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TWO-DIMENSIONAL IRROTATIONAL MIXED SUBSONIC AND SUPERSONIC FLOW OF A COMPRESSIBLE FLUID AND THE UPPER CRITICAL MACH NUMBER

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Washington
May 1946

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SUMMARY

The problem of flow of a compressible fluid past a body with subsonic flow at infinity is formulated by the hodograph method. The solution in the hodograph plane is first constructed about the origin by superposition of the particular integrals of the transformed equations of motion with a set of constants which would determine, in the limiting case, a known incompressible flow. This solution is then extended outside the circle of convergence by analytic continuation.

The previous difficulty of the Chaplygin method of slow convergence of the series has been overcome by using the asymptotic properties of the hypergeometric functions so that numerical solutions can be obtained without difficulty. It is emphasized that, for a solution covering the whole domain of the field of flow, both fundamental solutions of the hypergeometrical differential equation are required.

Explicit formulas for numerical calculations are given for the flow about a body, such as an elliptic cylinder, and for the periodic flow such as would exist over a wavy surface.

Numerical examples based on the incompressible flow solution of an elliptic cylinder of thickness ratio of 0.6 are computed for free-stream Mach numbers of 0.6 and 0.7.

The results of this investigation indicate an appreciable distortion in the shape of the bodies in compressible flow from that of incompressible flow, which necessitates a series of computations with various values of the geometric parameter in order that the desired body shapes can be selected for a given Mach number. It also is shown that the breakdown of irrotational flow depends solely upon the occurrence of limiting

lines, which, in turn, are dependent on the boundary conditions.

The numerical calculations show that at a free-stream Mach number of 0.6, irrotational supersonic flow exists up to a local Mach number of 1.25; whereas breakdown occurs at 1.22 for a Mach number of 0.7.

INTRODUCTION

When a flow of nonviscous incompressible fluid is irrotational, it is well known that the problem can be reduced to either the problem of Dirichlet or that of Neumann, and that there exists a unique solution for any given boundary conditions. When the fluid is nonviscous but compressible, the variation of density makes the mathematical problem very difficult and complex. In this case, a pure potential flow throughout the region is not always possible for a given body; this depends very much upon the condition at infinity. If a certain speed of the flow at infinity is reached, regions within the field of flow will be created in which the irrotational flow does not exist owing to the appearance of "limiting lines." Such regions were picturesquely designated as "forbidden regions" by Th. von Kármán (reference 1), and they appear when the local speed of the flow considerably exceeds the local speed of sound. It has been shown that the occurrence of limiting lines is directly connected with the breakdown of irrotational flow and with the resultant increase in drag of the body due to shock waves. In other words, if there is a limiting line in the field of flow, the isentropic irrotational flow must break down. However, the irrotational flow may break down before the appearance of limiting line due to the instability of the velocity field. On the other hand, shock waves can occur only in supersonic flow. Therefore, there is no danger of breakdown of isentropic flow if the whole field of flow is subsonic. Consequently, the Mach number corresponding to the first appearance of local speed equal to that of sound can be designated as the "lower critical Mach number"; and the Mach number corresponding to the first appearance of limiting lines can be designated as the "upper critical Mach number." The actual critical Mach number for a given body will be influenced by the boundary layer and hence the Reynolds number. However, it must lie between these two limiting critical values. (See reference 2.) Thus, knowledge of these critical speeds of the flow are essential for the design of efficient aerodynamic bodies.

To determine the critical Mach numbers, the general problem of flow of a compressible fluid about a given body must be solved. The often-used methods treating such a problem are Janzen-Rayleigh's method of successive approximations and Glauert-Prandtl's method of small perturbation. The latter method has been extended recently by both Hantzsche and Wendt (reference 3) and C. Kaplan (reference 4). Indeed, both methods yield valuable information regarding the effects of compressibility and are useful for many practical design problems, particularly the determination of the lower critical Mach number of a given body. But, so far as the general problem of limiting line and upper critical number is concerned, none seems to be adequate, owing to the doubtful convergence of such successive approximations at the required high Mach numbers.

An entirely different approach first was made by Molenbroek (reference 5) and Chaplygin (reference 6) by introducing the velocity components instead of the usual space coordinates as independent variables. The advantage of the method is that, instead of a nonlinear differential equation as is the case in the physical plane, it leads to a linear one in the velocity or hodograph plane. The particular solutions of this linear equation are found to be products of trigonometric functions of the angle of inclination of velocity vector and hypergeometric functions of the magnitude of the velocity vector. It is then possible to construct a general solution from the particular solutions of the differential equation. The difficulty, however, is that the character of the field in the physical plane to which the solution in the hodograph plane corresponds cannot be determined beforehand. This difficulty prevents the exact formulation of the boundary value problem in the hodograph plane. Chaplygin has overcome this handicap by first choosing a "suitable solution" in the hodograph plane and then proceeding to find the corresponding flow in the physical plane. The suitable solution is one which, in the limiting case of zero Mach number at infinity, becomes identical with the incompressible flow over a body similar to the body concerned. This will ensure the satisfaction of the proper boundary conditions in the physical plane. Furthermore, such a solution would be exact both for the subsonic and for the supersonic regions, as no approximation is introduced. Therefore, it is particularly suitable for the problem of determining the upper critical Mach number for a given body, as limiting lines occur only in mixed subsonic and supersonic flows. This method is followed in the present report, except for the introduction of the transformed potential function χ, for easy calculation of the space coordinates.

