NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

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A METHOD OF COMPUTING THE TRANSIENT TEMPERATURE OF THICK WALLS FROM ARBITRARY VARIATION OF ADIABATIC-WALL TEMPERATURE AND HEAT-TRANSFER COEFFICIENT

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SUMMARY

A method of calculating the temperature of thick walls has been developed in which are used relatively new concepts, such as the time series and the response to a unit triangle variation of surface temperature, together with essentially standard formulas for transient temperature and heat flow into thick walls. The method can be used without knowledge of the mathematical tools of its development. The method is particularly suitable for determining the wall temperature in one-dimensional thermal problems in aeronautics where there is a continuous variation of the heat-transfer coefficient and adiabatic-wall temperature. The method also offers a convenient means for solving the inverse problem of determining the heat-flow history when temperature history is known.

A series of diversified problems were solved by exact analysis as well as by the new method. A comparison of the results shows the new method to be accurate. The labor involved is very modest in consideration of the nature of the thick-wall temperature problem. Limiting solutions for the "infinitely thick" wall and for walls so thin that thermal lag can be neglected were also obtained.

INTRODUCTION

In aeronautical applications, external surfaces are heated by the impact and friction of the air. For cases in which the structural temperatures never reach equilibrium, the transient temperatures of the surfaces often govern the design; and it is necessary to be able to predict these temperatures.

Literature on transient temperatures in thick walls dates from the classical works of Fourier. Perhaps the most extensive work on the subject is given in reference 1. Most literature giving the solution to the transient temperatures in thick walls is based on the premise that the temperature history of one or more principal surfaces is known or given. Only a limited amount of literature is available relative to transient temperatures in thick walls under the influence of forced convection. The forced-convection equation for heat transfer in aeronautical applications is $q = h(T_{aw} - T)$, which states that the rate of heating q is proportional to the difference between the adiabatic-wall temperature T_{aw} and the wall temperature T. The coefficient of proportionality is the heat-transfer coefficient h. In the classical problem of the convection heating of a thick wall, h has been assumed to be constant. In the usual aeronautical application, the fact that h varies with time is the source of the difficulty in obtaining a solution.

The thick-wall case treated in this paper is the one governed by Fourier's classical partial differential equation

$$\frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}$$

In the case governed by this equation, the wall is composed of a homogeneous material and the temperature gradients and heat flow parallel to the surface are negligible. In one boundary relation for this case, the convective heat rate is equated to the heat absorbed by the wall or to the product of the conductivity and the temperature gradient in the wall at the heated surface; that is,

$$h(T_{aw} - T) = K\left(\frac{\partial T}{\partial x}\right)_w$$

Since h occurs as a product with T in this boundary equation, the usual procedures of operational calculus do not apply. When solutions for the temperatures of thick walls have been necessary in aeronautical work, the method generally used has been to divide the thick wall into a number of slabs in order to make a step-by-step numerical integration of Fourier's equation of heat flow. Since steps in both distance and time must be taken, the procedure is tedious and time consuming unless the use of a high-speed automatic computing machine is resorted to. If it is necessary to do the work without the use of such equipment, a method introduced by Schmidt (ref. 2) wherein some of the calculations are accomplished graphically may be used to reduce the labor to some extent. This method is known as the Schmidt plot method.

In the present paper a simple method is developed for the calculation of the temperature history of the surfaces of a thick wall or of any plane within the wall. The procedure is to select from a table a set of coefficients which depend on the physical properties of the wall. These coefficients and other data are substituted into explicit algebraic formulas to determine the temperature of the heated wall surface. If the heat-transfer coefficients are known, no guess or iteration procedure is required. As can be seen by the results of the example problems presented, the accuracy can be as good as is desired. For equal time-step sizes, the method is more accurate than more laborious numerical methods.

The simplicity of the results depends on two factors: One is the suppression of the variable x representing the distance into the wall by using an integrated form of Fourier's equation and assigning a value of x corresponding to the heated surface. The other is a mathematical device known as the time series introduced by Tustin (ref. 3). The time series is defined in appendix A. Reference 3 also introduced various manipulations of the series. The multiplication of two series is an important manipulation by means of which specific results can be generalized. Other writers (ref. 4, for example) have also presented various manipulations of the series.

The present paper is divided into two parts, analysis and application. The section on analysis includes a treatment of the determination of the temperature history for the special cases of the thermally thin wall and the infinitely thick wall as well as for the wall of intermediate thickness. The inverse problem of determining the heat flow corresponding to a known temperature history is also discussed. Although the method was set up for the purpose of predicting wall temperatures in engineering applications, it has also been found to be suitable for research applications wherein the transient skin temperature is measured and the heat-flow and heat-transfer-coefficient histories are deduced. Appendix A gives background material pertaining to the use of time series that may be an aid to a study of the analysis. Appendix B gives a summary of analytical temperature and heat-flow formulas used either as a basis of analysis or used in the solution of examples to test the accuracy of the present method. In the section on application the computing formulas are reviewed and several examples of their use are given. Because of the explicit nature of the temperature formulas, it is not necessary to study the analysis to use the results:

SYMBOLS

$A_0, A_1, A_2, \ldots A_m$ — dimensionless coefficients to determine heated-surface temperature history

b — slope of wall surface temperature with respect to time, $^{\circ}$F/hr

$B_0, B_1, B_2, \ldots B_m$	dimensionless coefficients to determine inside-wall temperature history
c	specific heat, Btu/lb-$^{\circ}$F
e	base of natural logarithms
$F(x,t)$	any function corresponding to reference slope $y = (1/\delta)t$
$F_y(t)$	any function corresponding to arbitrary control line $y = y(t)$
$F_{\Delta}(t)$	any function corresponding to unit triangle control line
G	heat capacity of wall, $\rho c l$, Btu/(sq ft)($^{\circ}$F)
ηh	heat-transfer coefficient at $x = 0$, Btu/(hr)(sq ft)($^{\circ}$F)
h	heat-transfer coefficient at $x = l$, Btu/(hr)(sq ft)($^{\circ}$F)
H	heat-transfer number, $h\delta\pi^2/16G$
H_{∞}	heat-transfer number for infinitely thick wall, $3h\sqrt{\pi k\delta}/8K$
k	diffusivity, $K/c\rho$, sq ft/hr
K	conductivity, (Btu)(ft)/(hr)(sq ft)($^{\circ}$F)
l	wall thickness, ft
M_1, M_2, M_3	memory coefficients, dimensionless
m	term designating time in multiples of basic interval δ
n	term number in infinite series
q	instantaneous heat-transfer rate due to uniform temperature rise of heated wall surface of 1° in time δ, Btu/(hr)(sq ft)
$\bar{q}_m$	average heat-transfer rate from time $(m - 1)\delta$ to $m\delta$ due to uniform temperature rise of heated wall surface of 1° in time δ, Btu/(hr)(sq ft)

q_{Δ}	heat-transfer rate corresponding to unit triangle variation of surface temperature
$\bar{q}_{\Delta,m}$	average heat-transfer rate from time $(m - 1)\delta$ to $m\delta$ due to unit triangle reference temperature variation of heated-wall surface, Btu/(hr)(sq ft)
$q_s(t)$	heat-flow history at heated surface due to unit temperature step of heated surface
r	radiation rate, Btu/(hr)(sq ft)
R	radiation term, $r\delta\pi^2/16G$, °F
R_∞	radiation term for infinitely thick wall, $3R\sqrt{\pi k\delta}/8K$, °F
t	time, hr
T	heated-wall-surface temperature, °F
$T_1, T_2, T_3, \ldots T_m$	T expressed as time series, °F
T_1	value of step in wall-surface temperature, °F
T_{aw}	adiabatic-wall temperature or effective boundary-layer temperature, °F
T_i	temperature of inside (unheated) surface or of any plane within wall, °F
V(x,t)	temperature response to unit step in T_{aw}, °F
x	distance through wall, ft
y	ordinate of control line or altitude of triangle
α_n	positive roots of auxiliary equation in analytical solution of wall temperature
δ	basic time interval in time series
η	ratio of heat-transfer coefficient at cooler wall surface to heat-transfer coefficient at heated wall surface
θ	difference in temperature between heated surface and any other plane due to unit triangle variation of heated surface, °F

θ_r difference in temperature between heated surface and any other plane due to uniform reference-temperature rise of 1° in time δ, $^{\circ}F$

ρ weight density, lb/cu ft

τ dummy time variable, hr

A prime denotes the derivative with respect to time.

ANALYSIS

TEMPERATURES ON OUTSIDE SURFACE

Problem

The wall considered in this paper is composed of a homogeneous material, and the temperature gradients and heat flow parallel to the surface are negligible. The transient temperatures of the heated or outside surface of the wall are determined by means of Fourier's equation which governs the heat flow through the wall:

$$\frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} \tag{1}$$

The wall properties expressed by the diffusivity k are constant. The boundary conditions are given by the statements that the flow of heat at the unheated face of the wall (where x is taken as zero) is zero, that is,

$$\left(\frac{\partial T}{\partial x}\right)_{x=0} = 0 \tag{2}$$

and that the heat transferred to the heated face is given by the relation

$$h(T_{aw} - T) = K(\partial T/\partial x)_{x=l} \tag{3}$$

For convenience, the initial temperature is taken as zero at zero time:

$$T = 0 \qquad (t = 0) \tag{4}$$

Since both h and T are functions of t and occur as a product in equation (3), the transform procedures of operational calculus do not apply. The problem may be stated in another form by means of an integral equation expressing the heat balance at the heated surface.

Let $q_s(t)$ be the heat-flow history at the heated wall surface at $x = l$ due to a unit step in that wall-surface temperature at $t = 0$. Then the heat flow $q(t)$ due to a temperature history $T(t)$ of the heated surface that is continuous and is zero when $t = 0$, but is otherwise an arbitrary variation, can be expressed by Duhamel's integral in the form indicated in equation (5). In this integral T' represents $\frac{d}{dt} T(t)$ and τ is a dummy time variable.

$$q(t) = \int_0^t T'(t - \tau) q_s(\tau) d\tau \tag{5}$$

A heat balance is formed at $x = l$ by equating the integral in equation (5) to the left member of the boundary condition expressed by equation (3):

$$h(T_{aw} - T) = \int_0^t T'(t - \tau) q_s(\tau) d\tau \tag{6}$$

The determination of T from equation (6) establishes the solution.

Method

The wall temperature T is determined, in general, from equation (6) for thermally thin, thick, and infinitely thick walls. The method first discussed is based on thick walls, and modifications of this method are introduced for the special cases of thermally thin and infinitely thick walls.

Thermally thick walls.-

Time series: In the calculation of wall temperature T for thick walls, the right member of equation (6) is replaced by the product of two time series. As explained in detail in appendix A, a time series is the value (here the ordinate) of a function of time at successive equal increments of time δ. Thus, any variation of wall surface temperature can be expressed as the series

$$T = T_1, T_2, T_3, \ldots T_m \tag{7}$$

When a wall surface has a unit triangle variation of surface temperature, the surface temperature increases from 0° at a constant rate to a value of 1° at the time $t = \delta$ and decreases at a constant rate to the value 0° at $t = 2\delta$. The heat flow corresponding to a unit triangle variation of surface temperature can be expressed by the series

$$q_\Delta = q_{\Delta,1}, q_{\Delta,2}, q_{\Delta,3}, \ldots q_{\Delta,m} \tag{8}$$

The product of equations (7) and (8) gives the instantaneous values of heat flow due to the temperature variation given by equation (7) and could be used to eliminate the integral in equation (6). However, a slight variation or refinement of the method is introduced which has been found to increase the accuracy of the results without increasing the labor involved.

If m represents the term number of a time series, the product $m\delta$ represents the corresponding time. The refinement consists in averaging the heat flow from the time $(m - 1)\delta$ to the time $m\delta$. Let $\bar{q}_{\Delta,m}$ represent the average heat flow over this interval due to a triangular variation in surface temperature. Then the heat-flow history due to the triangular temperature variation can be represented by the series

$$\bar{q}_\Delta = \bar{q}_{\Delta,1}, \bar{q}_{\Delta,2}, \bar{q}_{\Delta,3}, \ldots \bar{q}_{\Delta,m} \tag{9}$$

where $\bar{q}_{\Delta,1}$ is the average for the time 0 to δ, $\bar{q}_{\Delta,2}$ is the average for time δ to 2δ, and so forth.

The heat-flow history expressed as the average heat flow over successive increments δ, but due to the temperature variation (7), is given by the product of equations (7) and (9):

$$\bar{q} = \left(T_1, T_2, T_3, \ldots T_m\right)\left(\bar{q}_{\Delta,1}, \bar{q}_{\Delta,2}, \bar{q}_{\Delta,3}, \ldots \bar{q}_{\Delta,m}\right) \tag{10}$$

Such a multiplication actually gives the result by forming the proper superpositions, as demonstrated in appendix A.

