NACA TN 3748

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 3748

CALCULATION AND COMPILATION OF THE UNSTEADY-LIFT FUNCTIONS FOR A RIGID WING SUBJECTED TO SINUSOIDAL GUSTS AND TO SINUSOIDAL SINKING OSCILLATIONS

By Joseph A. Drischler

Langley Aeronautical Laboratory
Langley Field, Va.

Washington

October 1956

1W

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 3748

CALCULATION AND COMPILATION OF THE UNSTEADY-LIFT FUNCTIONS FOR A RIGID WING SUBJECTED TO SINUSOIDAL GUSTS AND TO SINUSOIDAL SINKING OSCILLATIONS

By Joseph A. Drischler

SUMMARY

The total lift responses of wings to sinusoidal gusts and to sinusoidal vertical oscillations are calculated from the response to gust penetration and to a sudden change in sinking velocity through use of the well-established reciprocal relations for unsteady flow. The cases considered are two-dimensional wings in incompressible, subsonic compressible, sonic, and supersonic flow; elliptical and rectangular wings in incompressible flow; wide rectangular and delta wings in supersonic flow; and delta wings of vanishingly low aspect ratio in incompressible and compressible flow. For most of the cases considered, closed-form expressions are given and the final results are presented in the form of plots of the square of the modulus of the lift coefficient for wings in a sinusoidally oscillating gust and in the form of the real and imaginary parts of the lift component for wings undergoing sinusoidal sinking oscillations. A summary table is presented as a guide to the scope and results of this paper; this table contains the figure and equation numbers for the types of flow and plan forms considered.

INTRODUCTION

Two of the factors required in the harmonic analysis of airplane response to continuous atmospheric turbulence are the unsteady-lift functions associated with sinusoidal vertical oscillations and with sinusoidal gusts. The unsteady-lift functions associated with a rigid wing undergoing sinusoidal translational oscillations have been derived in references 1 to 10 for two-dimensional wings in incompressible, subsonic compressible, sonic, and supersonic flow; for elliptical and rectangular wings in incompressible flow; for wide rectangular and delta wings in supersonic flow; and for very narrow delta wings in incompressible and compressible flow. Calculations of the unsteady-lift functions associated with rigid restrained wings in sinusoidal gusts seem to be nonexistent, with the exception of the work by Jones (ref. 6) for elliptical wings in

incompressible flow and by Garrick (ref. 11) and Sears (ref. 12) for wings in two-dimensional incompressible flow.

The purpose of this report is twofold - to compile the unsteady-lift functions associated with sinusoidal sinking oscillations and to derive the unsteady-lift functions associated with a rigid restrained wing in a sinusoidal gust. These latter functions are derived herein from existing unsteady-lift functions for a wing penetrating a sharp-edged gust by means of the reciprocal relation between the function for a wing in a sinusoidal gust and the function for a wing penetrating a unit sharp-edged gust. The reciprocal relation used was of the same type as that reported in reference 11.

The unsteady-lift functions associated with a rigid restrained wing in a sinusoidal oscillating gust are derived for two-dimensional wings in incompressible, subsonic compressible, sonic, and supersonic flow; for elliptical and rectangular wings in incompressible flow; and for wide rectangular and delta wings in supersonic flow. In addition, the indicial lift function for a wing penetrating a sharp-edged gust and the corresponding oscillatory lift function are derived for a delta wing of vanishing aspect ratio in compressible flow. The functions presented in this paper are total lift functions which include the circulatory and noncirculatory components.

In studies of the airplane response to atmospheric turbulence (see ref. 13, for instance), the unsteady-lift functions for a rigid wing in a sinusoidal gust usually appear in the form of the square of the modulus of lift coefficient, whereas the unsteady-lift functions for a wing undergoing sinusoidal sinking oscillations appear in the form of the individual in-phase and out-of-phase (real and imaginary, respectively) components of lift. Therefore, on this basis, all the results in this paper are presented in the figures in the forms mentioned. An index to the figures and equations or other sources of information for the unsteady-lift functions for the types of flow and wing plan forms considered herein is presented as a table.

SYMBOLS

A aspect ratio

a velocity of sound

$b(x)$ spanwise coordinate of leading edge of wing, measured from root chord, mx

$C(k)$ total lift coefficient for wing oscillating harmonically in pure translational motion, normalized to unity by its steady-state value, $F(k) + iG(k)$

$C(k)_{cir}$	circulatory component of $C(k)$
C_L	steady-state lift coefficient
C_{L_α}	wing lift-curve slope
c	root chord of wing
E	complete elliptic integral of second kind with modulus $\sqrt{1 - \left(\frac{4}{\pi A}\right)^2}$
$F(k)$	in-phase component of $C(k)$ (real part)
$f(k)$	Fresnel integral (see eq. (56))
$f_n(M,\overline{\omega})$	Schwarz function of order n (see eq. (62))
$G(k)$	out-of-phase component of $C(k)$ (imaginary part)
$\dot{h}_o$	amplitude of vertical velocity of wing
$J_n(k)$	Bessel function of first kind
k	reduced-frequency parameter, $\omega c/2V$
$k_1(s)$	lift coefficient for wing experiencing sudden change in sinking speed, normalized to unity by its steady-state lift
$k_1(s)_{cir}$	circulatory component of $k_1(s)$
$k_2(s)$	lift coefficient for wing penetrating sharp-edged gust, normalized to unity by its steady-state lift
$L_{g,ind}$	lift on rigid restrained wing penetrating sharp-edged gust
$L_{g,osc}$	total lift on rigid restrained wing in sinusoidal gust
L_{ind}	lift on rigid wing experiencing sudden change in sinking speed
L_{osc}	total lift on rigid wing oscillating harmonically in pure translational motion
$l(x)$	lift per unit length

M	Mach number
m	tangent of semiapex angle of delta wing
$\Delta p/q$	loading coefficient
q	dynamic pressure, $\frac{\rho V^2}{2}$
S	wing area
s	nondimensional distance traveled, root semichords
t	distance traveled by sound wave, at'
t'	time variable
V	forward velocity
W_o	amplitude of vertical gust velocity
x,y,z	coordinate axes, fixed on wing
$Y_n(k)$	Bessel function of second kind
$\beta = \sqrt{\lvert M^2 - 1 \rvert}$	
$\delta(s)$	unit impulse function or Dirac delta function
ρ	air density
$\varphi(k)$	total lift coefficient for wing immersed in harmonically oscillating gust, normalized to unity by its steady-state value
ω	circular frequency
$\bar{\omega} = 2M^2k/\beta^2$	
$\mathit{1}(s)$	unit jump function

PROCEDURE

Since this paper deals with the lift functions $C(k)$ and $\varphi(k)$, the lift due to sinusoidal sinking oscillations and the lift due to sinusoidal

gusts, respectively, a brief description of the total lift in terms of these functions and the method by which they were derived is in order.

For a rigid wing oscillating harmonically in pure translational motion, the total lift can be expressed as

$$L_{osc} = -qSC_{L_\alpha} \frac{\dot{h}_o e^{iks}}{V} C(k) \tag{1}$$

where k is the reduced-frequency parameter $\omega c/2V$, and $C(k)$ is a complex quantity $F(k) + iG(k)$. The real part of this quantity is associated with the in-phase component of lift and the imaginary part with the out-of-phase component of lift; the total lift functions include both circulatory and noncirculatory effects. For a rigid restrained wing in a sinusoidal gust the total lift can be expressed as

$$L_{g,osc} = -qSC_{L_\alpha} \frac{W_o e^{iks}}{V} \varphi(k) \tag{2}$$

The lift on a rigid wing experiencing a sudden acquisition of vertical velocity $\dot{h}_o$ can be expressed as

$$L_{ind} = -qSC_{L_\alpha} \frac{\dot{h}_o}{V} k_1(s) \tag{3}$$

and for a wing penetrating a sharp-edged gust of vertical velocity W_o, the lift can be expressed as

$$L_{g,ind} = -qSC_{L_\alpha} \frac{W_o}{V} k_2(s) \tag{4}$$

where $k_1(s)$ and $k_2(s)$ are the indicial lift functions for a wing given a sudden change in sinking speed and for a wing penetrating a sharp-edged gust, respectively.

The functions $C(k)$ and $k_1(s)$ are reciprocally related as shown in reference 11 by the following expressions:

$$C(k) \equiv F(k) + iG(k) = 1 + ik \int_0^\infty \left[k_1(s) - 1\right] e^{-iks} ds \tag{5}$$

$$k_1(s) = 1 + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{C(k) - 1}{k} e^{iks} dk \qquad (s > 0) \tag{6}$$

Similarly, $\varphi(k)$ and $k_2(s)$ are reciprocally related as follows:

$$\varphi(k) = 1 + ik\int_0^\infty \left[k_2(s) - 1\right]e^{-iks}ds \tag{7}$$

$$k_2(s) = 1 + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\varphi(k) - 1}{k} e^{iks}dk \qquad (s > 0) \tag{8}$$

In appendix A the functions $k_1(s)$ and $k_2(s)$ are given as obtained from various references for different types of flows and plan forms. The functions $C(k)$ and $\varphi(k)$ were obtained by means of equations (5) and (7), respectively, and are also presented in appendix A. The various types of flows and plan forms for which these functions were derived are discussed more fully in the following section.

PRESENTATION OF RESULTS

The unsteady-lift functions $k_1(s)$, $k_2(s)$, $C(k)$, and $\varphi(k)$ are presented in appendix A and in figures 1 to 20. These functions are given for two-dimensional wings in incompressible flow (figs. 1 and 2), subsonic compressible flow (figs. 7 and 8), sonic flow (figs. 11 and 12), and supersonic flow (figs. 13 and 14); for elliptical and rectangular wings in incompressible flow (figs. 3 to 6); for wide delta and rectangular wings in supersonic flow (figs. 15 to 20); and for delta wings of vanishing aspect ratio in incompressible and compressible flow (figs. 9 and 10).

The $C(k)$ functions, although derived by other authors for all the wings considered herein, were recalculated by means of equation (5) from existing $k_1(s)$ functions. The functions $C(k)$ as derived by use of equation (5) are in agreement with the functions derived by other authors.

The results are given by the equations in appendix A and the figures which contain plots of the modulus squared for the function $\varphi(k)$ $\left(\text{that is, } |\varphi(k)|^2\right)$ and the separated real and imaginary parts of the function $C(k)$ (that is, $F(k)$ and $G(k)$). As an aid to the reader, table I has been prepared as an index to the equation or reference identifying the functions $k_1(s)$, $k_2(s)$, $C(k)$, or $\varphi(k)$, the plan form and type of flow for which these functions were considered, and the figures where the functions $|\varphi(k)|^2$, $F(k)$, and $-G(k)$ are plotted.

CONCLUDING REMARKS

The total lift responses to sinusoidal sinking oscillations $C(k)$, and to sinusoidal gusts $\varphi(k)$, have been calculated through use of the well-established reciprocal relations for unsteady flow for two-dimensional wings in incompressible, subsonic compressible, sonic, and supersonic flow; for elliptical and rectangular wings in incompressible flow; for wide rectangular and delta wings in supersonic flow; and for delta wings of vanishingly low aspect ratio in incompressible and compressible flow. For most of the cases considered, closed-form expressions are given and the final results are presented in the form of plots of the square of the modulus of the lift coefficients for a wing in a sinusoidal gust, and the in-phase and out-of-phase lift components are presented for a wing undergoing sinusoidal sinking oscillations.

Certain gaps still exist in the knowledge of the unsteady-lift problem. For instance, there seems to be little or no information available for the swept wing. For rectangular wings in subsonic flow, and in supersonic flow for which the characteristic Mach lines intersect the side edges of the wing, the unsteady-lift problem remains unsolved, as it is for the delta wing for subsonic compressible and incompressible flow. Information on other wings with subsonic leading edges in supersonic flow is also missing.

Langley Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., June 8, 1956.

APPENDIX A

A COMPILATION OF THE FUNCTIONS $k_1(s)$, $k_2(s)$, $C(k)$, AND $\varphi(k)$

Two-Dimensional Wing in Incompressible Flow

The functions $k_1(s)$ and $k_2(s)$ have been derived by Wagner (ref. 14) and Von Kármán and Sears (ref. 15), respectively. Exponential approximations to these functions have been given by Robert T. Jones (ref. 6) and are now presented, together with the $C(k)$ and $\varphi(k)$ functions as given in references 1 and 11, respectively:

$$k_1(s) \approx 1.0 - 0.165e^{-0.045s} - 0.335e^{-0.300s} + \frac{1}{2}\delta(s) \tag{9}$$

$$k_2(s) \approx 1.0 - 0.236e^{-0.058s} - 0.513e^{-0.364s} - 0.171e^{-2.42s} \tag{10}$$

$$C(k) = C(k)_{cir} + \frac{ik}{2} \tag{11}$$

$$\varphi(k) = \left\{C(k)_{cir}\left[J_0(k) - iJ_1(k)\right] + iJ_1(k)\right\}e^{-ik} \tag{12}$$

where $C(k)$ is defined as the total lift function and $C(k)_{cir}$ represents the circulatory component of the lift and is given in reference 1 as

$$C(k)_{cir} = \frac{-J_1(k) + iY_1(k)}{-\left[J_1(k) + Y_0(k)\right] + i\left[Y_1(k) - J_0(k)\right]} \tag{13}$$

and

$$\left.\begin{aligned} C(k) &\approx \frac{1}{2}(1 + ik) \\ |\varphi(k)|^2 &\approx \frac{1}{2\pi k} \end{aligned}\right\} \quad (k >> 1.0) \tag{14}$$

The multiplier e^{-ik} (eq. 12) is not included in the function $\varphi(k)$ (ref. 11) because in reference 11 the time origin is the instant at which the gust reaches the midchord position of the wing, whereas in this paper the time origin is the instant at which the gust reaches the leading edge of the wing.

Elliptical Wing in Incompressible Flow

For an elliptical wing in incompressible flow, the following equations were derived from references 6, 16, and 17:

$$k_1(s)_{A=0} = 1(s) + \frac{4}{3}\delta(s) \tag{15}$$

$$k_1(s)_{A=3} \approx 1.0 - 0.283e^{-0.540s} + \frac{8\delta(s)}{3C_{L_\alpha}E\left[\sqrt{1 - \left(\frac{4}{3\pi}\right)^2}\right]} \tag{16}$$

$$k_1(s)_{A=6} \approx 1.0 - 0.361e^{-0.381s} + \frac{8\delta(s)}{3C_{L_\alpha}E\left[\sqrt{1 - \left(\frac{4}{6\pi}\right)^2}\right]} \tag{17}$$

where $E \equiv E\left[\sqrt{1 - \left(\frac{4}{\pi A}\right)^2}\right]$ is a complete elliptic integral of the second kind. Also,

$$\left.\begin{aligned} k_2(s)_{A=0} &= s(2 - s) \qquad (0 \leqq s \leqq 1.0) \\ k_2(s)_{A=0} &= 1.0 \qquad (s > 1.0) \end{aligned}\right\} \tag{18}$$

$$k_2(s)_{A=3} \approx 1.0 - 0.679e^{-0.558s} - 0.227e^{-3.20s} \tag{19}$$

$$k_2(s)_{A=6} \approx 1.0 - 0.448e^{-0.290s} - 0.272e^{-0.725s} - 0.193e^{-3.00s} \tag{20}$$

$$C(k)_{A=0} = 1 + \frac{4ik}{3} \tag{21}$$

$$C(k)_{A=3} \approx ik\left(\frac{1}{ik} - \frac{0.283}{0.540 + ik}\right) + \frac{8ik}{3EC_{L_\alpha}} \tag{22}$$

$$C(k)_{A=6} \approx ik\left(\frac{1}{ik} - \frac{0.361}{0.381 + ik}\right) + \frac{8ik}{3EC_{L_\alpha}} \tag{23}$$

$$\varphi(k)_{A=0} = ik\left(-\frac{2}{k^2} + \frac{2 - 2e^{-ik}}{ik^3}\right) \tag{24}$$

$$\varphi(k)_{A=3} \approx ik\left(\frac{1}{ik} - \frac{0.679}{0.558 + ik} - \frac{0.227}{3.20 + ik}\right) \tag{25}$$

$$\varphi(k)_{A=6} \approx ik\left(\frac{1}{ik} - \frac{0.448}{0.290 + ik} - \frac{0.272}{0.725 + ik} - \frac{0.193}{3.00 + ik}\right) \tag{26}$$

Since the functions $C(k)$ and $\varphi(k)$ have been derived from approximate expressions for aspect ratios of 3 and 6, no exact asymptotic expressions could be obtained. However, for the vanishing-aspect-ratio case, it can be shown that

$$\left.\begin{aligned} C(k) &\approx 1 + \frac{4ik}{3} \\ |\varphi(k)|^2 &\approx \frac{1}{k^2} \end{aligned}\right\} \quad (k >> 1.0) \tag{27}$$

(The magnitude of the Dirac delta function in equations (16) and (17) approaches $\frac{8}{6\pi}$ as $A \longrightarrow \infty$ for s in half root chords but reduces to $\frac{1}{2}$ for s in half mean geometric chords.)

Rectangular Wing in Incompressible Flow

The functions $k_1(s)_{cir}$, $k_2(s)$, and $C(k)_{cir}$ for rectangular wings of aspect ratio 4 and 6 have been calculated approximately and are tabulated in reference 7. The noncirculatory components of $k_1(s)$ are shown in reference 18 to be $\frac{0.860\pi}{C_{L_\alpha}}\delta(s)$ and $\frac{0.895\pi}{C_{L_\alpha}}\delta(s)$ for $A = 4$ and $A = 6$, respectively. In order to obtain the function $\varphi(k)$, an exponential approximation was made for the function $k_2(s)$. These expressions are now presented, together with the expression for the vanishing-aspect-ratio case obtained from reference 16 and the corresponding oscillatory functions $\varphi(k)$:

$$k_2(s)_{A=0} = 1(s) \tag{28}$$

$$k_2(s)_{A=4} \approx 1.0 - 0.391e^{-0.285s} - 0.609e^{-1.638s} \tag{29}$$

$$k_2(s)_{A=6} \approx 1.0 - 0.535e^{-0.299s} - 0.465e^{-2.00s} \tag{30}$$

$$\varphi(k)_{A=0} = 1.0 \tag{31}$$

$$\varphi(k)_{A=4} \approx ik\left(\frac{1}{ik} - \frac{0.391}{0.285 + ik} - \frac{0.609}{1.638 + ik}\right) \tag{32}$$

$$\varphi(k)_{A=6} \approx ik\left(\frac{1}{ik} - \frac{0.535}{0.299 + ik} - \frac{0.465}{2.00 + ik}\right) \tag{33}$$

Again, the asymptotic expressions could not be obtained except for the vanishing-aspect-ratio case:

$$\left.\begin{aligned} C(k) &\equiv 1 + 2ik \\ |\varphi(k)|^2 &\equiv 1 \end{aligned}\right\} \qquad (k \gg 1.0) \tag{34}$$

Two-Dimensional Wing in Subsonic Compressible Flow

For the two-dimensional wing in subsonic compressible flow, the following equations have been derived from references 2 and 3:

$$k_1(s)_{M=0.5} \approx 1.0 - 0.352e^{-0.0754s} - 0.261e^{-0.372s} + 0.669e^{-1.890s} \tag{35}$$

$$k_1(s)_{M=0.6} \approx 1.0 - 0.362e^{-0.0646s} - 0.504e^{-0.481s} + 0.714e^{-0.958s} \tag{36}$$

$$k_1(s)_{M=0.7} \approx 1.0 - 0.364e^{-0.0536s} - 0.405e^{-0.357s} - 0.419e^{-0.902s} \tag{37}$$

$$k_2(s)_{M=0.5} \approx 1.0 - 0.390e^{-0.0716s} - 0.407e^{-0.374s} - 0.203e^{-2.165s} \tag{38}$$

$$k_2(s)_{M=0.6} \approx 1.0 - 0.328e^{-0.0545s} - 0.430e^{-0.257s} - 0.242e^{-1.461s} \tag{39}$$

$$k_2(s)_{M=0.7} \approx 1.0 - 0.402e^{-0.0542s} - 0.461e^{-0.3125s} - 0.137e^{-1.474s} \tag{40}$$

$$\varphi(k)_{M=0.5} \approx ik\left(\frac{1}{ik} - \frac{0.390}{0.0716 + ik} - \frac{0.407}{0.374 + ik} - \frac{0.203}{2.165 + ik}\right) \tag{41}$$

$$\varphi(k)_{M=0.6} \approx ik\left(\frac{1}{ik} - \frac{0.328}{0.0545 + ik} - \frac{0.430}{0.257 + ik} - \frac{0.242}{1.461 + ik}\right) \tag{42}$$

$$\varphi(k)_{M=0.7} \approx ik\left(\frac{1}{ik} - \frac{0.402}{0.0542 + ik} - \frac{0.461}{0.3125 + ik} - \frac{0.137}{1.474 + ik}\right) \tag{43}$$

The function $C(k)$ has been calculated from the coefficients compiled in table I of reference 3.

The asymptotic expressions can be shown to be

$$\left.\begin{aligned} C(k) &\approx \frac{2\beta}{\pi M}\left(1 + i\,\frac{1 - M}{2kM}\right) \\ |\varphi(k)|^2 &\approx \frac{\beta^2}{\pi^2 M k^2} \end{aligned}\right\} \quad (k \gg 1.0) \tag{44}$$

provided it is assumed that the functions $k_1(s)$ and $k_2(s)$ and their first derivatives are continuous. (See appendix B.)

As shown in appendix B, the determination of the asymptotic behavior is dependent not only on the continuity of the function and its derivatives, but also on the value of the function and its derivatives at $s = 0$. Therefore, although the k_1 and k_2 functions have been determined numerically for $s > \frac{2M}{1 + M}$, the known exact expression for $s < \frac{2M}{1 + M}$ as given in reference 19 may be utilized to obtain the asymptotic expressions for $C(k)$ and $\varphi(k)$, provided it is assumed that the functions $k_1(s)$ and $k_2(s)$ and their first derivatives are continuous.

