular to and along the plate boundary, respectively
Q_1^*, Q_2^*	plate stresses defined by equations (67)
S	surface of plane face of plate
S_o	surface of plane face of plate before deformation
T	kinetic energy
t	time
U	potential energy
u, v, w	displacement components in x-, y-, and z-directions, respectively
$\bar{u}, \bar{v}, \bar{w}$	approximate displacement components of plate theory, which are functions of all three space variables
u_o, v_o, w_o	plate displacement components defined by equations (7), which are functions of two space variables only
V_o	plate volume before deformation
W	elastic strain energy
W_e	work done by external forces
x, y, z	original coordinates of particle (before deformation)
γ	wave number, $2\pi/L$

$\left.\begin{matrix}\epsilon_{11},\epsilon_{22},\epsilon_{33}\\ \epsilon_{23},\epsilon_{31},\epsilon_{12}\end{matrix}\right\}$ components of strain defined by equations (4) or (5)

λ,μ Lamé's constants of elasticity

ν Poisson's ratio

ξ,η,ζ final coordinates of particle (after deformation)

ρ mass density

$\left.\begin{matrix}\tau_{11},\tau_{22},\tau_{33}\\ \tau_{23},\tau_{31},\tau_{12}\end{matrix}\right\}$ components of stress defined by equations (6)

A prime indicates differentiation with respect to x; a bar indicates an approximation; a dot indicates differentiation with respect to time.

BASIC EQUATIONS OF THREE-DIMENSIONAL NONLINEAR THEORY OF ELASTICITY

The basic equations of the three-dimensional nonlinear theory of elasticity written down in this section are taken from Biot's work in the field (refs. 3 to 5). A more extensive account of nonlinear theories of elasticity may be found in references 12 and 13.

Deformation

Let the original coordinates x, y, and z of a particle attached to the material before deformation become

$$\left.\begin{aligned}\xi &= x + u\\ \eta &= y + v\\ \zeta &= z + w\end{aligned}\right\} \tag{1}$$

after deformation. The Cartesian rectangular coordinate system itself is considered to be fixed during this operation. The deformation of an infinitesimal region surrounding this particle is given by the linear transformation

$$\left.\begin{aligned}
d\xi &= \left(1 + \frac{\partial u}{\partial x}\right) dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz \\
d\eta &= \frac{\partial v}{\partial x} dx + \left(1 + \frac{\partial v}{\partial y}\right) dy + \frac{\partial v}{\partial z} dz \\
d\zeta &= \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \left(1 + \frac{\partial w}{\partial z}\right) dz
\end{aligned}\right\} \quad (2)$$

Using the notation

$$\left.\begin{aligned}
e_{xx} &= \frac{\partial u}{\partial x} \\
e_{yy} &= \frac{\partial v}{\partial y} \\
e_{zz} &= \frac{\partial w}{\partial z} \\
e_{yz} &= \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \\
e_{zx} &= \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\
e_{xy} &= \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \\
\omega_x &= \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \\
\omega_y &= \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \\
\omega_z &= \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)
\end{aligned}\right\} \quad (3)$$

and assuming these nine quantities to be small, the components of the strain tensor, including terms of first and second order, may be defined as

$$\left.\begin{aligned}
\epsilon_{11} &= e_{xx} + e_{xy}\omega_z - e_{zx}\omega_y + \frac{1}{2}\left(\omega_z^2 + \omega_y^2\right) \\
\epsilon_{22} &= e_{yy} + e_{yz}\omega_x - e_{xy}\omega_z + \frac{1}{2}\left(\omega_x^2 + \omega_z^2\right) \\
\epsilon_{33} &= e_{zz} + e_{zx}\omega_y - e_{yz}\omega_x + \frac{1}{2}\left(\omega_y^2 + \omega_x^2\right) \\
\epsilon_{23} &= e_{yz} + \frac{1}{2}\omega_x\left(e_{zz} - e_{yy}\right) + \frac{1}{2}\omega_y e_{xy} - \frac{1}{2}\omega_z e_{zx} - \frac{1}{2}\omega_y\omega_z \\
\epsilon_{31} &= e_{zx} + \frac{1}{2}\omega_y\left(e_{xx} - e_{zz}\right) + \frac{1}{2}\omega_z e_{yz} - \frac{1}{2}\omega_x e_{xy} - \frac{1}{2}\omega_z\omega_x \\
\epsilon_{12} &= e_{xy} + \frac{1}{2}\omega_z\left(e_{yy} - e_{xx}\right) + \frac{1}{2}\omega_x e_{zx} - \frac{1}{2}\omega_y e_{yz} - \frac{1}{2}\omega_x\omega_y
\end{aligned}\right\} \quad (4)$$

These strain components are referred to a local coordinate system (1, 2, and 3) originally parallel to the x-, y-, and z-directions, respectively, and undergoing the same rotation as the material at that particular point. This rotation is given, in first approximation, by a vector with components ω_x, ω_y, and ω_z, which will be adequate if these components are small as compared with unity.

The expressions (eqs. (4)) for the strain components, which are valid for elongations and shears which are small as compared with unity, contain linear and square terms in the components of rotation. The linear terms in the components of rotation are, however, third-order terms as may be readily seen by expressing the quantities e_{ij} in terms of the components of strain and by remembering that all quantities given in equations (3) are assumed to be of the first order (ref. 13, p. 52).

Thus, the expressions for the strain components simplify to

$$\left.\begin{aligned}
\epsilon_{11} &= e_{xx} + \frac{1}{2}\left(\omega_z^2 + \omega_y^2\right) \\
\epsilon_{22} &= e_{yy} + \frac{1}{2}\left(\omega_x^2 + \omega_z^2\right) \\
\epsilon_{33} &= e_{zz} + \frac{1}{2}\left(\omega_y^2 + \omega_x^2\right) \\
\epsilon_{23} &= e_{yz} - \frac{1}{2}\omega_y\omega_z \\
\epsilon_{31} &= e_{zx} - \frac{1}{2}\omega_z\omega_x \\
\epsilon_{12} &= e_{xy} - \frac{1}{2}\omega_x\omega_y
\end{aligned}\right\} \quad (5)$$

Even though the rotations and strains are small as compared with unity, they do not necessarily have the same magnitude. For this reason the squares of the rotation are retained.

Strain Energy

The variation of the total strain energy in an original volume V_o is given by

$$\delta W = \iiint_{V_o} \left(\tau_{11}\,\delta\epsilon_{11} + \tau_{22}\,\delta\epsilon_{22} + \tau_{33}\,\delta\epsilon_{33} + 2\tau_{23}\,\delta\epsilon_{23} + 2\tau_{31}\,\delta\epsilon_{31} + 2\tau_{12}\,\delta\epsilon_{12}\right)dx\,dy\,dz \tag{6}$$

where the τ_{ij} represent the components of a symmetric stress tensor taken per unit initial area before deformation with respect to locally rotated axes (1, 2, and 3) and as a function of the fixed original coordinates (x, y, and z).

The equations of equilibrium may be derived from the expression for the strain energy by the same process as followed in the linear theory. The components of strain are expressed in terms of displacement derivatives and the principle of virtual work, the static analogue of Hamilton's principle, is applied, followed by a partial integration.

PLATE EQUATIONS

Consider a plane plate of constant thickness h referred to an xyz Cartesian coordinate system, the xy-plane being the middle plane of the plate and the z-axis being directed normal to that plane. The approximate displacements $\bar{u}$, $\bar{v}$, and $\bar{w}$ characterizing the plate theory with large deflections are taken as

$$\left.\begin{aligned} \bar{u} &= u_o(x,y) - z\,\frac{\partial w_o}{\partial x} \\ \bar{v} &= v_o(x,y) - z\,\frac{\partial w_o}{\partial y} \\ \bar{w} &= w_o(x,y) \end{aligned}\right\} \tag{7}$$

These displacements clearly consist of two parts, namely, the transverse displacement w_o, which characterizes the classical plate theory, and the displacements u_o and v_o, which serve to describe the displacements in the plane of the plate. The displacements given by equations (7) are those of the Von Kármán plate theory (ref. 14) as was pointed out by Biot (ref. 5).

Using equations (7), the following approximations to the quantities given in equations (3) are obtained:

$$\left.\begin{aligned}
\bar{e}_{xx} &= \frac{\partial u_o}{\partial x} - z\,\frac{\partial^2 w_o}{\partial x^2} \\
\bar{e}_{yy} &= \frac{\partial v_o}{\partial y} - z\,\frac{\partial^2 w_o}{\partial y^2} \\
\bar{e}_{zz} &= 0 \\
\bar{e}_{yz} &= 0 \\
\bar{e}_{xz} &= 0 \\
\bar{e}_{xy} &= \frac{1}{2}\left(\frac{\partial v_o}{\partial x} + \frac{\partial u_o}{\partial y} - 2z\,\frac{\partial^2 w_o}{\partial x\,\partial y}\right) \\
\bar{\omega}_x &= \frac{\partial w_o}{\partial y} \\
\bar{\omega}_y &= -\frac{\partial w_o}{\partial x} \\
\bar{\omega}_z &= \frac{1}{2}\left(\frac{\partial v_o}{\partial x} - \frac{\partial u_o}{\partial y}\right)
\end{aligned}\right\} \tag{8}$$

Plate deformations may produce only large rotations $\bar{\omega}_x$ and $\bar{\omega}_y$ but not $\bar{\omega}_z$. Thus, in calculating the components of strain from equations (5) with the aid of equations (8), second-order terms containing $\bar{\omega}_z$ are neglected.

The approximate components of strain in the plate are then

$$\left.\begin{aligned}
\bar{\epsilon}_{11} &= \frac{\partial u_o}{\partial x} - z\,\frac{\partial^2 w_o}{\partial x^2} + \frac{1}{2}\left(\frac{\partial w_o}{\partial x}\right)^2 \\
\bar{\epsilon}_{22} &= \frac{\partial v_o}{\partial y} - z\,\frac{\partial^2 w_o}{\partial y^2} + \frac{1}{2}\left(\frac{\partial w_o}{\partial y}\right)^2 \\
\bar{\epsilon}_{33} &= \frac{1}{2}\left(\frac{\partial w_o}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial w_o}{\partial x}\right)^2 \\
\bar{\epsilon}_{23} &= 0 \\
\bar{\epsilon}_{31} &= 0 \\
\bar{\epsilon}_{12} &= \frac{1}{2}\left(\frac{\partial v_o}{\partial x} + \frac{\partial u_o}{\partial y} - 2z\,\frac{\partial^2 w_o}{\partial x\,\partial y} + \frac{\partial w_o}{\partial y}\,\frac{\partial w_o}{\partial x}\right)
\end{aligned}\right\} \tag{9}$$

Strain Energy

Inserting the approximate expressions for the strain components given in equations (9) into the expression for variation of the strain energy (eq. (6)) gives the following approximate expression for the variation of the strain energy $\delta\bar{W}$:

$$\begin{aligned}
\delta\bar{W} = \iiint_{V_o} \Bigg\{ &\tau_{11}\left[\delta\left(\frac{\partial u_o}{\partial x}\right) - z\,\delta\left(\frac{\partial^2 w_o}{\partial x^2}\right) + \frac{\partial w_o}{\partial x}\,\delta\left(\frac{\partial w_o}{\partial x}\right)\right] + \\
&\tau_{22}\left[\delta\left(\frac{\partial v_o}{\partial y}\right) - z\,\delta\left(\frac{\partial^2 w_o}{\partial y^2}\right) + \frac{\partial w_o}{\partial y}\,\delta\left(\frac{\partial w_o}{\partial y}\right)\right] + \\
&\tau_{33}\left[\frac{\partial w_o}{\partial x}\,\delta\left(\frac{\partial w_o}{\partial x}\right) + \frac{\partial w_o}{\partial y}\,\delta\left(\frac{\partial w_o}{\partial y}\right)\right] + \\
&\tau_{12}\left[\delta\left(\frac{\partial v_o}{\partial x}\right) + \delta\left(\frac{\partial u_o}{\partial y}\right) - 2z\,\frac{\partial^2 w_o}{\partial x\,\partial y} + \frac{\partial w_o}{\partial y}\,\delta\left(\frac{\partial w_o}{\partial x}\right) + \frac{\partial w_o}{\partial x}\,\delta\left(\frac{\partial w_o}{\partial y}\right)\right]\Bigg\}\,dx\,dy\,dz
\end{aligned} \tag{10}$$