For the flow around a body, Chaplygin's procedure will lead to a solution in the form of an infinite series, each term of which is a product of a trigonometric function and a hypergeometric function. To put the method on a firm foundation, it is necessary to establish the convergence of the infinite series. Chaplygin himself has done this for the subsonic region. Thus, only the extension to include the supersonic region remains to be completed. In part I of this report, the general properties of hypergeometric functions of large order are investigated in preparation for the proof of the convergence given in part II. The essential point in these parts is to establish the upper and lower bounds for the hypergeometric functions so that the sum of the infinite series can be discussed. It is appropriate to mention here that for the proper representation of the general solution in the hodograph plane, both fundamental solutions of the hypergeometric differential equation are required. This fact has not been considered by many of the previous investigators in this field. In other cases (reference 7) the investigator has chosen to work with only the first solution.

The general solution constructed by the Chaplygin method is really an existence theorem. The extremely slow convergence of the series makes numerical calculation very difficult, if not impossible. This, in fact, constitutes the main difficulty of the method. In part III of the present report, this difficulty is overcome by using the asymptotic properties of the hypergeometric functions. The result is the separation of the solution in the hodograph plane into two parts. One part is of closed form and is the product of a universal function of the velocity and the same solution as for incompressible flow but with a velocity distortion, or velocity correction. For instance, the first part of the stream function for the compressible flow is equal to the product of the universal function of velocity and the stream function for the incompressible flow with the magnitude of velocity modified by a given rule. The other part is an infinite series which converges rapidly everywhere except in a small region on both sides of a critical circle with a radius equal to $q = c$ in the hodograph plane. In practice, by using only a few terms of the infinite series, this zone of slow convergence can be limited to such a small interval that it is of no consequence. Thus the Chaplygin procedure is improved to a point where actual numerical calculations can be made without difficulty.

As a result of this part of the study it becomes clear

that by the mere substitution of a different speed scale, or velocity distortion, in the solution for an incompressible fluid, an accurate enough solution for the compressible flow cannot be obtained. For if this were the case, then not only the second part of the solution (the rapidly convergent series given by the present method) would be negligible, but also the value of the multiplying universal function of velocity in the first part of the solution would be unity. However, the value of the second part of the solution is not small compared with that of the first part for a speed near that of sound, and the value of the multiplying function of velocity is far from unity. In other words, the usual so-called hodograph method (reference 8) cannot, in general, yield satisfactory results, for mixed subsonic and supersonic flow. On the other hand, the present method does show that the second part of the solution is zero and the multiplying function in the first part takes the constant value of unity, if the isentropic exponent is equal to -1. This means that for this particular case, a simple speed distortion is sufficient. This is, of course, in accordance with the previous investigation of von Kármán (reference 1) and Tsien (reference 9) and L. Bers (reference 10).

Furthermore, the present method also shows that the rules of speed distortion for the first part of the solution can be used only for subsonic flow and that there is a singularity at the local sonic speed. For regions of supersonic flow, the first part of the solution involves both the incompressible stream function and the incompressible potential function. Thus even without considering the second part of the solution, there is no possibility of making the compressible stream lines coincide with those for incompressible flow in the hodograph plane by a simple stretching of the speed scale. The mathematical basis of this fact is the change in character of the differential equation from elliptic to hyperbolic in the transition from subsonic to supersonic flow. For the supersonic regions, it is not possible to use a real transformation of the velocity variable to convert the differential equation of flow to the Laplace equation, and thus make a simple connection between the compressible and the incompressible flows. This is one of the difficulties of the previously proposed hodograph method. In fact, writers using this method must generally limit their calculation to subsonic speeds. (See references 9, 10.) Now this limit is removed, and the whole field of mixed subsonic and supersonic flows can be treated at once with ease.

For the purely subsonic flow, the second part of the

solution is small compared with the first part and may be neglected. Furthermore, if only the zero streamline representing the body is considered, the universal multiplying function of velocity is of no importance. In other words, for this case, a simple speed distortion from the solution of incompressible flow is sufficient to give accurate enough results. However, the subject of the "best" velocity distortion rule in subsonic regions has been the subject of many discussions. (See references 1 and 8.) The present analysis is considered to settle this question. This is due to the fact that the present velocity distortion rule is obtained from the asymptotic properties of the hypergeometric functions, and that such properties are definite and unique. Therefore, the resultant velocity distortion rule is not the result of uncertain speculation. Furthermore, it is also the best rule, because the analysis implies that this rule will make the second part of the solution, or the correction terms, the smallest. This distortion rule is found to coincide with that of Temple and Yarwood. (See reference 11.)