In order to adjust the boundary condition expressed by equation (3) or the left side of equation (6) to represent an average flow of heat $\bar{q}_m$ over the interval δ, the average flow of heat from the boundary layer is approximated by the mean of the values at the beginning and end of the interval. Thus, for the interval ending at $t = m\delta$, $\bar{q}_m$ has the value

$$\bar{q}_m = \frac{1}{2}\left[h_m(T_{aw} - T)_m + h_{m-1}(T_{aw} - T)_{m-1}\right] \tag{11}$$

If radiation is important to the problem, it may be included. Let r_m be the rate of heat radiation per unit area at the time $m\delta$. Equation (11) may then be written

$$\bar{q}_m = \frac{1}{2}\left[h_m(T_{aw} - T)_m - r_m + h_{m-1}(T_{aw} - T)_{m-1} - r_{m-1}\right] \tag{11a}$$

With or without radiation, the heating history, or variation of $\bar{q}$, may be expressed by the series

$$\bar{q} = \bar{q}_1, \bar{q}_2, \bar{q}_3, \ldots \bar{q}_m \tag{12}$$

wherein each term has the value given by equation (11) or (11a). Equating the average heat flow given by equation (12) to that given by equation (10) yields

$$\bar{q}_1, \bar{q}_2, \bar{q}_3, \ldots \bar{q}_m = \left(T_1, T_2, T_3, \ldots T_m\right)\left(\bar{q}_{\Delta,1}, \bar{q}_{\Delta,2}, \bar{q}_{\Delta,3}, \ldots \bar{q}_{\Delta,m}\right) \tag{13}$$

In order to evaluate equation (13), the product in the right-hand member is expanded by algebraic multiplication and values of both members of the equation pertaining to equal time are equated. The following set of equations results (for simplicity, the radiation terms are not written in):

$$\left.\begin{aligned}
&\frac{1}{2}\left[h_1\left(T_{aw} - T\right)_1 + h_0\left(T_{aw} - T\right)_0\right] = T_1\bar{q}_{\Delta,1} \\
&\frac{1}{2}\left[h_2\left(T_{aw} - T\right)_2 + h_1\left(T_{aw} - T\right)_1\right] = T_2\bar{q}_{\Delta,1} + T_1\bar{q}_{\Delta,2} \\
&\frac{1}{2}\left[h_m\left(T_{aw} - T\right)_m + h_{m-1}\left(T_{aw} - T\right)_{m-1}\right] = T_m\bar{q}_{\Delta,1} + T_{m-1}\bar{q}_{\Delta,2} + \cdots + T_1\bar{q}_{\Delta,m}
\end{aligned}\right\} \quad (14)$$

Equations (14) can be rearranged to obtain the equivalent equations for T:

$$\left.\begin{aligned}
&T_1 = \frac{h_1T_{aw,1} + h_0T_{aw,0}}{2\bar{q}_{\Delta,1} + h_1} \\
&T_2 = \frac{h_2T_{aw,2} + h_1T_{aw,1} - h_1T_1 - 2T_1\bar{q}_{\Delta,2}}{2\bar{q}_{\Delta,1} + h_2} \\
&T_m = \frac{\left(hT_{aw}\right)_m + \left(hT_{aw} - hT\right)_{m-1} - 2\left(\bar{q}_{\Delta,2}T_{m-1} + \bar{q}_{\Delta,3}T_{m-2} + \cdots + \bar{q}_{\Delta,m}T_1\right)}{2\bar{q}_{\Delta,1} + h_m}
\end{aligned}\right\} \quad (15)$$

The values of $\bar{q}_{\Delta,m}$ must be derived.

Determination of average heat flow due to unit triangle variation of surface temperature: In order to obtain the average heat flow due to a unit triangle variation of surface temperature, the average heat-flow rate $\bar{q}_m$ due to a uniform increase in the surface temperature of 1° in each unit time δ must be obtained. The average is taken over the time δ by integrating the instantaneous heat-flow rate from $t = (m - 1)\delta$ to $t = m\delta$ and dividing by δ. This determination is carried out in appendix B. The result is

$$\bar{q}_m = \frac{8G}{\pi^2 \delta}\left(\frac{\pi^2}{8} + A_m - A_{m-1}\right) \tag{16}$$

In equation (16), A_m and A_{m-1} are the summations

$$\left.\begin{aligned} A_m &= \sum_{n=1}^{\infty} \frac{e^{-m(2n-1)^2 \frac{\pi^2}{4} \frac{k\delta}{l^2}}}{\frac{\pi^2}{4} \frac{k\delta}{l^2}(2n - 1)^4} \\ A_{m-1} &= \sum_{n=1}^{\infty} \frac{e^{-(m-1)(2n-1)^2 \frac{\pi^2}{4} \frac{k\delta}{l^2}}}{\frac{\pi^2}{4} \frac{k\delta}{l^2}(2n - 1)^4} \end{aligned}\right\} \tag{17}$$

and G is the heat capacity of the wall per square foot per $^{\circ}$F and is the product of weight density, specific heat, and wall thickness:

$$G = \rho c l \tag{18}$$

The average heat flow due to a unit triangle variation of surface temperature $\bar{q}_{\Delta,m}$ is obtained by the superposition of the heat flows $\bar{q}$ due to three linear variations of wall temperature as follows (for further details, see the development of equation (A1) in appendix A):

$$\bar{q}_{\Delta,m} = \bar{q}_m - 2q_{m-1} + q_{m-2} \tag{19}$$

Expanding equation (19) by substituting for $\bar{q}_m$ from equation (16) gives

$$\bar{q}_{\Delta,m} = \frac{8G}{\pi^2 \delta}\left[\left(\frac{\pi^2}{8} + A_m - A_{m-1}\right) - 2\left(\frac{\pi^2}{8} + A_{m-1} - A_{m-2}\right) + \left(\frac{\pi^2}{8} + A_{m-2} - A_{m-3}\right)\right] \tag{20}$$

Substituting in equation (20) successive values of m, starting with $m = 1$, ignoring any terms with negative subscripts, and collecting terms result in the following equations:

$$\left.\begin{aligned}
\bar{q}_{\Delta,1} &= \frac{8G}{\pi^2\delta}\left(\frac{\pi^2}{8} + A_1 - A_0\right) \\
\bar{q}_{\Delta,2} &= \frac{8G}{\pi^2\delta}\left(-\frac{\pi^2}{8} + A_2 - 3A_1 + 2A_0\right) \\
\bar{q}_{\Delta,3} &= \frac{8G}{\pi^2\delta}\left(A_3 - 3A_2 + 3A_1 - A_0\right) \\
\bar{q}_{\Delta,m} &= \frac{8G}{\pi^2\delta}\left(A_m - 3A_{m-1} + 3A_{m-2} - A_{m-3}\right)
\end{aligned}\right\} \tag{21}$$

For convenience, the quantities in parentheses can be tabulated. The quantities in parentheses usually retain significant values after the completion of the temperature triangle which created them. For this reason, in accordance with the notation of reference 3, they are called memory terms and are designated by the symbol M. With this notation, equations (21) become

$$\left.\begin{aligned}
\bar{q}_{\Delta,1} &= \frac{8G}{\pi^2\delta} M_1 \\
\bar{q}_{\Delta,2} &= \frac{8G}{\pi^2\delta} M_2 \\
\bar{q}_{\Delta,3} &= \frac{8G}{\pi^2\delta} M_3 \\
\bar{q}_{\Delta,m} &= \frac{8G}{\pi^2\delta} M_m
\end{aligned}\right\} \tag{21a}$$

Hence, the following equations for memory terms are established:

$$\left.\begin{aligned}
M_1 &= \frac{\pi^2}{8} + A_1 - A_0 \\
M_2 &= -\frac{\pi^2}{8} + A_2 - 3A_1 + 2A_0 \\
M_3 &= A_3 - 3A_2 + 3A_1 - A_0 \\
M_m &= A_m - 3A_{m-1} + 3A_{m-2} - A_{m-3}
\end{aligned}\right\} \tag{22}$$

Obviously, the values of M are combinations of the values of A. However, as is explained in greater detail in the section entitled "Application," it is usually not necessary to calculate the values of A and M since the values of M listed in table I may be used. The value of M decreases with increasing term number and sooner or later further terms can be neglected.

Resulting temperature formulas: Equations (21a) give the value of $\bar{q}_{\Delta,m}$ sought to complete equations (15); therefore, equations (21a) are substituted into equations (15). The result can be simplified by dividing through by $16G/\pi^2\delta$ and letting

$$H = \frac{h}{\left(\frac{16G}{\pi^2\delta}\right)} \tag{23}$$

If radiation is important, the appropriate terms are included by using equation (11a) rather than equation (11). Since equations (15) are being divided through by $16G/\pi^2\delta$, the radiation term R is defined as

$$R = \frac{r}{\left(\frac{16G}{\pi^2\delta}\right)} \tag{24}$$

With the substitutions of equations (23) and (24) in equations (15), the final results, including terms for radiation, are

$$\left.\begin{aligned}
T_1 &= \frac{H_1T_{aw,1} + H_0T_{aw,0} - R_1 - R_0}{M_1 + H_1} \\
T_2 &= \frac{H_2T_{aw,2} + H_1T_{aw,1} - H_1T_1 - M_2T_1 - R_2 - R_1}{M_1 + H_2} \\
T_m &= \frac{\left(HT_{aw}\right)_m + \left(HT_{aw} - HT\right)_{m-1} - M_2T_{m-1} - M_3T_{m-2} - \cdot \cdot \cdot - M_mT_1 - R_m - R_{m-1}}{M_1 + H_m}
\end{aligned}\right\} \tag{25}$$

Infinitely thick walls.-

General considerations: If a wall is thermally very thick and is heated rapidly so that the unheated side experiences little heating, it is convenient and accurate to assume that the wall is infinitely thick. The same formulas, equations (25), are used to compute the wall surface temperature. However, instead of the values of M for a particular wall or diffusion number, the values of M which are used are always a fixed set of numbers which are now derived. The values of H and R are also changed.

Determination of average heat flow due to unit triangle variation of surface temperature: The determination of the heat flow due to a unit triangle variation of surface temperature of an infinitely thick wall depends upon the instantaneous heat flow into the surface due to a unit rise in surface temperature in unit time. From page 110 of reference 4, the instantaneous heat flow is equivalent to $2K\sqrt{t}/\sqrt{\pi k}$. Since the heat-flow rate is proportional to the surface-temperature slope, the instantaneous heat transfer due to unit rise of surface temperature in the time δ is $2K\sqrt{t}/\delta\sqrt{\pi k}$. This expression is integrated with respect to t between the limits $(m - 1)\delta$ and $m\delta$. Dividing by δ gives, for the average heat-flow rate $\bar{q}_m$ over the interval δ terminating at $m\delta$,

$$\bar{q}_m = \frac{4K}{3\sqrt{\pi k\delta}}\left[m^{3/2} - (m - 1)^{3/2}\right] \tag{26}$$

The usual superposition required to change the result of the slope function to that of the unit triangle input function is accomplished by substituting equation (26) into equation (19):

$$\bar{q}_{\Delta,m} = \frac{4K}{3\sqrt{\pi k\delta}}\left[m^{3/2} - 3(m - 1)^{3/2} + 3(m - 2)^{3/2} - (m - 3)^{3/2}\right] \tag{27}$$

Resulting temperature formulas.- If the bracketed quantity in equation (27) is designated as the memory coefficient M_m, then

$$M_m = \left[m^{3/2} - 3(m - 1)^{3/2} + 3(m - 2)^{3/2} - (m - 3)^{3/2}\right] \tag{28}$$

A dimensionless heat-transfer coefficient (suggested by eq. (27)) is defined as

$$H_\infty = \frac{3h\sqrt{\pi k\delta}}{8K} \tag{29}$$

and a corresponding term for radiation is defined as

$$R_\infty = \frac{3r\sqrt{\pi k\delta}}{8K} \tag{30}$$

The substitution of equation (27) into equations (15) again results in equations (25). Hence, equations (25) are used to obtain the heated-surface temperature of the infinitely thick wall as well as of walls of intermediate thickness, except that M, H, and R for infinitely thick walls are defined by equations (28) to (30). Inspection of equations (28) to (30) indicates that the wall material properties and time-step size are expressed by equations (29) and (30), while the memory terms are invariant with wall properties or step size. Substituting successive integers for m from 1 to 20 into equation (28) gives the following corresponding values of M:

$$\left.\begin{array}{llll}
M_1 = 1.0 & M_6 = -0.040234 & M_{11} = -0.012874 & M_{16} = -0.006807 \\
M_2 = -0.171573 & M_7 = -0.029536 & M_{12} = -0.011069 & M_{17} = -0.006157 \\
M_3 = -0.289129 & M_8 = -0.022885 & M_{13} = -0.009650 & M_{18} = -0.005605 \\
M_4 = -0.103176 & M_9 = -0.018412 & M_{14} = -0.008511 & M_{19} = -0.005130 \\
M_5 = -0.059630 & M_{10} = -0.015232 & M_{15} = -0.007580 & M_{20} = -0.004719
\end{array}\right\} \tag{31}$$

These values of M, along with values of H_∞ and R_∞ from equations (29) and (30), can be used in all problems wherein the wall is so thick relative to the heating rates and times involved that the wall behaves as though it were infinitely thick.

Thin walls.- When a wall is thermally thin, the temperature drop through the wall becomes negligible and the problem is simplified by assuming that all interior temperatures are equal to the surface temperature. The heat absorbed by the wall during any time interval δ must be equal to the gain in enthalpy or total heat during this time. Hence,

$$\bar{q}_m = \frac{1}{\delta}\left(G_m T_m - G_{m-1} T_{m-1}\right) \tag{32}$$

Equating the average rate of gain in enthalpy as given by equation (32) to the average rate of heat transfer through the boundary layer as given by equation (11a) results in the following heat balance:

$$h_m\left(T_{aw} - T\right)_m - r_m + h_{m-1}\left(T_{aw} - T\right)_{m-1} - r_{m-1} = \frac{2}{\delta}\left[(GT)_m - (GT)_{m-1}\right] \tag{33}$$

Solving for T_m gives

$$\left.\begin{aligned}
T_1 &= \frac{h_1 T_{aw,1} + h_0 T_{aw,0} - h_0 T_0 + \frac{2}{\delta} G_0 T_0 - r_1 - r_0}{\frac{2}{\delta} G_1 + h_1} \\
T_2 &= \frac{h_2 T_{aw,2} + h_1 T_{aw,1} - h_1 T_1 + \frac{2}{\delta} G_1 T_1 - r_2 - r_1}{\frac{2}{\delta} G_2 + h_2} \\
T_m &= \frac{h_m T_{aw,m} + \left(hT_{aw} - hT + \frac{2}{\delta} GT\right)_{m-1} - r_m - r_{m-1}}{\frac{2}{\delta} G_m + h_m}
\end{aligned}\right\} \tag{34}$$

Any variation of G with temperature is accounted for by equations (34). If the wall properties do not change over the temperature range covered, obviously, G is a constant. If G is considered to be constant, equations (34) can be derived from equations (25) as follows: As the diffusion number $k\delta/l^2$ becomes large, all values of A approach the value $\pi^2/8$, and from equations (22) it is seen that the only memory terms not identically equal to zero are M_1 and M_2, which have the values $\pi^2/8$ and $-\pi^2/8$. Eliminating M from equations (25) and utilizing definitions (23) and (24) yield equations (34).

In equations (34) the terms h_0, $T_{aw,0}$, and T_0 have been retained since, unlike the thick-wall problem, it is here convenient for T_0 to have any value. These equations have considerable advantage since the need for temperature extrapolation is reduced, if not eliminated. The equations tend to give accurate results and, as is shown subsequently, are suitable for the use of relatively large time increments.

INSIDE TEMPERATURES

If the heated wall surface is called the outside surface, the temperatures at other parallel planes may be called inside temperatures. In particular, this paper is concerned with the inside surface temperature. According to the notation used in this paper, the inside surface is designated by $x/l = 0$, the outside surface by $x/l = 1$, and other planes by values of x/l between 0 and 1.