Vanishing-Aspect-Ratio Delta Wing in Incompressible and Compressible Flow

For a delta wing of vanishing aspect ratio in incompressible flow, the indicial lift functions have been obtained from reference 16 and are

$$k_1(s) = 1(s) + \frac{2}{3}\,\delta(s) \tag{45}$$

$$\left.\begin{aligned} k_2(s) &= \frac{s^2}{4} && (0 \leqq s \leqq 2) \\ k_2(s) &= 1.0 && (s > 2) \end{aligned}\right\} \tag{46}$$

The corresponding oscillatory functions are

$$C(k) = 1.0 + \frac{2}{3} ik \tag{47}$$

$$\varphi(k) = \frac{ie^{-2ik}}{k} - \frac{1}{2k^2}\left(1 - e^{-2ik}\right) \tag{48}$$

and

$$\left.\begin{aligned} C(k) &\approx 1.0 + \frac{2}{3} ik \\ \left|\varphi(k)\right|^2 &\approx \frac{1}{k^2} \end{aligned}\right\} \qquad (k \gg 1.0) \tag{49}$$

The k_1 function for the vanishing-aspect-ratio delta wing in compressible flow has been presented in reference 19. However, because of the difficulty of obtaining the oscillatory lift function $C(k)$ from this function, the results derived in reference 10 for the oscillatory lift function for the delta wing are presented instead. The k_2 function has been derived in appendix C of this report and the corresponding $\varphi(k)$ function was subsequently derived. The functions $\left|\varphi(k)\right|^2$ and $C(k)$ are presented in figures 9 and 10 for $Mm = M\frac{A}{4} = 0.1$ where m is the tangent of the semiapex angle of the delta wing. The asymptotic behavior of $\left|\varphi(k)\right|^2$ for $Mm = 0.1$ can be obtained if it is assumed that the $k_2(s)$ function and its first two derivatives are continuous. Based on the analysis obtained in appendix B and the k_2 function presented in appendix C, it can be shown that

$$\left|\varphi(k)\right|^2 \approx \frac{3.37}{k^4} \qquad (k \gg 1.0) \tag{50}$$

The asymptotic expression for the function $C(k)$, also based on the assumptions stated previously, can be shown to be

$$C(k) \approx \frac{2}{\pi mM}\left[1 + \frac{i(1 - 2mM)}{mMk}\right] \qquad (k \gg 1.0) \tag{51}$$

Two-Dimensional Wing in Sonic Flow

The functions $C_{L_\alpha}k_1(s)$ and $C_{L_\alpha}k_2(s)$ for a wing in two-dimensional sonic flow were obtained from the functions presented in reference 20 for a wing in two-dimensional supersonic flow by taking the limit as $M \longrightarrow 1.0$. The resulting expressions are

$$\left.\begin{aligned} C_{L_\alpha}k_1(s) &= 4 && (0 \leqq s \leqq 1.0) \\ C_{L_\alpha}k_1(s) &= \frac{4}{\pi}\left[2\sqrt{s - 1} + \cos^{-1}\left(\frac{s - 2}{s}\right)\right] && (s > 1.0) \end{aligned}\right\} \tag{52}$$

$$\left.\begin{aligned} C_{L_\alpha}k_2(s) &= 2s && (0 \leqq s \leqq 1.0) \\ C_{L_\alpha}k_2(s) &= \frac{2}{\pi}\, s \cos^{-1}\left(\frac{s - 2}{s}\right) + \frac{4}{\pi}\sqrt{s - 1} && (s > 1.0) \end{aligned}\right\} \tag{53}$$

The function $C_{L_\alpha}C(k)$ was obtained from reference 4 and $C_{L_\alpha}\varphi(k)$ was derived by the use of equation (7). These expressions are

$$C_{L_\alpha}C(k) = \frac{4(1 - i)}{\sqrt{2\pi k}}\, e^{-ik} + 4(1 + i)f(k) \tag{54}$$

$$C_{L_\alpha}\varphi(k) = \frac{2(1 - i)}{k}\, f(k) \tag{55}$$

where the Fresnel integral $f(k)$ is defined by

$$f(k) = \int_0^k \frac{e^{-ix}}{\sqrt{2\pi x}}\, dx \tag{56}$$

The unnormalized functions have been presented, since the theoretical value of C_{L_α} for two-dimensional sonic flow is infinite.

Two-Dimensional Wing in Supersonic Flow

For a two-dimensional wing in supersonic flow, the following equations have been derived from references 20 and 5:

$$\left.\begin{aligned}
k_1(s) &= \frac{\beta}{M} && \left(0 \leqq s \leqq \frac{2M}{1 + M}\right) \\
k_1(s) &= \frac{\beta}{\pi}\left[\frac{1}{M}\left(\frac{\pi}{2} + \sin^{-1}\frac{2M - Ms}{s}\right) + \frac{1}{\beta}\cos^{-1}\frac{2M^2 + s - sM^2}{2M} + \right. \\
&\qquad \left.\frac{1}{M}\sqrt{\frac{s^2}{4M^2} - \left(1 - \frac{s}{2}\right)^2}\right] && \left(\frac{2M}{M + 1} < s \leqq \frac{2M}{M - 1}\right) \\
k_1(s) &= 1.0 && \left(s > \frac{2M}{M - 1}\right)
\end{aligned}\right\} \tag{57}$$

$$\left.\begin{aligned}
k_2(s) &= \frac{\beta}{2M}\, s && \left(0 \leqq s \leqq \frac{2M}{M + 1}\right) \\
k_2(s) &= \frac{\beta}{2\pi M}\, s\left(\frac{\pi}{2} + \sin^{-1}\frac{2M - Ms}{s}\right) + \frac{1}{\pi}\cos^{-1}\frac{2M^2 + s - sM^2}{2M} \\
& && \left(\frac{2M}{M + 1} < s \leqq \frac{2M}{M - 1}\right) \\
k_2(s) &= 1.0 && \left(s > \frac{2M}{M - 1}\right)
\end{aligned}\right\} \tag{58}$$

$$C(k) = f_0(M,\bar{\omega}) + 2ik\left[f_0(M,\bar{\omega}) - f_1(M,\bar{\omega})\right] \tag{59}$$

$$\varphi(k) = f_0(M,\bar{\omega}) \tag{60}$$

where

$$\bar{\omega} = \frac{2M^2 k}{\beta^2} \tag{61}$$

and f_0 and f_1 are the Schwarz functions of order 0 and 1, the Schwarz function of order n being defined as

$$f_n(M,\bar{\omega}) = \int_0^1 x^n e^{-i\bar{\omega}x} J_0\left(\frac{\bar{\omega}x}{M}\right)dx \tag{62}$$

The functions $f_n(M,\bar{\omega})$ are tabulated for $M > 1.0$ in reference 21, for example. The asymptotic expressions are

$$\left.\begin{aligned} C(k) &\approx \frac{\beta}{M} \\ \left|\varphi(k)\right|^2 &\approx \frac{\beta^2}{4M^2k^2} \end{aligned}\right\} \qquad (k \gg 1.0) \tag{63}$$

Wide Delta Wing in Supersonic Flow

For a wide delta wing in supersonic flow, the following equations have been derived from references 22 and 8:

$$\left.\begin{aligned} k_1(s) &= \frac{\beta}{M}\left(1 + \frac{s^2}{8M^2}\right) && \left(0 < s \leqq \frac{2M}{M + 1}\right) \\ k_1(s) &= \frac{\beta}{\pi}\left[\left(\frac{1}{M} + \frac{s^2}{8M^3}\right)\left(\frac{\pi}{2} + \sin^{-1}\frac{2M - Ms}{s}\right) + \frac{1}{\beta}\cos^{-1}\frac{2M^2 - sM^2 + s}{2M} + \right. \\ &\qquad \left. \frac{1}{M}\left(\frac{3}{2} - \frac{s}{4}\right)\sqrt{\frac{s^2}{4M^2} - \left(1 - \frac{s}{2}\right)^2}\right] && \left(\frac{2M}{M + 1} < s \leqq \frac{2M}{M - 1}\right) \\ k_1(s) &= 1.0 && \left(s > \frac{2M}{M - 1}\right) \end{aligned}\right\} \tag{64}$$

$$\left.\begin{aligned} k_2(s) &= \frac{\beta}{4M}s^2 && \left(0 \leqq s \leqq \frac{2M}{M + 1}\right) \\ k_2(s) &= \frac{1}{\pi}\cos^{-1}\frac{2M^2 - sM^2 + s}{2M} + \frac{\beta}{4\pi M}s^2\cos^{-1}\frac{Ms - 2M}{s} - \\ &\qquad \frac{\beta}{\pi}\left(\frac{s}{2} - 1\right)^2\cosh^{-1}\frac{s}{M|s - 2|} && \left(\frac{2M}{M + 1} < s \leqq \frac{2M}{M - 1}\right) \\ k_2(s) &= 1.0 && \left(s > \frac{2M}{M - 1}\right) \end{aligned}\right\} \tag{65}$$

$$C(k) = 2\left\{f_0(M,\overline{\omega}) - f_1(M,\overline{\omega}) + ik\left[f_0(M,\overline{\omega}) - 2f_1(M,\overline{\omega}) + f_2(M,\overline{\omega})\right]\right\} \tag{66}$$

$$\varphi(k) = \frac{i}{k}\left[e^{-2ik}f_0\left(\frac{1}{M},\frac{\overline{\omega}}{M^2}\right) - f_0(M,\overline{\omega})\right] \tag{67}$$

and

$$\left.\begin{aligned} C(k) &\approx \frac{\beta}{M} \\ \left|\varphi(k)\right|^2 &\approx \frac{\beta^2\left(M^2 + 1\right)}{4M^2k^4} \end{aligned}\right\} \qquad (k \gg 1.0) \tag{68}$$

The function $f_0\left(\frac{1}{M},\frac{\varpi}{M^2}\right)$ was calculated by numerical integration of equation (62), inasmuch as no tabulations were available of the Schwarz function for $M < 1$.

Wide Rectangular Wing in Supersonic Flow

For a wide rectangular wing in supersonic flow, the following equations have been derived from references 23 and 9:

$$\left.\begin{aligned} &\left(\frac{4}{\beta} - \frac{2}{\beta^2 A}\right)k_1(s) = \frac{4}{M} - \frac{2}{M^2 A}\left(s - \frac{s^2}{4}\right) \qquad \left(0 \leqq s \leqq \frac{2M}{M + 1}\right) \\ &\left(\frac{4}{\beta} - \frac{2}{\beta^2 A}\right)k_1(s) = \frac{4}{\pi}\left[\frac{1}{M}\left(\frac{\pi}{2} + \sin^{-1}\frac{2M - Ms}{s}\right) + \frac{1}{\beta}\cos^{-1}\frac{2M^2 + s - sM^2}{2M} + \right. \\ &\qquad \left. \frac{1}{M}\sqrt{\frac{s^2}{4M^2} - \left(1 - \frac{s}{2}\right)^2}\right] - \frac{2}{\beta^2 A}\left[\frac{M - 1}{2M} + \frac{\beta^2}{2M^2}s - \frac{\beta^2(M - 1)}{8M^3}s^2\right] \\ &\qquad \left(\frac{2M}{M + 1} < s \leqq \frac{2M}{M - 1}\right) \\ &\left(\frac{4}{\beta} - \frac{2}{\beta^2 A}\right)k_1(s) = \frac{4}{\beta} - \frac{2}{\beta^2 A} \qquad \left(s > \frac{2M}{M - 1}\right) \end{aligned}\right\} \tag{69}$$

$$\left(\frac{4}{\beta} - \frac{2}{\beta^2 A}\right)k_2(s) = \frac{2s}{M} - \frac{s^2}{2M^2 A} \qquad \left(0 \leqq s \leqq \frac{2M}{M + 1}\right) \tag{70a}$$

$$\begin{aligned} \left(\frac{4}{\beta} - \frac{2}{\beta^2 A}\right)k_2(s) = {} & \frac{2s}{\pi M}\left[\frac{\pi}{2} + \sin^{-1}\left(\frac{2M - Ms}{s}\right)\right] + \frac{4}{\pi\beta}\cos^{-1}\left(\frac{2M^2 + s - sM^2}{2M}\right) - \\ & \frac{2}{\beta^2 A}\left(-\frac{M - 1}{2} + \frac{\beta^2}{2M}s - \frac{(M - 1)\beta^2 s^2}{8M^2}\right) \\ & \left(\frac{2M}{M + 1} < s \leqq \frac{2M}{M - 1}\right) \end{aligned} \tag{70b}$$

$$\left(\frac{4}{\beta} - \frac{2}{\beta^2 A}\right)k_2(s) = \frac{4}{\beta} - \frac{2}{\beta^2 A} \qquad \left(s > \frac{2M}{M - 1}\right) \tag{70c}$$

$$\left(\frac{4}{\beta} - \frac{2}{\beta^2 A}\right)C(k) = \frac{8}{\beta}\left\{\frac{1}{2} f_0(M,\bar{\omega}) + ik\left[f_0(M,\bar{\omega}) - f_1(M,\bar{\omega})\right]\right\} + \frac{1}{M^2k^2A}\left[2ik - 1 + e^{-i\bar{\omega}}\left(\cos\frac{\bar{\omega}}{M} + \frac{i}{M}\sin\frac{\bar{\omega}}{M}\right)\right] \tag{71}$$

$$\left(\frac{4}{\beta} - \frac{2}{\beta^2 A}\right)\varphi(k) = \frac{4}{\beta} f_0(M,\bar{\omega}) + \frac{1}{M^2k^2A}\left[1 - e^{-i\bar{\omega}}\left(\cos\frac{\bar{\omega}}{M} + iM\sin\frac{\bar{\omega}}{M}\right)\right] \tag{72}$$

and

$$\left.\begin{aligned} C(k) &\approx \frac{2 + \frac{i}{MAk}}{M\left(\frac{2}{\beta} - \frac{1}{\beta^2 A}\right)} \\ \left|\varphi(k)\right|^2 &\approx \frac{\frac{4}{k^2M^2}}{\left(\frac{4}{\beta} - \frac{2}{\beta^2 A}\right)^2} \end{aligned}\right\} \qquad (k \gg 1.0) \tag{73}$$

APPENDIX B

ASYMPTOTIC BEHAVIOR OF OSCILLATORY LIFT COEFFICIENTS AS DETERMINED FROM INDICIAL LIFT FUNCTIONS

In the following analysis it is shown that the asymptotic behavior of the oscillating lift coefficients can be determined from the initial behavior and the discontinuities in the derivatives of the indicial lift functions.

If $\psi(k)$ represents either $C(k)$ or $\varphi(k)$ and if $K(s)$ represents the corresponding function $k_1(s)$ or $k_2(s)$, then the reciprocal relation is

$$\psi(k) = ik \int_0^\infty K(s)e^{-iks}ds \tag{74}$$

If $K(s)$ has the following properties:

(1) $K(s)$ and all its derivatives up to and including $K^{(N)}(s)$ are continuous

(2) There is a sequence of points s_j (where $j = 1, 2, \ldots$) at which one or more of the derivatives $K^{(n)}(s)$ (where $n > N$) has a finite discontinuity

then by $N + 1$ successive integrations by parts, equation (74) can be expressed as

$$\psi(k) = \frac{ik}{(ik)^{N+1}} \int_0^\infty K^{(N+1)}(s)e^{-iks}ds + \sum_{n=0}^{N} \frac{K^{(n)}(0)}{(ik)^n} \tag{75}$$

For further integrations by parts, the discontinuities contribute terms of the form

$$\frac{1}{(ik)^n}\left[K^{(n)}\left(s_j+\right) - K^{(n)}\left(s_j-\right)\right]e^{-iks_j} \tag{76}$$

so that

$$\psi(k) = \sum_{n=0}^{N} \frac{K^{(n)}(0)}{(ik)^n} + \sum_{n=N+1}^{\infty} \frac{K^{(n)}(0) + \sum_{j=1}^{\infty}\left[K^{(n)}\left(s_j^+\right) - K^{(n)}\left(s_j^-\right)\right]e^{-iks_j}}{(ik)^n} \tag{77}$$

APPENDIX C

INDICIAL LIFT FUNCTION FOR DELTA WING OF VANISHING ASPECT RATIO IN COMPRESSIBLE FLOW PENETRATING A SHARP-EDGED GUST

The differential equation which governs the flow field for a very narrow delta wing has been shown in reference 19 to be

$$\phi_{yy} + \phi_{zz} = \phi_{tt} \tag{78}$$

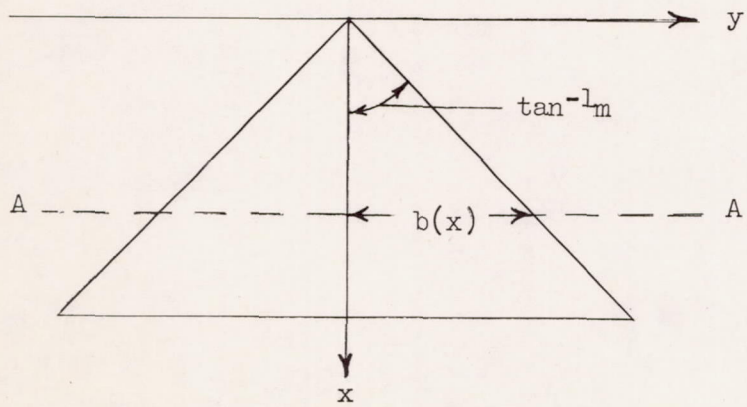
where the x, y, and z axes are fixed on the wing and the time variable is $t' = t/a$. The loading coefficient can be shown to be

$$\frac{\Delta p}{q} = \frac{4}{V}\left(\frac{1}{M}\phi_t + \phi_x\right) \tag{79}$$

and the boundary conditions associated with this wing penetrating a sharp-edged gust are

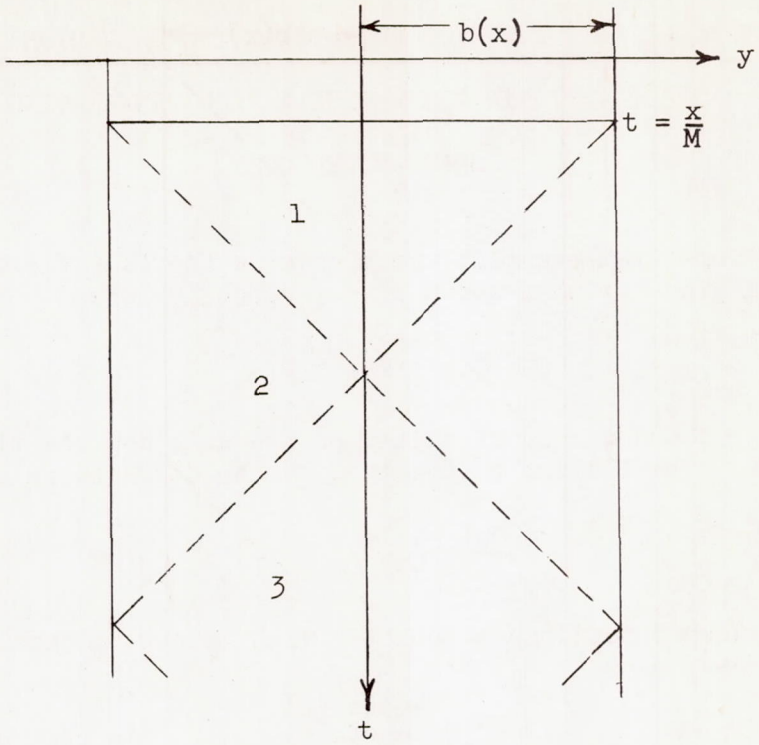
$$\left.\begin{aligned} W_u &= -W_o \qquad (x < Mt) \\ W_u &= 0 \qquad (x > Mt) \end{aligned}\right\} \tag{80}$$

where W_u is the induced vertical velocity on the wing. As in reference 19 the boundary conditions need only be satisfied over a span strip of the wing, since it was assumed that the velocity gradients in the y-, z-, and t-directions are independent of the gradient in the x-direction. See following sketch.



Sketch (a)

Sketch (b) then represents the problem under consideration:

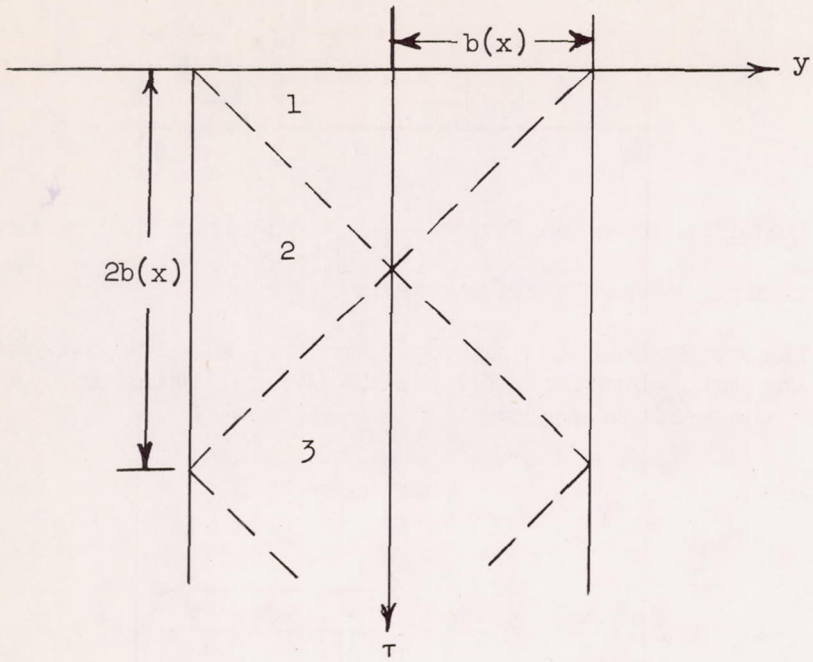


Sketch (b)

If the axes are transformed by the relation

$$\tau = t - \frac{x}{M} \tag{81}$$

then the problem can be represented by the following sketch:



Sketch (c)

In the lifting-surface analog, this corresponds to the problem of finding the velocity potential over a flat rectangular wing of low aspect ratio situated in a free stream at a Mach number equal to $\sqrt{2}$. Solutions to this problem are given in reference 19 and are now presented; the subscripts represent the region under consideration.

$$\left(\phi_\tau\right)_1 = -W_o \tag{82}$$

$$\left(\phi_\tau\right)_2 = -\frac{2W_o}{\pi}\tan^{-1}\sqrt{\frac{b(x) - |y|}{\tau - b(x) + |y|}} \tag{83}$$

$$\left(\phi_\tau\right)_3 = -\frac{2W_o}{\pi}\left[\tan^{-1}\sqrt{\frac{b(x) + y}{\tau - b(x) - y}} + \tan^{-1}\sqrt{\frac{b(x) - y}{\tau - b(x) + y}} - \frac{\pi}{2}\right] \tag{84}$$

and, for large values of τ (that is, for the higher number regions in sketch (c)),

$$\phi_\tau \approx -W_o f\left[\frac{\tau}{b(x)}\right]\sqrt{1 - \left[\frac{y}{b(x)}\right]^2} \tag{85}$$

where an integral equation for $f\left[\frac{\tau}{b(x)}\right]$ together with a tabulation of this function is given in reference 19.