For the purely supersonic flow, the second part of the solution is again small compared with the first part and may be neglected. In fact, the solution then can be reduced to that of the simple wave equation with the inclination of the velocity vector and the distorted velocity as independent variables. This is, of course, the counterpart of the fact that by a simple distortion in velocity, the differential equation for subsonic flows can be reduced to the Laplace equation. The usefulness of this new result for purely supersonic flow has yet to be exploited.

Once the general problem of mixed subsonic and supersonic flow around a body is solved, the determination of the upper critical Mach number or the Mach number for the first appearance of the limiting lines is a simple matter. This problem is discussed in part IV of the report. A simple method is developed, based on the properties of the limiting line as given by von Kármán (reference 1), Ringleb (reference 12), Tollmien (reference 13), and Tsien (reference 2).

To test the practicability of the method developed, two numerical examples are worked out in detail. However, in order to reduce the amount of computational work and in view of the limited time available, a slightly different procedure actually is used. This procedure is only approximate but is

believed to be sufficiently accurate in the supersonic region to give a satisfactory description of the most interesting features of such flows. The examples chosen are derived from the incompressible solution of an elliptic cylinder of thickness ratio 0.6. The free-stream Mach numbers of the compressible flow are 0.6 and 0.7 for these two examples. The first case gives a smooth flow over an "elliptic" cylinder of thickness ratio 0.42. The maximum local Mach number is approximately 1.25. Thus a considerable supersonic region exists. The second case gives a flow with limiting line.

Finally, it must be said that owing to the limitation of time, only the case of flow without circulation is investigated in detail. The explicit formulas for numerical calculation are given for two cases: (a) Flow around a body such as an ellipse, (b) periodic flow pattern such as that over a wavy surface. However, it is believed that more general cases can be studied by a slight extension of the present results and use of the same method of approach.

This investigation, conducted at the Guggenheim Aeronautics Laboratory, California Institute of Technology, was sponsored by and conducted with the financial assistance of the National Advisory Committee for Aeronautics.

NOTATIONS

The symbols used in this report are classified according to the following groups:

A. Physical Quantities

x, y Cartesian coordinates

u, v the velocity components

q the absolute value of the velocity vector

ϑ the inclination of the velocity vector with x-axis

ρ density of the fluid

ρ_0 density of the fluid at $q = 0$

p pressure within the fluid corresponding to ρ

p_0 pressure at $q = 0$

γ ratio of the specific heats

c the local speed of sound

c_0 the speed of sound at $q = 0$

U the value of q at infinity, assuming parallel to the x-axis. With subscript, however, it may be a function of τ.

B. Hydrodynamic Functions in the Physical Plane

$z = x + iy$

$W_0(z) = \varphi_0(x,y) + i\psi_0(x,y)$ complex potential for incompressible flow in z

φ_0 velocity potential for incompressible flow

ψ_0 stream function for incompressible flow

φ velocity potential for compressible flow

ψ stream function for compressible flow

C. Hydrodynamic Functions in the Hodograph Plane

$w = u - iv$

$W_0(w) = \varphi_0(u,v) + i\psi_0(u,v)$ complex potential for incompressible flow in w

$\varphi_0(u,v)$ velocity potential for incompressible flow

$\psi_0(u,v)$ stream function for incompressible flow

$\Lambda_0(w) = zw - W_0(w) = \chi_0(u,v) - i\sigma_0(u,v)$ transformed complex potential function

$\chi_0(u,v) = ux + vy - \varphi_0(x,y)$; $x = \dfrac{\partial\chi_0}{\partial u}$,

$y = \dfrac{\partial\chi_0}{\partial v}$ transformed potential function

$W(w;\tau)$ the complex potential function for compressible flow

$\psi(u,v) = \mathrm{Im}\left\{W(w;\tau)\right\}$ stream function for compressible flow

$\Lambda(w;\tau)$ transformed complex potential function for compressible flow

$\chi(u,v) = ux + vy - \varphi(x,y) = \mathrm{Rl}\left\{\Lambda(w;\tau)\right\}$ transformed potential function for compressible flow

$$\Theta_o(u,v) = \frac{\partial \chi_o}{\partial \vartheta}$$

$$\Omega_o(u,v) = \frac{\partial \sigma_o}{\partial \vartheta}$$

$\psi(q,\vartheta) = \psi_1(q,\vartheta) + \psi_2^{(l)}(q,\vartheta)$; $\psi_1(q,\vartheta)$ represents the contribution by the velocity distorsion; $\psi_2^{(l)}(q,\vartheta)$ stands for the transformed infinite series, where the superscript l may either mean i the inner, or o the outer solution. In the case of coordinates, the notation is exactly the same.