Consider a wall, initially at zero temperature, which has the heated surface $x = l$ raised at a reference temperature slope $T = (1/\delta)t$, while the surface $x = 0$ is insulated. The difference in temperature θ_r between the heated surface and any plane x is shown by equation (B11) of appendix B to be

$$\theta_r = \frac{16}{\pi^3}\frac{l^2}{k\delta}\left\{\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^3}\cos\left[(2n-1)\frac{\pi}{2}\frac{x}{l}\right] - \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^3}\cos\left[(2n-1)\frac{\pi}{2}\frac{x}{l}\right]e^{-(2n-1)^2\frac{\pi^2}{4}\frac{kt}{l^2}}\right\} \qquad (35)$$

A set of terms is defined to represent the summations in equation (35):

$$\left.\begin{aligned}
B_0 &= \frac{16}{\pi^3}\frac{l^2}{k\delta}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^3}\cos\left[(2n-1)\frac{\pi}{2}\frac{x}{l}\right] \\
B_1 &= \frac{16}{\pi^3}\frac{l^2}{k\delta}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^3}\cos\left[(2n-1)\frac{\pi}{2}\frac{x}{l}\right]e^{-(2n-1)^2\frac{\pi^2}{4}\frac{k}{l^2}\delta} \\
B_2 &= \frac{16}{\pi^3}\frac{l^2}{k\delta}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^3}\cos\left[(2n-1)\frac{\pi}{2}\frac{x}{l}\right]e^{-2(2n-1)^2\frac{\pi^2}{4}\frac{k}{l^2}\delta} \\
B_m &= \frac{16}{\pi^3}\frac{l^2}{k\delta}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^3}\cos\left[(2n-1)\frac{\pi}{2}\frac{x}{l}\right]e^{-m(2n-1)^2\frac{\pi^2}{4}\frac{k}{l^2}\delta}
\end{aligned}\right\} \tag{36a}$$

If the temperature difference across the entire wall is sought, then $x/l = 0$ and equations (36a) become

$$\left.\begin{aligned}
B_0 &= \frac{16}{\pi^3}\frac{l^2}{k\delta}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^3} = \frac{l^2}{2k\delta} \\
B_1 &= \frac{16}{\pi^3}\frac{l^2}{k\delta}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^3}\,e^{-(2n-1)^2\frac{\pi^2}{4}\frac{k}{l^2}\delta} \\
B_2 &= \frac{16}{\pi^3}\frac{l^2}{k\delta}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^3}\,e^{-2(2n-1)^2\frac{\pi^2}{4}\frac{k}{l^2}\delta} \\
B_m &= \frac{16}{\pi^3}\frac{l^2}{k\delta}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^3}\,e^{-m(2n-1)^2\frac{\pi^2}{4}\frac{k}{l^2}\delta}
\end{aligned}\right\} \tag{36b}$$

In either case, by using equations (36a) or (36b) equation (35) may be written

$$\theta_r = B_0 - B_m \tag{37}$$

By the usual superposition, in order to change the result of the reference slope function to that of a unit triangular input function, the temperature difference between the plane of the heated wall and the plane of consideration is

$$\theta = B_0 - B_m - 2(B_0 - B_{m-1}) + B_0 - B_{m-2} \tag{38}$$

Assigning integral values to m and simplifying give the equations corresponding to successive values of time:

$$\left.\begin{aligned}
\theta_1 &= -B_1 + B_0 \\
\theta_2 &= -B_2 + 2B_1 - B_0 \\
\theta_3 &= -B_3 + 2B_2 - B_1 \\
\theta_4 &= -B_4 + 2B_3 - B_2 \\
\theta_m &= -B_m + 2B_{m-1} - B_{m-2}
\end{aligned}\right\} \tag{39}$$

A sufficient number of values of θ for practical purposes are given in table I. Equations (39) mean that the temperature difference between the plane of the heated surface and any other plane due to a unit triangle temperature variation of the heated surface is given by the time series $\theta_t = \theta_1, \theta_2, \theta_3, \ldots \theta_m$. In addition, any general temperature of the heated surface can be represented as $T = T_1, T_2, T_3, \ldots T_m$. The temperature difference due to this general temperature variation is obtained by formally multiplying these two time series, and the inside temperature is obtained by subtracting the product from the heated-surface temperature T. Hence

$$\left.\begin{aligned}
T_{i,1} &= T_1 - \theta_1 T_1 \\
T_{i,2} &= T_2 - \left(\theta_1 T_2 + \theta_2 T_1\right) \\
T_{i,3} &= T_3 - \left(\theta_1 T_3 + \theta_2 T_2 + \theta_3 T_1\right) \\
T_{i,m} &= T_m - \left(\theta_1 T_m + \theta_2 T_{m-1} + \ldots + \theta_m T_1\right)
\end{aligned}\right\} \tag{40}$$

Computation of $T_{i,m}$ does not depend on prior computation of $T_{i,1}$, $T_{i,2}$, and so forth.

If the temperature distribution through the wall is required in a form which is analytical with respect to x and can be differentiated or integrated analytically with respect to x, the procedures outlined in appendix C should be followed.

CALCULATION OF HEAT FLOW FROM TEMPERATURE HISTORY

Temperature History of Outside Surface Known

If the heated-surface temperature history is known, the heat-flow history can be determined by substituting equations (21a) into equation (10) as follows:

$$\bar{q} = \frac{8G}{\pi^2\delta}\left(T_1,\ T_2,\ T_3,\ .\ .\ .\ T_m\right)\left(M_1,\ M_2,\ M_3,\ .\ .\ .\ M_m\right) \tag{41}$$

Multiplication shows that the mth term is given by

$$\bar{q}_m = \frac{8G}{\pi^2\delta}\left(M_1T_m + M_2T_{m-1} + .\ .\ . + M_mT_1\right) \tag{42}$$

The heat-flow history can be determined readily from a given temperature history of the heated surface by means of equation (42). The average heat flow over a small interval δ can be assumed to give the instantaneous rate at the center of the interval.

Temperature History of Inside Surface Known

If the temperature history of the outer surface or of a plane near the outer surface is known, the feasibility of accurately determining the heat flow is excellent. If the temperature history of the inside surface for a thermally thin wall (kt/l^2 large and hl/K small) is known, it is also feasible to determine the history of the heat flow into the outer surface. However, if the wall is thermally thick, relatively small changes in temperatures at the inside surface may make it difficult to reconstruct the temperature history and heat flow at the outer surface.

A rearrangement of equations (40) may be used to determine the outside-surface-temperature history from the inside-surface-temperature history. Rearranging equations (40) gives

$$\left.\begin{aligned} T_1 &= \frac{T_{i,1}}{1 - \theta_1} \\ T_2 &= \frac{T_{i,2} + \theta_2 T_1}{1 - \theta_1} \\ T_m &= \frac{T_{i,m} + \theta_2 T_{m-1} + \theta_3 T_{m-2} + \cdot \cdot \cdot + \theta_m T_1}{1 - \theta_1} \end{aligned}\right\} \tag{43}$$

The rate of heat flow at any time is then determined from equation (42).

APPLICATION

GENERAL CONSIDERATIONS

The section on application is devoted to demonstrating the solution of two types of problems: In the first type the heat-transfer coefficient and adiabatic-wall-temperature histories are given and the wall-temperature solutions are obtained. In the second type the temperature history is known and the heating-rate history is computed. In each example, the problem chosen was a special case, selected so that its solution could be and was obtained by an exact analytical method. The degree of exactness of the present method is demonstrated by comparing each result with the solution calculated by exact theory.

With one exception, radiation was a negligible consideration in the examples given. Although the method presented is well suited to accounting for radiation and includes terms for that purpose, radiation was neglected in all cases to make possible an exact analytical solution for comparison.

Equations for Heated Surface

If the temperature of the heated surface of the wall is required, equations (18) and (23) to (25) are used. For convenience, these equations are summarized as follows:

$$k = \frac{K}{c\rho} \qquad G = \rho c l \qquad H = \frac{h\delta\pi^2}{16G} \qquad R = \frac{\bar{r}\delta\pi^2}{16G}$$

$$T_1 = \frac{H_1T_{aw,1} + H_0T_{aw,0} - R_1 - R_0}{M_1 + H_1}$$

$$T_2 = \frac{H_2T_{aw,2} + \left(HT_{aw} - HT\right)_1 - M_2T_1 - R_2 - R_1}{M_1 + H_2}$$

$$T_3 = \frac{H_3T_{aw,3} + \left(HT_{aw} - HT\right)_2 - M_2T_2 - M_3T_1 - R_3 - R_2}{M_1 + H_3}$$

$$T_4 = \frac{H_4T_{aw,4} + \left(HT_{aw} - HT\right)_3 - M_2T_3 - M_3T_2 - M_4T_1 - R_4 - R_3}{M_1 + H_4}$$

$$T_m = \frac{H_mT_{aw,m} + \left(HT_{aw} - HT\right)_{m-1} - M_2T_{m-1} - M_3T_{m-2} - \cdots - M_mT_1 - R_m - R_{m-1}}{M_1 + H_m}$$

Whether the objective is to compute wall temperature or to compute the heat flow from a known wall temperature, the first steps involve the determination of the required memory coefficients. In order to minimize the labor involved, the recommended procedure is as follows: First, choose a tentative time interval δ which seems appropriate to the particular problem. (A review of the examples presented herein will give an idea of a reasonable value.) Then compute a tentative value of the dimensionless diffusion number $k\delta/l^2$. From table I or II pick a diffusion number close to the one tentatively computed. The memory coefficients M and inside temperature coefficients θ given in the table for this diffusion number are to be used, and they do not therefore have to be computed. An adjustment in the value of δ is made by multiplying it by the ratio of the tabular value of $k\delta/l^2$ selected to the value of $k\delta/l^2$ tentatively computed. Then compute from equations (18) and (23) the value of G and the values of H; if the radiation is important, R must be computed also (eq. (24)). The temperature history of the heated-wall surface is then found from equations (25). If the heating rate is being determined from a known temperature history, the values of H and R are not required. The procedure for this case is discussed in the section entitled "Example 6."

The temperature formulas were derived with the assumption that the initial wall temperature was zero in order to avoid writing $T - T_0$ numerous times in the formula. The simplest way to handle most problems is to subtract the amount that the initial wall temperature is above zero from both the wall and the adiabatic-wall temperatures. The last step in the problem is to add this amount to the solution.

Equations for Inside Surface

If the temperature of the unheated side of the wall is required, equations (40) are used. These equations are summarized for convenience:

$$T_{i,1} = T_1 - \theta_1 T_1$$

$$T_{i,2} = T_2 - \left(\theta_1 T_2 + \theta_2 T_1\right)$$

$$T_{i,3} = T_3 - \left(\theta_1 T_3 + \theta_2 T_2 + \theta_3 T_1\right)$$

$$T_{i,m} = T_m - \left(\theta_1 T_m + \theta_2 T_{m-1} + \ldots + \theta_m T_1\right)$$

Except for the case of the thermally thin wall, all equations were derived for constant material properties. For small changes in material properties with temperature, it appears reasonable to use an average value of the properties for the temperature range involved. For cases in which material properties vary, it seems possible that a more accurate answer might be obtained by varying the diffusion number or by varying the step size to keep the diffusion number constant; however, any consideration of such a technique is beyond the scope of this paper.

APPLICATION OF METHOD IN SPECIFIC EXAMPLES

The following illustrative examples were calculated before table I was prepared. Therefore, the values of the coefficients M and θ were computed for the particular walls and chosen time intervals δ of the examples. The values of M and θ used are all listed in table II, which may be considered as being supplementary to table I.

Example 1

Example 1(a).-

Problem: A copper wall which is 1/2 inch thick is initially at a temperature of 0° F. One surface is heated by a boundary layer while the other side is insulated. The effective boundary-layer temperature T_{aw} is initially 0° F but increases linearly at the rate of 1,000° F per second for 10 seconds. The heat-transfer coefficient remains constant at

$h = 100$ Btu/(hr)(sq ft)(°F). The conductivity K and diffusivity k of copper are taken as

$$K = 227(\text{Btu})(\text{ft})/(\text{hr})(\text{sq ft})(^\circ\text{F})$$

$$k = 4.41 \text{ sq ft/hr}$$

Find the temperature history of both wall surfaces.

Solution: The material properties are usually given in terms of the hour unit. However, since fast heating conditions may be more easily understood in terms of seconds, time is referred to in seconds and is converted to hours for use in the equations. For example, if

$$\delta = 1 \text{ sec} = \frac{1}{3600} \text{ hr}$$

then

$$\frac{k\delta}{l^2} = \frac{(4.41)(24)^2}{3600} = 0.7056$$

By using this dimensionless diffusion number, the values of M in column 2 of the following table are obtained from table II. The values of T_{aw} are listed in column 4. The value of G is given by the equation

$$G = \frac{Kl}{k} = \frac{227}{(24)(4.41)} = 2.1447 \text{ Btu/(sq ft)}(^\circ\text{F})$$

The value of $h = 100$ Btu/(hr)(sq ft)(°F) is converted to $H = 0.00800$. The use of columns 2 and 4 in equations (25) gives the heated-wall temperature in column 5. Using the inside-surface-temperature formulas, equations (40), and the values of θ in column 6 gives the values of T_i in column 7.

1	2	3	4	5	6	7
Term number	M	Time, sec	T_{aw}, °F	T, °F	θ, °F	T_i, °F
0	-------	0	0	0	--------	0
1	0.75160	1	1,000	11	0.580382	4
2	-.35256	2	2,000	36	-.474632	21
3	-.33055	3	3,000	74	-.087207	50
4	-.05648	4	4,000	124	-.015291	91
5	-.00990	5	5,000	186	-.002681	144
6	-.00174	6	6,000	260	-.000470	210
7	-.00030	7	7,000	346	-.000082	288
8	-.00005	8	8,000	443	-.000015	377
9	-.00001	9	9,000	552	-.000002	478
10	-.00000	10	10,000	673	-.000000	591

The wall-surface-temperature curves of T and T_i are shown in figure 1(a). For comparison, the results calculated by the theoretically exact formula (eq. (B10) of appendix B) are shown. This formula is

$$T = T_{aw} - 2b\frac{l^2}{k}\sum_{n=1}^{\infty}\frac{\cos\left(\frac{x}{l}\alpha_n\right)\left(1 - e^{-\frac{kt}{l^2}\alpha_n^2}\right)}{\alpha_n^2\left(1 + \frac{hl}{K} + \frac{K}{hl}\alpha_n^2\right)\cos\alpha_n} \tag{44}$$

where α_n represents the positive roots of the auxiliary equation

$$\alpha_n \tan\alpha_n = \frac{hl}{K} \tag{44a}$$

Setting $b = 3,600,000$ °F/hr and setting $x = l$ and $x = 0$ in equation (44) result in the values for outside and inside temperatures plotted as circles and squares in figure 1. The comparison shows that accurate results are obtained by the present method, the maximum difference between methods being about 1°.