Solving equations (82) to (85) for the velocity potential ϕ and substituting into equation (79) yields the following loading coefficients for the various regions:

$$\left(\frac{\Delta p}{q}\right)_1 = 0 \tag{86}$$

$$\left(\frac{\Delta p}{q}\right)_2 = \frac{8W_o m}{\pi V}\sqrt{\frac{\tau - b(x) + |y|}{b(x) - |y|}} \tag{87}$$

$$\left(\frac{\Delta p}{q}\right)_3 = \frac{8W_o m}{\pi V}\left[\sqrt{\frac{\tau - b(x) - y}{b(x) + y}} + \sqrt{\frac{\tau - b(x) + y}{b(x) - y}}\right] \tag{88}$$

$$\left(\frac{\Delta p}{q}\right)_{\tau>>2b(x)} = \frac{4W_o m}{V}\left\{\frac{1}{\sqrt{1 - \left[\frac{y}{b(x)}\right]^2}}\int_0^{\tau/b(x)} f(\eta)d\eta - \frac{\tau}{b(x)} f\left[\frac{\tau}{b(x)}\right]\sqrt{1 - \left[\frac{y}{b(x)}\right]^2}\right\} \tag{89}$$

The corresponding lift coefficient per unit length is

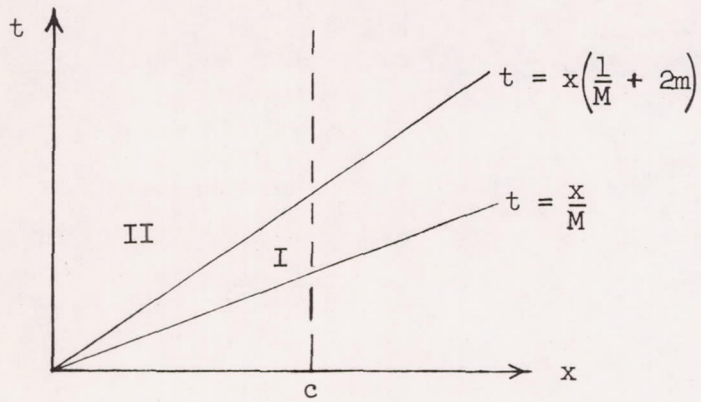
$$\left.\begin{aligned}
l\left[\frac{\tau}{b(x)}\right] &= \frac{1}{b(x)}\int_0^{b(x)-\tau}\left(\frac{\Delta p}{q}\right)_1 dy + \frac{1}{b(x)}\int_{b(x)-\tau}^{b(x)}\left(\frac{\Delta p}{q}\right)_2 dy && (\tau < b(x)) \\
l\left[\frac{\tau}{b(x)}\right] &= \frac{1}{b(x)}\int_0^{b(x)-\tau}\left(\frac{\Delta p}{q}\right)_3 dy + \frac{1}{b(x)}\int_{b(x)-\tau}^{b(x)}\left(\frac{\Delta p}{q}\right)_2 dy && (b(x) < \tau < 2b(x)) \\
l\left[\frac{\tau}{b(x)}\right] &= \frac{1}{2b(x)}\int_{-b(x)}^{b(x)}\left(\frac{\Delta p}{q}\right)_{\tau>>2b(x)} dy && (\tau >> 2b(x))
\end{aligned}\right\} \tag{90}$$

Evaluation of equation (90) yields

$$l\left(\frac{\tau}{b(x)}\right)_{I} = \frac{4W_{o}m}{V}\frac{\tau}{b(x)} \qquad (\tau < 2b(x)) \tag{91a}$$

$$l\left(\frac{\tau}{b(x)}\right)_{II} = \frac{2\pi W_{o}m}{V}\left\{\int_{0}^{\tau/b(x)} f(\eta)d\eta - \frac{\tau}{2b(x)} f\left[\frac{\tau}{b(x)}\right]\right\} \qquad (\tau \gg 2b(x)) \tag{91b}$$

where the subscripts indicate the regions in the xt-plane where $l(x)$ applies. See following sketch.



Sketch (d)

A plot of equations (91a) and (91b) is presented in figure 21 where it can be seen that an error of about 3 percent exists at $\frac{\tau}{b(x)} = 2$ because of the assumption made in equation (85).

The lift coefficient for the wing is obtained from the equation

$$C_L = \frac{1}{c}\int_{0}^{c} \frac{2b(x)}{mc}\, l\left[\frac{\tau}{b(x)}\right]dx \tag{92}$$

Substituting the appropriate expression from equations (91a) and (91b) into equation (92) gives

$$\left.\begin{aligned}
C_L(t) &= \frac{2}{mc^2}\left\{\int_{Mt/1+2Mm}^{Mt} b(x)l\left[\frac{\tau}{b(x)}\right]_I dx + \int_0^{Mt/1+2Mm} b(x)l\left[\frac{\tau}{b(x)}\right]_{II} dx\right\} \\
&\qquad\qquad \left(t < \frac{c}{M}\right) \\
C_L(t) &= \frac{2}{mc^2}\left\{\int_{Mt/1+2Mm}^{c} b(x)l\left[\frac{\tau}{b(x)}\right]_I dx + \int_0^{Mt/1+2Mm} b(x)l\left[\frac{\tau}{b(x)}\right]_{II} dx\right\} \\
&\qquad\qquad \left(\frac{c}{M} < t < c\left(\frac{1 + 2Mm}{M}\right)\right) \\
C_L(t) &= \frac{2}{mc^2}\int_0^c b(x)l\left[\frac{\tau}{b(x)}\right]_{II} dx \qquad \left(t > c\left(\frac{1 + 2Mm}{M}\right)\right)
\end{aligned}\right\} \quad (93)$$

Evaluating equation (93) and making the substitution

$$s = \frac{2Vt'}{c} = \frac{2Mt}{c}$$

gives

$$\left.\begin{aligned}
k_2(s) &= \frac{C_L}{\frac{\pi A}{2}\frac{W_o}{V}} = \frac{s^2}{4}\left[\frac{8Mm}{\pi(1 + 2Mm)^2} + \frac{1}{(1 + 2Mm)^2}\int_0^2 f(\eta)d\eta + \int_2^\infty \frac{f(\eta)d\eta}{(1 + Mm\eta)^3}\right] \\
&\qquad\qquad (s \leqq 2) \\
k_2(s) &= -\frac{2}{\pi Mm} + \frac{2}{\pi Mm}s + \left[k_2(2) - \frac{2}{\pi Mm}\right]\frac{s^2}{4} \qquad (2 < s \leqq 2 + 4Mm) \\
k_2(s) &= \int_0^{\frac{s-2}{2Mm}} f(\eta)d\eta + \frac{s^2}{4}\int_{\frac{s-2}{2Mm}}^\infty \frac{f(\eta)d\eta}{(1 + Mm\eta)^3} \qquad (s > 2 + 4Mm)
\end{aligned}\right\} \quad (94)$$

Equation (94) has been evaluated numerically for $Mm = \frac{MA}{4} = 0.1$ and the results are presented in figure 22, together with the results for $M = 0$ as given in reference 17. It might be of interest to note that for $M = 0$ equation (94) reduced to the equation given in reference 17.

REFERENCES

1. Theodorsen, Theodore: General Theory of Aerodynamic Instability and the Mechanism of Flutter. NACA Rep. 496, 1935.

2. Mazelsky, Bernard, and Drischler, Joseph A.: Numerical Determination of Indicial Lift and Moment Functions for a Two-Dimensional Sinking and Pitching Airfoil at Mach Numbers 0.5 and 0.6. NACA TN 2739, 1952.

3. Mazelsky, Bernard: Numerical Determination of Indicial Lift of a Two-Dimensional Sinking Airfoil at Subsonic Mach Numbers From Oscillatory Lift Coefficients With Calculations for Mach Number 0.7. NACA TN 2562, 1951.

4. Nelson, Herbert C., and Berman, Julian H.: Calculations on the Forces and Moments for an Oscillating Wing-Aileron Combination in Two-Dimensional Potential Flow at Sonic Speed. NACA Rep. 1128, 1953. (Supersedes NACA TN 2590.)

5. Garrick, I. E., and Rubinow, S. I.: Flutter and Oscillating Air-Force Calculations for an Airfoil in a Two-Dimensional Supersonic Flow. NACA Rep. 846, 1946. (Supersedes NACA TN 1158.)

6. Jones, Robert T.: The Unsteady Lift of a Wing of Finite Aspect Ratio. NACA Rep. 681, 1940.

7. Jones, W. Prichard: Aerodynamic Forces on Wings in Non-Uniform Motion. R. & M. No. 2117, British A.R.C., Aug. 1945.

8. Miles, John W.: On Harmonic Motion of Wide Delta Airfoils at Supersonic Speeds. NAVORD Rep. 1234 (NOTS 294), U. S. Naval Ord. Test Station (Inyokern, Calif.), June 13, 1950.

9. Miles, John W.: The Oscillating Rectangular Airfoil at Supersonic Speeds. NAVORD Rep. 1170 (NOTS 226), U. S. Naval Ord. Test Station (Inyokern, Calif.), July 21, 1949.

10. Merbt, H., and Landahl, M.: The Oscillating Wing of Low Aspect Ratio - Results and Tables of Auxiliary Functions. KTH AERO TN 31, Roy. Inst. of Tech., Div. of Aero. (Stockholm, Sweden), Jan. 15, 1954.

11. Garrick, I. E.: On Some Fourier Transforms in the Theory of Non-Stationary Flows. Proc. Fifth Int. Cong. Appl. Mech. (Cambridge, Mass., 1938), John Wiley & Sons, Inc., 1939, pp. 590-593.

12. Sears, William R.: Some Aspects of Non-Stationary Airfoil Theory and Its Practical Application. Jour. Aero. Sci., vol. 8, no. 3, Jan. 1941, pp. 104-108.

13. Houbolt, John C., and Kordes, Eldon E.: Structural Response to Discrete and Continuous Gusts of an Airplane Having Wing-Bending Flexibility and a Correlation of Calculated and Flight Results. NACA Rep. 1181, 1954. (Supersedes NACA TN 3006; also contains essential material from TN 2763 and TN 2897.)

14. Wagner, Herbert: Über die Entstehung des dynamischen Auftriebes von Tragflügeln. Z.a.M.M., Bd. 5, Heft 1, Feb. 1925, pp. 17-35.

15. Von Kármán, Th., and Sears, W. R.: Airfoil Theory for Non-Uniform Motion. Jour. Aero. Sci., vol. 5, no. 10, Aug. 1938, pp. 379-390.

16. Drischler, Joseph A.: Approximate Indicial Lift Functions for Several Wings of Finite Span in Incompressible Flow As Obtained From Oscillatory Lift Coefficients. NACA TN 3639, 1956.

17. Jones, W. Prichard: Theoretical Determination of the Aerodynamic Inertias of an Elliptic Plate in Still Air. R. & M. No. 1953, British A.R.C., Mar. 17, 1942.

18. Jones, W. Prichard: Calculation of Additional Mass and Inertia Coefficients for Rectangular Plates in Still Air. R. & M. No. 1947, British A.R.C., Nov. 8, 1941.

19. Lomax, Harvard, Heaslet, Max. A., Fuller, Franklyn B., and Sluder, Loma: Two- and Three-Dimensional Unsteady Lift Problems in High-Speed Flight. NACA Rep. 1077, 1952. (Supersedes NACA TN 2403 by Lomax, Heaslet, Sluder and TN 2387 by Lomax, Heaslet, Fuller; contains material from TN 2256 by Lomax, Heaslet, Fuller.)

20. Heaslet, Max. A., and Lomax, Harvard: Two-Dimensional Unsteady Lift Problems in Supersonic Flight. NACA Rep. 945, 1949. (Supersedes NACA TN 1621.)

21. Huckel, Vera: Tabulation of the f_λ Functions Which Occur in the Aerodynamic Theory of Oscillating Wings in Supersonic Flow. NACA TN 3606, 1956.

22. Miles, John W.: Transient Loading of Wide Delta Airfoils at Supersonic Speeds. Jour. Aero. Sci., vol. 18, no. 8, Aug. 1951, pp. 543-554.

23. Miles, John W.: Transient Loading of Supersonic Rectangular Airfoils. Jour. Aero. Sci., vol. 17, no. 10, Oct. 1950, pp. 647-652.

TABLE I.- INDEX TO FIGURES, EQUATIONS, AND REFERENCES

Type of flow	Wing plan form	C(k)	φ(k)	C(k) ≡ F(k) + iG(k) in figure -	$\lvert\varphi(k)\rvert^2$ in figure -
Incompressible	Two dimensional	Eq. (11)	Eq. (12)	1	2
Incompressible	Elliptical A = 0 A = 3 A = 6 A = ∞	Eq. (21) Eq. (22) Eq. (23) Eq. (11)	Eq. (24) Eq. (25) Eq. (26) Eq. (12)	$(a)_3$	4
Incompressible	Rectangular A = 0 A = 4 A = 6 A = ∞	Ref. 16 Ref. 7 Eq. (11)	Eq. (31) Eq. (32) Eq. (33) Eq. (12)	$(a)_5$	6
Subsonic compressible: M = 0 M = 0.5 M = 0.6 M = 0.7	Two dimensional	Eq. (11) Ref. 3	Eq. (12) Eq. (41) Eq. (42) Eq. (43)	7	8
Compressible and incompressible: M = 0 $\frac{MA}{4} = 0.1$	Vanishingly low aspect-ratio delta	Eq. (47) Ref. 10	Eq. (48) (b)	9	10
Sonic	Two dimensional	Eq. (54)	Eq. (55)	11	12
Supersonic: $M = \frac{10}{9}, \frac{10}{7}, 2$, and $\frac{10}{3}$	Two dimensional	Eq. (59)	Eq. (60)	13	14
Supersonic: $M = \frac{10}{9}, \frac{10}{7}, 2$, and $\frac{10}{3}$	Wide delta	Eq. (66)	Eq. (67)	15	16
Supersonic: $M = \frac{10}{7}$, 2, and $\frac{10}{3}$ $M = \frac{10}{7}$ M = 2 $M = \frac{10}{3}$	Wide rectangular A = 1 A = 2 A = 4 A = 1, 2, 4, and ∞	Eq. (71)	Eq. (72)	17(a) 17(b) 17(c) 19(a) 19(b) 19(c)	18(a) 18(b) 18(c) 20(a) 20(b) 20(c)

[a]Only circulatory component of lift plotted.

[b]Calculated numerically from $k_2(s)$ function given in appendix C.

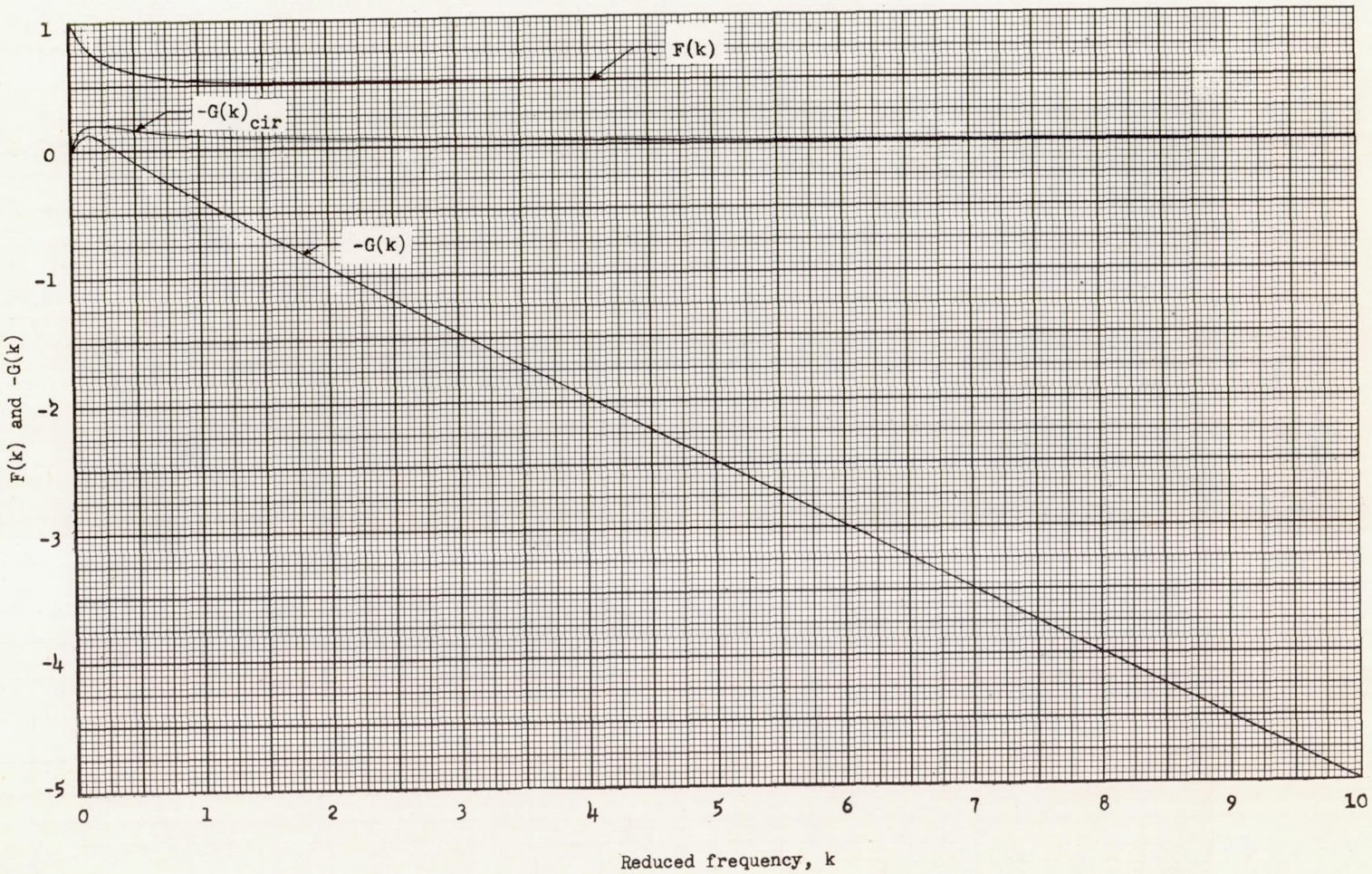


Figure 1.- The functions C(k) for a wing in two-dimensional incompressible flow.

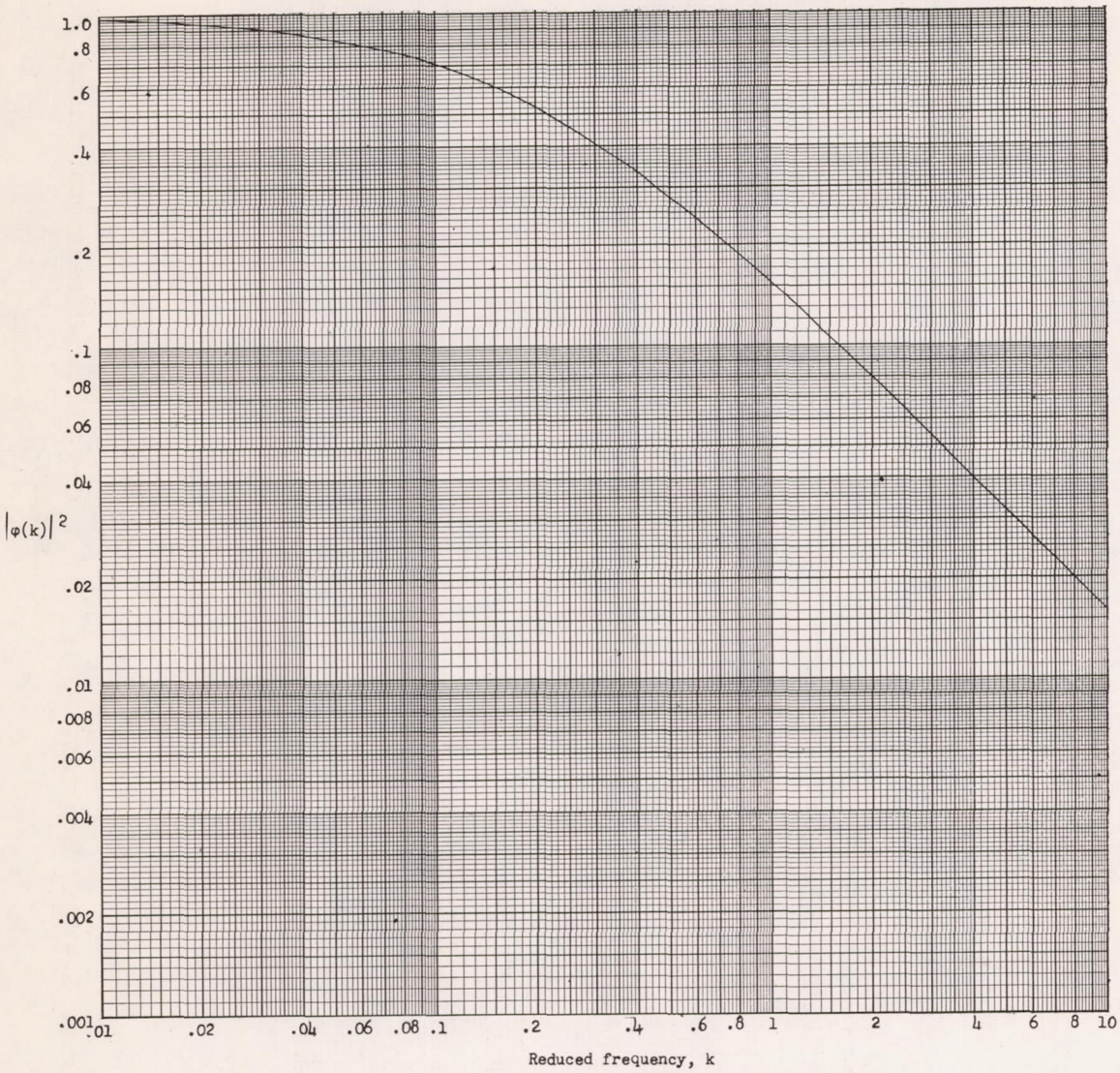


Figure 2.- The function $|\varphi(k)|^2$ for a wing in two-dimensional incompressible flow.