$$G_\nu^{(\alpha)}(\tau) = \underline{F}_\nu(\tau)\Delta B_n^{(\alpha)} + \frac{B_n \Delta \underline{F}_\nu(\tau)}{f(\tau_1)T^\nu(\tau_1)}$$

$$\tilde{G}_\nu^{(\alpha)}(\tau) = \tilde{\underline{F}}_\nu(\tau)\Delta\tilde{B}_n^{(\alpha)} + \frac{\tilde{B}_n \Delta \tilde{\underline{F}}_\nu(\tau)}{f(\tau_1)T^\nu(\tau_1)}$$

$$\tilde{G}_{\nu,1}^{(\alpha)}(\tau) = \frac{\nu - 1}{\nu + 1}\tilde{\underline{F}}_{\nu,1}(\tau)\Delta\tilde{B}_n^{(\alpha)} + \frac{\tilde{B}_n \Delta \tilde{\underline{F}}_{\nu,1}(\tau)}{f(\tau_1)T^\nu(\tau_1)}$$

D. Parameters and Variables

ν positive rational numbers

m, n positive integers

α denotes 1 or 2 when used as superscript with a bracket

or $\alpha = \sqrt{\dfrac{\gamma + 1}{\gamma - 1}}$

β denotes the dependence on β when used as subscript

or $\beta = \dfrac{1}{\gamma - 1}$

$\lambda = \dfrac{2(2\beta)^{\alpha/2}}{(1 + \alpha)^{\alpha}} \dfrac{1}{\sqrt{2\beta\tau_1}} \dfrac{1}{T(\tau_1)}$ the ratio of the distorted speed to that at infinity

$\tau = \dfrac{1}{2\beta} \dfrac{q^2}{c_0^2}$

$\mu = \cos^{-1}\sqrt{\dfrac{\alpha^2\tau - 1}{2\beta\tau}}$

ξ, η With superscript or subscript they denote some functions of τ or stand for the two families of the characteristic parameters $\vartheta + \omega(\tau)$, $\vartheta - \omega(\tau)$ of the partial differential equations for $\psi(q, \vartheta)$ or $\chi(q, \vartheta)$.

ζ complex variable or $\zeta(\tau)$ a function of τ

$M_1 = \dfrac{U}{c_1}$ the Mach number at infinity

$\tau_1 = \dfrac{1}{2\beta} \dfrac{U^2}{c_0^2}$,

ϵ geometrical parameter of the body

Δ Laplacian or difference between exact and approximate values of a function or a constant

E. Hypergeometric Functions

a, b, c parameters of the hypergeometric functions. In particular, a_ν, b_ν, c_ν are defined by (29).

$\underline{F}_\nu(\tau) = F(a_\nu, b_\nu; c_\nu; \tau)$ first integral of the hypergeometric equation associated with the stream function

$$\underline{F}_{-\nu}(\tau) = F(1 + \alpha_\nu - c_\nu, 1 + b_\nu - c_\nu; 2 - c_\nu; \tau)$$

$$F_\nu(\tau) = \frac{\pi\, \tau^{-\nu}}{(2\beta c_o^{\,2})^\nu\, \Gamma(c_\nu - 1)\Gamma(c_\nu)} \left[\frac{\Gamma(a_\nu)\Gamma(b_\nu)}{\Gamma(1+a_\nu-c_\nu)\Gamma(1+b_\nu-c_\nu)} \tau^\nu\, \underline{F}_\nu(\tau) - \frac{\Gamma(c_\nu)}{\Gamma(2-c_\nu)} \underline{F}_{-\nu}(\tau) \right]$$ second integral of the same equation

$$\underline{G}_\nu(\tau) = q^{2\nu}\, F_\nu(\tau)$$

$$\underline{F}_{\nu,1}(\tau) = F(1+a_\nu, 1+b_\nu; 1+c_\nu; \tau)$$

$$\underline{F}_\nu^{(r)}(\tau) = \underline{F}_\nu(\tau)/\underline{F}_\nu(\tau_1)$$

$$\underline{F}_{\nu,1}^{(r)}(\tau) = \underline{F}_{\nu,1}(\tau)/\underline{F}_\nu(\tau_1)$$

$$\underline{F}^*_\nu(\tau) = \underline{F}_\nu(\tau) + i\, F_\nu(\tau)$$

$$R_\nu(\tau) = \left| \underline{F}^*_\nu(\tau) \right|$$

$$\Phi_\nu(\tau) = \arg \underline{F}^*_\nu(\tau)$$

If any function or a constant is associated with $\chi(q, \vartheta)$, it will be marked on top by a symbol ~, such as $\underline{F}_\nu(\tau)$.