Example 1(b).-

Problem: The conditions for example 1(b) are the same as those for example 1(a), except that the copper wall is 3 inches thick, or $l = 1/4$ foot.

Solution: Since this wall is so thick, more highly transient conditions prevail throughout the heating period. A value of δ of 1/2 second or 1/7200 hour was therefore used. With the same procedure as used before, the results of the present method are given as continuous lines in figure 1(b), while the results from equation (44) are given by the symbols. The agreement is evident.

Example 2

Problem: Example 2 illustrates the principal advantage of the present method; that is, its capability of handling arbitrary variations of T_{aw} and h. Neither is it necessary to know a mathematical formulation for these variations.

A 1/2-inch copper wall which is initially at a temperature of zero is aerodynamically heated on one side and insulated on the other. The time histories of T_{aw} and h are given at 1/2-second intervals in the following table:

Time, sec	T_{aw}, °F	h, Btu/(hr)(sq ft)(°F)	Time, sec	T_{aw}, °F	h, Btu/(hr)(sq ft)(°F)
0	0	36	5.5	5,391	68.4
.5	1,365	41.4	6.0	5,356	66.6
1.0	2,485	45.0	6.5	5,255	63.9
1.5	3,388	48.6	7.0	5,107	60.0
2.0	4,094	52.2	7.5	4,831	55.8
2.5	4,620	55.8	8.0	4,335	52.2
3.0	4,932	60.0	8.5	3,654	48.6
3.5	5,119	63.9	9.0	2,769	45.0
4.0	5,263	66.6	9.5	1,658	41.4
4.5	5,345	68.4	10.0	297.5	36.0
5.0	5,387	69.0			

Find the temperature history of both wall surfaces.

Solution: The heating conditions are severe and continuously transient, with the boundary-layer temperature rising and falling over 5,000° in 10 seconds. A computing interval δ smaller than that used in the first problem is therefore used. Let δ = 1/2 second = 1/7200 hour. The resulting wall-surface-temperature curves are drawn in figure 2. The circles and squares give the surface temperatures computed by a theoretically exact procedure. Comparison shows the present method to be accurate.

Example 3

Problem: Example 3 is the same as example 2 except that the wall is 3 inches thick and the effective boundary-layer temperatures are slightly different. The initial value of T_{aw} is 0. The subsequent values of T_{aw} are given at 0.5-second intervals by the following time series: T_{aw} = 1,365, 2,484, 3,386, 4,088, 4,609, 4,915, 5,094, 5,227, 5,296, 5,325, 5,315, 5,265, 5,149, 4,986, 4,694, 4,184, 3,489, 2,591, 1,471, 100. Find the temperature history of both wall surfaces.

Solution (a) - thick-wall solution: The value of δ was taken as 1/2 second. By following standard procedure, the wall-surface-temperature curves shown in figure 3 were obtained. Again the symbols give the results of exact theory.

It should be noted that the "exact solution" for examples 2 and 3 is not actually an alternate method of solution for any practical problem but gives a solution to the particular problems only. The "solution" was obtained by working in reverse; that is, a heat flow was assumed and the corresponding boundary-layer characteristics were computed. A truly alternate method of solution is now considered, however.

Solution (b) - infinitely thick-wall solution: Since the thermal lag of a 3-inch copper wall is so great when subjected to the rapid heating specified by this problem, it appears reasonable to obtain the heated-surface temperature by assuming that the wall is infinitely thick. The memory coefficients are the same for all infinitely thick walls and are given by equation (31). The same temperature formulas, the same values of T_{aw}, and the same values of h are used as before, but the values of H_∞ are given by equation (29). The results calculated by this method are listed along with those from the thick-wall solution (a).

Time, sec	T, °F, calculated by -	
	Thick-wall solution	Infinitely thick-wall solution
1	12.8	12.8
2	35.3	35.3
3	61.3	61.3
4	87.0	87.0
5	109.1	109.1
6	124.8	124.8
7	132.3	132.3
8	130.6	130.6
9	119.5	119.4
10	100.0	99.8

The consistency of the alternate methods for a thermally thick wall is evident. The reason for the close agreement may be found in figure 3, which shows that the unheated surface of the 3-inch wall rose to only 14° F.

Example 4

Problem: The most severe test of the present method would occur if there were a large instantaneous increase of T_{aw}. While this condition could hardly happen in flight, it might happen if a research model were suddenly immersed in a high-stagnation-temperature jet. Let a 1/2-inch copper wall, initially at a temperature of zero, be instantly subjected to an effective boundary-layer temperature of 5,000° F on one surface while no heat transfer occurs on the other surface. The heat-transfer coefficient is 100 Btu/(hr)(sq ft)(°F). Solve for the temperature history of both wall surfaces for 10 seconds.

Solution: In this case not only is there a very high transient-temperature condition initially but the instantaneous increase in T_{aw} does not lend itself to approximation by the unit triangle. The simplest procedure is to take small steps for the first few seconds to minimize the errors introduced. In order to help circumvent the difficulty of calculation, an excellent method of approximating the wall surface temperature for the first or first few small steps is to use the following formula from page 109 of reference 4, which gives the temperature on the surface of an infinitely thick wall for a constant flow of heat at the surface:

$$T = \frac{2hT_{aw}}{K}\sqrt{\frac{kt}{\pi}} \tag{45}$$

The values of δ used were $\delta = 1/8$ second for 2 seconds, then $\delta = 1/2$ second for 8 seconds. Since the use of an equation based on an infinite wall is permissible for a 1/2-inch copper wall for at least 1/4 second, the values of T were computed by equation (45) for the first two 1/8-second steps, then by the usual equations. The results are presented in figure 4. Since the inside temperature T_i depends on the outside temperature T and not directly on T_{aw}, there is no particular difficulty of approximation in obtaining T_i. Accordingly, in obtaining T_i, 1/8-second steps were taken for 1 second to define the highly transient part of the curve, then 1/2-second steps for the remaining 9 seconds.

A theoretically exact solution to this problem was obtained by equation (B7) of appendix B. The results of applying this equation are given by the symbols in figure 4, which shows that agreement was obtained.

Example 5

Problem: A 1/16-inch Inconel wall is heated by a high-temperature jet. If $T_{aw} = 5{,}000^{\circ}$ F and $h = 50$ Btu/(hr)(sq ft)($^{\circ}$F), determine the skin-temperature history for 15 seconds. Neglect radiation.

Solution: The heat capacity of a 1/16-inch Inconel wall was assumed to be $G = 0.3229$ Btu/(sq ft)($^{\circ}$F). The example was worked three times with values of δ of 1, 2, and 5 seconds to show how sensitive the "thin wall" formula is to step size. Substituting the given constants into equation (34) yielded the results given by the symbols in figure 5. For this example the exact theory is shown by the solid line. Evidently, coarse steps are permissible with this formula.

Example 6

Example 6(a).-

Problem: The temperature history of the heated surface of a 1/2-inch copper wall initially at zero temperature is given by the following time series in which the temperatures are separated by 1/2 second: $T = 4.8$, 13.2, 25.2, 40.2, 58.2, 77.9, 99.4, 122.6, 146.3, 169.8, 193.0, 214.8, 235.3, 253.2, 268.8, 281.5, 290.8, 296.7, 298.0, 297.5. The inside surface is insulated. Determine the history of heat flow into the heated surface from the given surface-temperature history.

Solution: If the time interval used is sufficiently small, the average rate of heat flow over the interval is a good approximation to the rate of heat flow at the center of the interval. Equation (42), which gives the average rate of heat flow over the interval ending at $t = m\delta$, may be used. For a value of δ of 1/2 second, the values of M are given in table II. Substituting in equation (42) gives the rate of heat flow plotted as circles in figure 6. The solid curve gives the theoretically exact instantaneous rate of heat flow for comparison. The results from equation (42) are seen to be precise. The system yielding instantaneous heat flow, mentioned previously, would seem to be a natural one for the present problem; however, the results obtained by that system were found to be inferior to those presented.

Example 6(b).-

Problem: The corresponding inside-surface-temperature history of the same wall is given by the following time series in which the temperatures are separated by 1/2 second: $T_i = 0.8$, 4.4, 11.7, 22.4, 36.0, 52.6, 71.6, 92.3, 114.2, 137.4, 160.6, 183.7, 205.4, 226.4, 245.3, 261.6, 275.2, 285.6, 292.4, 296.3. Determine the history of heat flow into the outer surface by using only the given inside temperatures.

Solution: Equations (43) may be used to determine the outside-surface-temperature history from the inside-temperature history. Then, the rate of heat flow at any time is determined as in the solution for example 6(a). The factor $1/(1 - \theta_1)$ in equations (43) may be thought of as a magnification factor. Large values of this factor tend to cause an instability in the computed temperatures. In this example, if δ is taken as 1/2 second or 1/7200 hour, the value of $k\delta/l^2$ is 0.3528, θ_1 is 0.80478, and the magnification factor is 5.1. On substitution in equations (43), an oscillation of period 2δ builds up in the values of T, very slowly at first and to either side of the correct answer as a mean value, but in a divergent manner so that by 4 seconds the amplitude is 6°. For a value of δ of 1 second, the value of $k\delta/l^2$ is 0.7056, θ_1 is 0.58038, and the magnification factor is 2.4. In this instance, an oscillation in T of period 2δ and maximum amplitude of 1.5° occurred. Substituting these values of T, without fairing, into equation (42) gives the results shown by the square symbols in figure 6. If a larger value of δ were used, the oscillation would be damped out but the accuracy would suffer because of a lack of definition of the rapidly varying heating rate. The particular case demonstrated is therefore a marginal one for the determination of heat flow from the temperatures of the inside surface. The instability is found to disappear for thermally thinner walls and conversely to increase rapidly for thicker walls.

CONCLUDING REMARKS

Formulas to facilitate the determination of the transient surface temperatures of thick walls from an arbitrary variation of adiabatic-wall temperature and heat-transfer coefficient have been developed. Formulas to facilitate the determination of heat flow from an arbitrary variation of wall surface temperature were also obtained. The numerical applications given demonstrate a high degree of accuracy for the present method.

Langley Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., June 18, 1957.

APPENDIX A

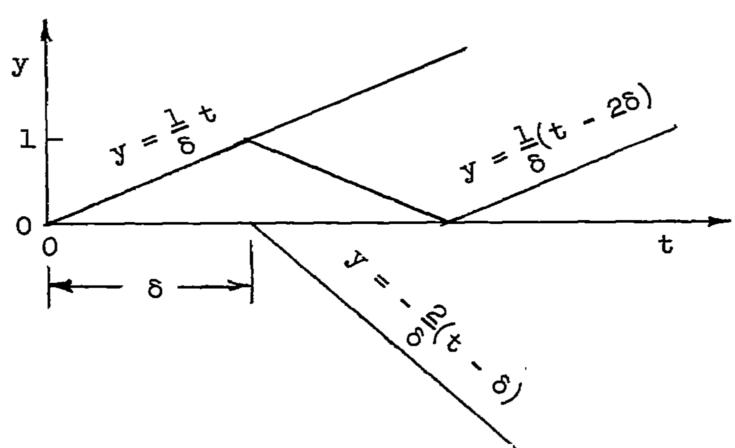
REVIEW OF SUPERPOSITION AND TIME SERIES

TIME SERIES

A time series may be defined as a series of numbers or quantities which represent the values of a function of time at successive equal intervals of time. According to the notation of Tustin (ref. 3), each quantity is separated from the others by a comma since the values corresponding to different times are not added together. The quantity at zero time is zero. The first quantity recorded represents the value of the function at the end of the first time interval. The time interval used is arbitrary and its size is selected to obtain the accuracy required in the solution of a specific problem. The symbol for the time interval is δ. Thus, the series $y = y_1, y_2, y_3, \ldots y_m$ represents the values of the function y at the times $\delta, 2\delta, 3\delta, \ldots m\delta$.

THE UNIT TRIANGLE

A unit triangle is an isosceles triangle which has an altitude of unit magnitude and a base of 2δ, or two time intervals. Since δ is an arbitrary time interval, the unit triangle is accordingly arbitrary. A plot of a unit triangle centered at $t = \delta$ is shown in sketch 1, where y represents magnitude or altitude.



Sketch 1.

The slopes of the sides of the triangle depend on the value of δ and are equal to $\pm 1/\delta$. Three lines may be superimposed to represent the unit triangle. The equations of the three lines which may be added to represent the unit triangle are $y = (1/\delta)t$, $y = -(2/\delta)(t - \delta)$, and $y = (1/\delta)(t - 2\delta)$. Although the triangle terminates at $t = 2\delta$, the values of t in the equations for y can go on to infinity since the values of y add to 0 beyond $t = 2\delta$.

The Function Corresponding to a Reference Line

Let $F(x,t)$ be the solution to a boundary-value problem specified by a linear partial differential equation and the linear boundary condition $y = (1/\delta)t$, where y is the value of F or one of its derivatives or integrals at some fixed value of x. Because of the linearity of the problem, the magnitude of the solution is directly proportional to the magnitude of the slope $1/\delta$ of the boundary condition. For example, if $y = (2/\delta)t$, the corresponding function representing the solution is $2F(x,t)$. Again, if $y = -(2/\delta)t$, the corresponding function is $-2F(x,t)$. The slope of the line $y = (1/\delta)t$ can thus be used as a reference magnitude. This slope is the same as that of the left side of the unit triangle.

The value of t in $F(x,t)$ is always identical with the value of t in the boundary condition $y = (1/\delta)t$. Thus, if the origin is shifted so that $y = (1/\delta)(t - 2\delta)$, then the corresponding function is $F(x,t-2\delta)$. Particular solutions of a linear differential equation can always be added in linear combinations to satisfy more general boundary conditions. If a and b are constants, and $aF(x,t)$ corresponds to the boundary condition $y_1 = (a/\delta)t$, and $bF(x,t-\delta)$ corresponds to $y_2 = (b/\delta)(t - \delta)$, then the function corresponding to the sum of the two lines $y = y_1 + y_2$ is $F = aF(x,t) + bF(x,t-\delta)$. Let an additional property of F, as well as of y, be that it assumes the value 0 for any time less than 0. The range of time of interest is therefore from 0 to ∞.