Since the z-dependence of all the components of strain has been made explicit by equations (7), an integration through the thickness h of the plate from $z = -\frac{h}{2}$ to $z = \frac{h}{2}$ can be performed immediately with the result:

$$
\begin{aligned}
\delta\overline{W} = \iint_{S_o} \Bigg\{ & N_1\,\delta\left(\frac{\partial u_o}{\partial x}\right) - M_1\,\delta\left(\frac{\partial^2 w_o}{\partial x^2}\right) + N_1\frac{\partial w_o}{\partial x}\,\delta\left(\frac{\partial w_o}{\partial x}\right) + \\
& N_2\,\delta\left(\frac{\partial v_o}{\partial y}\right) - M_2\,\delta\left(\frac{\partial^2 w_o}{\partial y^2}\right) + N_2\frac{\partial w_o}{\partial y}\,\delta\left(\frac{\partial w_o}{\partial y}\right) + \\
& N_3\frac{\partial w_o}{\partial x}\,\delta\left(\frac{\partial w_o}{\partial x}\right) + N_3\frac{\partial w_o}{\partial y}\,\delta\left(\frac{\partial w_o}{\partial y}\right) + \\
& N_{12}\left[\delta\left(\frac{\partial w_o}{\partial x}\right) + \delta\left(\frac{\partial u_o}{\partial y}\right) + \frac{\partial w_o}{\partial y}\,\delta\left(\frac{\partial w_o}{\partial y}\right) + \frac{\partial w_o}{\partial x}\,\delta\left(\frac{\partial w_o}{\partial y}\right)\right] - \\
& 2M_{12}\,\delta\left(\frac{\partial^2 w_o}{\partial x\,\partial y}\right)\Bigg\}\,dx\,dy
\end{aligned}
\tag{11}
$$

the integration being extended over the original surface S_o of the plate before deformation. In this process one is led automatically to the following definitions of plate stresses:

$$
\left.
\begin{aligned}
N_1 &= \int_{-h/2}^{h/2} \tau_{11}\,dz & \qquad M_1 &= \int_{-h/2}^{h/2} \tau_{11} z\,dz \\
N_2 &= \int_{-h/2}^{h/2} \tau_{22}\,dz & \qquad M_2 &= \int_{-h/2}^{h/2} \tau_{22} z\,dz \\
N_3 &= \int_{-h/2}^{h/2} \tau_{33}\,dz & \qquad M_{12} &= \int_{-h/2}^{h/2} \tau_{12} z\,dz \\
N_{12} &= \int_{-h/2}^{h/2} \tau_{12}\,dz & &
\end{aligned}
\right\}
\tag{12}
$$

Noting that $\delta\left(\frac{\partial u_o}{\partial x}\right) = \frac{\partial}{\partial x}\,\delta u_o$, $\delta\left(\frac{\partial^2 w_o}{\partial x^2}\right) = \frac{\partial^2}{\partial x^2}\,\delta w_o$, and so forth, it is possible, in expression (11), to bring out the factors δu_o, δv_o, and δw_o by partial integration obtaining

$$\begin{aligned}
\delta\overline{W} = \iint_{S_o} \Bigg[&-\frac{\partial N_1}{\partial x}\,\delta u_o - \frac{\partial^2 M_1}{\partial x^2}\,\delta w_o - \frac{\partial}{\partial x}\left(N_1 \frac{\partial w_o}{\partial x}\right)\delta w_o - \\
&\frac{\partial N_2}{\partial y}\,\delta v_o - \frac{\partial^2 M_2}{\partial y^2}\,\delta w_o - \frac{\partial}{\partial y}\left(N_2 \frac{\partial w_o}{\partial y}\right)\delta w_o - \\
&\frac{\partial}{\partial x}\left(N_3 \frac{\partial w_o}{\partial x}\right)\delta w_o - \frac{\partial}{\partial y}\left(N_3 \frac{\partial w_o}{\partial y}\right)\delta w_o - \\
&\frac{\partial N_{12}}{\partial x}\,\delta v_o - \frac{\partial N_{12}}{\partial y}\,\delta u_o - \frac{\partial}{\partial x}\left(N_{12} \frac{\partial w_o}{\partial y}\right)\delta w_o - \\
&\frac{\partial}{\partial y}\left(N_{12} \frac{\partial w_o}{\partial x}\right)\delta w_o - 2\,\frac{\partial^2 M_{12}}{\partial x\,\partial y}\,\delta w_o\Bigg] dx\,dy + \\
&\oint \Bigg[N_n\,\delta u_n + N_{ns}\,\delta u_s + N_n \frac{\partial w_o}{\partial n}\,\delta w_o + N_{ns} \frac{\partial w_o}{\partial s}\,\delta w_o - \\
&M_n \frac{\partial(\delta w_o)}{\partial n} + \left(2\,\frac{\partial M_{ns}}{\partial s} + \frac{\partial M_n}{\partial s}\right)\delta w_o\Bigg] ds
\end{aligned} \tag{13}$$

Herein, use has been made of

$$-\oint M_{ns}\,\frac{\partial}{\partial s}\,\delta w_o\,ds = \oint \frac{\partial M_{ns}}{\partial s}\,\delta w_o\,ds \tag{14}$$

Work of External Forces

Let f_x, f_y, and f_z denote the x-, y-, and z-components, respectively, of the external force (traction) per unit original area acting at the boundary. The (virtual) work done by these external forces is

$$\delta W_e = \iint_S \left(f_x\,\delta u + f_y\,\delta v + f_z\,\delta w\right) dS \tag{15}$$

the integral being extended over the whole surface of the body.

In the case of a plate, the total surface consists of the two plane faces $z = \pm\frac{h}{2}$ and the cylindrical surfaces bounding the plate. The expression for the work is therefore

$$\delta W_e = \left[\iint \left(f_x\ \delta u + f_y\ \delta v + f_z\ \delta w\right) dx\ dy\right]_{z=-h/2}^{z=h/2} +$$

$$\oint \int_{-h/2}^{h/2} \left(f_n\ \delta u_n + f_s\ \delta u_s + f_z\ \delta w + f_n \frac{\partial w}{\partial n}\ \delta w + f_s \frac{\partial w}{\partial s}\ \delta w\right) dz\ ds \tag{16}$$

The line integral is to be taken around all (external and internal) cylindrical boundaries of the plate. Subscripts n and s indicate components referred to coordinates n and s measured normal to and along the boundary.

Substituting for the displacements u, v, and w their approximations $\bar{u}$, $\bar{v}$, and $\bar{w}$ from equations (7), the work of external forces acting on the plate is given by

$$\delta \overline{W}_e = \left(\iint \left\{f_x\left[\delta u_o - z\ \delta\left(\frac{\partial w_o}{\partial x}\right)\right] + f_y\left(\delta v_o - z\ \frac{\partial w_o}{\partial y}\right) + f_z\ \delta w_o\right\} dx\ dy\right)_{z=-h/2}^{z=h/2} +$$

$$\oint \int_{-h/2}^{h/2} \left\{f_n\left[\delta u_{on} - z\ \delta\left(\frac{\partial w_o}{\partial n}\right)\right] + f_s\left[\delta u_{os} - z\ \delta\left(\frac{\partial w_o}{\partial s}\right)\right] + f_z\ \delta w_o + \right.$$

$$\left.\left(f_n \frac{\partial w_o}{\partial n} + f_s \frac{\partial w_o}{\partial s}\right) \delta w_o\right\} dz\ ds \tag{17}$$

Employing the notation

$$\left.\begin{aligned}
F_x &= \left(f_x\right)_{-h/2}^{h/2} \\
m_x &= \left(zf_x\right)_{-h/2}^{h/2} \\
N_n^* &= \int_{-h/2}^{h/2} f_n\,dz \\
M_n^* &= \int_{-h/2}^{h/2} f_n z\,dz \\
F_y &= \left(f_y\right)_{-h/2}^{h/2} \\
m_y &= \left(zf_y\right)_{-h/2}^{h/2} \\
N_{ns}^* &= \int_{-h/2}^{h/2} f_s\,dz \\
Q_n &= \int_{-h/2}^{h/2} f_z\,dz \\
q &= \left(f_z\right)_{-h/2}^{h/2} \\
M_{ns}^* &= \int_{-h/2}^{h/2} f_s z\,dz
\end{aligned}\right\} \tag{18}$$

expression (17) may be rewritten as

$$\begin{aligned}
\delta\overline{W}_e = \iint &\left[F_x\,\delta u_o - m_x\,\delta\left(\frac{\partial w_o}{\partial x}\right) + F_y\,\delta v_o - m_y\,\delta\left(\frac{\partial w_o}{\partial y}\right) + q\,\delta w_o\right]dx\,dy\, + \\
&\oint\left[N_n^*\,\delta u_{o_n} - M_n^*\,\delta\left(\frac{\partial w_o}{\partial n}\right) + N_{ns}^*\,\delta u_{o_s} - M_{ns}^*\,\delta\left(\frac{\partial w_o}{\partial s}\right) + Q_n\,\delta w_o\, + \right. \\
&\left. N_n^*\,\frac{\partial w_o}{\partial n}\,\delta w_o + N_{ns}^*\,\frac{\partial w_o}{\partial s}\,\delta w_o\right]ds
\end{aligned} \tag{19}$$

Noting that

$$\iint \left[m_x \, \delta\left(\frac{\partial w_o}{\partial x}\right) + m_y \, \delta\left(\frac{\partial w_o}{\partial y}\right)\right] dx \, dy = \oint m_n \, \delta w_o \, ds - \iint \left(\frac{\partial m_x}{\partial x} + \frac{\partial m_y}{\partial y}\right) \delta w_o \, dx \, dy \tag{20}$$

the work of external forces may be expressed as

$$\begin{aligned}\delta \overline{W}_e = &\iint \left[F_x \, \delta u_o + F_y \, \delta v_o + \left(q + \frac{\partial m_x}{\partial x} + \frac{\partial m_y}{\partial y}\right) \delta w_o\right] dx \, dy + \\ &\oint \left[N_n^* \, \delta u_{o_n} + N_{ns}^* \, \delta u_s - M_n^* \, \delta\left(\frac{\partial w_o}{\partial n}\right) + \frac{\partial M_{ns}^*}{\partial s} \delta w_o - \right. \\ &\left. m_n \, \delta w_o + Q_n \, \delta w_o + N_n^* \frac{\partial w_o}{\partial n} \delta w_o + N_{ns}^* \frac{\partial w_o}{\partial s} \delta w_o\right] ds\end{aligned} \tag{21}$$

Kinetic Energy

The kinetic energy T of a body occupying before deformation a volume V_o and having a mass density ρ is given by

$$T = \frac{1}{2} \iiint_{V_o} \rho\left(\dot{u}^2 + \dot{v}^2 + \dot{w}^2\right) dx \, dy \, dz \tag{22}$$

The dot indicates differentiation with respect to time t. Using $\dot{\bar{u}}$, $\dot{\bar{v}}$, and $\dot{\bar{w}}$ in the above expression, an approximate form of the kinetic energy $\overline{T}$ in a plate is obtained as

$$\overline{T} = \frac{1}{2} \iiint_{V_o} \rho\left[\left(\dot{u}_o - z \frac{\partial \dot{w}_o}{\partial x}\right)^2 + \left(\dot{v}_o - z \frac{\partial \dot{w}_o}{\partial y}\right)^2 + \dot{w}_o^{\,2}\right] dx \, dy \, dz \tag{23}$$

which can be integrated through the thickness h of the plate yielding, under the simplifying assumption that ρ is independent of z,

$$\overline{T} = \frac{1}{2} \iint \rho\left[h\dot{u}_o^{\,2} + \frac{h^3}{12}\left(\frac{\partial \dot{w}_o}{\partial x}\right)^2 + h\dot{v}_o^{\,2} + \frac{h^3}{12}\left(\frac{\partial \dot{w}_o}{\partial y}\right)^2 + h\dot{w}_o^{\,2}\right] dx \, dy \tag{24}$$