PART I

DIFFERENTIAL EQUATIONS OF COMPRESSIBLE FLOW AND PROPERTIES OF THEIR PARTICULAR SOLUTIONS

1. Equations of Motion

It is proposed to study the irrotational steady motion of an inviscid nonconducting compressible fluid in an infinitely extended domain containing a cylindrical body with its axis perpendicular to the constant velocity at infinity. The flow is then two-dimensional. Let x and y be the Cartesian coordinates and u and v the velocity components parallel to the x- and the y-axis. The dynamical equations governing such a motion, in the absence of body force, are

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = - \frac{\partial p}{\partial x} \tag{1}$$

$$\rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} = - \frac{\partial p}{\partial y} \tag{2}$$

Here p is the pressure and ρ the density of the fluid, both being continuous functions of x and y. In addition, the following equation of continuity must be satisfied:

$$\frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) = 0 \tag{3}$$

Furthermore, since the velocity is constant at infinity, the flow is irrotational there. Then, according to Thomson's theorem, if the pressure is a function of the density alone, the flow will remain irrotational; that is,

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \tag{4}$$

In the case of flow of an inviscid nonconducting gas, the thermodynamic change of state of the gas is adiabatic. If

the flow is assumed to be continuous, excluding shock waves, then the relation between p and ρ must be that of an isentropic process:

$$p = \text{constant} \quad \rho^{\gamma} \tag{5}$$

where γ is the ratio of the specific heats.

As in the case of incompressible flow, there are more equations than the number of the variables. However, by virtue of equations (4) and (5), the dynamical equations (1) and (2) reduce to a single differential equation and can be integrated easily to give a relation between the pressure and the magnitude q of the velocity; namely,

$$p = p_0 \left\{1 - \frac{\gamma - 1}{2} \frac{q^2}{c_0^2}\right\}^{\frac{\gamma}{\gamma-1}}, \quad \text{with} \quad q^2 = u^2 + v^2 \tag{6}$$

Here p_0 and c_0 are respectively the pressure and the speed of sound at the stagnation point $q = 0$ and $c = \sqrt{\frac{dp}{d\rho}}$. It is possible to obtain a similar relation between ρ and q by means of equation (5):

$$\rho = \rho_0 \left\{1 - \frac{\gamma - 1}{2} \frac{q^2}{c_0^2}\right\}^{\frac{1}{\gamma-1}} \tag{7}$$

where ρ_0 denotes the value of ρ at $q = 0$.

After integrating the dynamical equations, the velocities u and v can be determined from the kinematic conditions specified by equations (3) and (4). By eliminating ρ from equation (3), the result is

$$\left(1 - \frac{u^2}{c^2}\right)\frac{\partial u}{\partial x} - \frac{2uv}{c^2}\frac{\partial u}{\partial y} + \left(1 - \frac{v^2}{c^2}\right)\frac{\partial v}{\partial y} = 0 \tag{8}$$

where $c^2 = \gamma p/\rho$ and thus can be calculated in terms of the speed by equations (6) and (7). It is of interest to note that the equation of continuity (8) now, unlike the case of incompressible flow, becomes dependent on the dynamical equations and, consequently, is nonlinear. This change in the

character of the fundamental equation makes the direct solution of the problem in space coordinates very difficult.

2. Transformation of the Differential Equations

The assumption of irrotationality implies the existence of a velocity-potential for such a flow. If this function is introduced to eliminate u and v, equations (4) and (8) would give rise to a nonlinear partial differential equation of the second order. The problem is further complicated by the possible appearance of supersonic regions, or regions where the speed of flow is larger than the local sonic speed. This means that for some part of the domain, the equation is of the elliptic type; while in the other part, it is of the hyperbolic type. Thus the equation not only is nonlinear but also is of mixed type, and there is as yet no successful method to deal with it directly in the physical plane. Molenbroek (reference 5) and Chaplygin (reference 6) made some progress in solving the problem by transforming the equations from the physical to the hodograph plane in which u and v are taken as the independent variables. If this is done, the differential equations become linear and thus can be solved by well-known methods.

Let the transformation be defined by

$$u = u(x,y) \tag{9}$$

$$v = v(x,y) \tag{10}$$

If u and v are continuous functions of x and y with continuous partial derivatives, and if the Jacobian $\left(\frac{\partial(x,y)}{\partial(u,v)}\right)$ is finite and nonvanishing, a unique inverse transformation exists. Under these conditions, equations (8) and (4) are easily transformed into

$$\left(1 - \frac{u^2}{c^2}\right)\frac{\partial y}{\partial v} + \frac{2uv}{c^2}\frac{\partial x}{\partial v} + \left(1 - \frac{v^2}{c^2}\right)\frac{\partial x}{\partial u} = 0 \tag{11}$$

$$\frac{\partial x}{\partial v} - \frac{\partial y}{\partial u} = 0 \tag{12}$$

Corresponding to $\varphi(x,y)$ in the physical plane, there is introduced here a function $\chi(u,v)$ defined by

$$\chi = xu + yv - \varphi; \quad x = \frac{\partial\chi}{\partial u}, \quad y = \frac{\partial\chi}{\partial v} \tag{13}$$

While equation (12) is satisfied identically, equation (11) becomes

$$\left(1 - \frac{u^2}{c^2}\right)\frac{\partial^2\chi}{\partial v^2} + \frac{2vu}{c^2}\frac{\partial^2\chi}{\partial v\partial u} + \left(1 - \frac{v^2}{c^2}\right)\frac{\partial^2\chi}{\partial u^2} = 0 \tag{14}$$

As c is a function of q alone, the equation for $\chi(u,v)$ is then linear. From equation (13) it is recognized that if $\chi(u,v)$ is known, a one-to-one correspondence between the space coordinates and the velocity components can be easily established.