The Function Corresponding to a Triangle

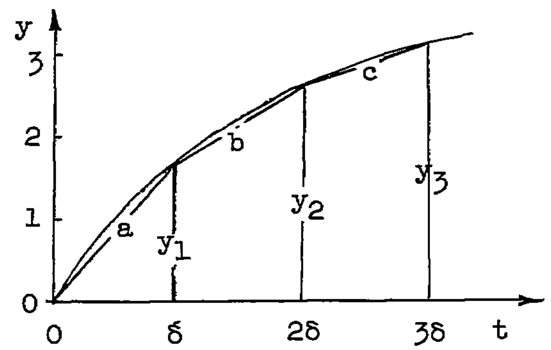
Now consider a function $F_\Delta(t)$ dependent on the lines of the unit triangle in sketch 1 for its value just as $F(t)$ is related to the line $y = (1/\delta)t$. In consideration of the three lines by which the unit triangle may be replaced, $y = (1/\delta)t$, $y = -(2/\delta)(t - \delta)$, and $y = (1/\delta)(t - 2\delta)$, the three corresponding solutions or functions of time are $F(t)$, $-2F(t - \delta)$, and $F(t - 2\delta)$. Because of the additive nature of solutions, the solution corresponding to the complete triangle may be defined as the sum of the solutions for the lines which compose it:

$$F_{\Delta}(t) = F(t) - 2F(t - \delta) + F(t - 2\delta) \tag{A1}$$

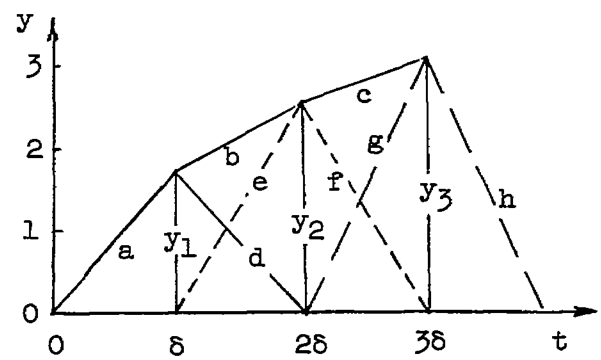
In equation (A1) each term has a value of zero for negative values of its argument.

Representation of a General Curve by Triangles

The curve A-B in sketch 2(a) is any arbitrary continuous function in the plane y,t which can be faired through its ordinate values 0, y_1, y_2, and y_3. The curve may be well approximated by a series of chords such as a, b, and c if a suitable spacing δ is used.



(a) The curve A-B



(b) The synthesis of A-B

Sketch 2.

Draw the lines d, e, f, g, and h of sketch 2(b) to form the isosceles triangles with sides a and d, e and f, and g and h. Since the sum of two straight lines is a straight line, it is clear that, if the letters which designated the lines are now used to designate the equation of the line, then $d + e = b$ and $f + g = c$. Of course, the line a is the first half of the first triangle as well as the first chord of the curve A-B. In designating the ordinates $y_1, y_2, y_3, \ldots y_m$, the altitudes of three triangles whose sides add up to the chords of the curve A-B are simultaneously designated. The ordinate series $y_1, y_2, y_3, \ldots y_m$, with spacing δ, is a time-series approximation of the curve A-B. In this case, as well as elsewhere in this report, each ordinate of a series is understood to represent the altitude of an isosceles triangle with a base width of 2δ.

Arbitrary Control Functions

Suppose that the curve A-B, or $y(t)$, is to serve as an arbitrary reference or control function for a corresponding function $F_y(t)$. It is desired to determine $F_y(t)$ in terms of functions corresponding to unit triangles, such as $F_\Delta(t)$, given by equation (A1). The function $F_\Delta(t)$ is a function corresponding to a unit triangle centered at $t = \delta$. If $F_\Delta(t)$ is multiplied by y_1, there results the function corresponding to the triangle with sides a and d of sketch 2(b). By superposition, the function corresponding to the three triangles centered at δ, 2δ, and 3δ is

$$F_y(t) = y_1F_\Delta(t) + y_2F_\Delta(t - \delta) + y_3F_\Delta(t - 2\delta) \tag{A2}$$

Then, since the three triangles add up to the chords of the curve A-B, this is the function, in ordinary algebraic form, corresponding exactly to the chords of the general reference curve $y = y_1, y_2, y_3, \ldots y_m$. In order to put equation (A2) in time-series form, let $F_\Delta(t) = a_1, a_2, a_3, \ldots a_m$, with spacing δ. Substituting in (A2) and placing terms for the same time in columns and adding yield

$$\begin{array}{lllll}
y_1F_\Delta(t) & = y_1a_1, & y_1a_2, & y_1a_3, & \ldots \\
y_2F_\Delta(t - \delta) & = \ 0\ , & y_2a_1, & y_2a_2, & \ldots \\
y_3F_\Delta(t - 2\delta) & = \ 0\ , & 0\ , & y_3a_1, & \ldots \\
\hline
F_y(t) & = y_1a_1, & \left(y_1a_2 + y_2a_1\right), & \left(y_1a_3 + y_2a_2 + y_3a_1\right), & \ldots
\end{array} \tag{A3}$$

The result shown by equation (A3) is obviously that which is obtained by formal algebraic multiplication of $a_1, a_2, a_3, \ldots$ by $y_1, y_2, y_3, \ldots$ as follows:

$$
\begin{array}{llllll}
F_\Delta(t) & = & a_1, & a_2, & a_3, & \cdots \\
y(t) & = & y_1, & y_2, & y_3, & \cdots \\
\hline
 & & y_1a_1, & y_1a_2, & y_1a_3, & \cdots \\
 & & & y_2a_1, & y_2a_2, & \cdots \\
 & & & & y_3a_1, & \cdots \\
\hline
\end{array}
$$

$$
F_y(t) = y_1a_1,\ \left(y_1a_2 + y_2a_1\right),\ \left(y_1a_3 + y_2a_2 + y_3a_1\right),\ \cdots \tag{A4}
$$

Therefore,

$$
F_y(t)_s = \left[F_\Delta(t)_s\right]\left[y(t)_s\right] \tag{A5}
$$

where the subscript s denotes time-series form.

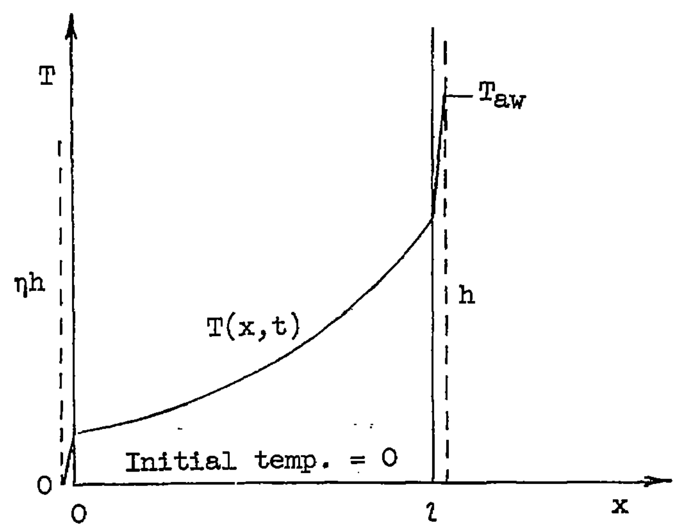
Equation (A5) states a simple theorem which was first given in reference 1. It was developed with $y(t)$ having a value of 0 at $t = 0$ and $F_\Delta(t)$ also having a value of 0 at $t = 0$. If either or both of the series had values at $t = 0$, multiplication as in equation (A5) would not be sufficient to obtain F_y.

APPENDIX B

SUMMARY OF ANALYTICAL TEMPERATURE FORMULAS FOR THICK WALLS

CONSTANT FLUID TEMPERATURES

Consider an infinite wall of thickness l and an initial temperature of zero. (See sketch 3.) Let the wall be suddenly contacted at the face $x = l$ by a fluid of temperature T_{aw} while the face at $x = 0$



Sketch 3

remains exposed to a fluid of zero temperature. Let the heat-transfer coefficient at the face $x = l$ have the value h and the heat-transfer coefficient at the face $x = 0$ have the value ηh. Let the wall have uniform physical properties which are invariant with time.

The flow of heat within the wall is governed by Fourier's equation for transient heat flow, which states that the rate of increase of temperature is proportional to the rate of change of temperature gradient:

$$\frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} \tag{B1}$$

The constant of proportionality, called the diffusion coefficient, is equal to the ratio of the conductivity of the material to the heat capacity as represented by the product of specific heat and weight density as follows:

$$k = \frac{K}{c\rho}$$

The boundary equation for the wall at $x = l$ is obtained by equating the heat transfer between the fluid at temperature T_{aw} and the wall at temperature T to the rate of heat transfer in the wall at $x = l$:

$$h\left(T_{aw} - T_{x=l}\right) = K\left(\frac{\partial T}{\partial x}\right)_{x=l} \tag{B2}$$

A similar boundary equation is written for the wall $x = 0$.

$$\eta h T_{x=0} = K\left(\frac{\partial T}{\partial x}\right)_{x=0} \tag{B3}$$

The initial condition is a statement that the initial temperature of the wall is zero:

$$(T)_{t=0} = 0 \tag{B4}$$

The simultaneous solution of partial differential equations (B1) to (B4) is an infinite series.

$$T = T_{aw} \frac{1 + \eta \frac{hx}{K}}{1 + \eta + \eta \frac{hl}{K}} - 2T_{aw} \sum_{n=1}^{\infty} \frac{\left[\cos\left(\frac{x}{l} \alpha_n\right) + \frac{\eta}{\alpha_n} \frac{hl}{K} \sin\left(\frac{x}{l} \alpha_n\right)\right] e^{-\alpha_n^2 \frac{kt}{l^2}}}{\left[\frac{K}{hl} \alpha_n^2 - \eta \frac{hl}{K} - 3(1 + \eta)\right] \cos \alpha_n + \left[\left(\frac{4K}{hl} + 1 + \eta\right)\alpha_n - 2 \frac{\eta}{\alpha_n} \frac{hl}{K}\right] \sin \alpha_n} \tag{B5}$$

The parameters α_n are angles which are the positive roots of the equation

$$\tan \alpha_n = \frac{\alpha_n(1 + \eta)}{\alpha_n^2 \frac{K}{hl} - \eta \frac{hl}{K}} \tag{B5a}$$

Two dimensionless numbers of physical significance are present in equation (B5): the diffusion number kt/l^2, and the conductance or Nusselt number hl/K. The first term of equation (B5) gives the steady-state solution or equilibrium condition.

Special Case $\eta = 0$

The special case of a plate that is heated by convection on one face while the heat transfer on the other face is so small as to be negligible is important. Setting the heat-transfer coefficient at the unheated face equal to zero corresponds to making the assumption that the plate is perfectly insulated at that face. If the plate is perfectly insulated at the face $x = 0$, substitute $\eta = 0$ in equations (B5) and (B5a) and utilize the following relation to eliminate $\sin \alpha_n$,

$$\alpha_n \tan \alpha_n = \frac{hl}{K} \tag{B6}$$

where α_n represents the positive roots of the equation. These substitutions yield the following expression for temperature:

$$T = T_{aw} - 2T_{aw} \sum_{n=1}^{\infty} \frac{\cos\left(\frac{x}{l} \alpha_n\right) e^{-\frac{kt}{l^2} \alpha_n^2}}{\left(1 + \frac{hl}{K} + \frac{K}{hl} \alpha_n^2\right) \cos \alpha_n} \tag{B7}$$

Equations (B6) and (B7) may be found on page 100 of reference 1.

Special Case $\eta = 0$ and $h = \infty$

The special case in which $\eta = 0$ and $h = \infty$ corresponds physically to perfect insulation on one face and a known initial temperature on the other face. This case is developed by setting $h = \infty$ in equations (B6) and (B7). Equation (B6) becomes

$$\alpha_n \tan \alpha_n = \infty$$

$$\tan \alpha_n = \infty$$

or, since α_n assumes the sequence $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$, and so forth,

$$\alpha_n = (2n - 1)\frac{\pi}{2}$$

Equation (B7) reduces to

$$T = T_1 - \frac{4}{\pi} T_1 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n - 1} \cos\left[(2n - 1)\frac{\pi}{2} \frac{x}{l}\right] e^{-(2n-1)^2 \frac{\pi^2}{4} \frac{kt}{l^2}} \tag{B8}$$

Equation (B8) may be found in standard references (ref. 4, page 196, problem 6, for example). In this equation, T_1 is the step in wall-surface temperature used in place of T_{aw}.

VARIABLE FLUID TEMPERATURE

Arbitrary Variation of T_{aw}

Equation (B7) is an exact solution of a thick-wall boundary-layer heating problem which is suitable for checking the accuracy of the present method of computing wall temperatures. Another method, which is more general, can be obtained by letting the adiabatic-wall temperature vary in a known manner, $T_{aw} = F(t)$. Let $V(x,t)$ be the variation of wall temperature due to a 1° step in adiabatic-wall temperature. If the initial wall temperature is zero, the wall temperature is given by Duhamel's formula as

$$T(x,t) = \int_0^t F'(\tau)V(x,t-\tau)d\tau \tag{B9}$$

where F' is the derivative of F with respect to t and τ is a dummy time variable.

Linear Variation of T_{aw}

In the linear variation of T_{aw}, $T_{aw} = bt$, or, in the notation of Duhamel's formula, $F = bt$ and $F' = b$. Substituting T from equation (B7), with $T_{aw} = 1$, into equation (B9) and performing the integration give

$$T(x,t) = bt - \frac{2bl^2}{k} \sum_{n=1}^{\infty} \frac{\cos\left(\frac{x}{l}\alpha_n\right)\left(1 - e^{-\frac{kt}{l^2}\alpha_n^2}\right)}{\alpha_n^2\left(1 + \frac{hl}{K} + \frac{K}{hl}\alpha_n^2\right)\cos\alpha_n} \tag{B10}$$

Equation (B6) is used to obtain α_n.