The variation of the kinetic energy $\delta\overline{T}$ is found to be

$$\delta\overline{T} = \iint\left(\rho h\dot{u}_o\,\delta\dot{u}_o + \rho h\dot{v}_o\,\delta\dot{v}_o - \frac{\rho h^3}{12}\frac{\partial^2\dot{w}_o}{\partial x^2}\,\delta\dot{w}_o - \frac{\rho h^3}{12}\frac{\partial^2\dot{w}_o}{\partial y^2}\,\delta\dot{w}_o + \rho h\dot{w}_o\,\delta\dot{w}_o\right)dx\,dy + \oint \rho\,\frac{h^3}{12}\frac{\partial\dot{w}_o}{\partial n}\,\delta\dot{w}_o\,ds \tag{25a}$$

By integrating partially the above expression with respect to time and setting, as usual, the variations at the beginning and end of the time interval equal to zero, the following equation is obtained:

$$\delta\overline{T} = \iint\rho\left(-h\ddot{u}_o\,\delta u_o - h\ddot{v}_o\,\delta v_o + \frac{h^3}{12}\frac{\partial^2\ddot{w}_o}{\partial x^2}\,\delta w_o + \frac{h^3}{12}\frac{\partial^2\ddot{w}_o}{\partial y^2}\,\delta w_o - h\ddot{w}_o\,\delta w_o\right)dx\,dy - \oint\frac{\rho h^3}{12}\frac{\partial\ddot{w}_o}{\partial n}\,\delta w_o\,ds \tag{25b}$$

Equations of Motion

Hamilton's principle is now applied to derive the equations of motion. The principle states that for an arbitrary time interval $t_1 - t_2$

$$\delta\int_{t_1}^{t_2} K\,dt = 0 \tag{26}$$

where K is the kinetic potential; in general,

$$K = T - U \tag{27}$$

U being the potential energy. In the present case of the plate, the potential energy consists of two parts, namely, the approximate internal strain energy $\overline{W}$ and the approximate potential energy of the external forces $-\overline{W}_e$. Hence,

$$\overline{K} = \overline{T} - \overline{W} + \overline{W}_e \tag{28}$$

and

$$\delta\overline{K} = \delta\overline{T} - \delta\overline{W} + \delta\overline{W}_e \tag{29}$$

Using equations (25), (13), and (21) for $\delta\overline{T}$, $\delta\overline{W}$, and $\delta\overline{W}_e$, respectively, the following expression for the variation of the kinetic potential is obtained:

$$\delta K = \iint \rho \left(-h\ddot{u}_o\ \delta u_o - h\ddot{v}_o\ \delta v_o + \frac{h^3}{12} \frac{\partial^2 \ddot{w}_o}{\partial x^2}\ \delta w_o + \frac{h^3}{12} \frac{\partial^2 \ddot{w}_o}{\partial y^2}\ \delta w_o - \right.$$

$$\left. h\ddot{w}_o\ \delta w_o \right) dx\ dy - \oint \frac{\rho h^3}{12} \frac{\partial \ddot{w}_o}{\partial n}\ \delta w_o\ ds + \iint \left[\frac{\partial N_1}{\partial x}\ \delta u_o + \frac{\partial^2 M_1}{\partial x^2}\ \delta w_o + \right.$$

$$\frac{\partial}{\partial x}\left(N_1 \frac{\partial w_o}{\partial x}\right) \delta w_o + \frac{\partial N_2}{\partial y}\ \delta v_o + \frac{\partial^2 M_2}{\partial y^2}\ \delta w_o + \frac{\partial}{\partial y}\left(N_2 \frac{\partial w_o}{\partial y}\right) \delta w_o +$$

$$\frac{\partial}{\partial x}\left(N_3 \frac{\partial w_o}{\partial x}\right) \delta w_o + \frac{\partial}{\partial y}\left(N_3 \frac{\partial w_o}{\partial y}\right) \delta w_o + \frac{\partial N_{12}}{\partial x}\ \delta v_o + \frac{\partial N_{12}}{\partial y}\ \delta u_o +$$

$$\left. \frac{\partial}{\partial x}\left(N_{12} \frac{\partial w_o}{\partial y}\right) \delta w_o + \frac{\partial}{\partial y}\left(N_{12} \frac{\partial w_o}{\partial x}\right) \delta w_o + 2 \frac{\partial^2 M_{12}}{\partial x\ \partial y}\ \delta w_o \right] dx\ dy -$$

$$\oint \left[N_n\ \delta u_n + N_{ns}\ \delta u_s + N_n \frac{\partial w_o}{\partial n}\ \delta w_o + N_{ns} \frac{\partial w_o}{\partial s}\ \delta w_o - M_n \frac{\partial (\delta w_o)}{\partial n} + \right.$$

$$\left. \left(2 \frac{\partial M_{ns}}{\partial s} + \frac{\partial M_n}{\partial n}\right) \delta w_o \right] ds + \iint \left[F_x\ \delta u_o + F_y\ \delta v_o + \right.$$

$$\left. \left(q + \frac{\partial m_x}{\partial x} + \frac{\partial m_y}{\partial y}\right) \delta w_o \right] dx\ dy + \oint \left[N_n^*\ \delta u_n + N_{ns}^*\ \delta u_s + M_n^*\ \delta\left(\frac{\partial w}{\partial n}\right) + \right.$$

$$\left. \left(Q_n - m_n + \frac{\partial M_{ns}^*}{\partial s} + N_n^* \frac{\partial w_o}{\partial n} + N_{ns}^* \frac{\partial w_o}{\partial s}\right) \delta w_o \right] ds = 0 \qquad (30)$$

This variation of the kinetic potential δK must be equal to zero for arbitrary values of δu_o, δv_o, and so forth. The three stress equations of motion are obtained from the integrand of the double integral in equation (30), obtained by equating to zero the coefficients of δu_o, δv_o, and δw_o, are

$$\left.\begin{aligned}
&\frac{\partial N_1}{\partial x} + \frac{\partial N_{12}}{\partial y} + F_x = \rho h\ddot{u}_o \\
&\frac{\partial N_2}{\partial y} + \frac{\partial N_{12}}{\partial x} + F_y = \rho h\ddot{v}_o \\
&\frac{\partial^2 M_1}{\partial x^2} + \frac{\partial^2 M_2}{\partial y^2} + 2\frac{\partial^2 M_{12}}{\partial x\, \partial y} + \frac{\partial}{\partial x}\left(N_1 \frac{\partial w_o}{\partial x}\right) + \frac{\partial}{\partial y}\left(N_2 \frac{\partial w_o}{\partial y}\right) + \\
&\frac{\partial}{\partial x}\left(N_3 \frac{\partial w_o}{\partial x}\right) + \frac{\partial}{\partial y}\left(N_3 \frac{\partial w_o}{\partial y}\right) + \frac{\partial}{\partial x}\left(N_{12} \frac{\partial w_o}{\partial y}\right) + \frac{\partial}{\partial y}\left(N_{12} \frac{\partial w_o}{\partial x}\right) + \\
&q + \frac{\partial m_x}{\partial x} + \frac{\partial m_y}{\partial y} = \rho h\ddot{w}_o - \frac{\rho h^3}{12}\frac{\partial^2 \ddot{w}_o}{\partial x^2} - \frac{\rho h^3}{12}\frac{\partial^2 \ddot{w}_o}{\partial y^2}
\end{aligned}\right\} \tag{31}$$

The integrands of the line integrals serve to identify the starred with the corresponding unstarred quantities (such as $N_n^* = N_n$) and permit the establishment of a relationship between the transverse shear force and other plate stresses as follows:

$$Q_n = \frac{\partial M_n}{\partial n} + \frac{\partial M_{ns}}{\partial s} + m_n + \frac{\rho h^3}{12}\frac{\partial \ddot{w}_o}{\partial n} \tag{32}$$

In addition, the integrand of the last line integral may be employed to establish the boundary conditions. For a system of linear differential equations, appropriate initial and boundary conditions are those which are sufficient to assure a unique solution. The classical uniqueness theorem of the linear theory of elasticity, using energy considerations, is given by Neumann (ref. 6, p. 176). Its analogy for a linear plate theory, including the effects of rotatory inertia and shear, was established by Mindlin (ref. 1).

Similar arguments may be used in the present nonlinear case. Since the superposition principle does not hold, the appropriate boundary conditions will ascertain not a unique solution but merely the nonviolation of the principle of conservation of energy. These boundary conditions to be specified on the edges of the plate are determined by any of the combinations which contains one member of each of the four products of the last line integral in equation (30).

The first two equations of the system (eqs. (31)) describe motions in the plane of the plate and possess the same form as the equations of the plane stress problem. The third equation describes the transverse motions of the plate. The first three terms are those of the classical plate theory. The following six terms represent the influence of large rotations and are nonlinear. Each of these six terms, for example, the first, $\frac{\partial}{\partial x}\left(N_1 \frac{\partial w_o}{\partial x}\right)$, may be written as the sum of two terms; that is,

$$\frac{\partial}{\partial x}\left(N_1 \frac{\partial w_o}{\partial x}\right) = \frac{\partial N_1}{\partial x}\frac{\partial w_o}{\partial x} + N_1 \frac{\partial^2 w_o}{\partial x^2} \tag{33}$$

Inasmuch as $\partial w_o/\partial x$ represents a component of rotation, it is recognized that the nonlinear terms in the stress equation of motion of the present plate theory are precisely of the same nature as those in the three-dimensional nonlinear theory of elasticity. Following the terminology of Biot (ref. 4) terms of the type $\frac{\partial N_1}{\partial x}\frac{\partial w_o}{\partial x}$ may be called buoyancy terms, because they arise from some kind of buoyancy due to the deformation of an element in its own stress field and depend on the stress gradient, while terms of the type $N_1 \frac{\partial^2 w_o}{\partial x^2}$ have their origin in change of direction of an element due to curvature and may be called curvature terms.

The two terms containing the total transverse normal stress N_3 will be neglected in conformance with the type of motions to be described by the plate equations for which this stress is negligible, just as in the classical plate theory. The remaining three terms on the left-hand side of the third equation of motion (eqs. (30)) are the forcing terms, q being the net normal pressure, while m_x and m_y are the external moments, which are customarily neglected in a plate theory. They are retained here merely for the sake of completeness.

On the right-hand side of the equation are found the inertia terms, the first $\rho h\ddot{w}_o$ being the transverse inertia term of the classical plate theory while $\frac{\rho h^3}{12}\frac{\partial^2 \ddot{w}_o}{\partial x^2}$ and $\frac{\rho h^3}{12}\frac{\partial^2 \ddot{w}_o}{\partial y^2}$ are the rotatory inertia terms which are discussed in reference 1.

There exist several other possible ways to deduce plate equations (31) from the general theory, two of which are outlined briefly as follows:

First, the three-dimensional equations could be written down using the components of strain given by expressions (5), which would yield, after substitution of plate displacements (eqs. (7)) and appropriate integration, the plate equations (31).

A second alternative consists in following the same procedure which leads to equations (31) but temporarily considering the rotations ω_x and ω_y as being independent of the transverse displacement w_o in performing the variation of the energy expressions. This would lead to five equations of motion, with five unknowns u_o, v_o, ω_x, ω_y, and w_o. The equations would contain, in addition to the plate stresses defined by equations (12), the plate shears Q_x and Q_y. Expressing ω_x and ω_y subsequently as derivatives of w_o and eliminating the shears Q_x and Q_y from the equations of motion lead to equations (31), two intermediate relationships in the x- and y-directions being those analogous to equation (32).

Comparing equations (31) with the Von Kármán plate equations as found, for example, in reference 15, it is noticed that the essential difference consists in the fact that the Von Kármán equations do not contain the buoyancy terms, but only the curvature terms, as the nonlinear terms of the stress equations. Thus, a mere addition of inertia and forcing terms to the Von Kármán equations will not result in equations (31). However, in the absence of inertia terms, equations (31) reduce readily to the ones of Von Kármán.

Comparing equations (31) with those given by Love (p. 558 of ref. 6), the absence, again, of the buoyancy terms is noted and, in addition, the occurrence of nonlinear terms containing the transverse shear force. The discussion of the significance of these latter terms is postponed to the last section of this report.