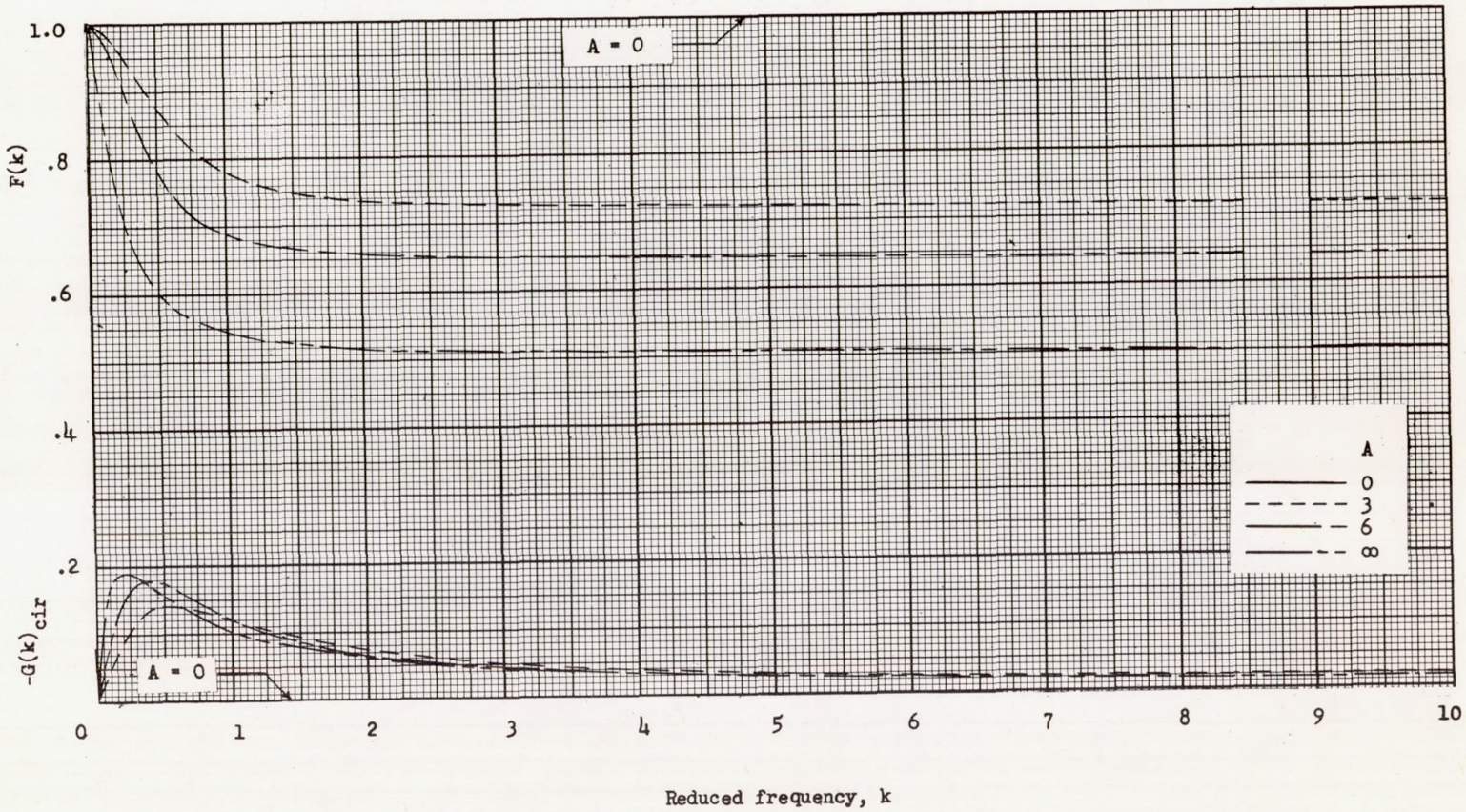


Figure 3.- The functions $C(k)_{cir}$ for elliptical wings of various aspect ratios in incompressible flow.

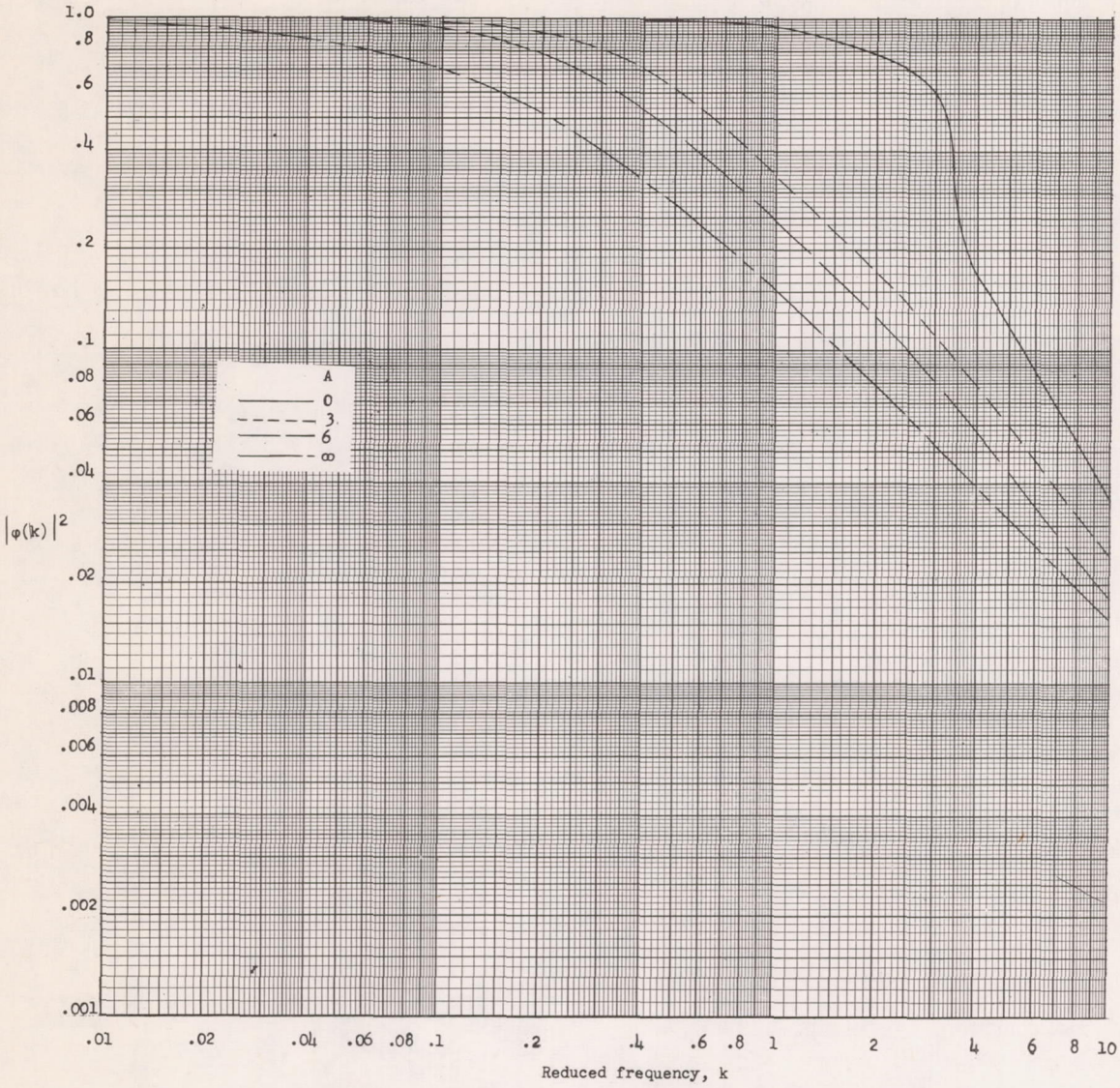


Figure 4.- The functions $|\varphi(k)|^2$ for elliptical wings of various aspect ratios in incompressible flow.

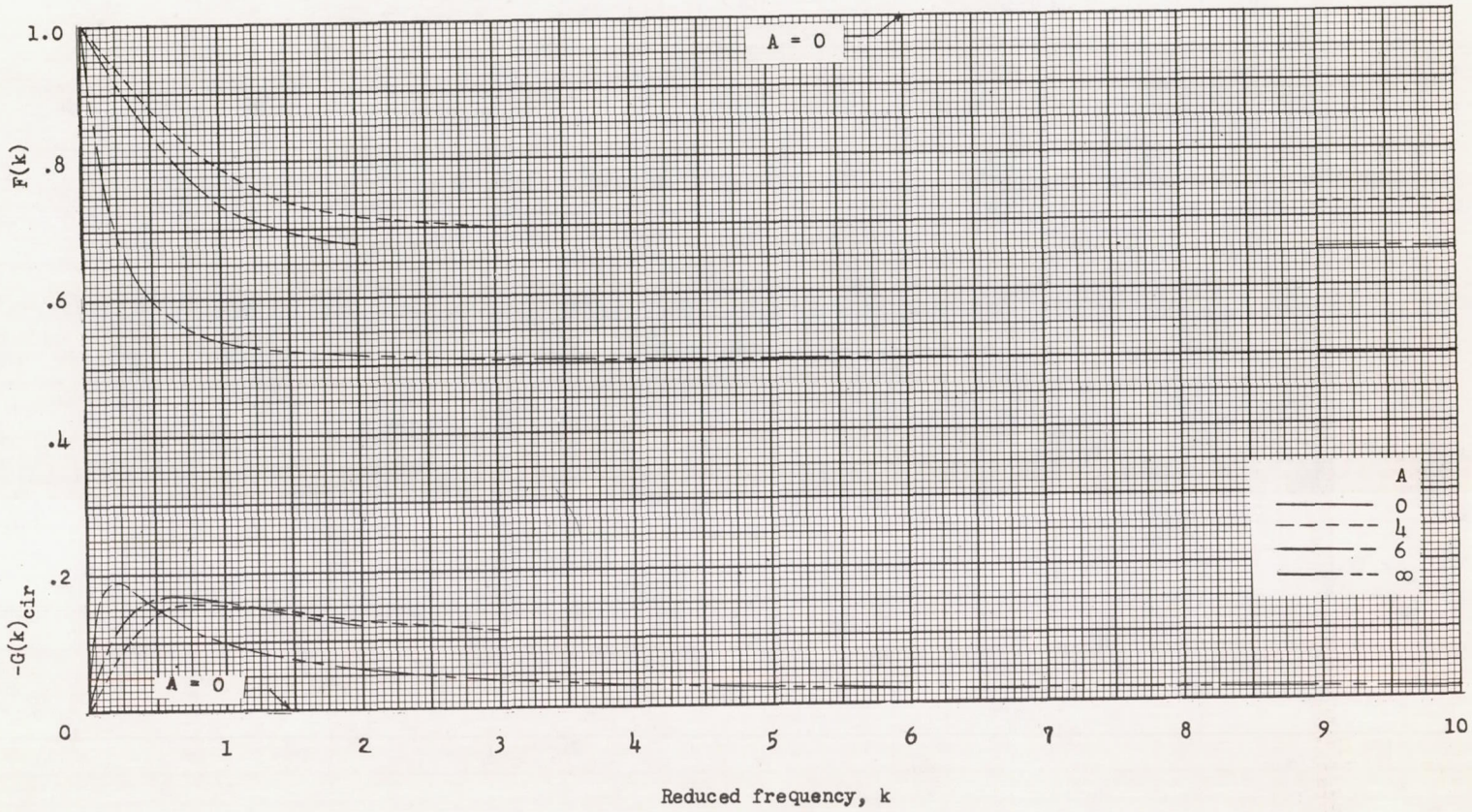


Figure 5.- The functions $C(k)_{cir}$ for rectangular wings of various aspect ratios in incompressible flow.

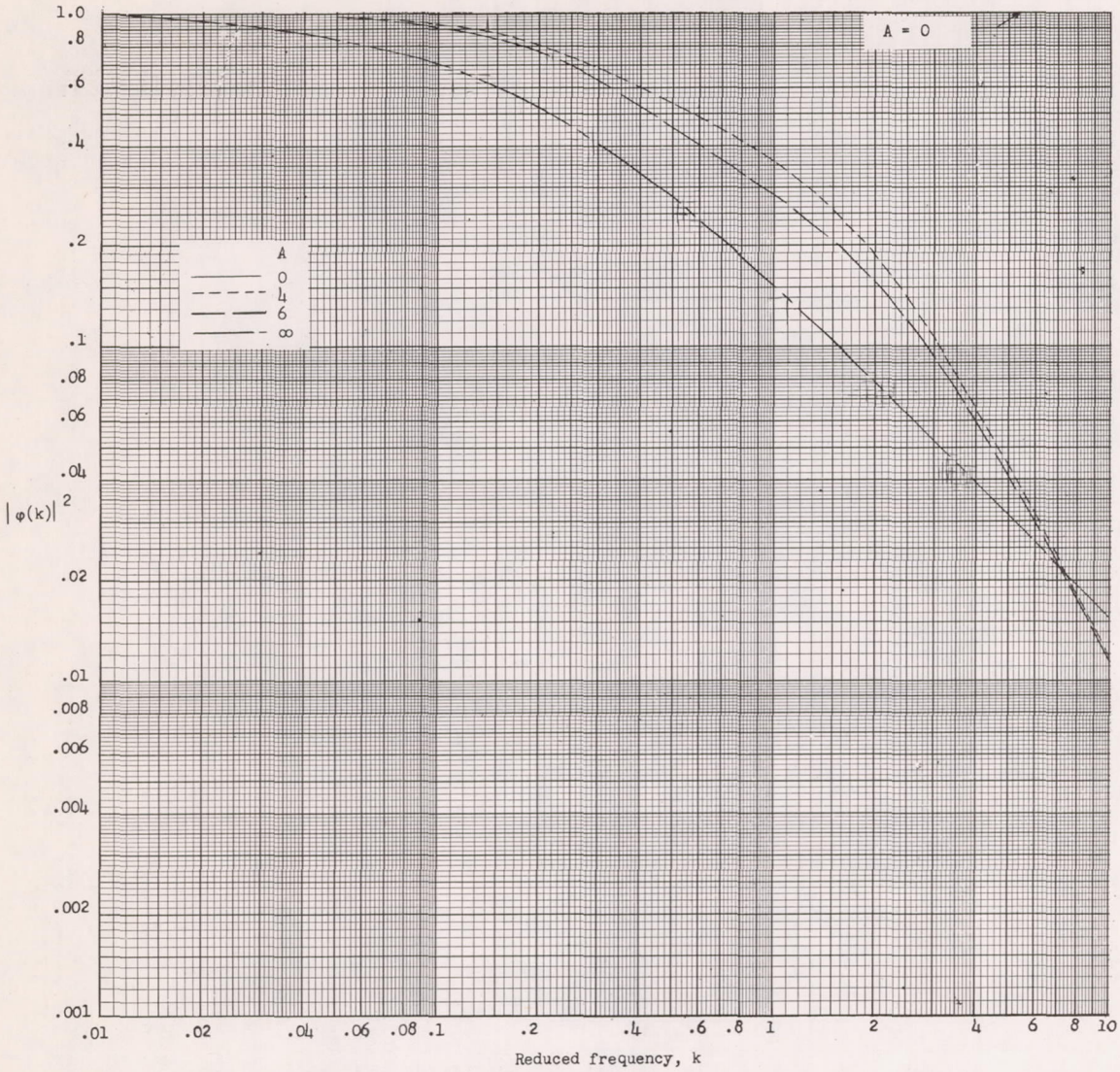


Figure 6.- The functions $|\varphi(k)|^2$ for rectangular wings of various aspect ratios in incompressible flow.

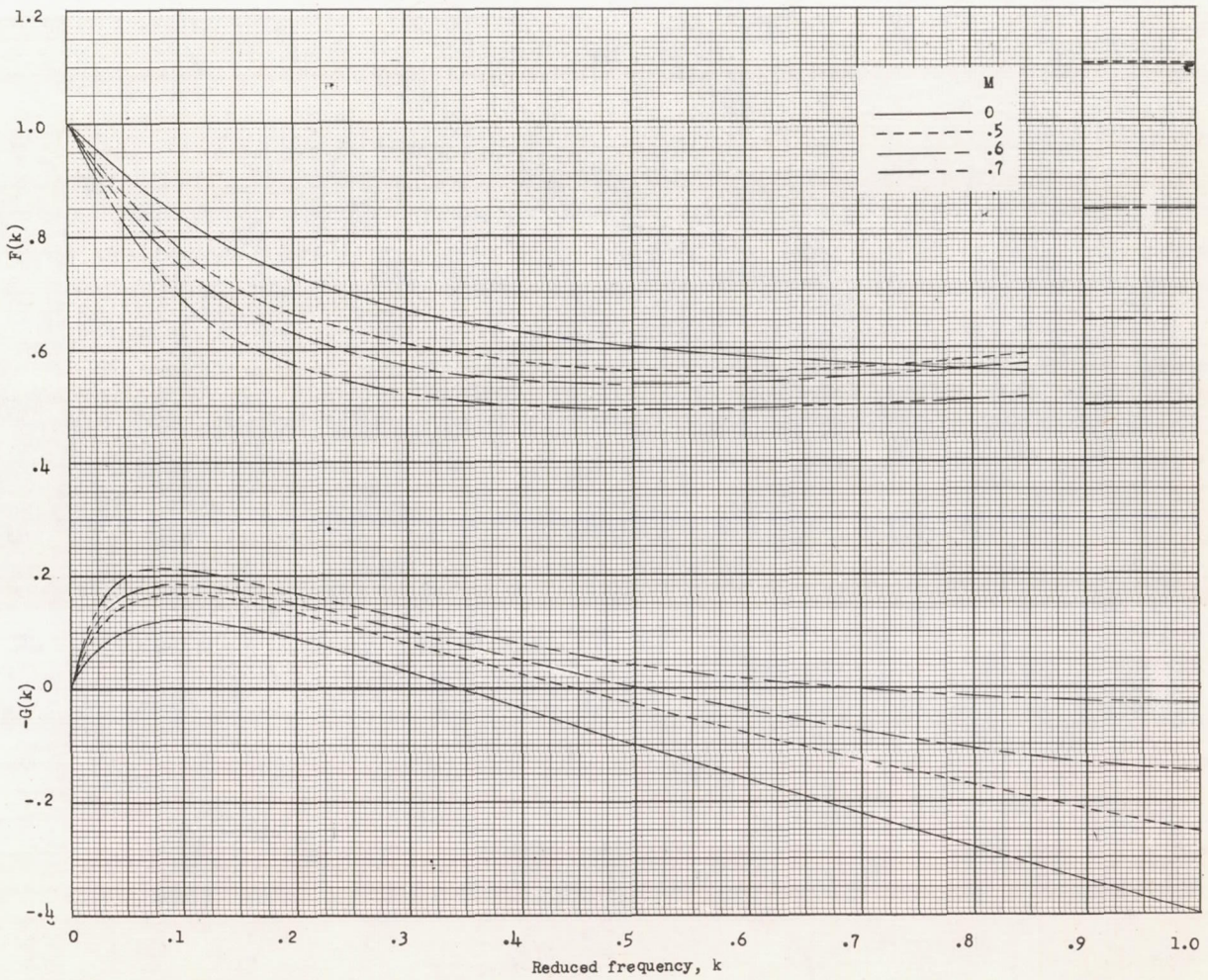


Figure 7.- The functions C(k) for a wing in two-dimensional subsonic flow.