Linear Variation of Wall Surface Temperature

One of the fundamental equations on which is based the time-series development of surface-temperature equations is one expressing the transient wall temperature due to an increase of 1° in temperature of one wall surface during each interval of time δ. The assumptions are made that the initial temperature is zero and that one wall is insulated. From these considerations, F in Duhamel's formula is $F = t/\delta$ and $F' = 1/\delta$. Substituting this derivative and T from equation (B8), with $T_1 = 1$, into equation (B9) and performing the integration give the following equation for the temperature at any plane due to a surface temperature rise of 1° in the interval δ:

$$T = \frac{t}{\delta} - \frac{16}{\pi^3}\frac{l^2}{k\delta} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cos\left[(2n-1)\frac{\pi}{2}\frac{x}{l}\right]}{(2n-1)^3}\left(1 - e^{-(2n-1)^2\frac{\pi^2}{4}\frac{k}{l^2}t}\right) \tag{B11}$$

The heat flow at any point within the wall due to 1° rise of wall surface temperature in the time δ is obtained by multiplying the conductivity by the temperature gradient. Differentiating equation (B11) with respect to x and multiplying by K yield

$$q = \frac{8}{\pi^2}\frac{lK}{k\delta} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sin\left[(2n-1)\frac{\pi}{2}\frac{x}{l}\right]}{(2n-1)^2}\left(1 - e^{-(2n-1)^2\frac{\pi^2}{4}\frac{k}{l^2}t}\right)$$

In order to eliminate k in the coefficient, substitute for k its definition $k = K/\rho c$. Then $lK/k\delta = \rho c l/\delta$. The product $\rho c l$ is the heat capacity of the wall per unit area per degree and is represented by the symbol G. In order to obtain the heat flow at the heated surface, where the heat balance is to be made, let $x = l$. Hence, the instantaneous heat flow due to a uniformly increasing surface temperature of 1° per unit of time δ is simplified to

$$q = \frac{8G}{\pi^2\delta}\sum_{n=1}^{\infty}\frac{1 - e^{-(2n-1)^2 \frac{\pi^2}{4}\frac{kt}{l^2}}}{(2n - 1)^2} \tag{B12}$$

In order to find the average flow of heat over the interval δ, equation (B12) is integrated with respect to time between the limits $(t - \delta)$ and t. This integration gives the total heat flow through the surface during the interval. On dividing by δ, the average rate of flow for the interval is obtained. The result is

$$\bar{q} = \frac{8}{\pi^2}\frac{G}{\delta}\sum_{n=1}^{\infty}\left[\frac{1}{(2n - 1)^2} + \frac{e^{-(2n-1)^2 \frac{\pi^2}{4}\frac{k}{l^2}t}}{\frac{\pi^2}{4}\frac{k}{l^2}\delta(2n - 1)^4} - \frac{e^{-(2n-1)^2 \frac{\pi^2}{4}\frac{k}{l^2}(t-\delta)}}{\frac{\pi^2}{4}\frac{k}{l^2}\delta(2n - 1)^4}\right] \tag{B13}$$

The summation $\sum_{n=1}^{\infty}\frac{1}{(2n - 1)^2}$ is a constant with the value $\pi^2/8$. In working with time series only integral increments of time (δ, 2δ, 3δ, . . . $m\delta$) are used. In equation (B13), therefore, t may have any value $m\delta$, where m is an integer. For convenience, and to systematize results, the following identities are defined:

$$\left.\begin{aligned}
A_0 &\equiv \frac{4l^2}{\pi^2 k\delta} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} \\
A_1 &\equiv \frac{4l^2}{\pi^2 k\delta} \sum_{n=1}^{\infty} \frac{e^{-(2n-1)^2 \frac{\pi^2}{4} \frac{k}{l^2}\delta}}{(2n-1)^4} \\
A_2 &\equiv \frac{4l^2}{\pi^2 k\delta} \sum_{n=1}^{\infty} \frac{e^{-2(2n-1)^2 \frac{\pi^2}{4} \frac{k}{l^2}\delta}}{(2n-1)^4} \\
A_m &\equiv \frac{4l^2}{\pi^2 k\delta} \sum_{n=1}^{\infty} \frac{e^{-m(2n-1)^2 \frac{\pi^2}{4} \frac{k}{l^2}\delta}}{(2n-1)^4} \\
A_{m-1} &\equiv \frac{4l^2}{\pi^2 k\delta} \sum_{n=1}^{\infty} \frac{e^{-(m-1)(2n-1)^2 \frac{\pi^2}{4} \frac{k}{l^2}\delta}}{(2n-1)^4}
\end{aligned}\right\} \tag{B14}$$

Equation (B13) then becomes

$$\bar{q}_m = \frac{8}{\pi^2} \frac{G}{\delta}\left(\frac{\pi^2}{8} + A_m - A_{m-1}\right) \tag{B15}$$

for the time interval ending at $t = m\delta$. Equation (B15) is used in forming the heat balance at the heated wall surface in the derivation of wall temperature formulas.

APPENDIX C

ANALYTICAL TEMPERATURE DISTRIBUTION

Equations (40) are the formulas most convenient for determining the temperature of the inside surface or of any plane within the surface. For the determination of the temperature of planes within the wall, equations (36a) are ordinarily evaluated numerically to obtain θ for a given kt/l^2 and x/l. However, it may be necessary to obtain the temperature distribution through the wall at some instant of time due to an arbitrary surface-temperature history in a form that is analytical with respect to x and which can be differentiated or integrated with respect to x (for example, in the derivation of a general formula for the thermal stress distribution or maximum stress). This temperature distribution is obtained by substituting equations (36a) into equations (39), substituting equations (39) into equations (40), and collecting terms. The results are

$$T_{i,1} = T_1 - \frac{16}{\pi^3}\frac{l^2}{k\delta} T_1 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^3} \cos\left[(2n-1)\frac{\pi}{2}\frac{x}{l}\right]\left[1 - e^{-(2n-1)^2 \frac{\pi^2}{4}\frac{k\delta}{l^2}}\right]$$

$$T_{i,2} = T_2 - \frac{16}{\pi^3}\frac{l^2}{k\delta} \sum_{n=1}^{\infty}\left\{\frac{(-1)^{n-1}\cos\left[(2n-1)\frac{\pi}{2}\frac{x}{l}\right]}{(2n-1)^3}\right\}\left[T_2 - T_1 + \left(2T_1 - T_2\right)e^{-(2n-1)^2 \frac{\pi^2}{4}\frac{k\delta}{l^2}} - T_1 e^{-2(2n-1)^2 \frac{\pi^2}{4}\frac{k\delta}{l^2}}\right]$$

$$T_{i,3} = T_3 - \frac{16}{\pi^3}\frac{l^2}{k\delta} \sum_{n=1}^{\infty}\left\{\frac{(-1)^{n-1}\cos\left[(2n-1)\frac{\pi}{2}\frac{x}{l}\right]}{(2n-1)^3}\right\}\left[T_3 - T_2 + \left(-T_3 + 2T_2 - T_1\right)e^{-(2n-1)^2 \frac{\pi^2}{4}\frac{k\delta}{l^2}} + \left(2T_1 - T_2\right)e^{-2(n-1)^2 \frac{\pi^2}{4}\frac{k\delta}{l^2}} - T_1 e^{-3(2n-1)^2 \frac{\pi^2}{4}\frac{k\delta}{l^2}}\right]$$

$$T_{i,m} = T_m - \frac{16}{\pi^3}\frac{l^2}{k\delta}\sum_{n=1}^{\infty}\left\{\frac{(-1)^{n-1}\cos\left[(2n - 1)\frac{\pi}{2}\frac{x}{l}\right]}{(2n - 1)^3}\right\}\left[T_m - T_{m-1} + \left(-T_m + 2T_{m-1} - T_{m-2}\right)e^{-(2n-1)^2\frac{\pi^2}{4}\frac{k\delta}{l^2}} + \left(-T_{m-1} + 2T_{m-2} - T_{m-3}\right)e^{-2(2n-1)^2\frac{\pi^2}{4}\frac{k\delta}{l^2}} + \ldots + \left(2T_1 - T_2\right)e^{-(m-1)(2n-1)^2\frac{\pi^2}{4}\frac{k\delta}{l^2}} - T_1 e^{-m(2n-1)^2\frac{\pi^2}{4}\frac{k\delta}{l^2}}\right]$$

Note that $T_{i,1}$, $T_{i,2}$, $T_{i,3}$, and $T_{i,m}$ can be computed independently. The use of these equations involves considerable labor, however, because all terms in the summation must be summed in unison.

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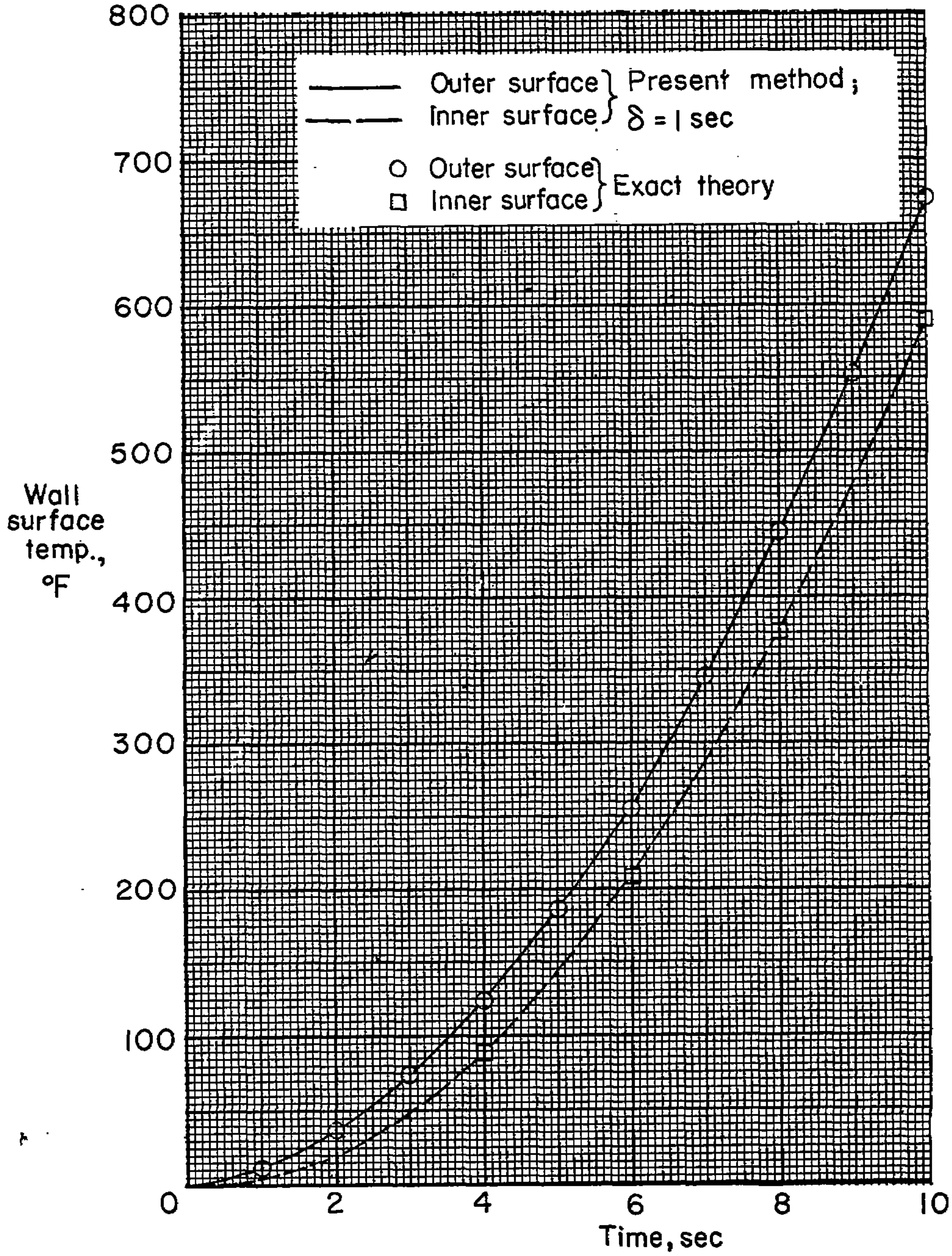
TABLE I.- VALUES OF M AND θ[a]

$k\delta/l^2$ / m	0.01	0.02	0.05	0.1	0.5	1.0	2.0	5.0
	M							
1	0.09281491	0.13125147	0.20752122	0.29347746	0.64728217	0.85683729	1.02954122	1.15145422
2	-.01594178	-.02252780	-.03560872	-.05052075	-.22817193	-.51142943	-.82682879	-1.06920825
3	-.02682329	-.03794254	-.06002358	-.08722726	-.30052435	-.31662004	-.20127594	-.08224562
4	-.00957530	-.01354153	-.02166582	-.03696546	-.08405219	-.02634648	-.00142616	-.00000036
5	-.00553400	-.00782681	-.01320028	-.02628367	-.02447704	-.00223431	-.00001026	-.00000000
6	-.00373390	-.00528488	-.00983262	-.02025561	-.00712803	-.00018948	-.00000007	
7	-.00274116	-.00389295	-.00809228	-.01579610	-.00207577	-.00001607		
8	-.00212389	-.00304410	-.00695594	-.01233887	-.00060449	-.00000136		
9	-.00170888	-.00249269	-.00608368	-.00964054	-.00017604	-.00000012		
10	-.00141415	-.00211866	-.00535620	-.00753254	-.00005126	-.00000001		
11	-.00119613	-.00185531	-.00472749	-.00588550	-.00001493			
12	-.00103005	-.00166301	-.00417648	-.00459860	-.00000435			
13	-.00090099	-.00151704	-.00369098	-.00359308	-.00000127			
14	-.00079666	-.00140173	-.00326234	-.00280743	-.00000037			
15	-.00071651	-.00130718	-.00288362	-.00219357	-.00000011			
16	-.00064769	-.00122688	-.00254890	-.00171393	-.00000003			
17	-.00059289	-.00115664	-.00225306	-.00133917	-.00000001			
18	-.00054714	-.00109385	-.00199156	-.00104635				
19	-.00050882	-.00103656	-.00176041	-.00081756				
20	-.00047652	-.00098373	-.00155610	-.00063880				
21	-.00044898	-.00093449	-.00137549	-.00049912				
22	-.00042534	-.00088826	-.00121584	-.00038998				
23	-.00040474	-.00084473	-.00107473	-.00030471				
24	-.00038678	-.00080357	-.00094999	-.00023809				
25	-.00037081	-.00076453	-.00083973	-.00018602				
26	-.00035655	-.00072753	-.00074227	-.00014535				
27	-.00034370	-.00069238	-.00065612	-.00011357				
28	-.00033195	-.00065894	-.00057997	-.00008874				
29	-.00032118	-.00062716	-.00051265	-.00006933				
30	-.00031123	-.00059693	-.00045315	-.00005417				
	θ							
1	1.00000000	0.99999990	0.99956261	0.98873107	0.69945338	0.45623848	0.24814437	0.09999955
2	-.00000019	-.00019242	-.02166158	-.12553131	-.48642965	-.41618822	-.24630212	-.09999908
3	-.00002493	-.00333067	-.06963678	-.17636952	-.15098863	-.03665380	-.00182900	-.00000045
4	-.00033480	-.01195026	-.09012900	-.14882949	-.04396967	-.00310842	-.00001315	-.00000000
5	-.00144190	-.02167827	-.08988005	-.11749067	-.01280454	-.00026361	-.00000009	
6	-.00344274	-.02904777	-.08284990	-.09193126	-.00372885	-.00002236	-.00000000	
7	-.00595604	-.03361672	-.07435643	-.07184409	-.00108589	-.00000190		
8	-.00854572	-.03599626	-.06609621	-.05613647	-.00031622	-.00000016		
9	-.01092536	-.03687745	-.05854671	-.04386205	-.00009209	-.00000001		
10	-.01296009	-.03679326	-.05179171	-.03427135	-.00002682	-.00000000		
11	-.01461454	-.03611078	-.04579381	-.02677771	-.00000781			
12	-.01590635	-.03507182	-.04048318	-.02092260	-.00000227			
13	-.01687671	-.03383231	-.03578599	-.01634774	-.00000066			
14	-.01757368	-.03249148	-.03163302	-.01277320	-.00000019			
15	-.01804424	-.03111188	-.02796174	-.00998026	-.00000006			
16	-.01833036	-.02973243	-.02471644	-.00779802	-.00000002			
17	-.01846822	-.02837688	-.02184780	-.00609293	-.00000000			
18	-.01848810	-.02705930	-.01931206	-.00476067				
19	-.01841482	-.02578770	-.01707064	-.00371972				
20	-.01826876	-.02456606	-.01508936	-.00290636				
21	-.01806620	-.02339609	-.01333804	-.00227088				
22	-.01782041	-.02227782	-.01178998	-.00177434				
23	-.01754197	-.02121042	-.01042159	-.00138637				
24	-.01723931	-.02019254	-.00921203	-.00108323				
25	-.01691916	-.01922244	-.00814285	-.00084638				
26	-.01658696	-.01829829	-.00719776	-.00066131				
27	-.01624686	-.01741809	-.00636236	-.00051671				
28	-.01590228	-.01658000	-.00562392	-.00040373				
29	-.01555588	-.01578204	-.00497119	-.00031545				
30	-.01520973	-.01502235	-.00439422	-.00024646				