Stress-Displacement Relations

The stress equations of motion (eqs. (31)) represent three equations relating 10 unknown functions, namely, the 3 displacements and the 7 plate stresses. Therefore, just as in the three-dimensional theory, additional equations, here numbering seven, are needed to complete the problem. They should relate plate stresses to plate displacements and are developed from the three-dimensional relations, as was done before for a linear plate theory by Mindlin (ref. 1). Assuming the material to be isotropic, the stress-strain relations expressing Hooke's law are

$$\left.\begin{aligned}
\tau_{11} &= \lambda\left(\epsilon_{11} + \epsilon_{22} + \epsilon_{33}\right) + 2\mu\epsilon_{11} \\
\tau_{22} &= \lambda\left(\epsilon_{11} + \epsilon_{22} + \epsilon_{33}\right) + 2\mu\epsilon_{22} \\
\tau_{33} &= \lambda\left(\epsilon_{11} + \epsilon_{22} + \epsilon_{33}\right) + 2\mu\epsilon_{33} \\
\tau_{12} &= 2\mu\epsilon_{12} \\
\tau_{23} &= 2\mu\epsilon_{23} \\
\tau_{31} &= 2\mu\epsilon_{31}
\end{aligned}\right\} \qquad (34)$$

where the stresses and strains are those introduced by expression (6), and λ and μ are Lamé's constants of elasticity. The third equation of the set (eqs. (34)) is now solved for ϵ_{33} and substituted into the first and second, resulting in

$$\tau_{11} = \frac{E}{1 - \nu^2}\epsilon_{11} + \frac{E\nu}{1 - \nu^2}\epsilon_{22} + \frac{\lambda}{\lambda + 2\mu}\tau_{33}$$

$$\tau_{22} = \frac{E}{1 - \nu^2}\epsilon_{22} + \frac{E\nu}{1 - \nu^2}\epsilon_{11} + \frac{\lambda}{\lambda + 2\mu}\tau_{33}$$

Performing the integrations indicated in equations (12) and neglecting again $N_3 = \int_{-h/2}^{h/2} \tau_{33}\,dz$ and, in addition, $\int_{-h/2}^{h/2} \tau_{33}z\,dz$ results in the following plate stress-displacement relations after substitution of the strain-displacement relations in equations (9):

$$\left.\begin{aligned}
M_1 &= -D\left(\frac{\partial^2 w_o}{\partial x^2} + \nu\frac{\partial^2 w_o}{\partial y^2}\right) \\
M_2 &= -D\left(\frac{\partial^2 w_o}{\partial y^2} + \nu\frac{\partial^2 w_o}{\partial x^2}\right) \\
M_{12} &= -D(1 - \nu)\frac{\partial^2 w_o}{\partial x\,\partial y}
\end{aligned}\right\}$$

(equation continued on next page)

$$
\left.
\begin{aligned}
N_1 &= \frac{Eh}{1 - \nu^2}\left[\frac{\partial u_o}{\partial x} + \frac{1}{2}\left(\frac{\partial w_o}{\partial x}\right)^2 + \nu\,\frac{\partial v_o}{\partial y} + \nu\,\frac{1}{2}\left(\frac{\partial w_o}{\partial y}\right)^2\right] \\
N_2 &= \frac{Eh}{1 - \nu^2}\left[\frac{\partial v_o}{\partial y} + \frac{1}{2}\left(\frac{\partial w_o}{\partial y}\right)^2 + \nu\,\frac{\partial u_o}{\partial x} + \nu\,\frac{1}{2}\left(\frac{\partial w_o}{\partial x}\right)^2\right] \\
N_{12} &= Gh\left(\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} + \frac{\partial w_o}{\partial x}\,\frac{\partial w_o}{\partial y}\right)
\end{aligned}
\right\} \tag{35}
$$

In relations (35) E is Young's modulus, G is the shear modulus, ν is Poisson's ratio, and D is the plate modulus expressed as

$$
D = \frac{Eh^3}{12\left(1 - \nu^2\right)} = \frac{Gh^3}{6(1 - \nu)} \tag{36}
$$

The stress-displacement relations (eqs. (35)) coincide in every respect with those of the Von Kármán plate theory (ref. 14) and also with those of the more general theory of Love (ref. 6), which are, however, derived in a different manner without making use of Hooke's law for a three-dimensional solid.

STRAIGHT-CRESTED WAVES

As an application of the plate theory derived above, the problem of propagation of free waves in an infinite plate is now investigated. Assuming the straight-crested harmonic wave to be propagated in the x-direction, the equations of motion are simplified by letting $v_o = 0$ and $\partial/\partial y = 0$.

In the absence of external loads the one-dimensional equations are, from plate equations (31),

$$
\left.
\begin{aligned}
N_1{}' &= \rho h \ddot{u}_o \\
M_1{}'' + \frac{\partial}{\partial x}\left(N_1 w_o{}'\right) &= \rho h \ddot{w}_o - \frac{\rho h^3}{12}\,\ddot{w}_o{}''
\end{aligned}
\right\} \tag{37}
$$

Primes indicate differentiation with respect to x. The corresponding stress-displacement relations are, from equations (35),

$$\left.\begin{aligned} N_1 &= \frac{Eh}{1 - \nu^2}\left[u_o' + \frac{1}{2}(w_o')^2\right] \\ M_1 &= -Dw_o'' \end{aligned}\right\} \tag{38}$$

so that the displacement equations of motion take the form

$$\left.\begin{aligned} &\frac{Eh}{1 - \nu^2}\left(u_o'' + w_o'w_o''\right) = \rho h\ddot{u}_o \\ &-Dw_o^{iv} + \frac{Eh}{1 - \nu^2}\left[u_o''w_o' + u_o'w_o'' + \frac{3}{2}w_o''(w_o')^2\right] + \frac{\rho h^3}{12}\ddot{w}_o'' = \rho h\ddot{w}_o \end{aligned}\right\} \tag{39}$$

It is noted that the first of the two displacement equations of motion (eqs. (39)) describes essentially the compressional motions in a plate, which are, however, coupled to the transverse motions because of the presence of a (nonlinear) term containing derivatives of transverse displacement w_o. The second equation governs the transverse (flexural) motions, which appear to be coupled to compressional motions by the presence of two terms containing derivatives of the longitudinal displacement u_o.

Neither the form (eqs. (37)) of the stress equations of motion nor the form (eqs. (39)) of the displacement equations of motion permits clear recognition of the nature of the coupling effect contained in these coupling terms. A deeper insight into this coupling phenomenon in both sets of equations (37) and (39) may be gained, however, with the aid of a redefinition of the plate stress N_1.

In view of the stress-displacement relation contained in the first equation of the set (38), N_1 may be expressed as the sum of two components, namely,

$$N_1 = N_x - N_{zx}\omega_y \tag{40}$$

where N_x is the classical component defined as

$$N_x = \frac{Eh}{1 - \nu^2}\frac{\partial u_o}{\partial x} \tag{41}$$

and N_{zx} is defined as

$$N_{zx} = \frac{Eh}{2(1 - \nu^2)}\frac{\partial w_o}{\partial x} \tag{42}$$

The rotation component ω_y appearing in equation (40) is considered, temporarily, as being independent of the transverse displacement w_o. The resolution of N_1, defined by equation (40), is convenient because it separates the terms containing u_o and w_o. In terms of the newly introduced plate stresses N_x and N_{zx}, the stress equations of motion (eqs. (37)) may be written as

$$\left.\begin{aligned} &N_x' - \frac{\partial}{\partial x}\left(N_{zx}\omega_y\right) = \rho h\ddot{u}_o \\ &M_1'' - \frac{\partial}{\partial x}\left(N_x\omega_y\right) - \frac{\partial}{\partial x}\left(N_{zx}\omega_y^2\right) = \rho h\ddot{w}_o - \frac{\rho h^3}{12}\frac{\partial^2\ddot{w}_o}{\partial x^2} \end{aligned}\right\} \tag{43}$$

and the displacement equations of motion (eqs. (39)), as

$$\left.\begin{aligned} &\frac{Eh}{1 - \nu^2} u_o'' - \frac{Eh}{2\left(1 - \nu^2\right)}\frac{\partial}{\partial x}\left(\omega_y \frac{\partial w_o}{\partial x}\right) = \rho h\ddot{u}_o \\ &-Dw_o^{iv} - \frac{Eh}{1 - \nu^2}\frac{\partial}{\partial x}\left(\omega_y \frac{\partial u_o}{\partial x}\right) - \frac{Eh}{2\left(1 - \nu^2\right)}\frac{\partial}{\partial x}\left(\omega_y^2 \frac{\partial w_o}{\partial x}\right) = \rho h\ddot{w}_o - \frac{\rho h^3}{12}\ddot{w}_o'' \end{aligned}\right\} \tag{44}$$

The coupling terms, the second term in each of the equations of the set (43) or (44), are thus clearly exhibited. As a consequence of a large rotation ω_y, the plate stress N_x enters the equation of transverse motions, while the plate stress N_{zx}, which depends on w_o only, enters the equation of compressional motions.

The coupling operator in equations (44) is recognized to be the same in both equations, namely,

$$\frac{Eh}{1 - \nu^2}\frac{\partial}{\partial x}\left(\omega_y\right) = \frac{Eh}{1 - \nu^2}\frac{\partial \omega_y}{\partial x} + \frac{Eh}{1 - \nu^2}\omega_y\frac{\partial}{\partial x} \tag{45}$$

thus making the operator matrix on the two dependent variables u_o and w_o symmetric. Thus, it appears that the operator matrix is symmetric, in a generalized sense, even in the present nonlinear theory just as it should be in any linear theory, as was pointed out, for example, in references 16 and 17. Moreover, it is observed that the coupling operator contains both a curvature term and a buoyancy term.

In addition, it is noticed that the effect of a large rotation is apparent not only in the coupling terms, which contain the rotation to the first power, but also in a term occurring in the second equation, which contains the rotation to the second power.

Equations (39) are now rewritten in a more convenient form. The first is solved for $w_o'w_o''$ and substituted into the second. If c_p denotes the phase velocity of compressional waves in a plate, then

$$c_p^2 = \frac{E}{\rho(1 - \nu^2)} \tag{46}$$

and equations (39) may be written as

$$\left.\begin{aligned} &c_p^2\left(u_o'' + w_o'w_o''\right) = \ddot{u}_o \\ &-c_p^2\left(\frac{h^2}{12} w_o^{iv} + \frac{1}{2} u_o''w_o' - u_o'w_o''\right) + \frac{3}{2} \ddot{u}_o w_o' + \frac{h^2}{12} \ddot{w}_o'' = \ddot{w}_o \end{aligned}\right\} \tag{47}$$

The solution of this coupled set of two partial, homogeneous, nonlinear, differential equations is sought in the form of harmonic waves

$$\left.\begin{aligned} u_o(x,t) &= A \sin \bar{\gamma} (x - \bar{c}t) \\ w_o(x,t) &= B \cos \gamma (x - ct) \end{aligned}\right\} \tag{48}$$

where A and B are the amplitudes, $\bar{\gamma}$ and γ, the wave numbers, and $\bar{c}$ and c, the phase velocities. Substitution of this trial solution into the first of the equations of motion reveals that, for periodic motions,

$$\bar{\gamma} = 2\gamma \tag{49}$$

$$\bar{c} = c \tag{50}$$

$$8A\gamma\left(1 - \frac{c^2}{c_p^2}\right) = B^2\gamma^2 \tag{51}$$

The first term in equation (51) represents the influence of compressional stiffness, the second, the influence of longitudinal inertia, and the term on the right-hand side, the influence of transverse stiffness. A sketch of the deformed plate is given in figure 1. Substitution of equations (48) into the second equation of the set (47) results in the following equation:

$$\left[-c_p^2\left(\frac{h^2}{12}\gamma^4 + 2A\gamma^3\right) + c^2\left(3A\gamma^3 + \frac{h^2\gamma^4}{12} + \gamma^2\right)\right] \cos \gamma (x - ct) -$$

$$3A\gamma^3 c^2 \cos 3\gamma (x - ct) = 0 \tag{52}$$

In obtaining equation (52), use was made of the relation

$$\cos^3 \alpha = \frac{1}{4} \cos 3\alpha + \frac{3}{4} \cos \alpha \tag{53}$$

Equation (52) is similar to an intermediate form of the Duffing equation which governs, for example, motions of a single mass attached to a nonlinear spring and which is discussed in detail in reference 18. Discussion of nonlinear problems governed by partial differential equations may be found in references 19 to 21.

Confining attention to periodic solutions, the term (eq. (52)) with $\cos 3\gamma (x - ct)$ may be disregarded as compared with the term $\cos \gamma (x - ct)$, provided that the amplitude of the former is small as compared with the amplitude of the latter. This restriction is discussed in detail below. The velocity equation is then

$$\frac{c^2}{c_p^2}\left(1 + \frac{h^2\gamma^2}{12} + 3A\gamma\right) = \frac{h^2\gamma^2}{12} + 2A\gamma \tag{54}$$

The first term on the left-hand side of velocity equation (54) represents the influence of transverse inertia on the wave velocity, the second term, the influence of rotatory inertia, and the third term, the influence of longitudinal inertia. On the right-hand side, the first term represents the influence of flexural stiffness and the second term, the influence of compressional stiffness.