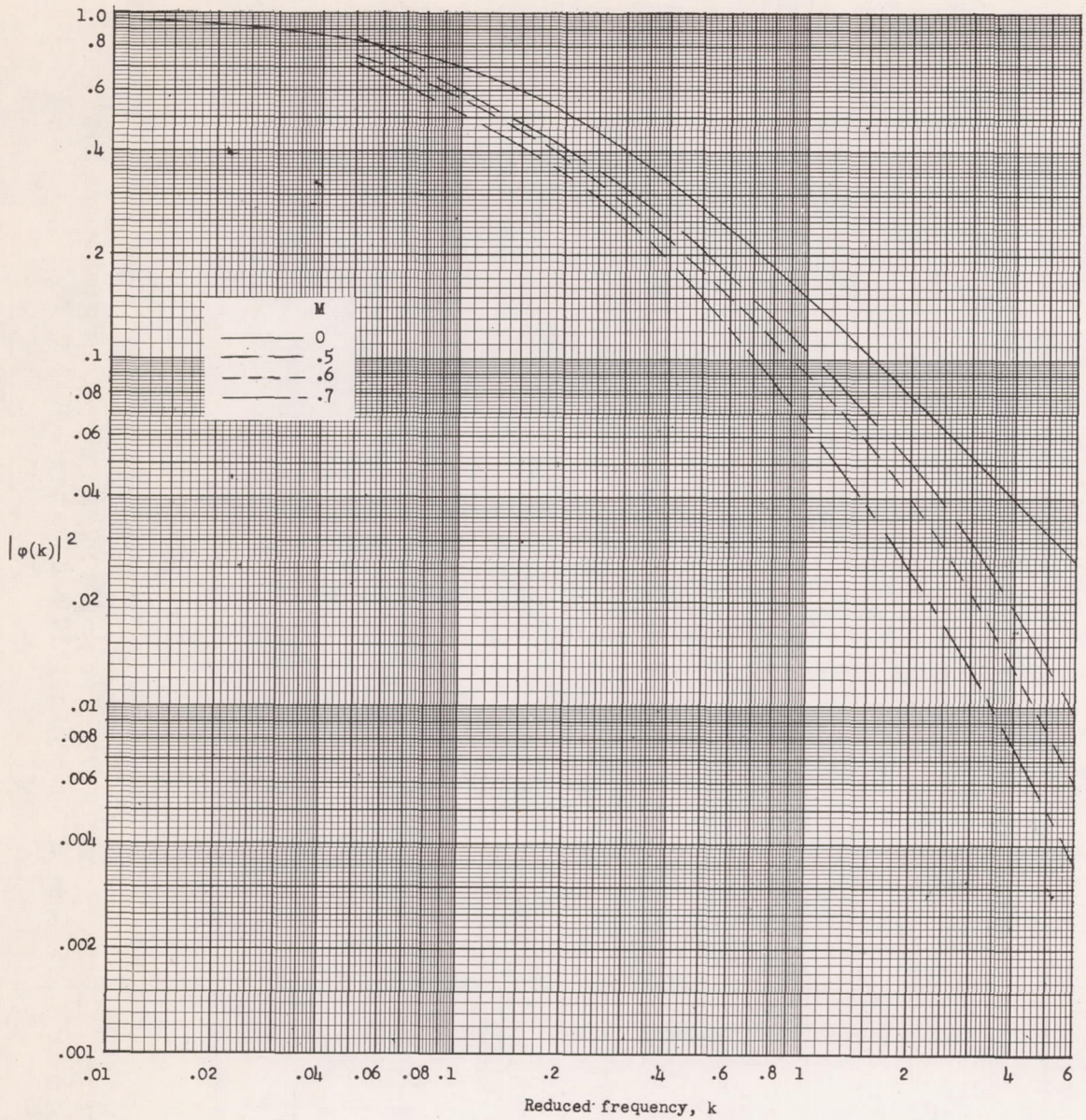


Figure 8.- The functions $|\varphi(k)|^2$ for a wing in two-dimensional subsonic compressible flow.

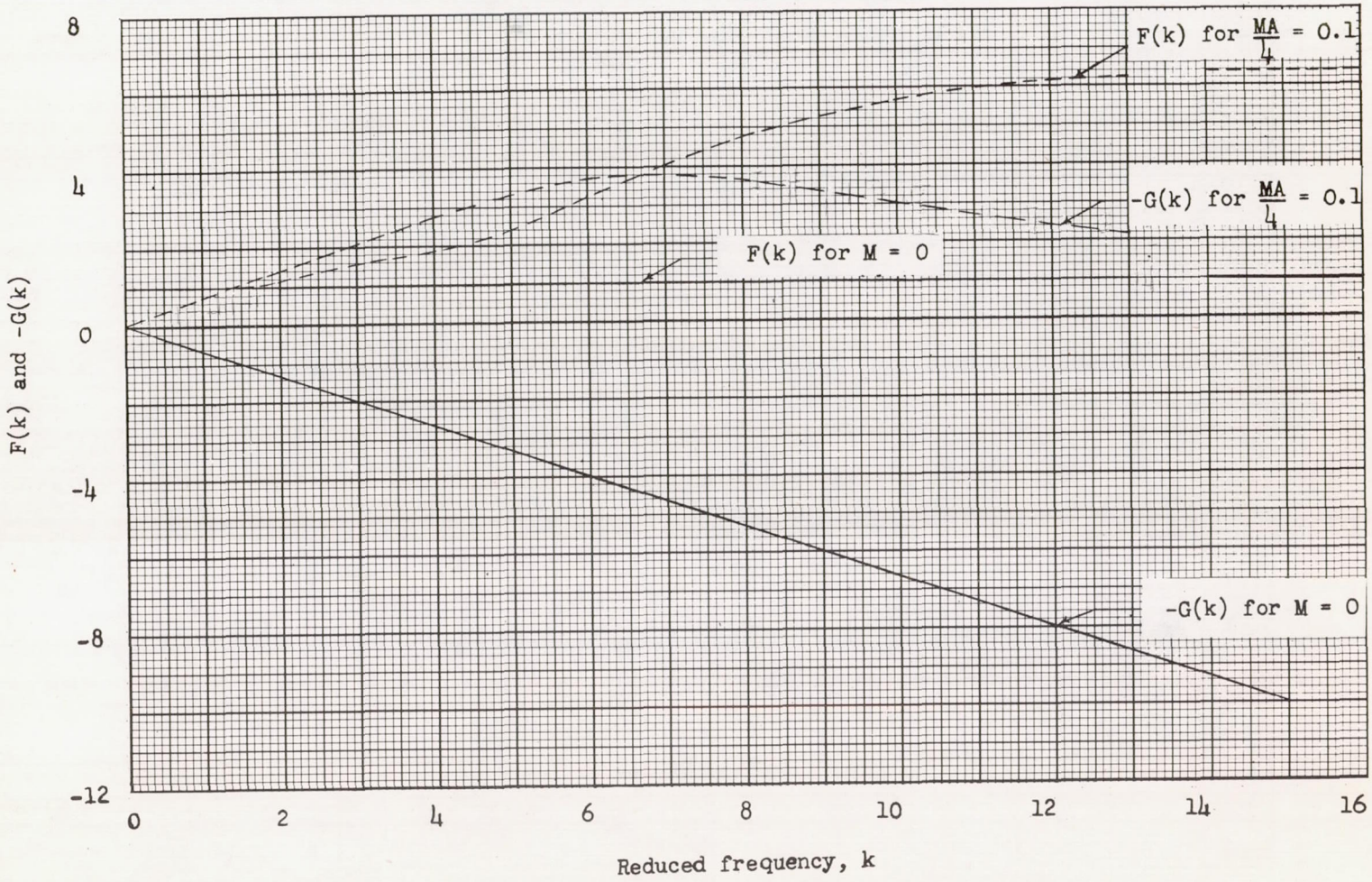


Figure 9.- The functions C(k) for delta wings of vanishingly small aspect ratios in incompressible and compressible flow.

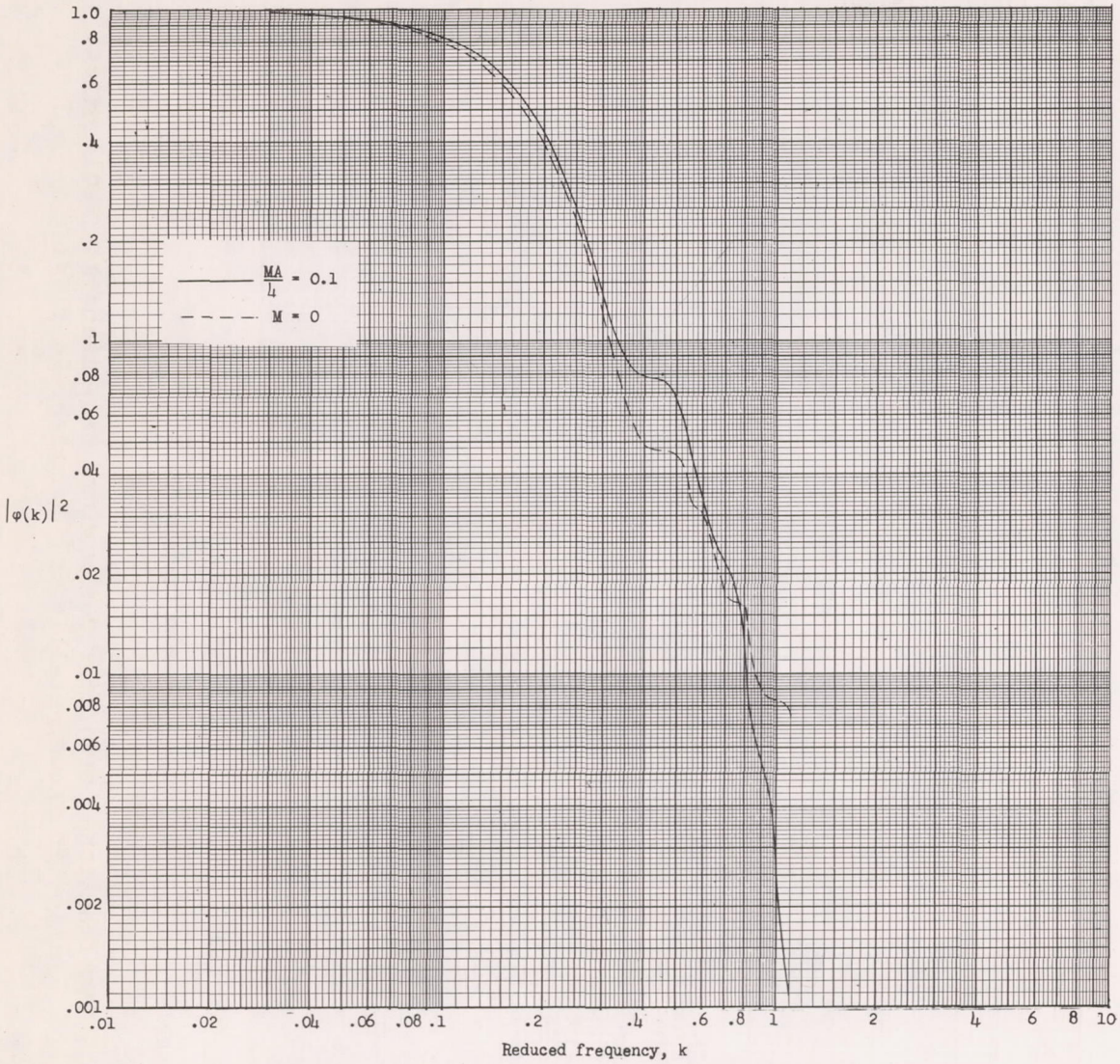


Figure 10.- The functions $|\varphi(k)|^2$ for a delta wing of vanishingly small aspect ratio in incompressible and compressible flow.

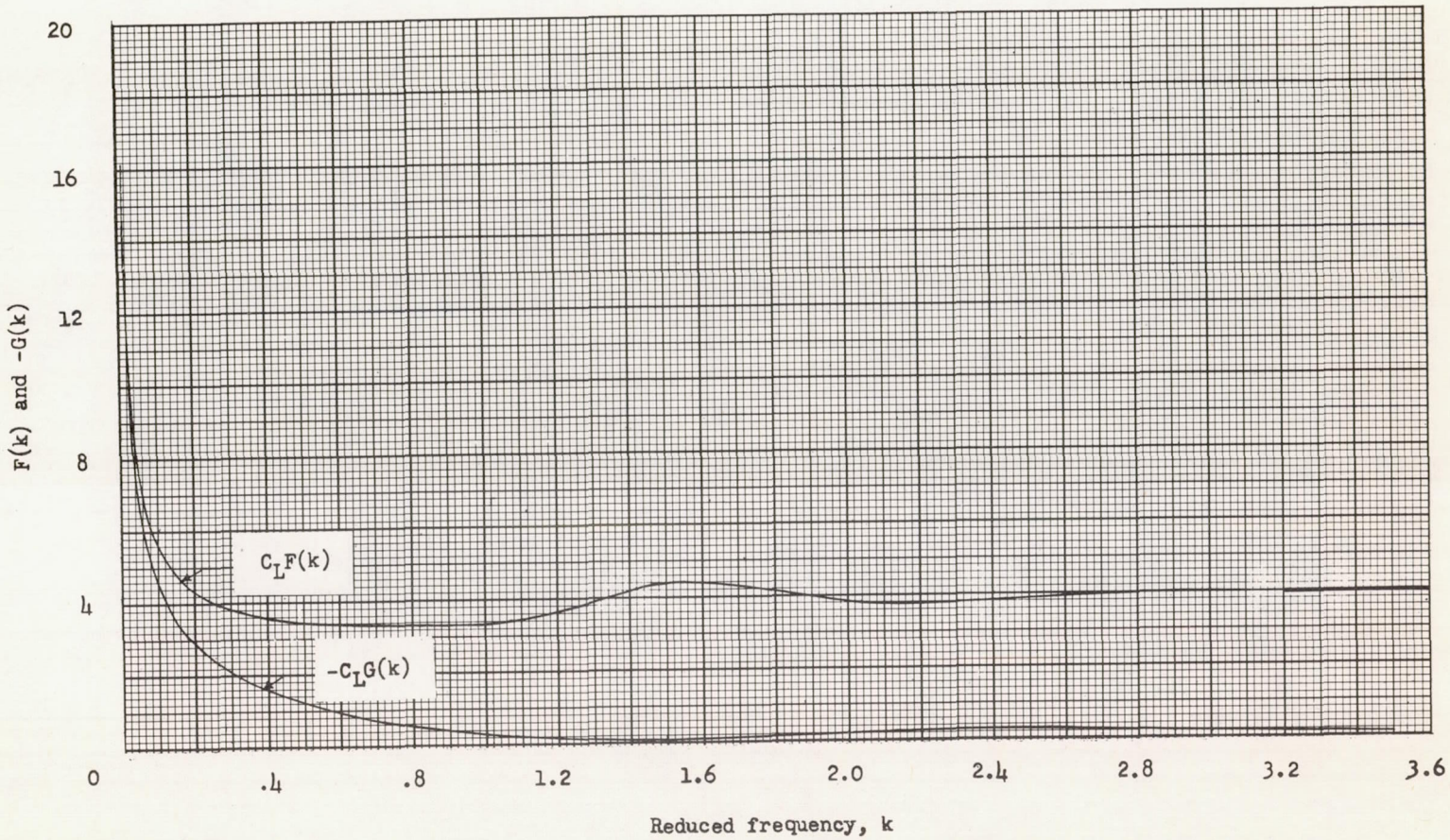


Figure 11.- The function $C_LC(k)$ for a wing in two-dimensional sonic flow.

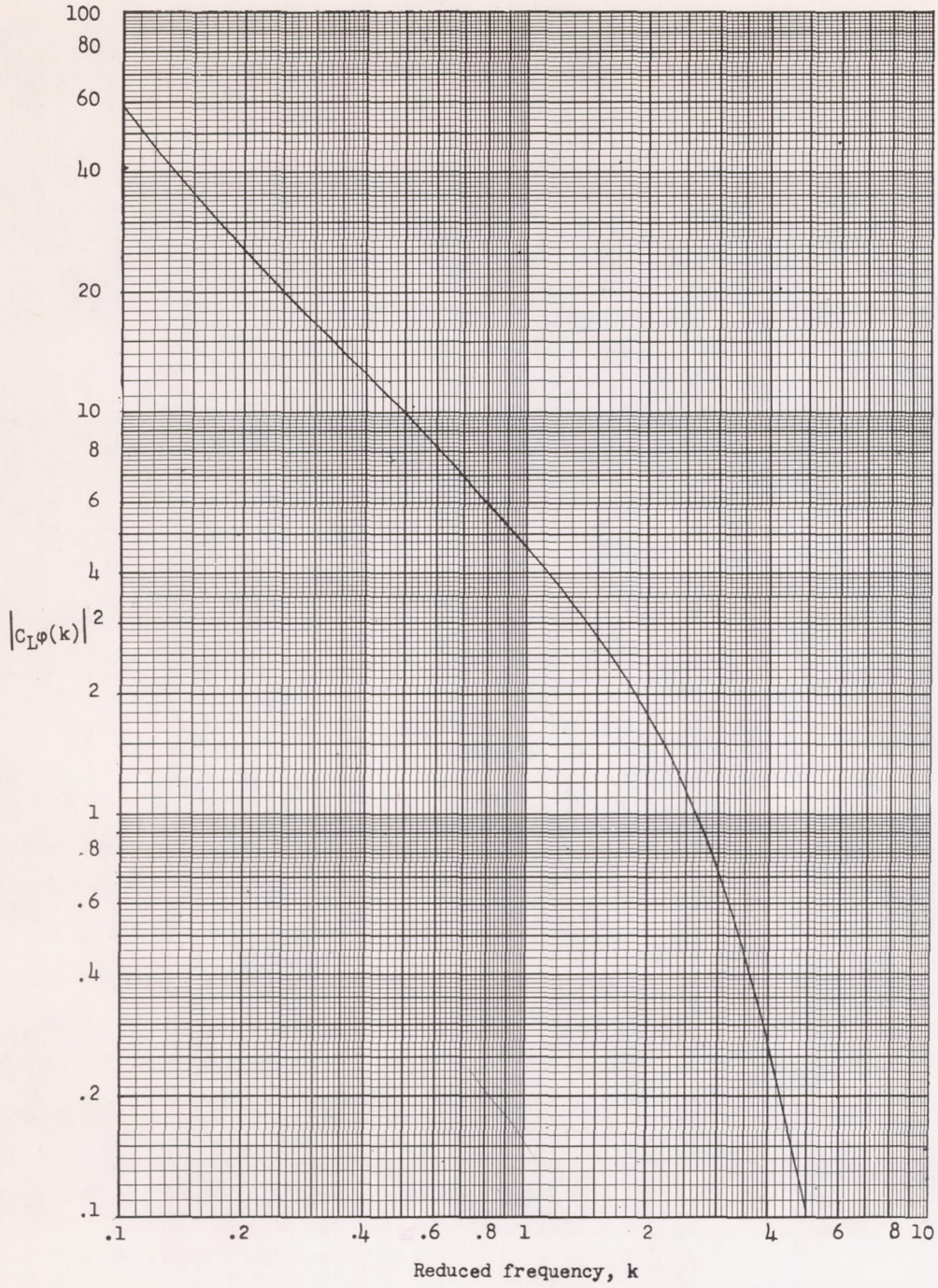


Figure 12.- The function $|C_L\varphi(k)|^2$ for a wing in two-dimensional sonic flow.

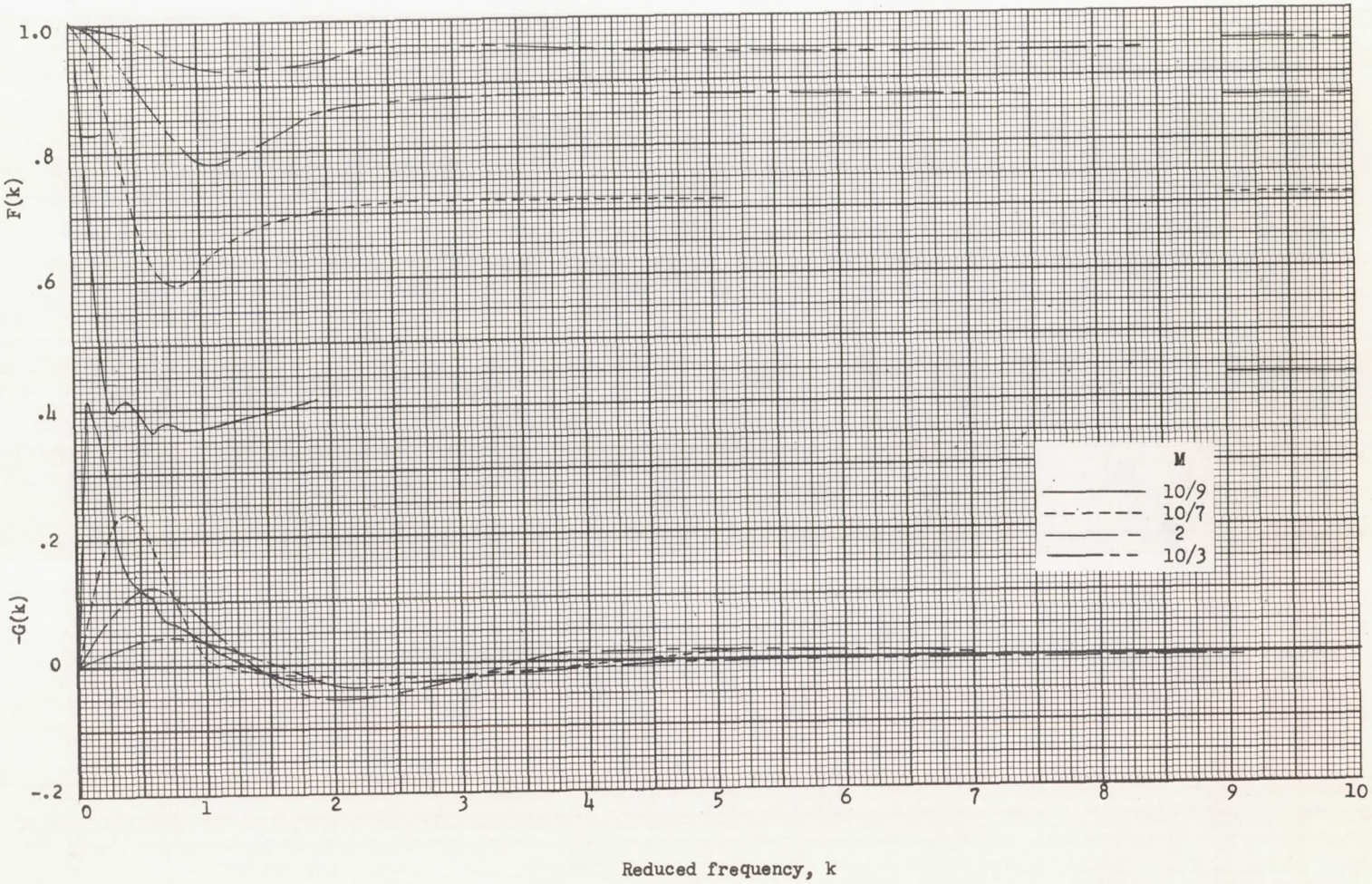


Figure 13.- The functions C(k) for a wing in two-dimensional supersonic flow.