[a]For additional value of $k\delta/l^2$ between 0.1 and 1.0, see table II.

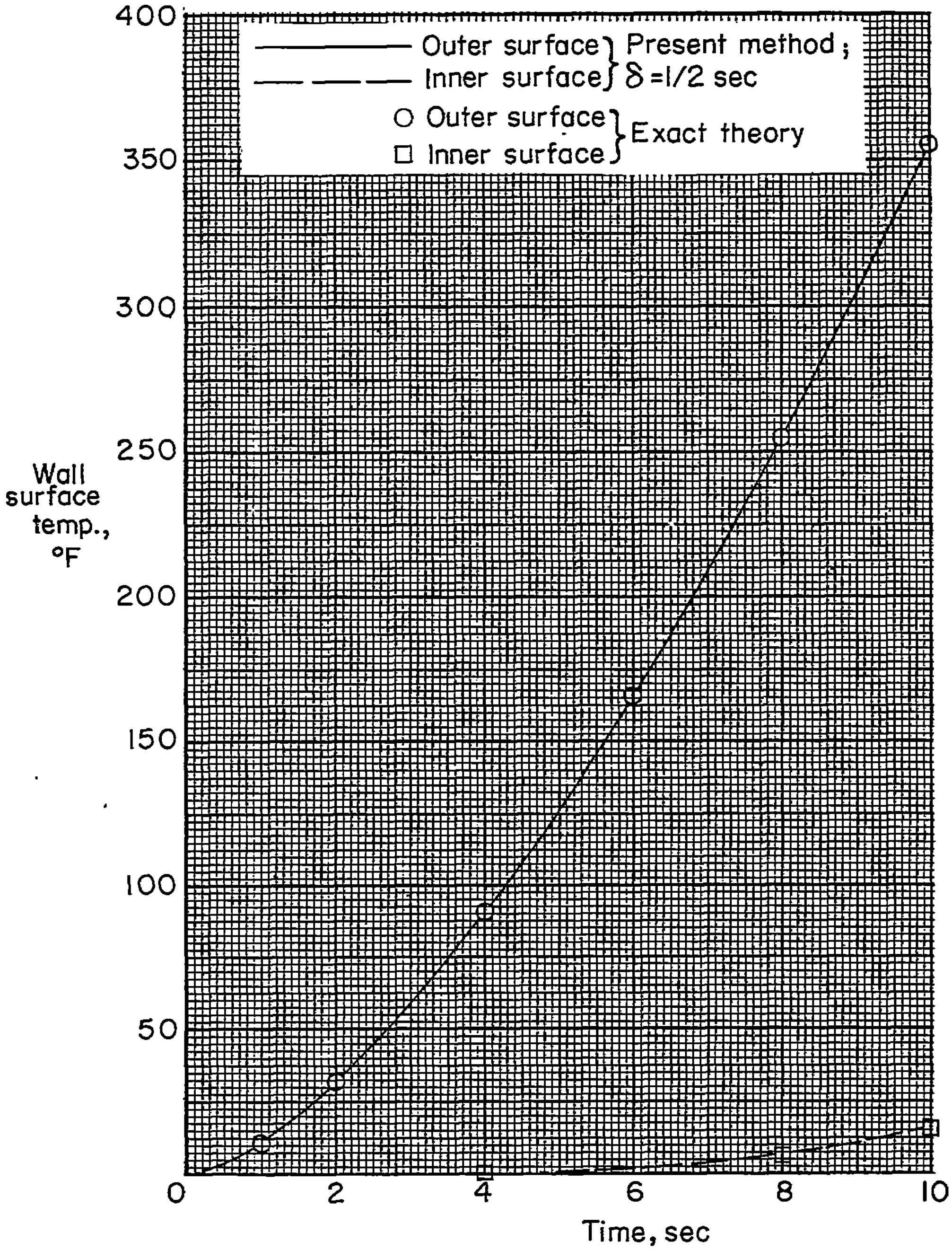
TABLE II.- VALUES OF M AND θ USED IN ILLUSTRATIVE EXAMPLES

$k\delta/l^2$ / m	0.00980	0.08820	0.17640	0.35280	0.70560
			M		
1	0.091882	0.275619	0.389750	0.549116	0.751602
2	-.015782	-.047357	-.071018	-.144143	-.352559
3	-.026554	-.080945	-.133036	-.242444	-.330551
4	-.009479	-.032753	-.065871	-.094474	-.056482
5	-.005478	-.022958	-.042292	-.039558	-.009904
6	-.003696	-.017994	-.027361	-.016565	-.001737
7	-.002714	-.014408	-.017705	-.006936	-.000305
8	-.002103	-.011580	-.011457	-.002905	-.000053
9	-.001692	-.009314	-.007414	-.001216	-.000009
10	-.001400	-.007493	-.004798	- 000509	-.000002
11	-.001184	-.006027	-.003105	-.000213	
12	-.001019	-.004848	-.002009	-.000089	
13	-.000891	-.003900	-.001300	-.000037	
14	-.000789	-.003137	-.000841	-.000016	
15	-.000707	-.002524	-.000544	-.000007	
16	-.000640	-.002030	-.000352	-.000003	
17	-.000585	-.001633	-.000228	-.000001	
18	-.000539	-.001314	-.000148	-.000000	
19	-.000501	-.001057	-.000095		
20	-.000469	-.000850	-.000062		
			θ		
1	0.999992	0.992899	0.943645	0.804779	0.580382
2	.000008	-.098507	-.277731	-.448794	-.474632
3	-.000021	-.157664	-.233674	-.206910	-.087207
4	-.000290	-.141628	-.152510	-.086651	-.015291
5	-.001288	-.116097	-.098715	-.032684	-.002681
6	-.003146	-.093525	-.063879	-.015194	-.000470
7	-.005529	-.075429	-.041337	-.006362	-.000082
8	-.008028	-.060636	-.026749	-.002664	-.000015
9	-.010355	-.048778	-.017309	-.001116	-.000002
10	-.012371	-.039239	-.011201	-.000467	-.000000
11	-.014028	-.031565	-.007248	-.000196	
12	-.015340	-.025391	-.004690	-.000082	
13	-.016338	-.020426	-.003035	-.000034	
14	-.017068	-.016431	-.001964	-.000014	
15	-.017574	-.013217	-.001271	-.000006	
16	-.017897	-.010632	-.000822	-.000003	
17	-.018070	-.008553	-.000532	-.000001	
18	-.018124	-.006880	-.000344	-.000000	
19	-.018081	-.005535	-.000223		
20	-.017966	-.004452	-.000144		



(a) 1/2-inch-thick wall.

Figure 1.- Example 1. Temperatures of copper-wall surfaces. Adiabatic-wall temperature varies linearly from 0° to $10{,}000^{\circ}$ in 10 seconds; $h = 100$ Btu/(hr)(sq ft)($^{\circ}$F).



(b) 3-inch-thick wall.

Figure 1.- Concluded.

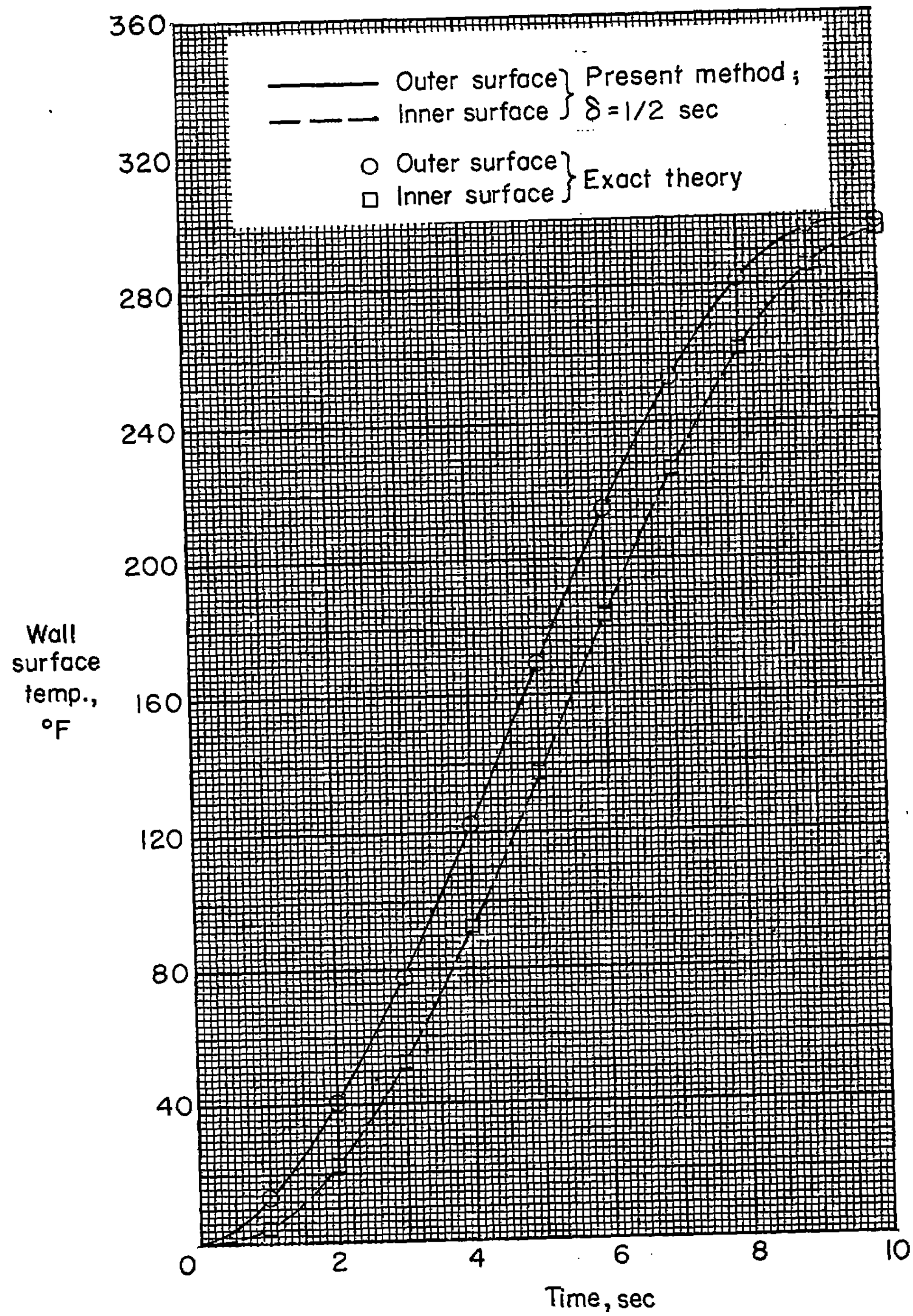


Figure 2.- Example 2. Temperatures of 1/2-inch copper wall heated according to assigned history of h and T_{aw}.