However, it is also clear that this function is inconvenient for obtaining the streamlines and the flow in the physical plane. To solve this part of the problem, a plan may be adopted similar to Chaplygin's by introducing both the potential function $\varphi(x,y)$ and the stream function $\psi(x,y)$ defined by:

$$u = \frac{\partial\varphi}{\partial x}, \quad v = \frac{\partial\varphi}{\partial y} \tag{15}$$

$$\rho u = \rho_o \frac{\partial\psi}{\partial y}, \quad \rho v = -\rho_o \frac{\partial\psi}{\partial x} \tag{16}$$

From these definitions are obtained immediately the following equivalent relations:

$$d\varphi = u dx + v dy \tag{17}$$

$$\rho_o d\psi = -\rho v dx + \rho u dy \tag{18}$$

For the subsequent calculations, it was found convenient to introduce the polar coordinates in the hodograph plane defined by:

$$u = q \cos \theta, \quad v = q \sin \theta \tag{19}$$

where θ is the inclination of the velocity vector to the x-axis. Functions dx and dy can be solved for from equations (17) and (18). As dx and dy are exact differentials, the conditions of integrability then give:

$$\frac{\partial \varphi}{\partial q} = -\frac{\rho_o}{\rho}\left(1 - \frac{q^2}{c^2}\right)\frac{1}{q}\frac{\partial \psi}{\partial \theta} \tag{20}$$

$$\frac{1}{q}\frac{\partial \varphi}{\partial \theta} = \frac{\rho_o}{\rho}\frac{\partial \psi}{\partial q} \tag{21}$$

By eliminating φ between equations (20) and (21), an equation for ψ is obtained:

$$q^2 \frac{\partial^2 \psi}{\partial q^2} + \left(1 + \frac{q^2}{c^2}\right) q \frac{\partial \psi}{\partial q} + \left(1 - \frac{q^2}{c^2}\right)\frac{\partial^2 \psi}{\partial \theta^2} = 0 \tag{22}$$

Equation (14) can also be transformed in polar coordinates. The procedure is straightforward and yields

$$q^2 \frac{\partial^2 \chi}{\partial q^2} + \left(1 - \frac{q^2}{c^2}\right) q \frac{\partial \chi}{\partial q} + \left(1 - \frac{q^2}{c^2}\right)\frac{\partial^2 \chi}{\partial \theta^2} = 0 \tag{23}$$

There is an additional relation between χ and φ derived from equation (13):

$$\varphi = q\chi_q - \chi \tag{24}$$

Since φ is connected with ψ, this relation ensures that ψ and χ are properly connected and represent the same flow pattern in the physical plane. It can be thus considered as the equation of compatibility. Equations (22), (23), and (24) are the three fundamental equations in the present problem dealing with the two-dimensional flow of a compressible fluid.

3. The Particular Solutions of the Differential Equations

As the differential equations for $\psi(q,\theta)$ and $\chi(q,\theta)$ are linear, a general solution can certainly be built by superimposing the particular integrals of the equations. To obtain the particular integrals, let $\psi(q,\theta)$ and $\chi(q,\theta)$ be of the following forms:

$$\psi(q,\theta) = q^{\nu}\ \psi_{\nu}(q)\ e^{i\nu\theta}$$

$$\chi(q,\theta) = q^{\nu}\ \chi_{\nu}(q)\ e^{i\nu\theta}$$

where ν is any real number. By substituting in equations (22) and (23), the equations satisfied by $\psi_{\nu}(q)$ and $\chi_{\nu}(q)$ are:

$$q^2 \frac{d^2\psi_{\nu}}{dq^2} + \left(2\nu + 1 + \frac{q^2}{c^2}\right) q \frac{d\psi_{\nu}}{dq} + \nu(\nu + 1) \frac{q^2}{c^2} \psi_{\nu} = 0 \quad (25)$$

$$q^2 \frac{d^2\chi_{\nu}}{dq^2} + \left(2\nu + 1 - \frac{q^2}{c^2}\right) q \frac{d\chi_{\nu}}{dq} + \nu(\nu - 1) \frac{q^2}{c^2} \chi_{\nu} = 0 \quad (26)$$

Now each of these equations can be further reduced by changing the independent variable. The appropriate transformation is found to be

$$\tau = \frac{1}{2\beta} \frac{q^2}{c_o^2}, \qquad \text{with} \quad \beta = \frac{1}{\gamma - 1}$$