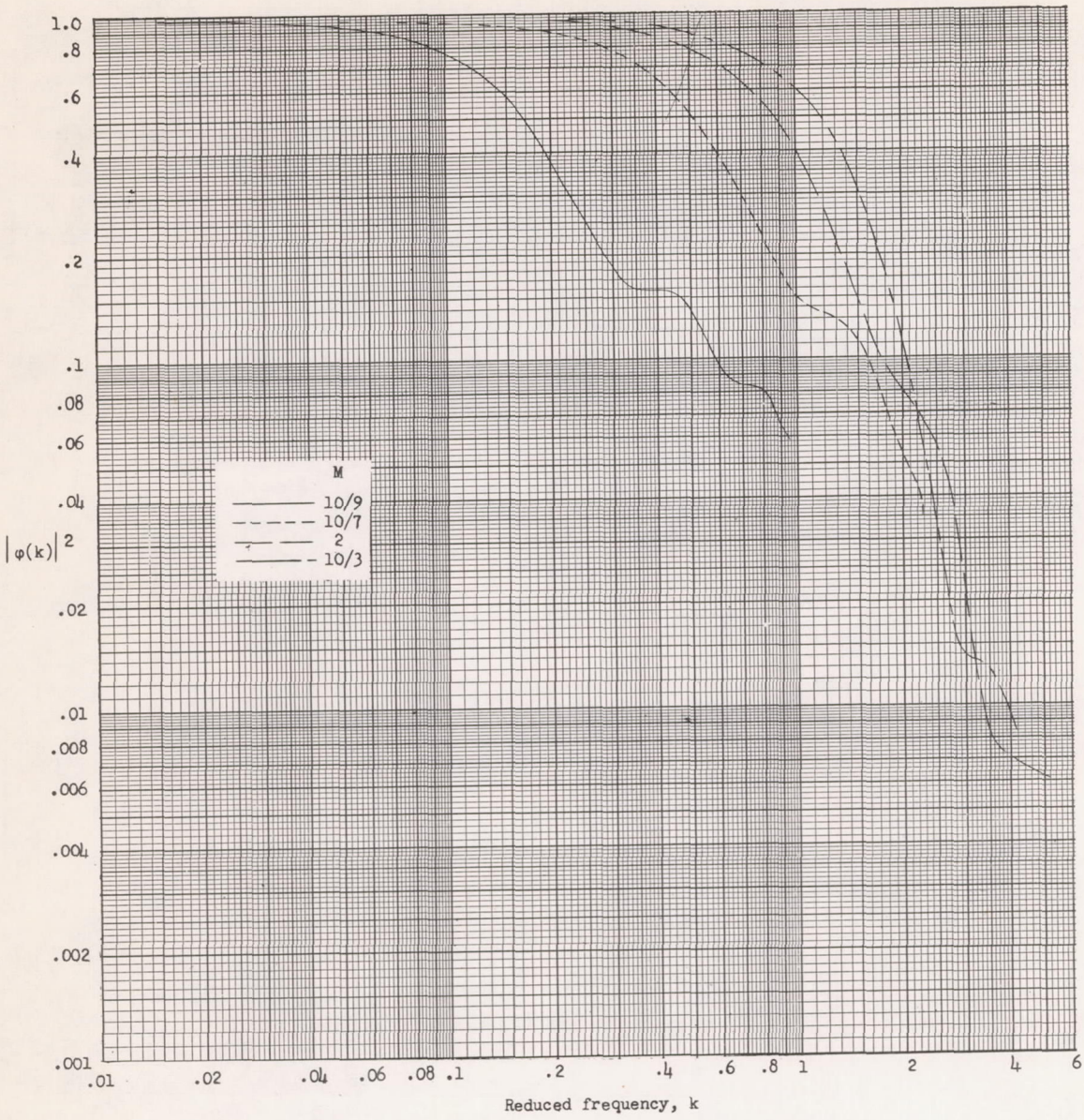


Figure 14.- The functions $|\varphi(k)|^2$ for a wing in two-dimensional supersonic flow.

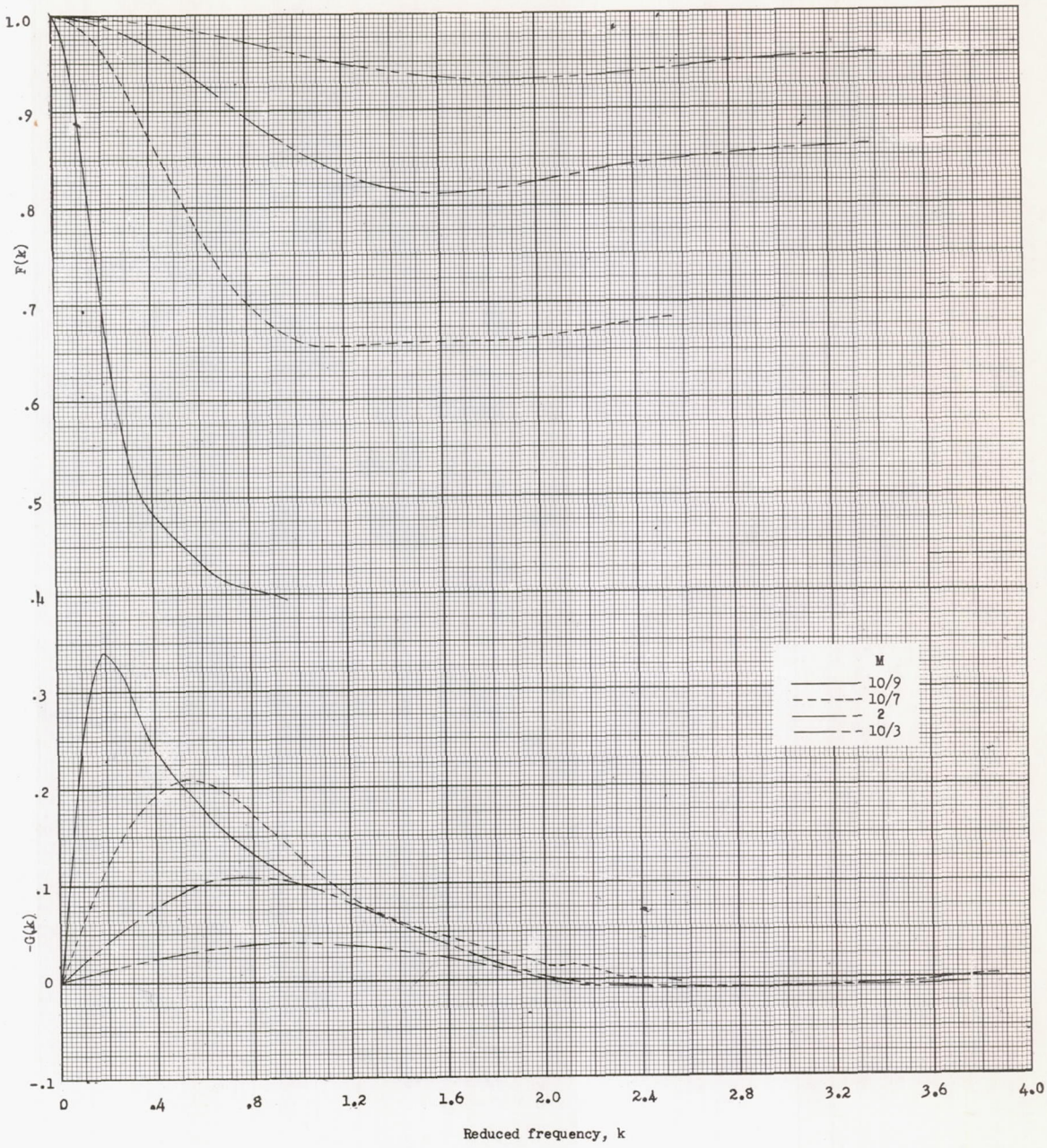


Figure 15.- The functions C(k) for a wide delta wing in supersonic flow.

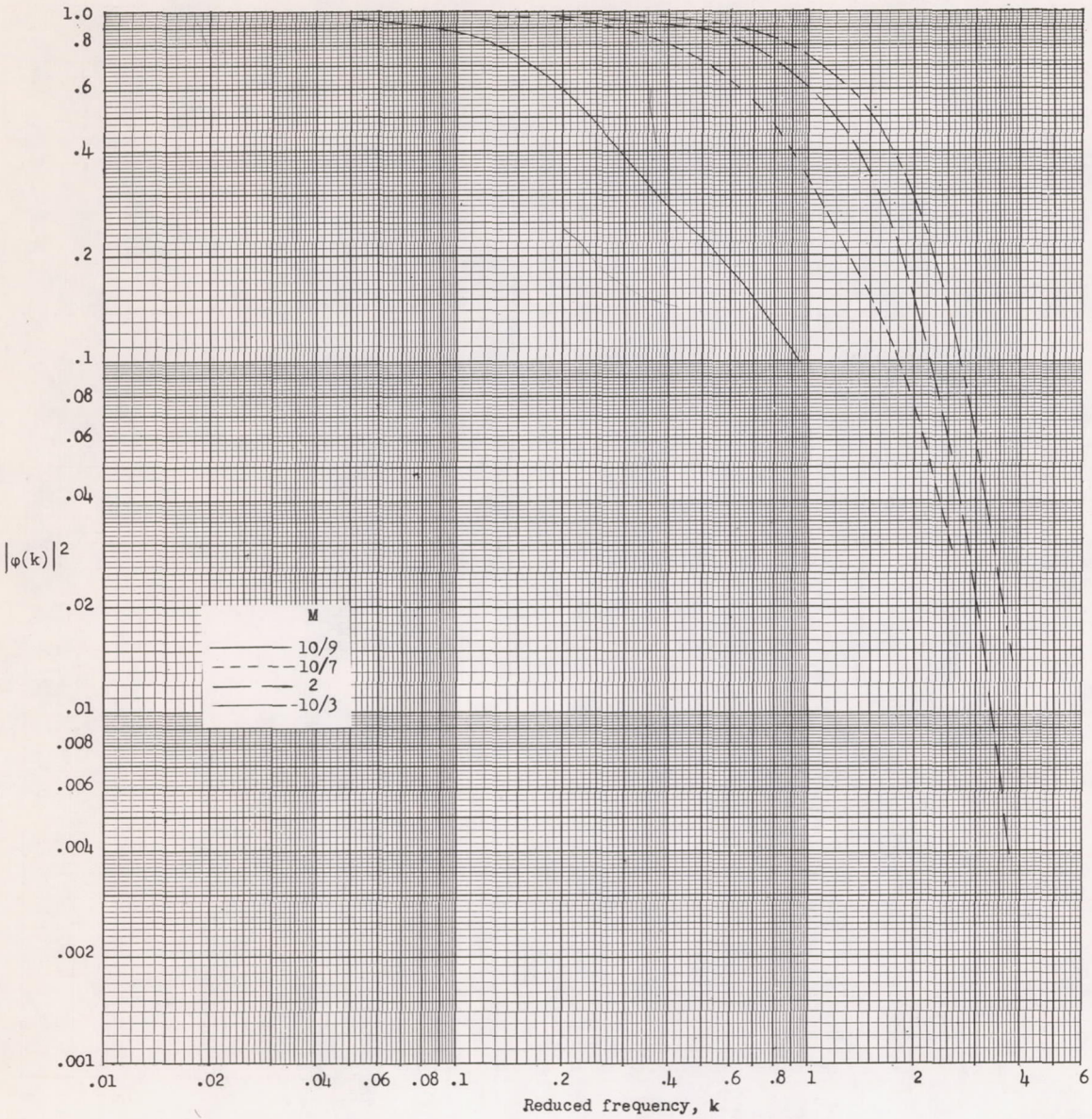
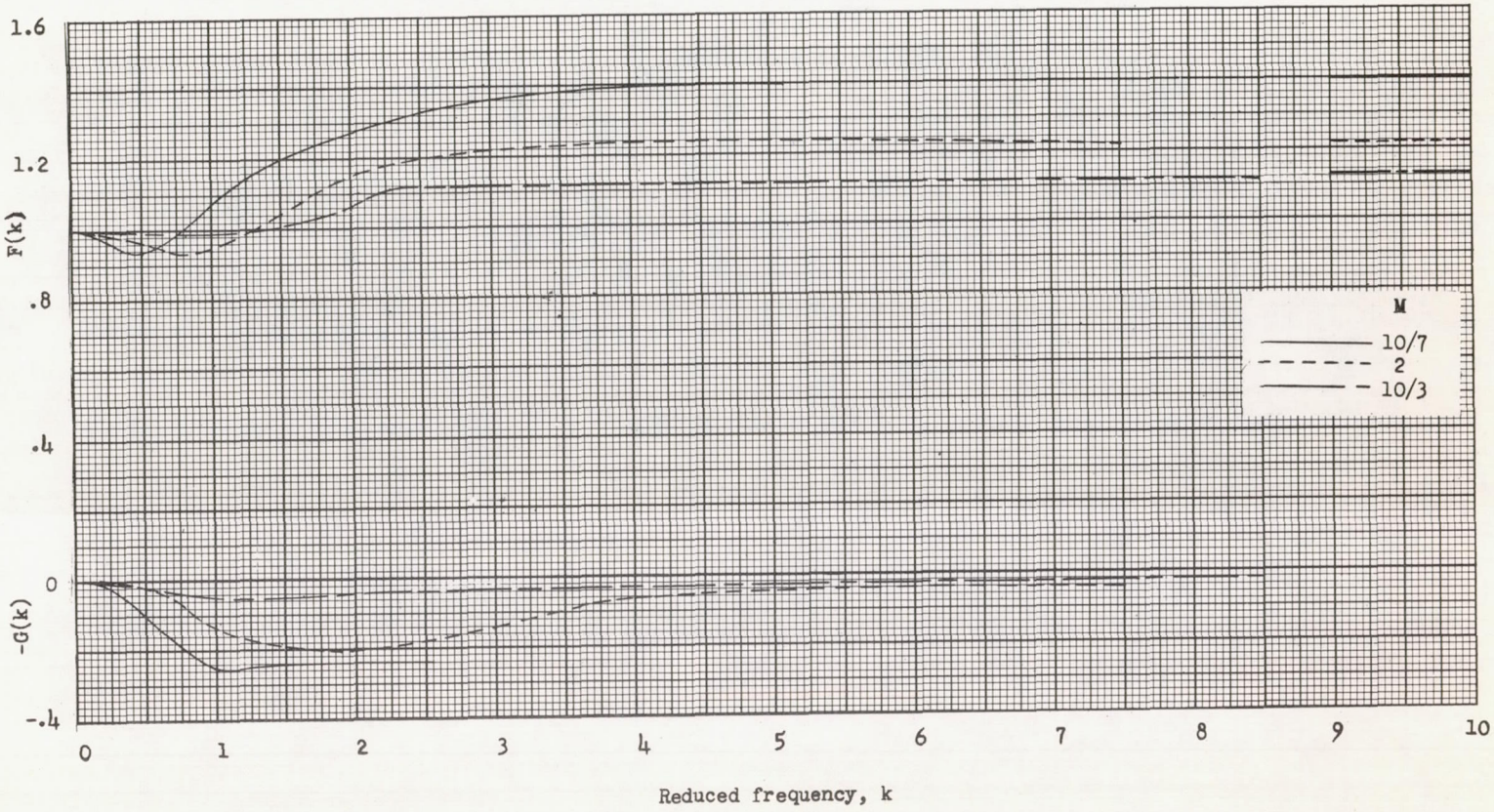
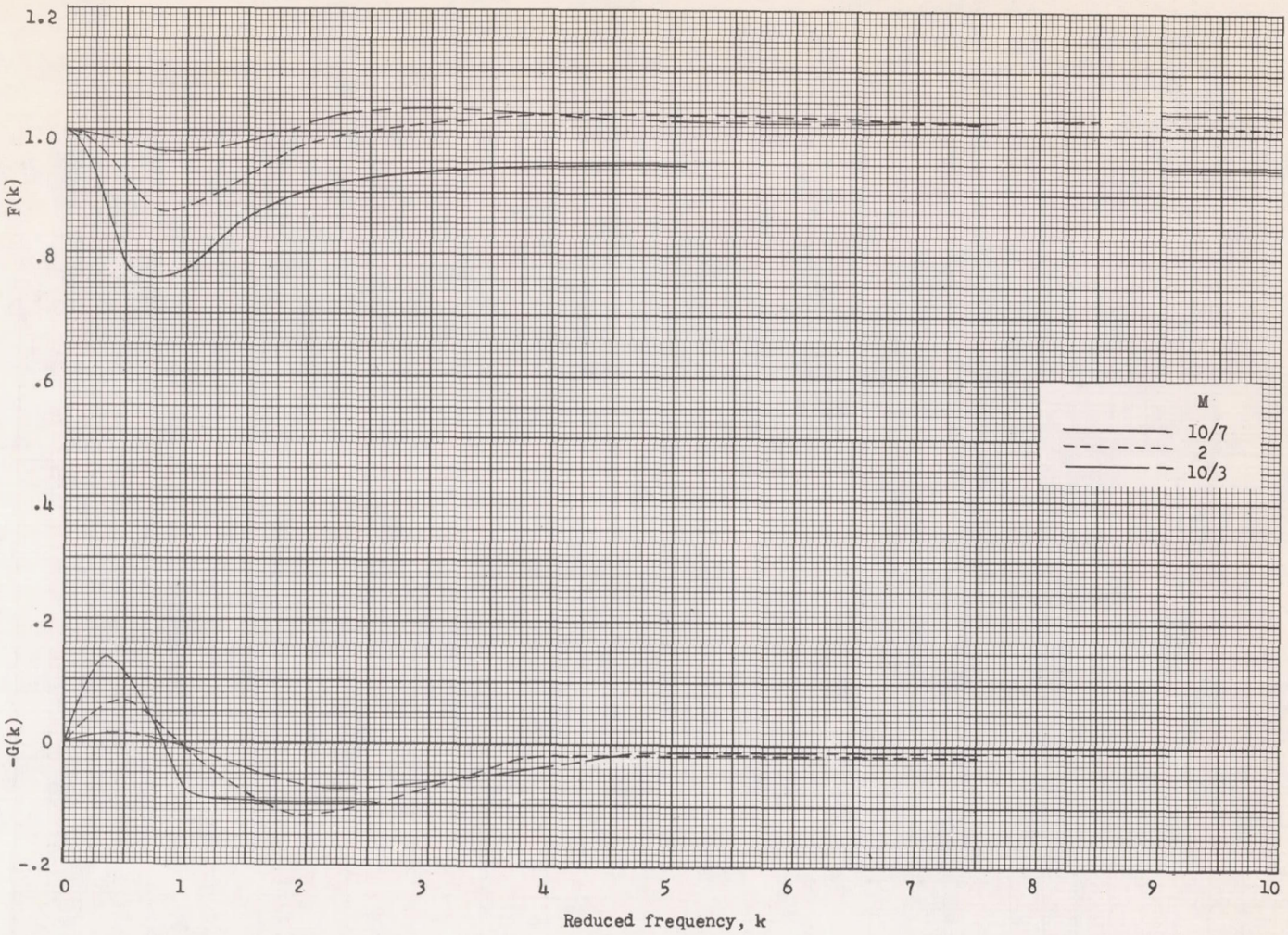


Figure 16.- The functions $|\varphi(k)|^2$ for a wide delta wing in supersonic flow.



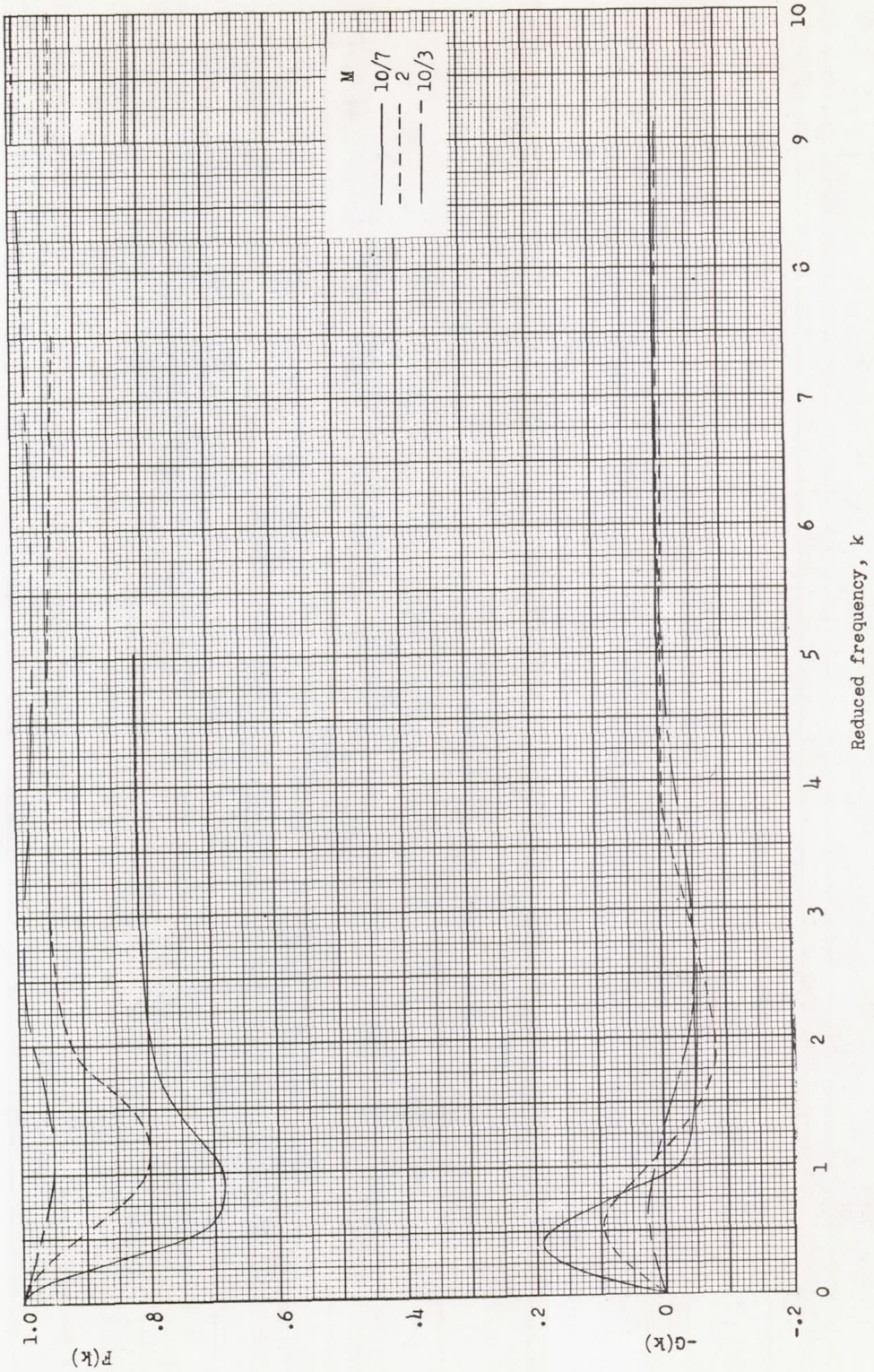
(a) A = 1.