Eliminating $A\gamma$ in equations (51) and (54), a single biquadratic equation

$$\left(\frac{c}{c_p}\right)^4\left(1 + \frac{h^2\gamma^2}{12}\right) - \left(\frac{c}{c_p}\right)^2\left(1 + \frac{h^2\gamma^2}{6} + \frac{3}{8} B^2\gamma^2\right) + \frac{h^2\gamma^2}{12} + \frac{B^2\gamma^2}{4} = 0 \tag{55}$$

governing the phase velocity c is obtained, which is seen to depend on two parameters, namely, $h\gamma$ and $B\gamma$. Since

$$\gamma = \frac{2\pi}{L}$$

where L is the wave length, $h\gamma$ represents the ratio of plate thickness to wave length, while $B\gamma$ equals the maximum rotation $\omega_y = -\frac{\partial w_o}{\partial x}$.

This velocity, or frequency, equation represents a typical two-mode coupled system similar, for example, to the one of reference 2. While in reference 2 the coupling between the two modes was established through Poisson's ratio and thus depended on material properties, the coupling is effected presently through a kinematical quantity, namely, by taking into account large rotations. A coupling due to gravity between the same two types of waves has been discussed in reference 3. The two modes may be uncoupled by omitting the terms in equation (55) containing $B\gamma$, which yields

$$\left(\frac{c^2}{c_p^2} - 1\right)\left(\frac{c^2}{c_p^2} + \frac{c^2}{c_p^2}\frac{h^2\gamma^2}{12} - \frac{h^2\gamma^2}{12}\right) = 0 \tag{56}$$

Thus, one mode, which describes compressional motions in the plate, has the velocity

$$\frac{c^2}{c_p^2} = 1 \tag{57}$$

while the second, the flexural mode, has a velocity

$$\frac{c^2}{c_p^2} = \frac{\frac{h^2\gamma^2}{12}}{1 + \frac{h^2\gamma^2}{12}} \tag{58}$$

which has been discussed in reference 1. If the influence of rotatory inertia is neglected, equation (58) simplifies to

$$\frac{c^2}{c_p^2} = \frac{h^2\gamma^2}{12} \tag{59}$$

Several graphs have been plotted to exhibit more clearly the coupling effect. Figure 2(a) shows the uncoupled and coupled velocity curves for a constant value of the ratio of transverse amplitude B to the wave length L, which is a measure of the rotation and which has been chosen to be equal to 0.1 in the case where the rotatory inertia is neglected. Figure 2(b) gives the same curves, including, however, the effect of rotatory inertia. It is seen that the coupled curves approach the velocity curves of uncoupled motions for large values of h/L but not for

$h/L = 0$. This deficiency is due, most probably, to the fact that the equations derived are not valid for the full range of the parameters involved. These inherent limitations are discussed subsequently.

Noting that $\frac{B}{L} = \frac{B}{h}\frac{h}{L}$, similar curves can be drawn for a constant value of B/h, which may be taken as a measure for the deflection. Figure 3(a) shows the two coupled velocity curves, drawn for $\frac{B}{h} = 1$, with the effect of rotatory inertia being neglected. The coupled curves approach the uncoupled curves for $\frac{h}{L} = 0$ but not for large values of h/L. Figure 3(b) gives the coupled and uncoupled velocity curves for the same constant value of B/h, but with the effect of rotatory inertia included, and again a similar behavior of the coupled curves with respect to the uncoupled ones is noted.

The range of validity of the velocity equation, however, is limited to specific ranges of the parameters involved in view of the simplifying assumptions underlying both the plate theory and the solution of the wave-propagation problem. These limitations are now examined in detail.

At the outset, in considering the possible deformations, the rotation was assumed to be not too large, so that it could be represented by a vector with components ω_x, ω_y, and ω_z. This is admissible only if these components are small as compared with unity.

In the case of waves, described in equations (48), propagated in a plate, the maximum value of the component of rotation ω_y is given by

$$|\omega_y|_{max} = B\gamma$$

which has to be small as compared with unity. Moreover, it was assumed that the components of strain are small as compared with unity, which implies that in the expressions for the strains all the nonlinear terms are of third and higher order, except the squares and products of rotation, which are of the second order. Such a typical nonlinear term not vanishing in the one-dimensional case is $\frac{1}{2}\omega_y e_{xx}$, which occurs in the expression for the strain ϵ_{31} in the set of equations (4). Another such term is $\frac{1}{2}e_{xx}^2$, which occurs in the expression for the strain ϵ_{11} if the strains are not assumed to be small (ref. 4, eqs. (1-12)).

Thus, the assumption of small strain requires the smallness, as compared with unity, of the coefficient $e_{xx} \equiv \frac{\partial u}{\partial x}$. In the case of the present plate theory, the coefficient $\bar{e}_{xx} \equiv \frac{\partial \bar{u}}{\partial x}$ is

$$\frac{\partial \bar{u}}{\partial x} = \frac{\partial u_o}{\partial x} - z \frac{\partial^2 w_o}{\partial x^2}$$

and consists of a compressional part $\frac{\partial u_o}{\partial x}$ and a flexural part $z \frac{\partial^2 w_o}{\partial x^2}$, the maximum of which is $\frac{h}{2} \frac{\partial^2 w_o}{\partial x^2}$. Since the coefficient $\bar{e}_{xx}$ has to be small throughout the thickness of the plate, each part must be small as compared with unity. In the case of waves, this implies that both $A\gamma$ (compressional part) and $\frac{h}{2} B\gamma^2$ (flexural part) should be small as compared with unity.

The significance of the smallness of the flexural part is not quite clear yet, in view of the fact that $B\gamma$ is small by itself. However, it should be pointed out that, in the expression for the strain, the squares of rotation contain terms with $B^2\gamma^2$ which are retained, while terms of the type $\frac{1}{2} \omega_y e_{xx}$ which contain $\frac{h\gamma}{4} B^2\gamma^2$ are neglected.

Thus, clearly, this implies that $h\gamma$ is small as compared with unity, and since $\gamma = 2\pi/L$, where L is the wave length, that h/L is small as compared with unity. Hence, the solution will be valid only for waves which are long as compared with plate thickness.

To summarize the restrictions of the wave solution inherent in the present plate theory, the smallness of the components of strain implies the smallness of $A\gamma$ and $h\gamma$, while the smallness of the rotations implies the smallness of $B\gamma$, all compared with unity.

Attention is drawn next to limitations inherent in the neglect of the term with $\cos 3\gamma(x - ct)$ as compared with the term with $\cos \gamma(x - ct)$ in equation (52). This neglect is justified only if the amplitude of the former is small as compared with the amplitude of the latter. The ratio of these amplitudes, which ought to be small, is calculated to be equal to $3A\gamma$. But this requirement coincides with one of the requirements imposed by the smallness of strain discussed previously. Thus, the approximations introduced in the wave solution do not introduce any additional limitations.

The solutions (51) and (54) are now examined in the light of the limitations just discussed. Solving equation (51) for $A\gamma$ and obtaining

$$A\gamma = \frac{B^2\gamma^2}{8} \frac{1}{1 - \frac{c^2}{c_p{}^2}}$$

it is noted that $\frac{1}{1 - c^2/c_p{}^2}$ should be small, as compared with unity, for a given value of $B\gamma$. The curve of $A\gamma$, as a function of c/c_p, exhibits a resonance phenomenon of a type similar to that in the case of forced vibrations of a single mass. The phase velocity c should be either smaller or larger than the compressional wave velocity in a plate c_p.

In equation (54) the terms with $\frac{h^2\gamma^2}{12}$ and $3A\gamma$ on the left-hand side may be omitted, in accordance with the limitations imposed, since these terms are added to unity. Thus, relation (54) simplifies to

$$\frac{c^2}{c_p{}^2} = \frac{h^2\gamma^2}{12} + 2A\gamma \tag{60}$$

That is, in the range of its validity, the phase velocity depends only upon transverse inertia and the flexural and compressional stiffness, while compressional and rotatory inertias are negligible. But it has already been established that $h\gamma$ and $A\gamma$ are small as compared with unity; hence, from equation (60), $\frac{c^2}{c_p{}^2}$ is small as compared with unity. Thus, equation (51) simplifies to

$$8A\gamma = B^2\gamma^2 \tag{61}$$

Eliminating $A\gamma$ from equation (60) with the aid of equation (61) gives the velocity equation

$$\frac{c^2}{c_p{}^2} = \frac{h^2\gamma^2}{12} + \frac{B^2\gamma^2}{4} = \frac{\gamma^2}{4}\left(\frac{h^2}{3} + B^2\right) = \frac{\pi^2}{L^2}\left(\frac{h^2}{3} + B^2\right) \tag{62}$$

in which each term has to be small as compared with unity. The result is thus obtained that, within the range of its validity, for a given wave length, the velocity is affected in the same manner by the deflection B as by the plate thickness divided by $\sqrt{3}$.

The velocity given by equation (62) is plotted in figure 4 for various values of the parameter B/L, which represents a measure of the rotation. Since $\frac{B}{L} = \frac{B}{h}\frac{h}{L}$, equation (62) may be rewritten as

$$\frac{c^2}{{c_p}^2} = \frac{\pi^2}{3}\frac{h^2}{L^2}\left(1 + 3\frac{B^2}{h^2}\right) \tag{63}$$

where the parameter B/h may be taken as a measure of the deflection. The velocity given by equation (63) is plotted in figure 5 for various values of the parameter B/h.

In the treatment of nonlinear systems with one degree of freedom, it is customary to study the so-called response curves, that is, curves relating the amplitude to the frequency (ref. 18). Since, in the present case, the velocity may be said to play the role of the frequency, B/h was plotted as a function of c/c_p for various values of h/L, which is now considered a parameter. These curves, given in figure 6, are qualitatively the same as the response curves of systems governed by Duffing's equation (ref. 18) which demonstrates still further a certain resemblance of the present nonlinear continuous system with infinitely many degrees of freedom to a nonlinear single-degree-of-freedom system.

It is worthwhile to point out that, if the terms containing longitudinal inertia are neglected in equations (47) by letting $\ddot{u}_o = 0$, equations (61) and (62) represent exact solutions of equations (47). It should also be observed that the contribution of the buoyancy term in the second equation (47) is contained only in this longitudinal inertia term.

A VARIANT OF PLATE EQUATIONS

In this last section of the present report, a large-deflection plate theory is derived the equation for which includes the influence of terms containing the first power of the rotation in the expressions for the components of strain, that is, expressions (4) for the strains will be used and not expressions (5). Even though it has been recognized previously that these terms are of higher order than the second, the development is worthwhile because it appears that better insight is gained in appraising other large-deflection plate and beam theories, particularly those derived by Love (ref. 6, p. 558) and Eringen (ref. 7).