By expanding the gas to zero pressure, or vacuum, the maximum velocity is obtained. Equation (6) shows that the maximum speed is $q_{max} = \sqrt{\frac{2}{\gamma - 1}}\ c_o$. Therefore, the maximum value of τ is unity, Similarly, it is found that for the speed of the flow equal to the local sonic speed, $\tau = \frac{1}{2\beta + 1}$, equations (25) and (26) then become

$$\tau(1 - \tau)\psi_{\nu}''(\tau) + \left[c_{\nu} - (a_{\nu} + b_{\nu} + 1)\tau\right]\psi_{\nu}'(\tau) - a_{\nu}b_{\nu}\psi_{\nu}(\tau) = 0 \quad (27)$$

$$\tau(1 - \tau)\chi_\nu''(\tau) + \left[c_\nu - (a_\nu + \beta + b_\nu + \beta + 1)\tau\right]\chi_\nu'(\tau)$$

$$- (a_\nu + \beta)(b_\nu + \beta)\chi_\nu(\tau) = 0 \qquad (28)$$

where

$$a_\nu + b_\nu = \nu - \beta, \quad a_\nu b_\nu = -\frac{1}{2}\beta\nu(\nu + 1), \quad \text{and} \quad c_\nu = \nu + 1 \qquad (29)$$

These are the hypergeometric equations, of which equation (27) was first obtained by Chaplygin in 1904. (See reference 6.) The differential equation of this type has three regular singularities at 0, 1, and $+\infty$. If the differences of the two exponents at the respective singularities; namely, $c - 1$, $a - b$, $a + b - c$, are not integers or zero, the two fundamental independent solutions are $F(a,b;\ c;\ \tau)$ and $\tau^{1-c}\ F(1 + a - c,\ 1 + b - c;\ 2 - c;\ \tau)$. They are single-valued and regular in the whole plane with a cut from $+1$ to $+\infty$. The function $F(a,b;\ c;\ \tau)$ known as the hypergeometric function of general parameters a, b, and c, is defined by the hypergeometric series which is absolutely and uniformly convergent when $|\tau| < 1$, provided $R\ell(c - a - b) > 0$. For $|\tau| \geq 1$, analytic continuation has to be used. Furthermore, it is normalized so that at $\tau = 0$

$$F(a,b;\ c;\ 0) = 1 \qquad (30)$$

Hence, the particular solutions of equation (27) are

$$F(a_\nu, b_\nu; c_\nu; \tau), \quad \tau^{1-c_\nu}\ F(1 + a_\nu - c_\nu,\ 1 + b_\nu - c_\nu;\ 2 - c_\nu; \tau) \qquad (31)$$

The particular solutions of equations (28) are

$$F(a_\nu + \beta;\ b_\nu + \beta;\ c_\nu;\ \tau), \quad \tau^{1-c_\nu}\ F(1 + a_\nu + \beta - c_\nu,\ 1 + b_\nu + \beta - c_\nu;\ 2 - c_\nu;\ \tau) \qquad (32)$$

Here a_ν, b_ν, and c_ν are parameters defined by equation (29).

When ν is a positive integer while a_ν and b_ν remain as they are, the second integral will reduce to a constant multiple of the first one. This case was first studied by Gauss (reference 14), who found a second integral involving a logarithmic term by considering the limiting value of the integrals given as ν tends to an integral value. The method has been further developed by Tannery (reference 15) and Goursat (reference 16). However, the form regarded as conventional nowadays was that obtained by Frobenius' general method. According to this method, the pair of fundamental solutions of a hypergeometric equation are

$$F(a,b;\ n+1;\ \tau),\quad K_n\,\tau^{-n}\left\{\tau^n F(a,b;\ n+1;\ \tau)\log\tau + \tau^n Q_n^{(1)}(a,b;\ \tau) + P_{n-1}^{(1)}(\tau)\right\} \tag{33}$$

when $c_n = n + 1$, n being a positive integer; and

$$Q_n^{(1)}(a,b;\tau) = \frac{\Gamma(n+1)}{\Gamma(a)\Gamma(b)}\sum_0^\infty \frac{\Gamma(a+m)\Gamma(b+m)}{\Gamma(m+1)\Gamma(n+1+m)}\Psi(a,b;\ m)\,\tau^m$$

$$P_{n-1}^{(1)}(\tau) = (-1)^{n+1}\frac{\Gamma(n+1)}{\Gamma(a)\Gamma(b)}\sum_0^{n-1}(-1)^m\frac{\Gamma(a-n+m)\Gamma(b-n+m)\Gamma(n-m)}{\Gamma(m+1)}\tau^m \tag{34}$$

$$\Psi(a,b;\ m) = \sum_{r=0}^{m-1}\left[\frac{1}{a+r}+\frac{1}{b+r}-\frac{1}{n+1+r}\right]-\sum_{r=1}^{m}\frac{1}{r}$$

Here a,b may be either a_n, b_n or $a_n+\beta$, $b_n+\beta$ defined in equation (29) according to whether the system (33) is referred to as solutions of equation (27) or (28). And K_n can be determined so that the product of the second integral and q^{2n} satisfies the condition (30).