Figure 17.- The effect of Mach number on the functions C(k) for a wide rectangular wing in supersonic flow.



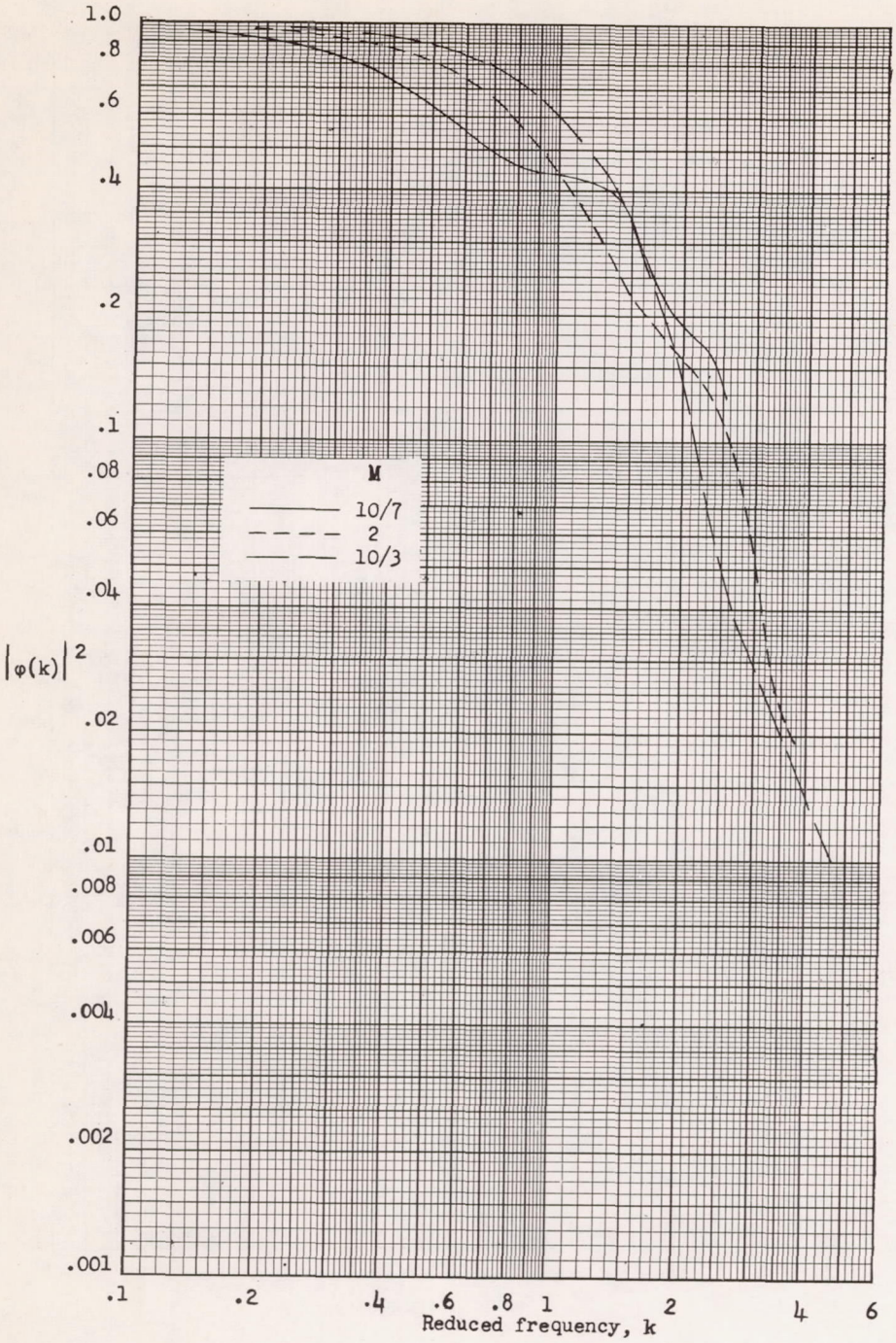
(b) A = 2.

Figure 17.- Continued.



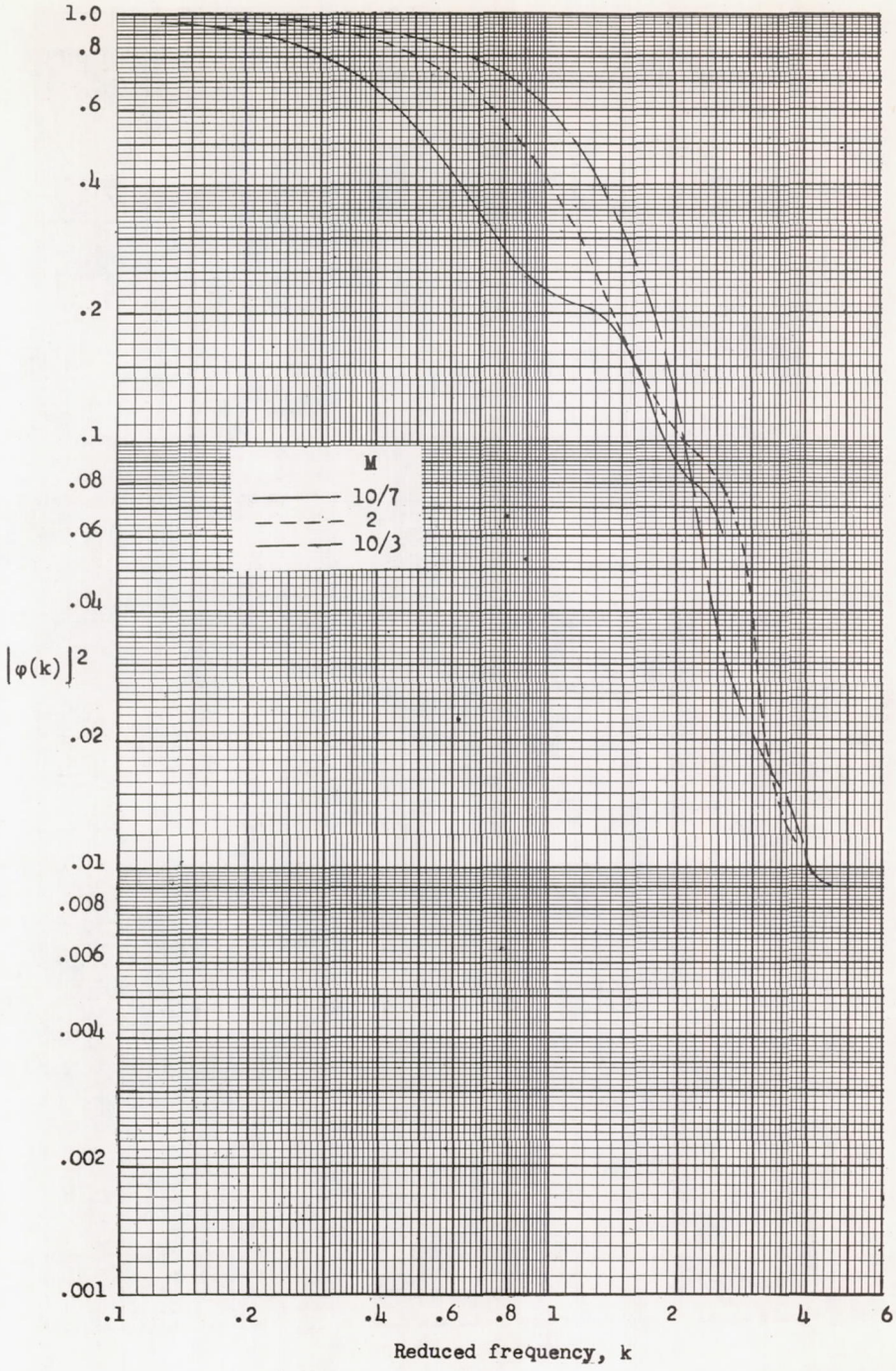
(c) A = 4.

Figure 17.- Concluded.



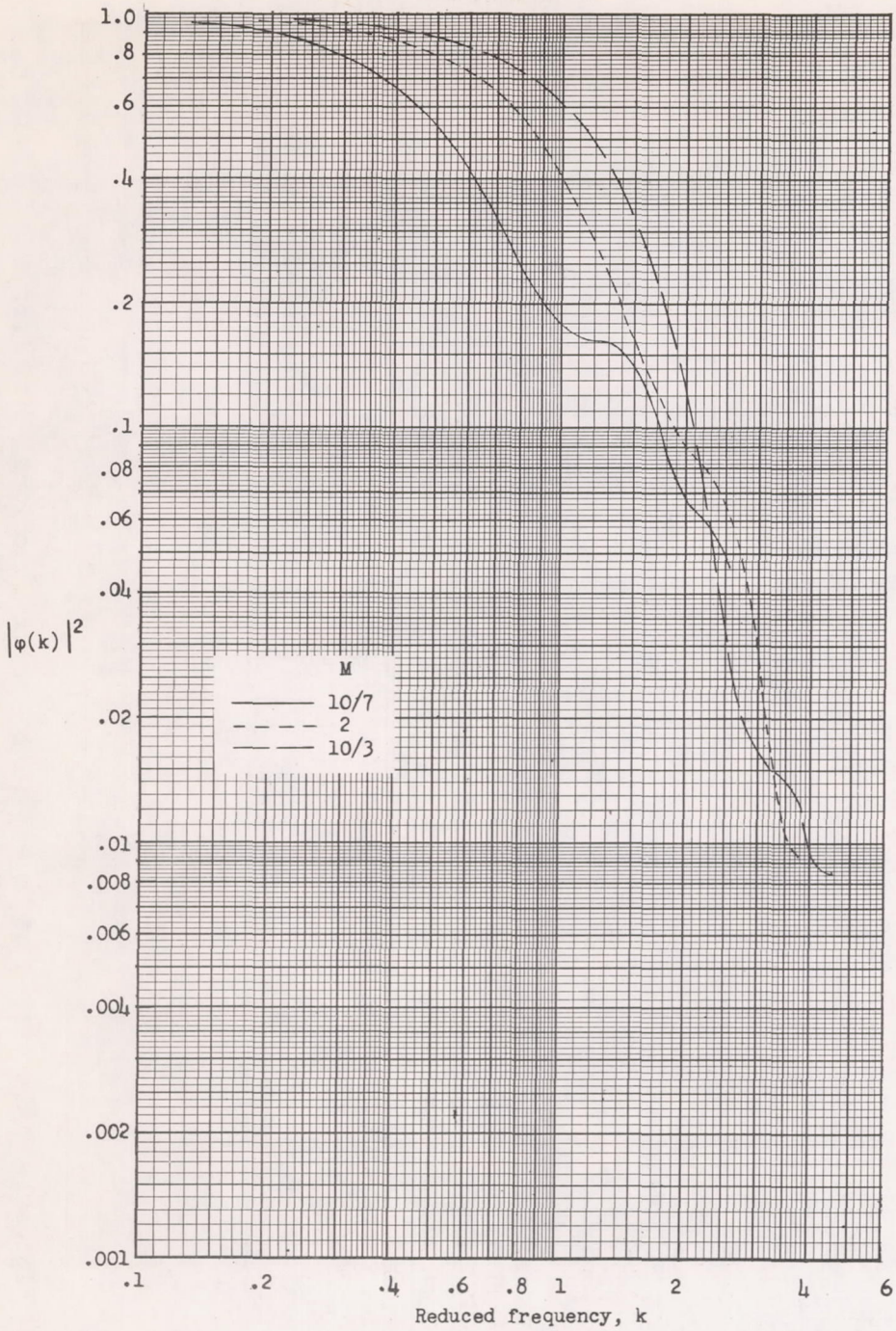
(a) A = 1.

Figure 18.- The effect of Mach number on the functions $|\varphi(k)|^2$ for a rectangular wing in supersonic flow.



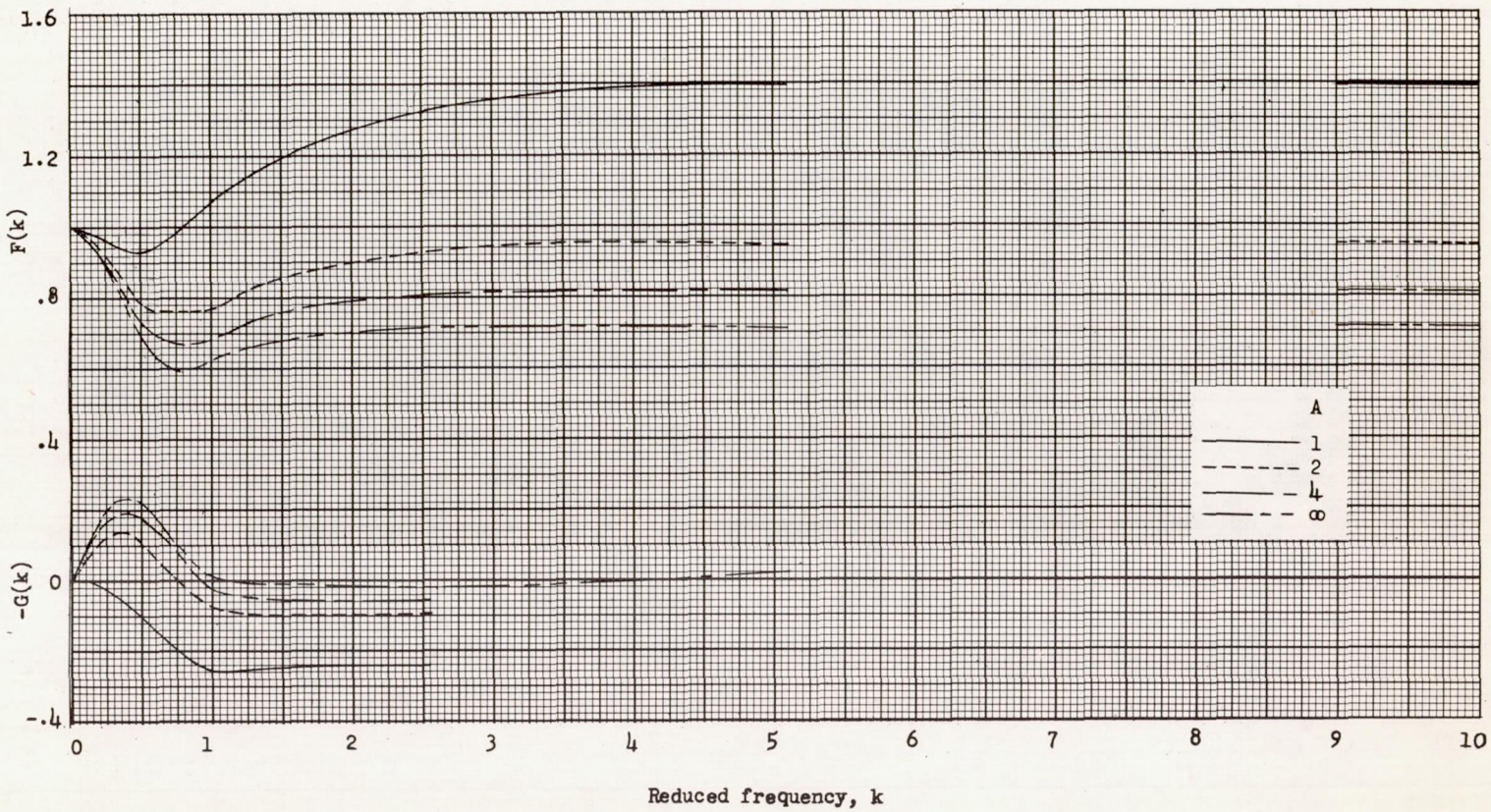
(b) A = 2.

Figure 18.- Continued.



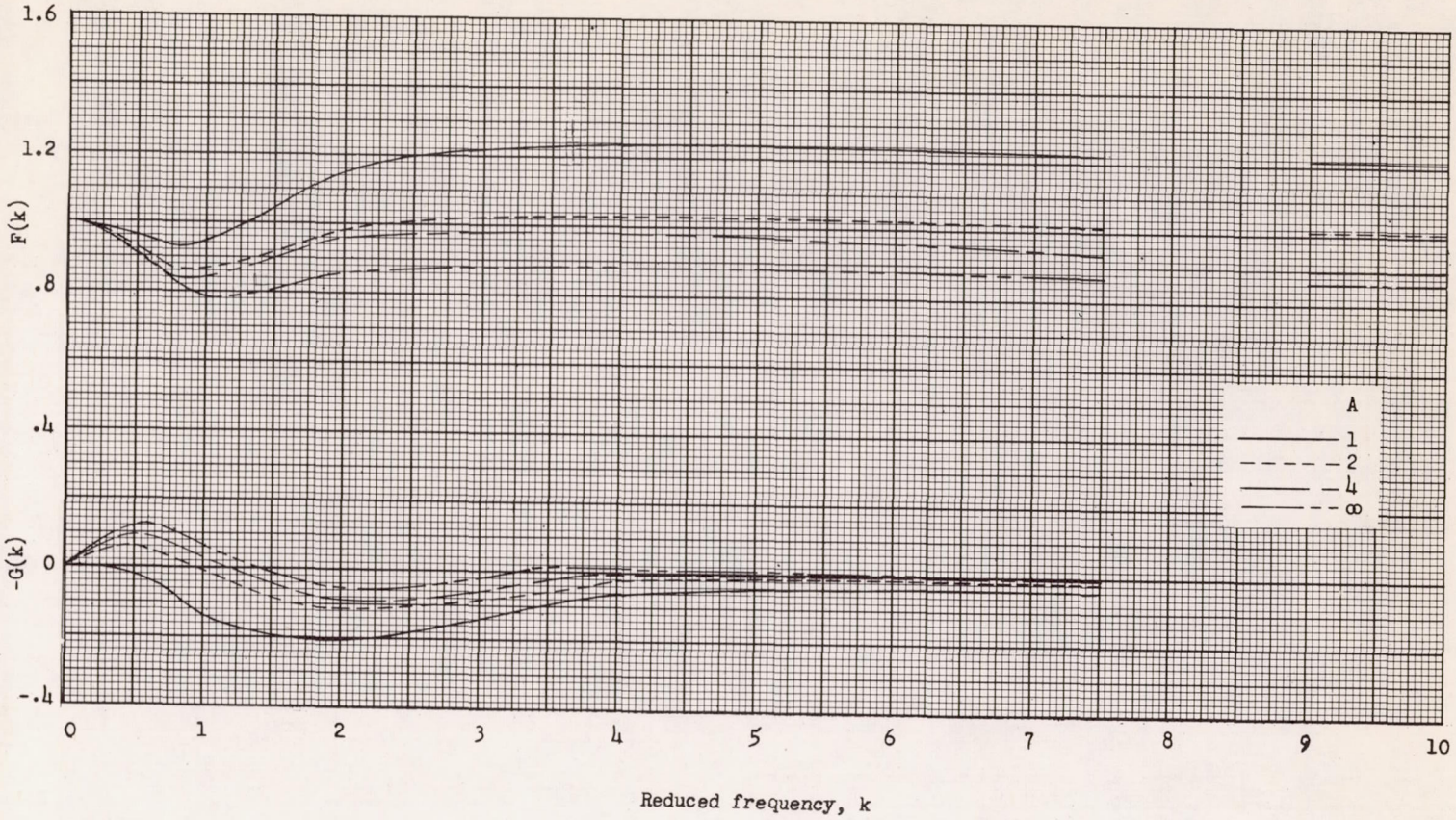
(c) A = 4.

Figure 18.- Concluded.