In view of the fact, however, that the assumptions as to displacements are still those given by equations (7), that is, $e_{xz} = e_{yz} = e_{zz} = 0$

and ω_z is small, the influence of linear terms in ω_x and ω_y appears only in two components of strain, namely, ϵ_{31} and ϵ_{23}, which are now

$$\left.\begin{aligned}
\epsilon_{23} &= -\frac{1}{2}\omega_x e_{yy} + \frac{1}{2}\omega_y\left(e_{xy} - \omega_z\right) \\
\epsilon_{31} &= \frac{1}{2}\omega_y e_{xx} - \frac{1}{2}\omega_x\left(e_{xy} + \omega_z\right)
\end{aligned}\right\} \tag{64}$$

or, using the displacements given by equations (7),

$$\left.\begin{aligned}
\epsilon_{23} &= -\frac{1}{2}\frac{\partial w_o}{\partial y}\left(\frac{\partial v_o}{\partial y} - z\frac{\partial^2 w_o}{\partial y^2}\right) - \frac{1}{2}\frac{\partial w_o}{\partial x}\left(\frac{\partial u_o}{\partial y} - z\frac{\partial^2 w_o}{\partial x\,\partial y}\right) \\
\epsilon_{31} &= -\frac{1}{2}\frac{\partial w_o}{\partial x}\left(\frac{\partial u_o}{\partial x} - z\frac{\partial^2 w_o}{\partial x^2}\right) - \frac{1}{2}\frac{\partial w_o}{\partial y}\left(\frac{\partial w_o}{\partial x} - z\frac{\partial^2 w_o}{\partial x\,\partial y}\right)
\end{aligned}\right\} \tag{65}$$

Assuming, as was done before, that the derivatives of u_o and v_o with respect to the space variables are small as compared with unity and using the same procedure which led to the equations of motion (31), the following homogeneous equations of motion are obtained:

$$\left.\begin{aligned}
&-\frac{\partial N_1}{\partial x} + \frac{\partial}{\partial x}\left(Q_1{}^*\frac{\partial w_o}{\partial x}\right) + \frac{\partial}{\partial y}\left(Q_2{}^*\frac{\partial w_o}{\partial x}\right) - \frac{\partial N_{12}}{\partial y} = -\rho h\ddot{u}_o \\
&-\frac{\partial N_2}{\partial y} + \frac{\partial}{\partial y}\left(Q_2{}^*\frac{\partial w_o}{\partial y}\right) + \frac{\partial}{\partial x}\left(Q_1{}^*\frac{\partial w_o}{\partial y}\right) - \frac{\partial N_{12}}{\partial x} = -\rho h\ddot{u}_o \\
&-\frac{\partial^2 M_1}{\partial x^2} - \frac{\partial^2 M_2}{\partial y^2} - 2\frac{\partial^2 M_{12}}{\partial x\,\partial y} - \frac{\partial}{\partial x}\left(N_1\frac{\partial w_o}{\partial x}\right) - \frac{\partial}{\partial y}\left(N_2\frac{\partial w_o}{\partial y}\right) - \\
&\frac{\partial}{\partial x}\left(N_{12}\frac{\partial w_o}{\partial y}\right) - \frac{\partial}{\partial y}\left(N_{12}\frac{\partial w_o}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial M_{23}}{\partial y}\frac{\partial w_o}{\partial y}\right) + \frac{\partial}{\partial x}\left(\frac{\partial M_{23}}{\partial y}\frac{\partial w_o}{\partial x}\right) + \\
&\frac{\partial}{\partial x}\left(\frac{\partial M_{31}}{\partial x}\frac{\partial w_o}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial M_{31}}{\partial x}\frac{\partial w_o}{\partial y}\right) = -\rho h\ddot{w}_o
\end{aligned}\right\} \tag{66}$$

In these equations, N_1, N_2, N_{12}, M_1, M_2, and M_{12} are the plate stresses defined by the integrals in equations (12). The transverse shear forces $Q_1{}^*$, $Q_2{}^*$ are now defined by

$$\left.\begin{aligned} Q_1^* &= \int_{-h/2}^{h/2} \tau_{31}\, dz \\ Q_2^* &= \int_{-h/2}^{h/2} \tau_{32}\, dz \end{aligned}\right\} \tag{67}$$

and no longer by equation (32) as in the previous theory. Moments of transverse shear stresses M_{31} and M_{32} are automatically defined as

$$\left.\begin{aligned} M_{31} &= \int_{-h/2}^{h/2} \tau_{31} z\, dz \\ M_{32} &= \int_{-h/2}^{h/2} \tau_{32} z\, dz \end{aligned}\right\} \tag{68}$$

and do not occur in any linear plate theory. The static terms in equations (66) could have also been obtained by a direct integration, through the thickness of the plate, of the corresponding three-dimensional equations, which were given by Biot (ref. 4, eqs. (5.4)).

Comparing equations (66) with those given by Love, it is recognized that the first two equations of each set are almost identical, except that in reference 6 only shear curvature terms of the type $Q_1^* \dfrac{\partial^2 w_o}{\partial x^2}$ occur, while equations (66) contain both shear curvature and shear buoyancy terms, the latter being typified by the term $\dfrac{\partial Q_1^*}{\partial x} \dfrac{\partial w_o}{\partial x}$.

Thus, the origin of the nonlinear terms in the first two stress equations of Love's theory is due to a retention of first-power terms of rotations in the expressions for the components of strain. It has been shown previously in this report that they are of a higher order than the second and may thus be omitted. In reference 6, it is merely stated that it will generally be sufficient to omit these terms.

The last equation of the set (66) contains, in addition to the terms found in the corresponding equation of the set (31), a series of terms containing the moments M_{23} and M_{31}, which are missing in Love's equation.

In order to compare Eringen's beam equations (ref. 7) with those of the present study, the plate equations (66) are reduced to beam equations by letting $\ddot{v}_o = 0$ and $\partial/\partial y = 0$. They are

$$\left.\begin{aligned} &\frac{\partial N_1}{\partial x} - \frac{\partial}{\partial x}\left(Q_1{}^* \frac{\partial w_o}{\partial x}\right) = \rho h\ddot{u}_o \\ &\frac{\partial^2 M_1}{\partial x^2} + \frac{\partial}{\partial x}\left(N_1 \frac{\partial w_o}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial M_{31}}{\partial x} \frac{\partial w_o}{\partial x}\right) = \rho h\ddot{w}_o \end{aligned}\right\} \tag{69}$$

Next, Eringen's equations are reduced to describe small strains and moderately large rotations by considering $\partial u/\partial x$ to be small as compared with unity, letting the cosine of the angle of rotation equal unity, and assuming that the sine equals the angle itself. They are, in the notation used herein,

$$\left.\begin{aligned} &\frac{\partial N_1}{\partial x} - \frac{\partial}{\partial x}\left(Q_1{}^* \frac{\partial w_o}{\partial x}\right) = \rho h\ddot{u}_o \\ &\frac{\partial^2 M_1}{\partial x^2} + \frac{\partial}{\partial x}\left(N_1 \frac{\partial w_o}{\partial x}\right) - \frac{\partial}{\partial x}\left(Q_1{}^* \frac{\partial^2 w_o}{\partial x^2}\right) = \rho h\ddot{w}_o \end{aligned}\right\} \tag{70}$$

Thus, it is seen that Eringen's term $Q_1{}^* \frac{\partial^2 w_o}{\partial x^2}$ is expressed in the present theory as $\frac{\partial M_{31}}{\partial x} \frac{\partial w_o}{\partial x}$.

Columbia University,
New York, N. Y., October 3, 1955.

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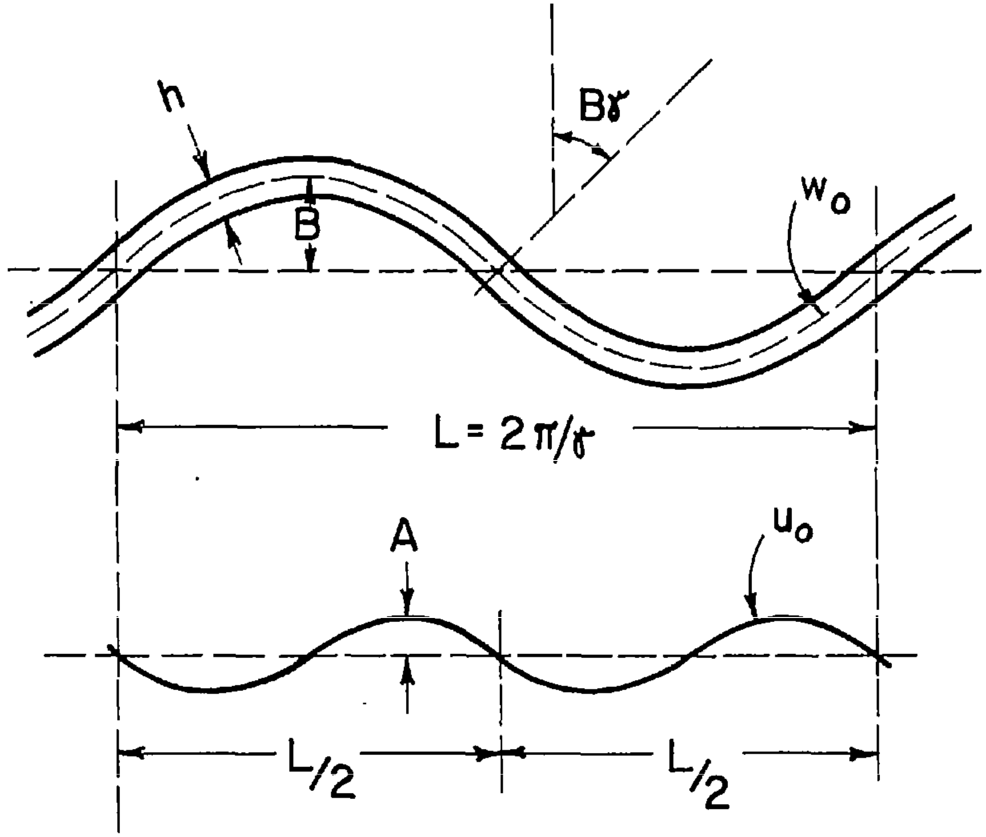
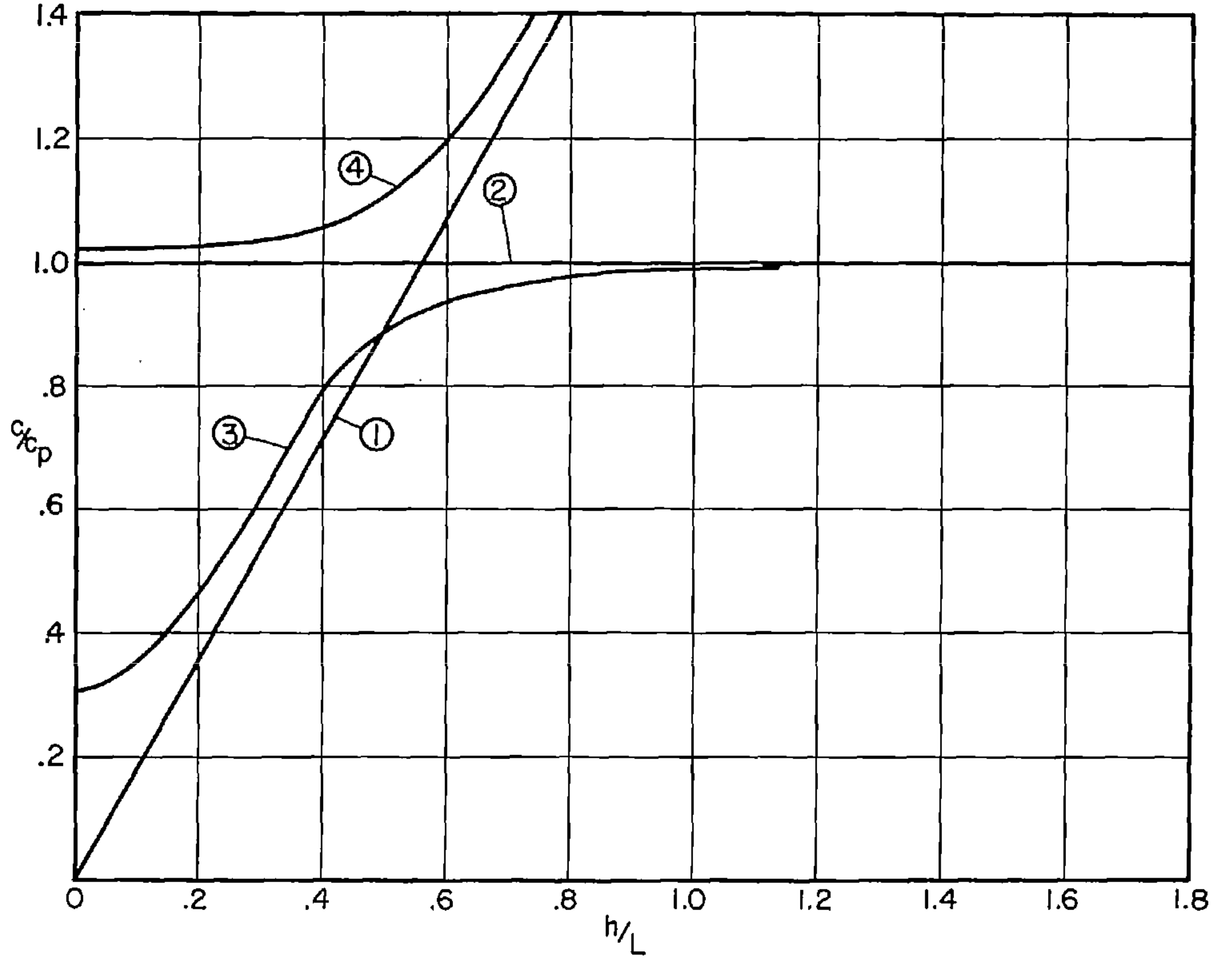
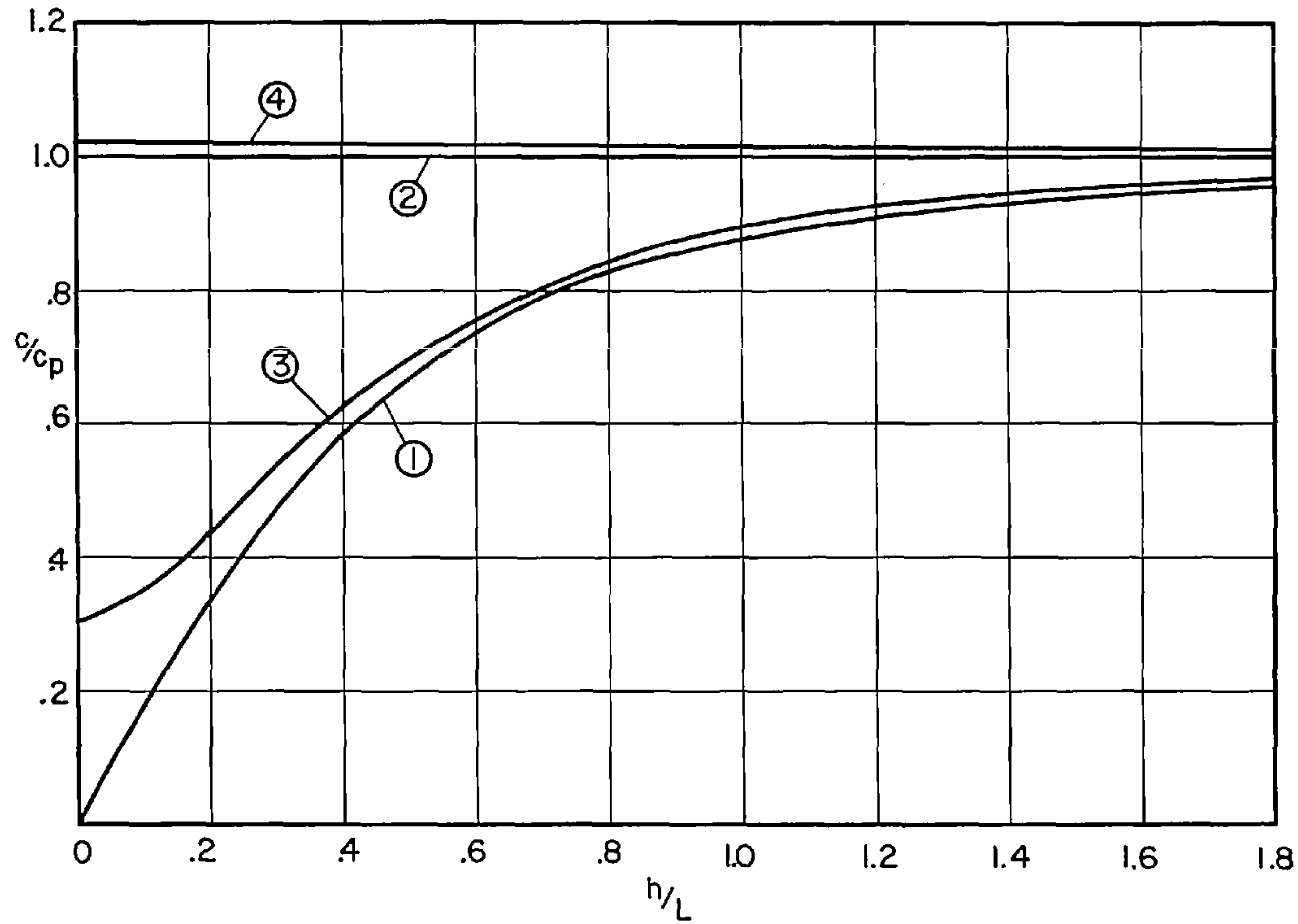