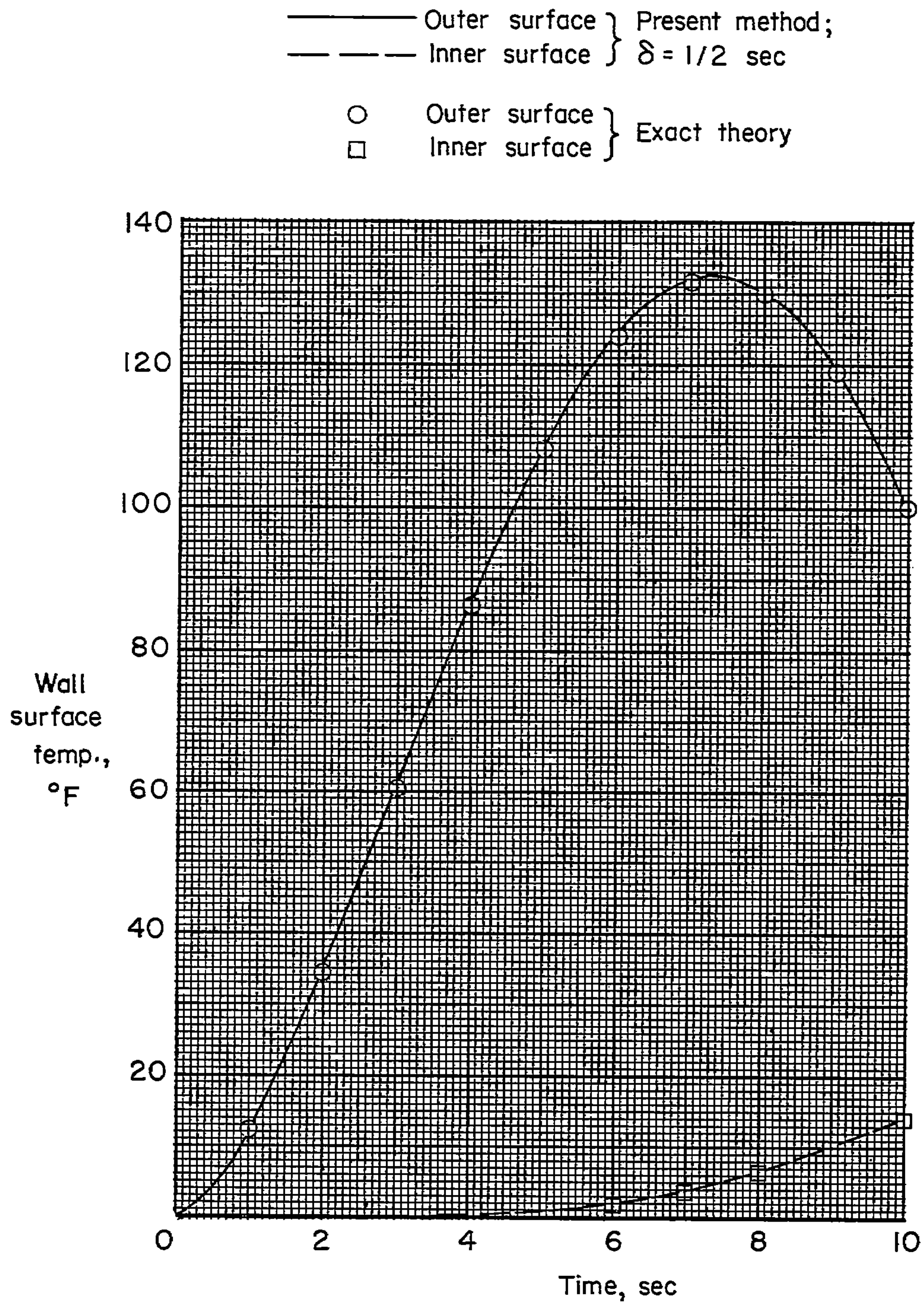


Figure 3.- Example 3. Temperatures of 3-inch copper wall heated according to assigned history of h and T_{aw}.

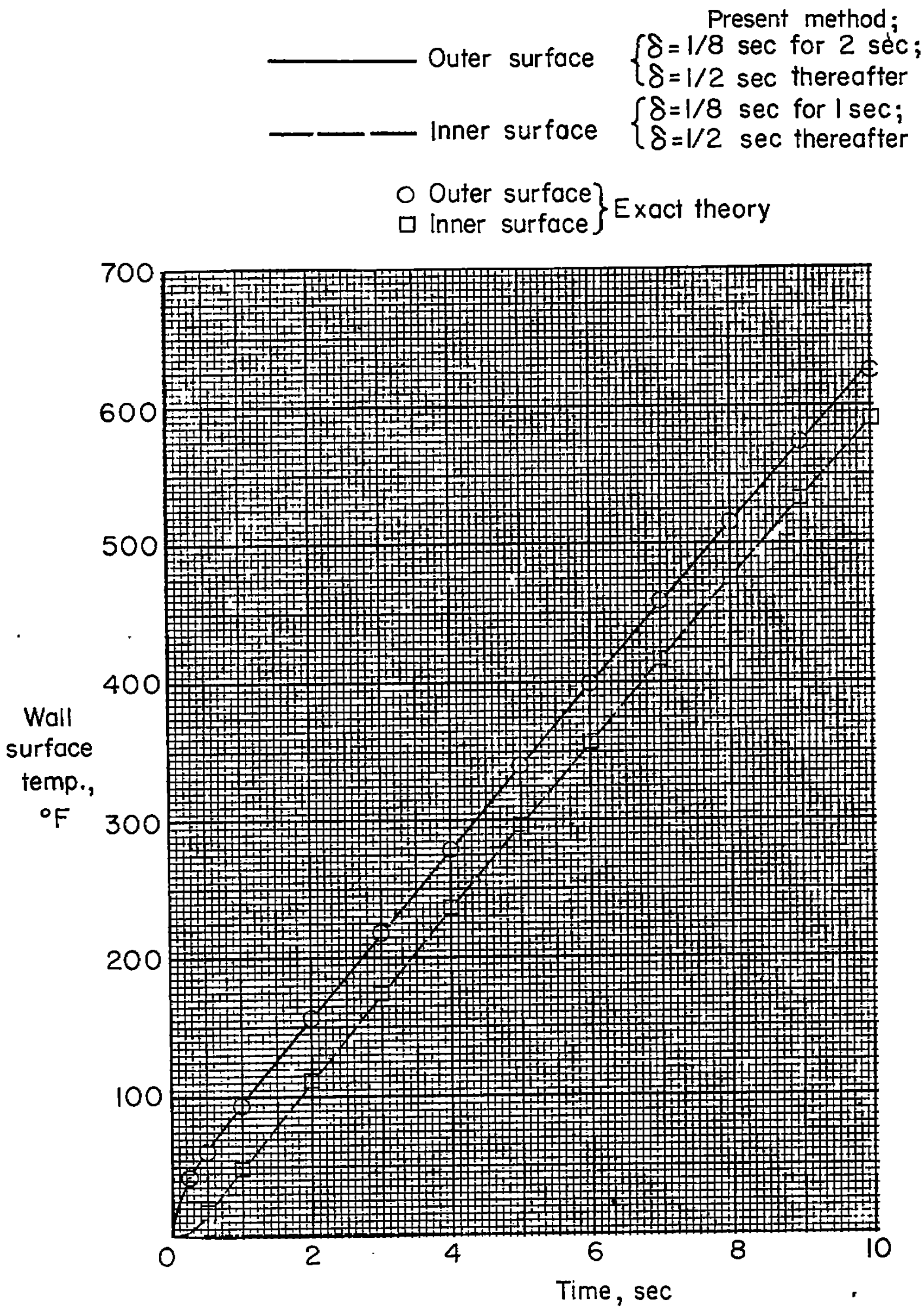


Figure 4.- Example 4. Temperatures of 1/2-inch copper wall after application of 5,000 °F jump in gas temperature. h = 100 Btu/(hr)(sq ft)(°F).

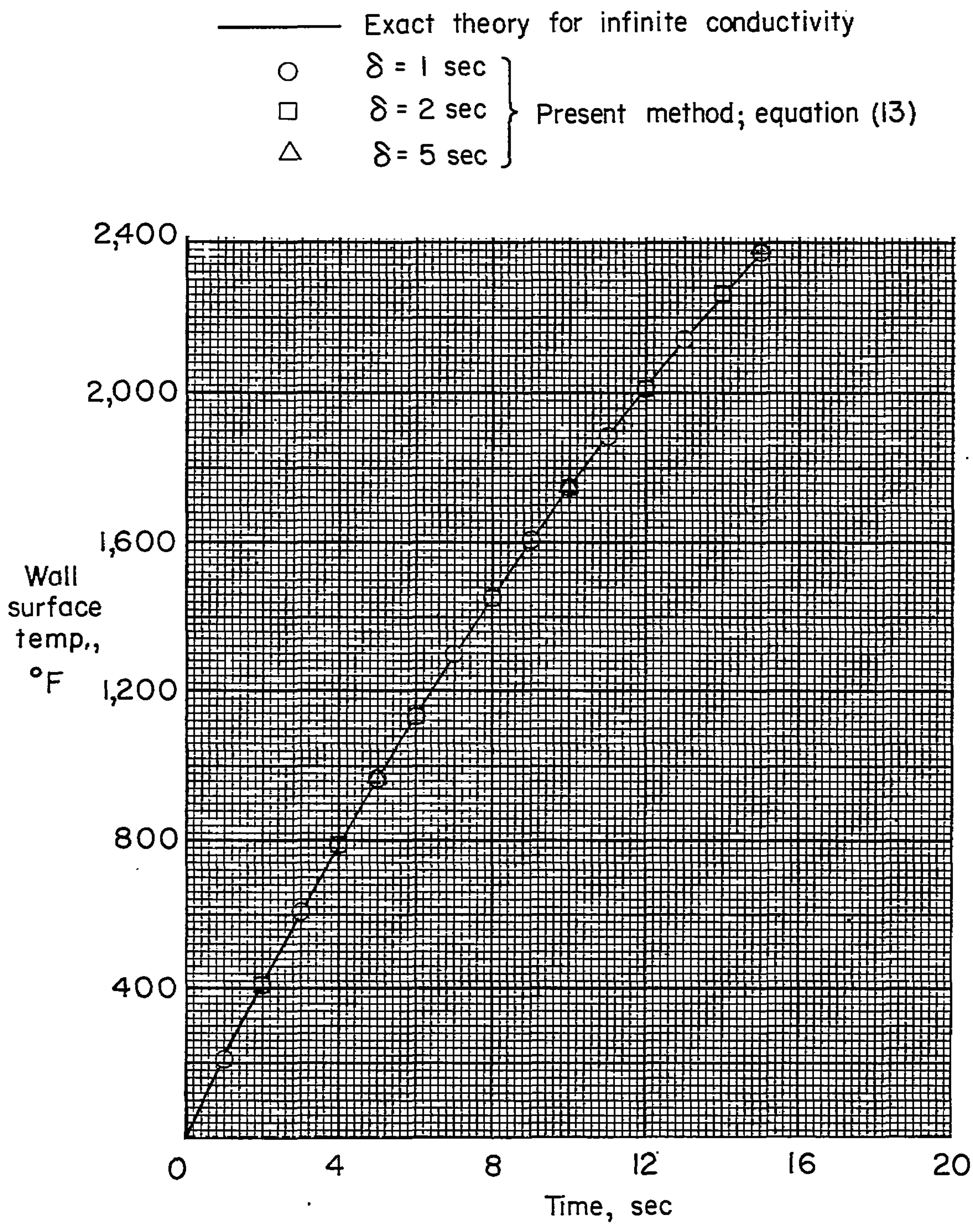


Figure 5.- Example 5. Temperatures of 1/16-inch Inconel wall. T_{aw} = 5,000 °F; h = 50 Btu/(hr)(sq ft)(°F).

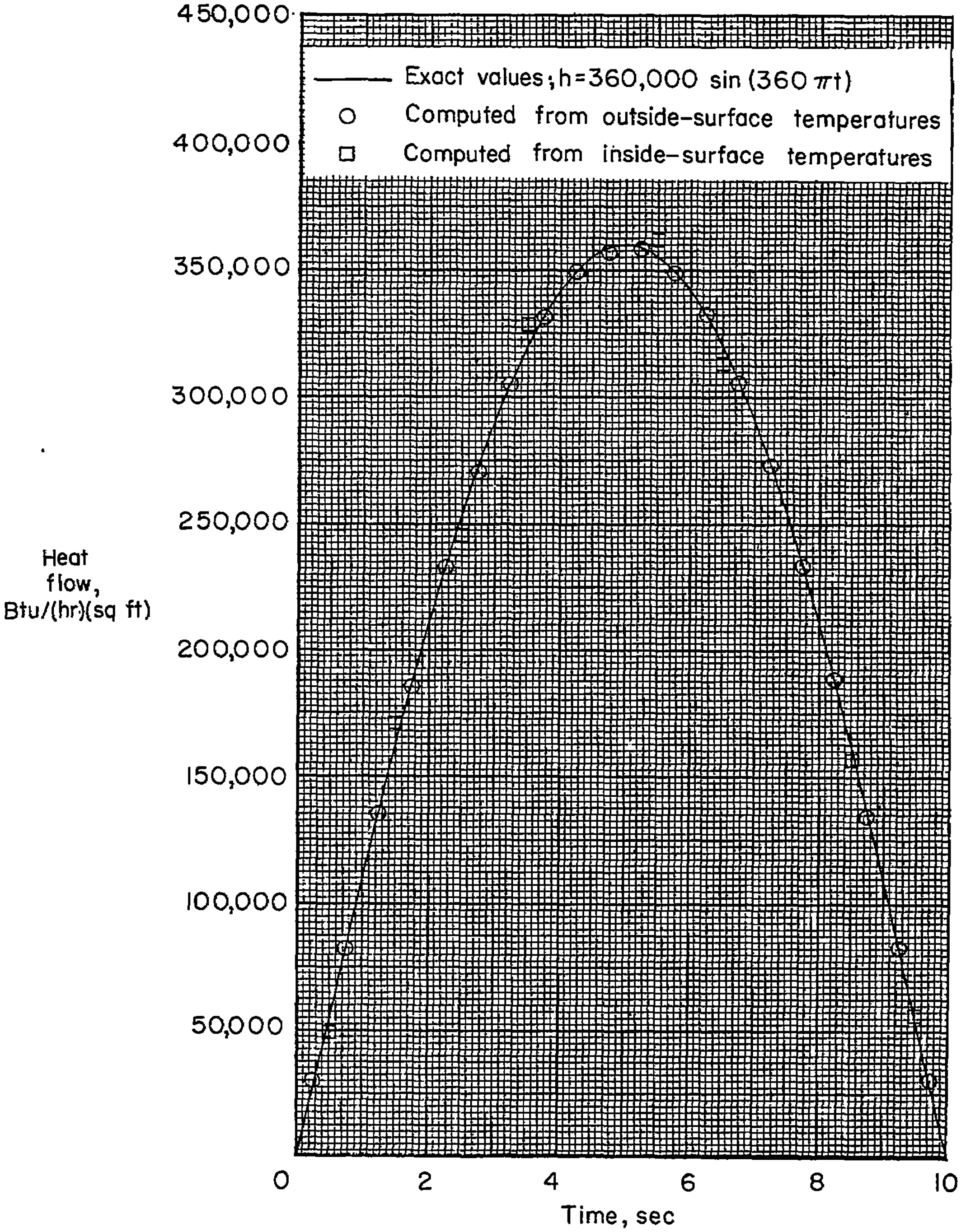


Figure 6.- Example 6. Rate of heat flow into heated (outer) surface of 1/2-inch copper wall computed from temperature history of outer surface and from temperature history of inner surface.