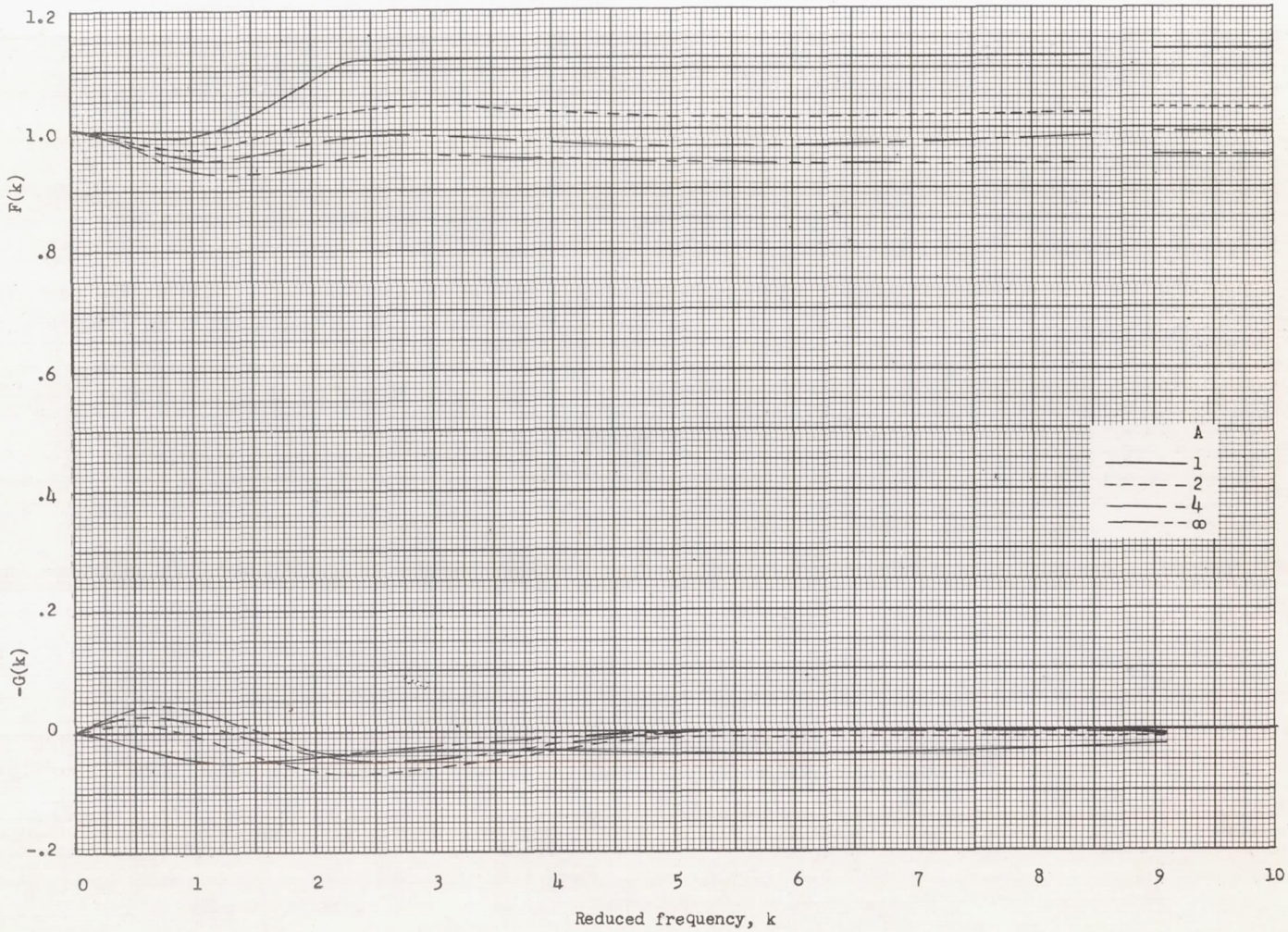
(a) M = 10/7.

Figure 19.- The effect of aspect ratio on the functions C(k) for a rectangular wing in supersonic flow.



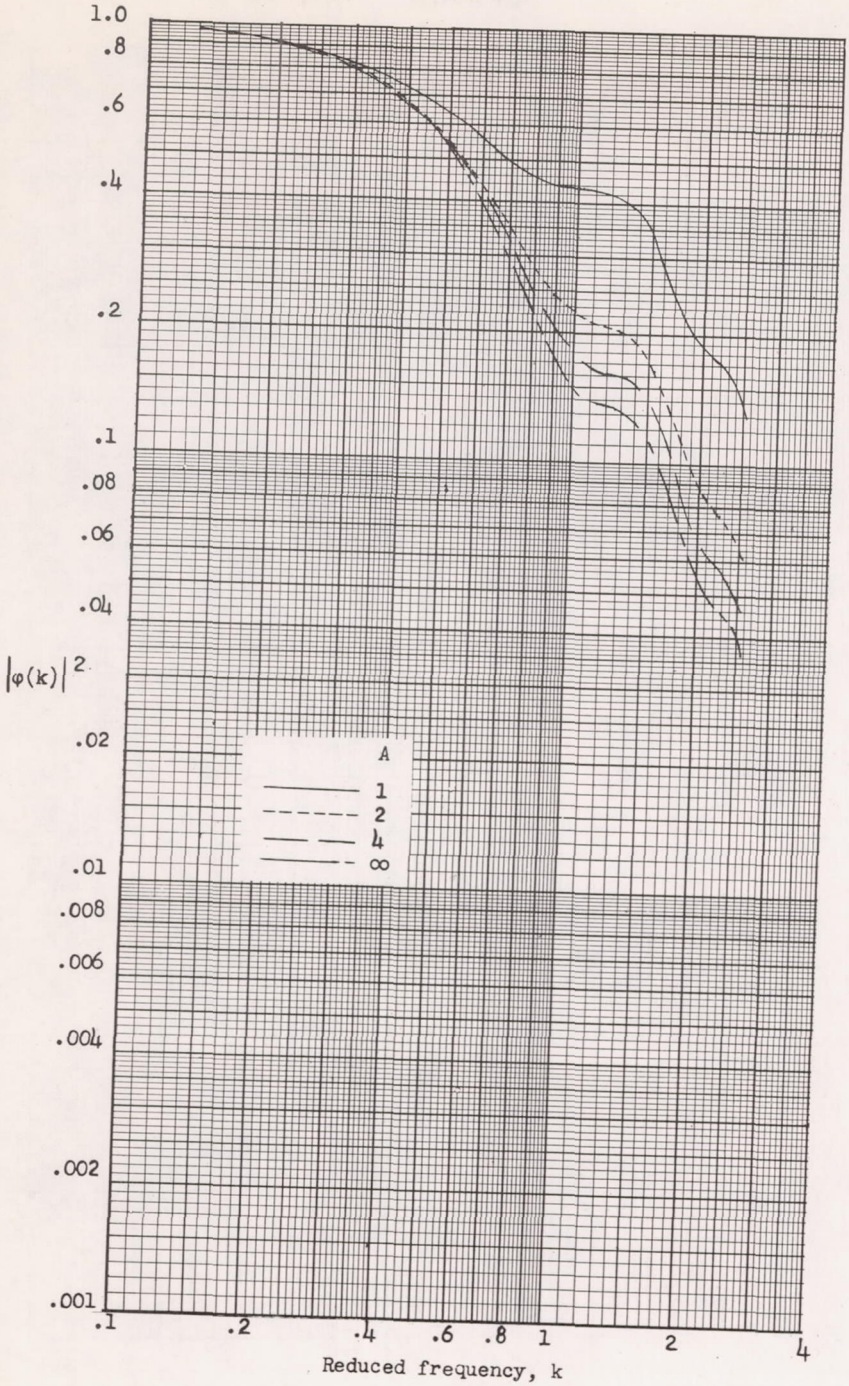
(b) M = 2.

Figure 19.- Continued.



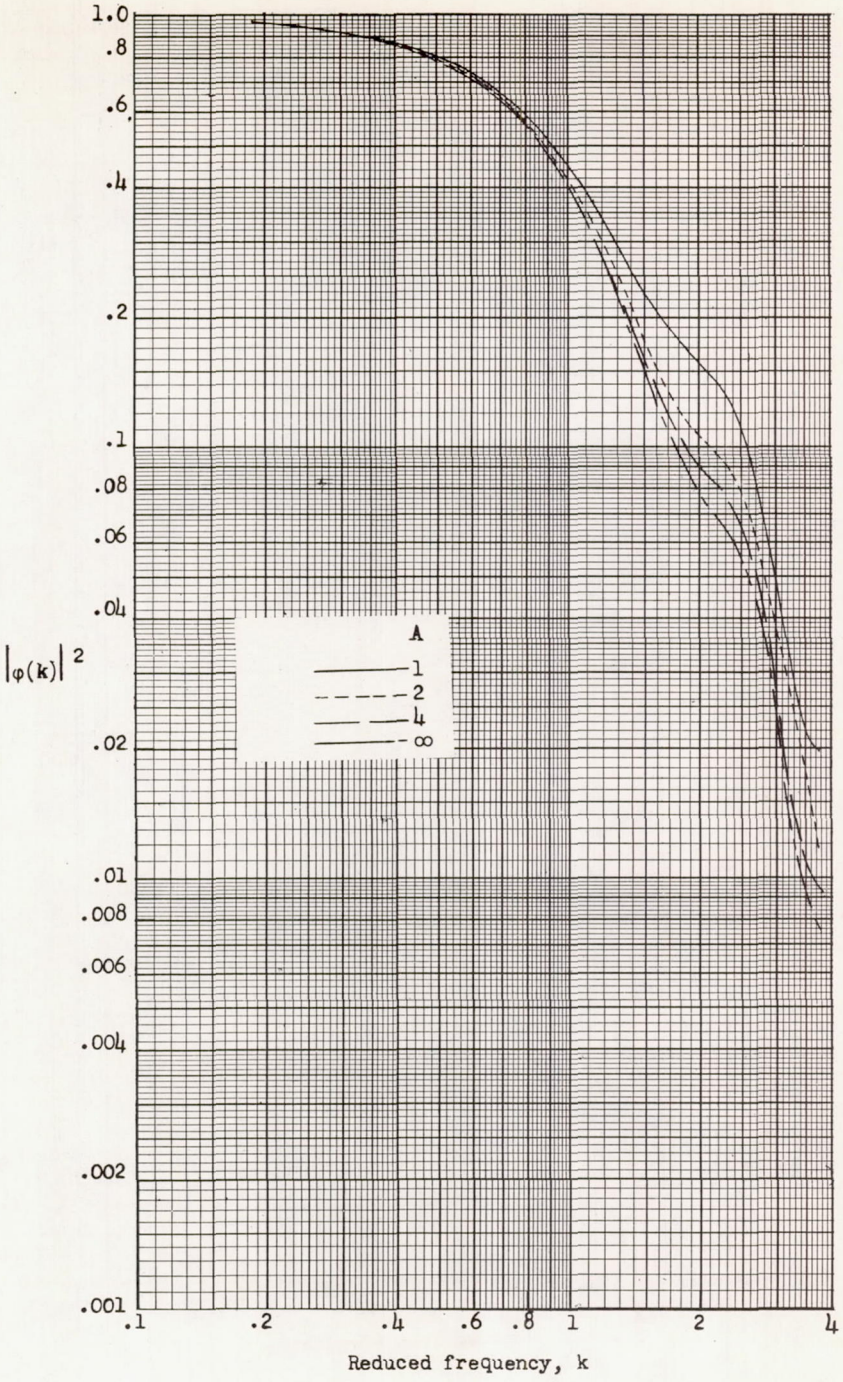
(c) M = 10/3.

Figure 19.- Concluded.



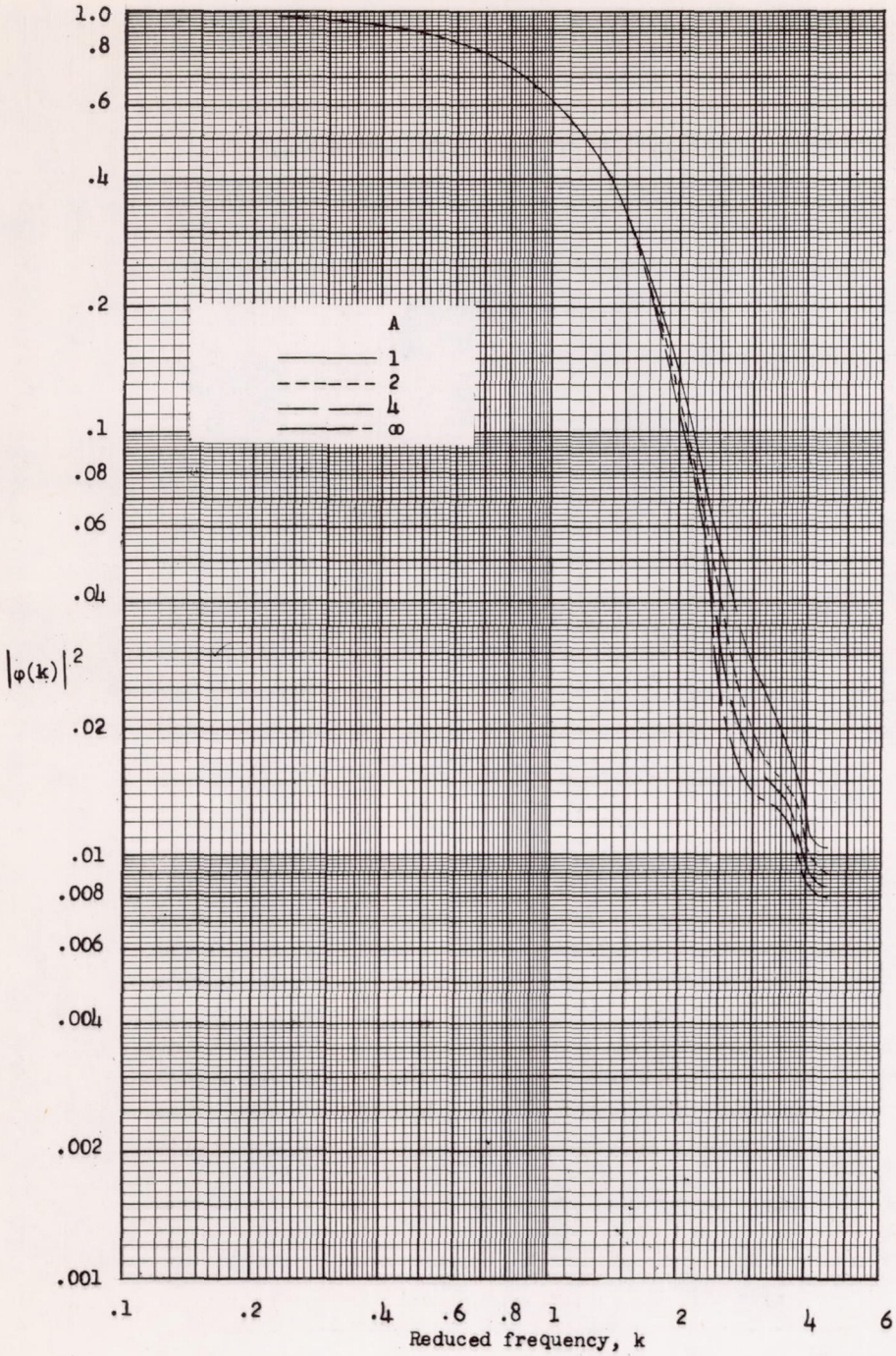
(a) M = 10/7.

Figure 20.- The effect of aspect ratio on the functions $|\varphi(k)|^2$ for a rectangular wing in supersonic flow.



(b) M = 2.

Figure 20.- Continued.



(c) M = 10/3.

Figure 20.- Concluded.

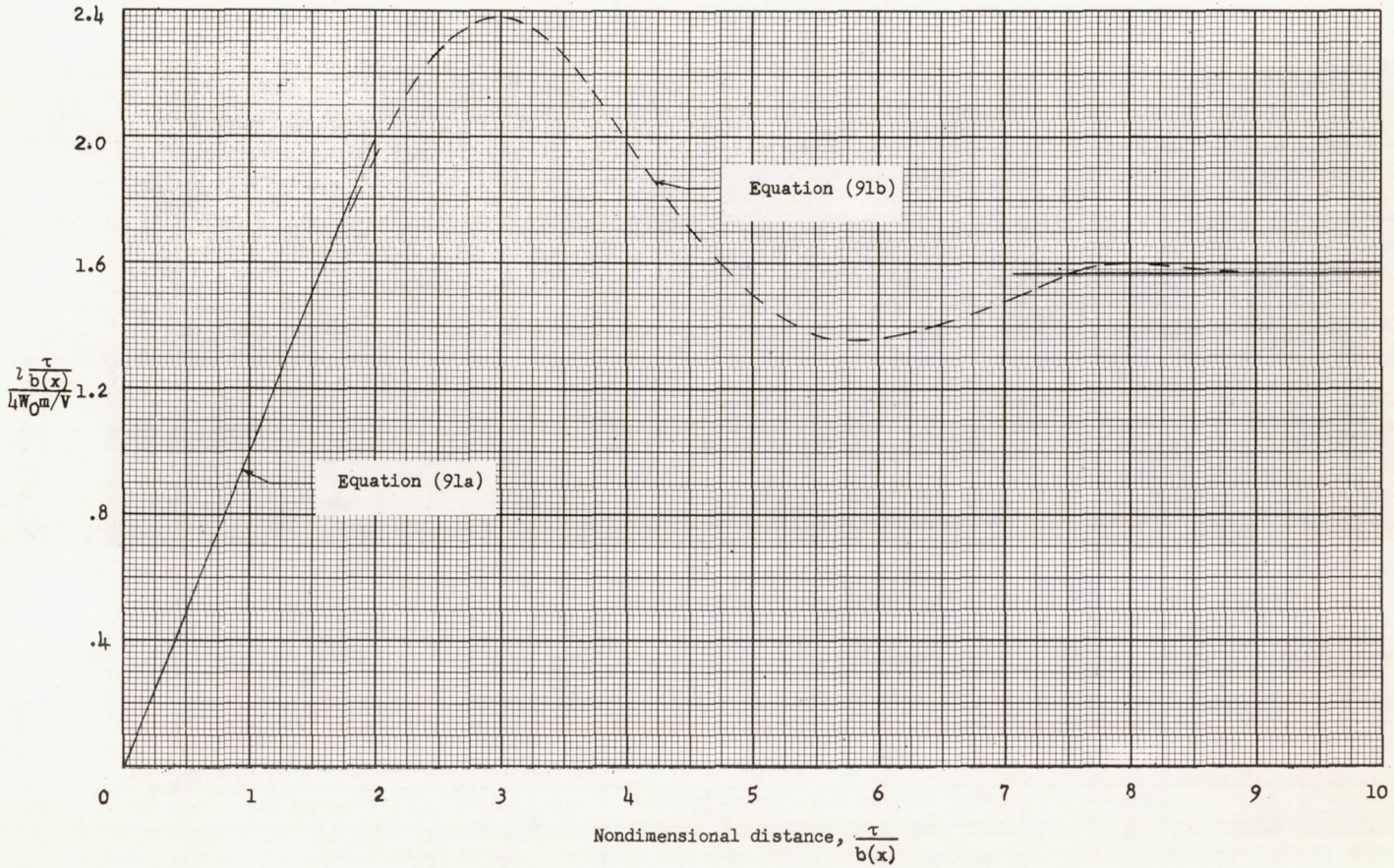


Figure 21.- Growth of lift on a spanwise strip of a narrow delta wing penetrating a sharp-edged gust.

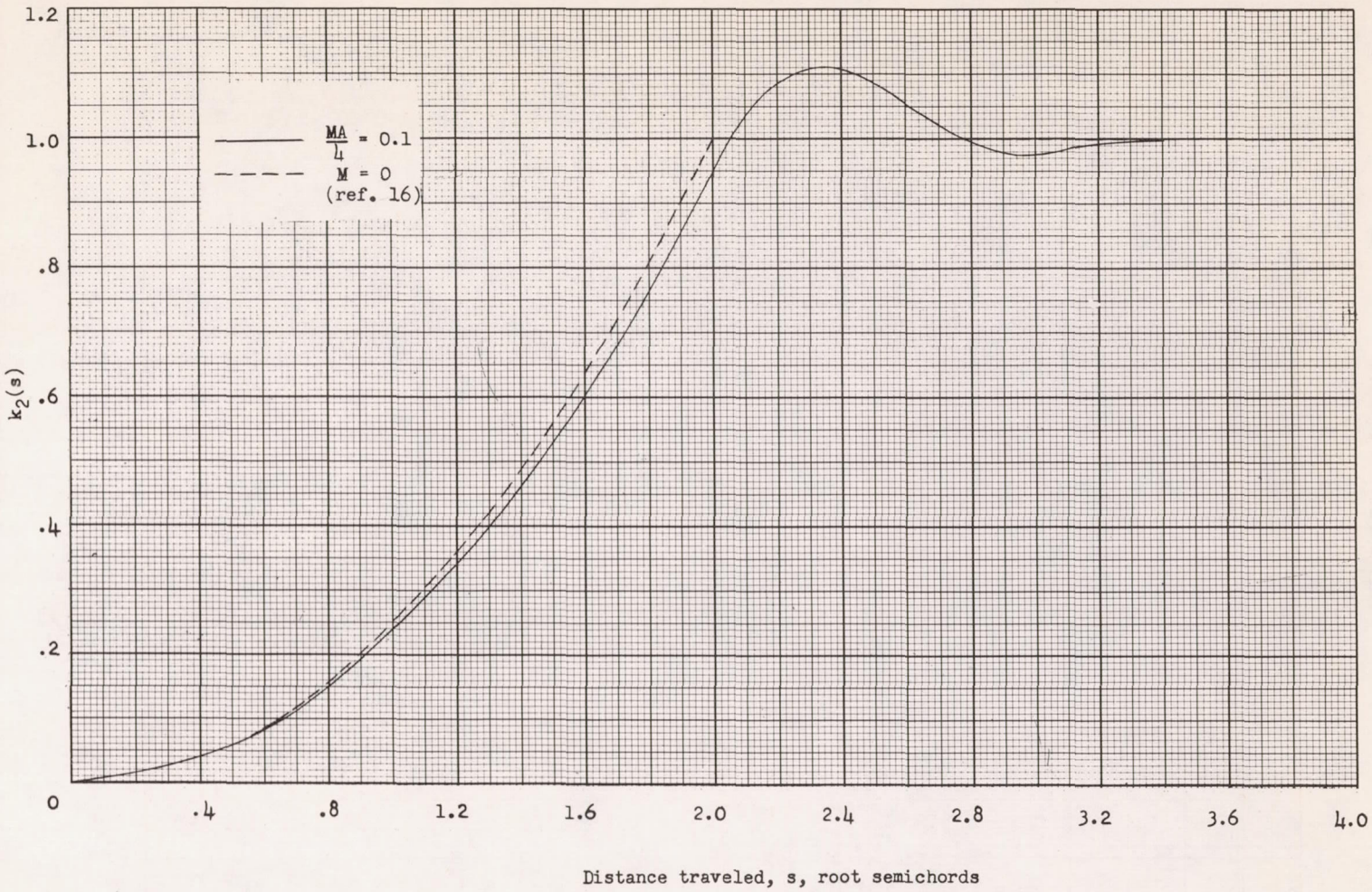


Figure 22.- The indicial lift function $k_2(s)$ for a delta wing of vanishingly small aspect ratio in incompressible and compressible flow.

NACA - Langley Field, Va.