Figure 1.- Straight-crested waves in plate. Transverse displacement w_o, amplitude B; longitudinal displacement u_o, amplitude A.



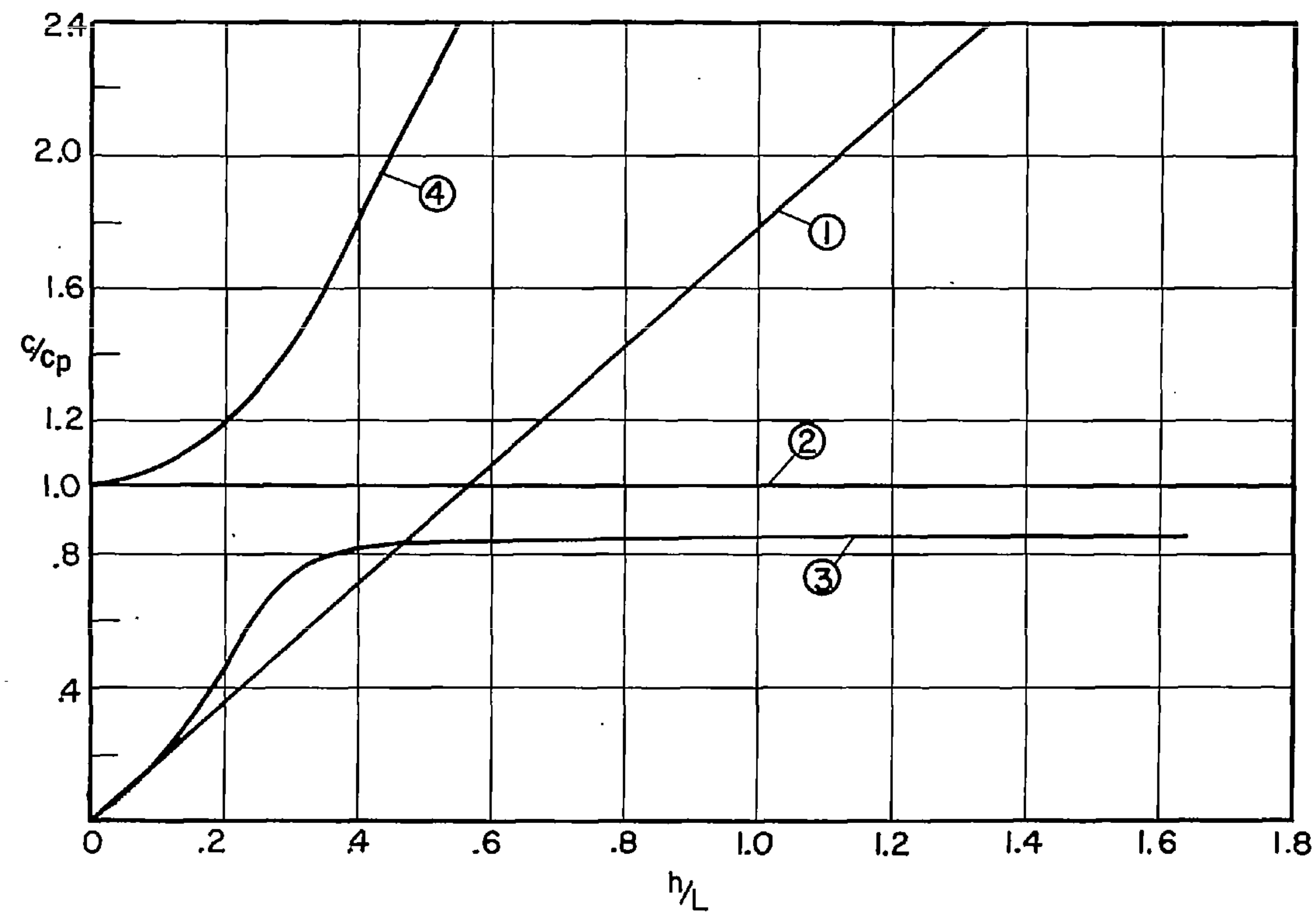
(a) Classical theory, (1) flexural waves, (2) longitudinal waves; coupled motions according to theory presented herein (B/L = 0.1), (3) lower branch, (4) upper branch.

Figure 2.- Plot showing dependence of wave velocity on wave length and influence of large rotations.



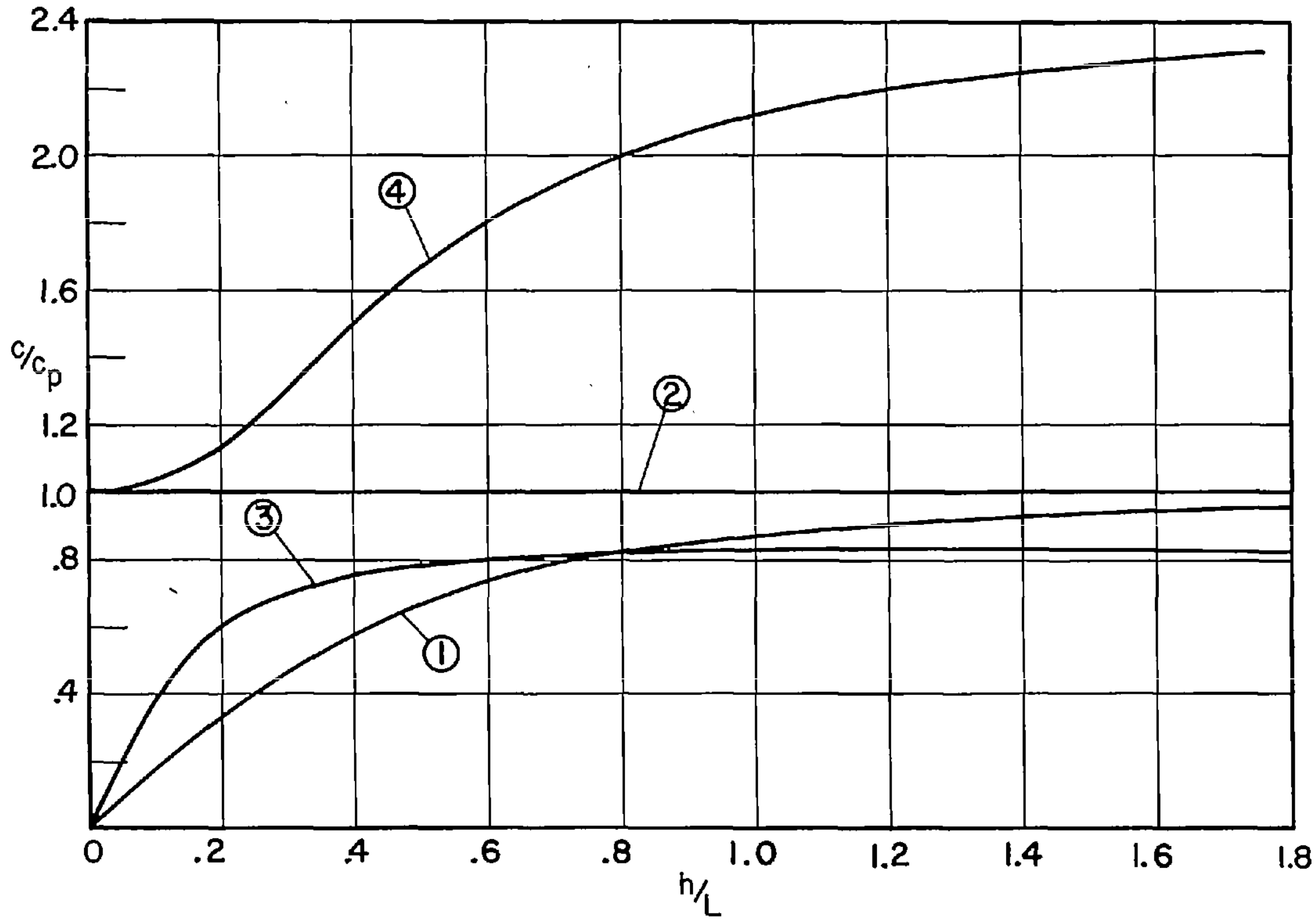
(b) Classical theory with rotatory inertia correction added, ① flexural waves; classical theory, ② longitudinal waves; coupled motions according to theory presented herein (B/L = 0.1), ③ lower branch, ④ upper branch.

Figure 2.- Concluded.



(a) Classical theory, 1 flexural waves, 2 longitudinal waves; coupled motions according to theory presented herein (B/h = 1), 3 lower branch, 4 upper branch.

Figure 3.- Plot showing dependence of wave velocity on wave length and influence of large deflections.



(b) Classical theory with rotatory inertia added, 1 flexural waves; classical theory, 2 longitudinal waves; coupled motions according to theory presented herein (B/h = 1), 3 lower branch, 4 upper branch.

Figure 3.- Concluded.

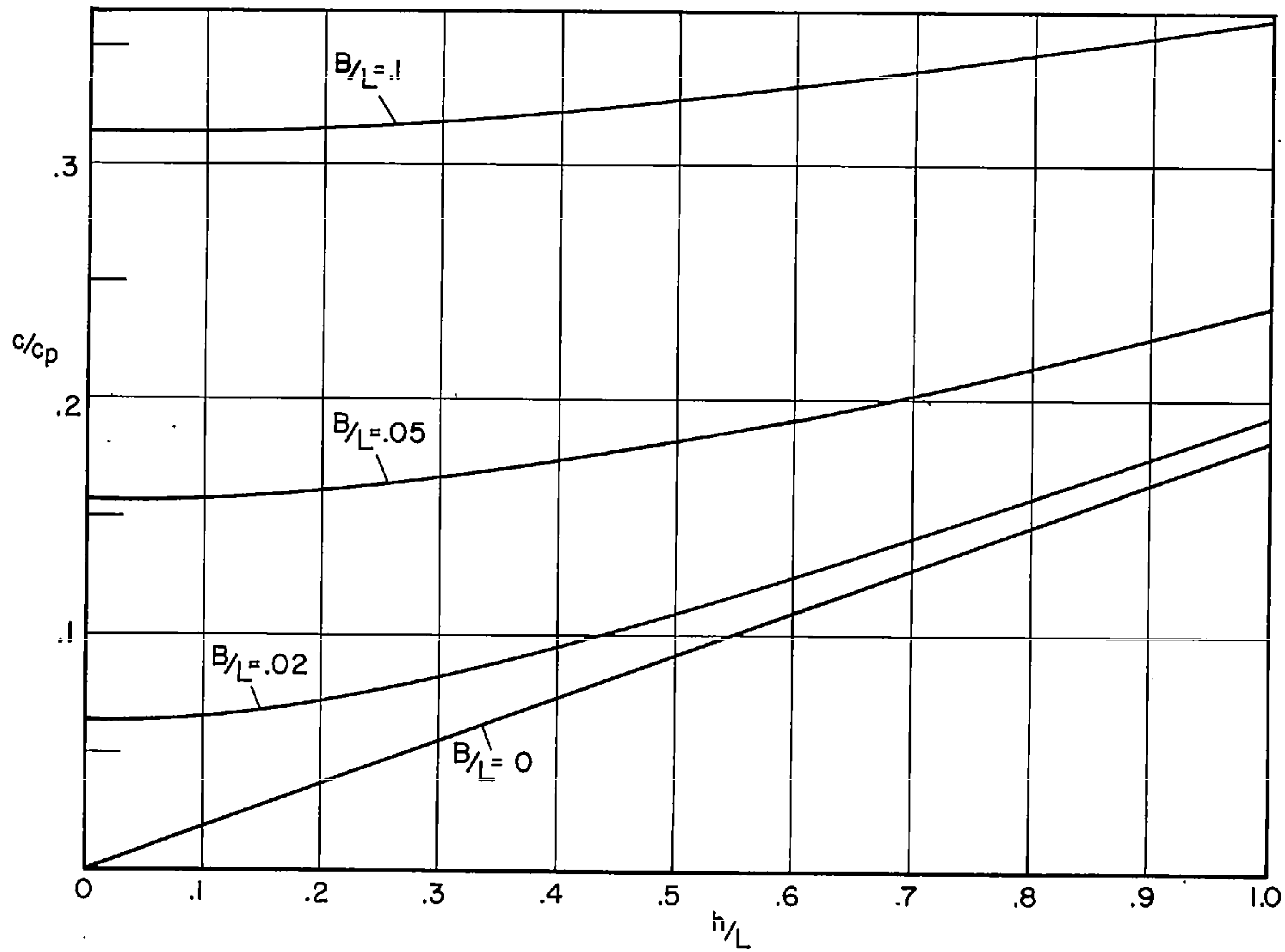


Figure 4.- Plot showing dependence of wave velocity on wave length and influence of large rotations for various values of B/L.

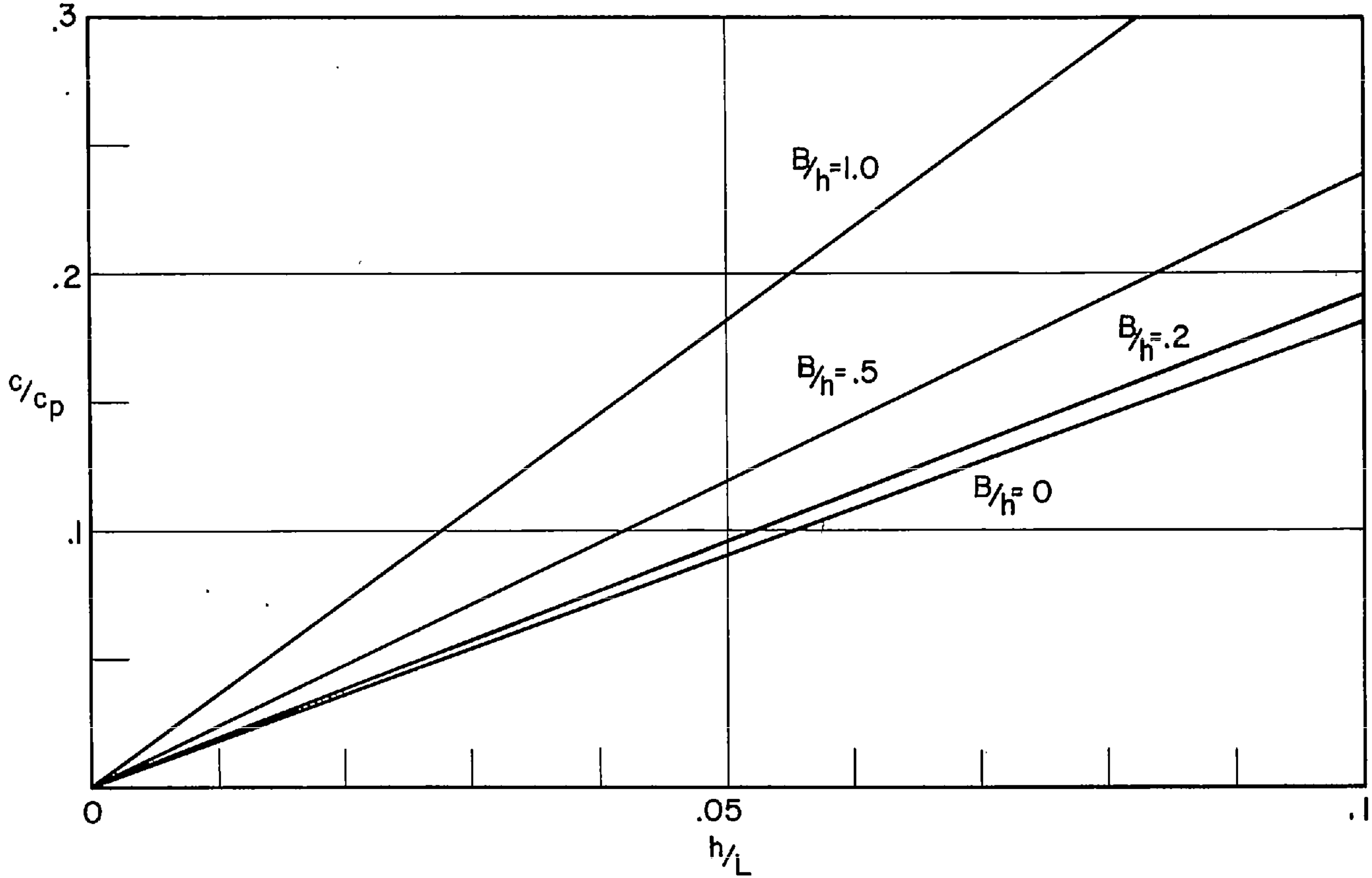


Figure 5.- Plot showing dependence of wave velocity on wave length and influence of large deflections for various values of B/h.

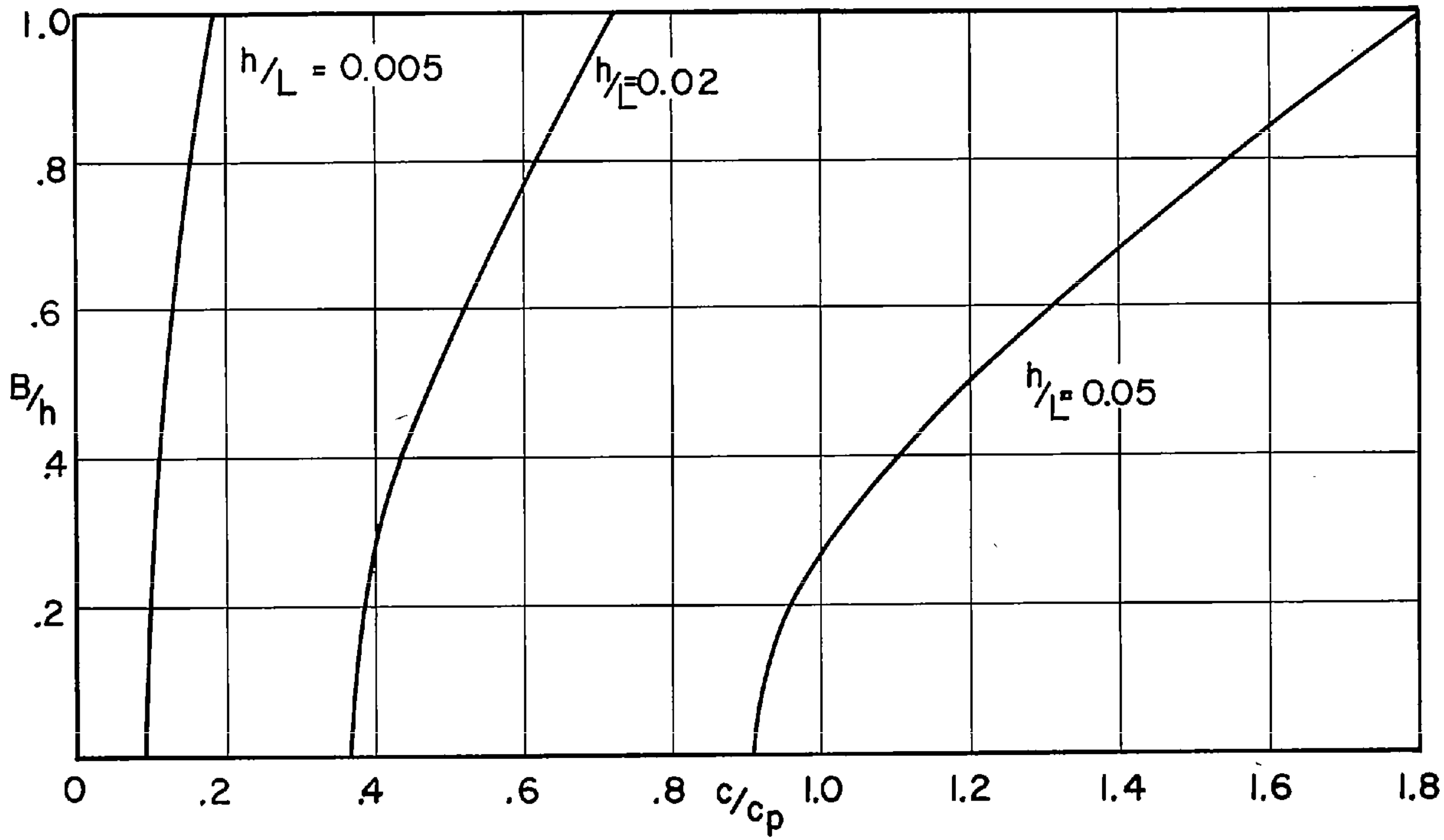


Figure 6.- Response curves showing dependence of amplitude on wave velocity and influence of wave length.