In view of the fact that the second integral in (33) does not constitute a family of solutions with the second integral given in (31) or (32), it is very desirable to define a new function as second integral which will be continuous in ν as well as in τ. Let $\underline{F}_\nu(\tau)$ denote the first integral

$F(a,b;\ c_\nu;\ \tau)$. As a second integral, take the linear combination of the solutions:

$$F_\nu(\tau) = K_\nu \left\{\Gamma(1 - c_\nu)\Gamma(a)\Gamma(b)\underline{F}_\nu(\tau) + \Gamma(1 - c_\nu)\Gamma(1 + a - c_\nu)\Gamma(1 + b - c_\nu)\tau^{1-c_\nu}\,\underline{F}_{-\nu}(\tau)\right\} \tag{35}$$

where

$$\underline{F}_{-\nu}(\tau) = F(1 + a - c_\nu,\ 1 + b - c_\nu;\ 2 - c_\nu;\ \tau)$$

This is evidently a solution and valid for all values of ν. The constant K_ν is determined subject to the following condition:

$$q^{2\nu}\,F_\nu(\tau) = 1 \quad \text{for} \quad \tau = 0 \tag{36}$$

The value of K_ν then is found to be

$$K^{-1} = (2\beta c_0)^\nu\,\Gamma(c_\nu - 1)\Gamma(1 + a - c_\nu)\Gamma(1 + b - c_\nu)$$

Using the relation

$$\Gamma(c_\nu)\Gamma(1 - c_\nu) = \pi \csc c_\nu\pi$$

equation (35), when multiplied by $q^{2\nu}$, will define a new function $\underline{G}_\nu(\tau)$: $a,\ b \neq -n$

$$\underline{G}_\nu(\tau) = \frac{\pi}{\sin c_\nu\pi}\left[\frac{\Gamma(a)\Gamma(b)\ \tau^\nu\ \underline{F}_\nu(\tau)}{\Gamma(c_\nu)\Gamma(c_\nu - 1)\Gamma(1 + a - c_\nu)\Gamma(1 + b - c_\nu)} - \frac{\underline{F}_{-\nu}(\tau)}{\Gamma(c_\nu - 1)\Gamma(2 - c_\nu)}\right] \tag{37}$$

When ν takes integral values, the expression in the bracket vanishes; however, the limit of the ratio exists:

$$\underline{G}_n(\tau) = \lim_{\nu \to n} \underline{G}_\nu(\tau) \tag{38}$$

The usual definition of the limit of a quotient gives

$$\underline{G}_n(\tau) = (-1)^{n+1}\left[\frac{\partial}{\partial\nu}\frac{\Gamma(a)\Gamma(b)\ \tau^{\nu}\ \underline{F}_{\nu}(\tau)}{\Gamma(\nu+1)\Gamma(\nu)\Gamma(a-\nu)\Gamma(b-\nu)} - \frac{\partial}{\partial\nu}\frac{\underline{F}_{-\nu}(\tau)}{\Gamma(\nu)\Gamma(1-\nu)}\right]_{\nu=n}$$

By considering separately the first n terms in $\underline{F}_{-\nu}(\tau)$, as $\Gamma(1-\nu)$ has poles at $\nu = n$, a straightforward reduction yields:

$$\underline{G}_n(\tau) = \underline{C}_n\ \tau^n \log\tau\ \underline{F}_n(\tau) + \tau^n Q_n^{(2)}(\tau) + P_{n-1}^{(2)}(\tau) \qquad (39)$$

where

$$Q_n^{(2)}(\tau) = \frac{(-1)^{n+1}}{\Gamma(n)\Gamma(-n+a)\Gamma(-n+b)}\sum_{m=0}^{\infty}\Big[\psi(a+m) + \psi(b+m) - \psi(c_n+m) - \psi(m+1)\Big]\frac{\Gamma(a+m)\Gamma(b+m)}{\Gamma(c_n+m)\Gamma(m+1)}\tau^m$$

$$P_{n-1}^{(2)}(\tau) = \frac{1}{\Gamma(n)\Gamma(a-n)\Gamma(b-n)}\sum_{m=0}^{n-1}(-1)^m\frac{\Gamma(a-n+m)\Gamma(b-n+m)\Gamma(n-m)}{\Gamma(m+1)}\tau^m$$

$$\underline{C}_n = \frac{(-1)^{n+1}\Gamma(a)\Gamma(b)}{\Gamma(n)\Gamma(n+1)\Gamma(a-n)\Gamma(b-n)}$$

and $\psi(\xi)$ denotes the derivative of $\log\Gamma(\xi)$. It can be seen that the difference between (33) and (39) lies only in a constant multiple of the first integral which has been absorbed in $Q_n^{(2)}(\tau)$.

In the following discussions, the two fundamental solutions of the hypergeometric differential equation will be taken as $\underline{F}_{\nu}(\tau)$ and $q^{-2\nu}\ \underline{G}_{\nu}(\tau)$. The normalization conditions given by (30) and (36) are chosen for the continuous passage of a compressible to an incompressible flow. Ultimately, these functions are again defined in terms of power series which are absolutely and uniformly convergent within the domain $|\tau| < 1$. However, since the maximum value of τ attainable by the fluid is unity, the continuation of the solutions beyond the unit circle will not be discussed here.

Thus $\underline{F}_\nu(\tau)$ and $q^{-2\nu}\underline{G}_\nu(\tau)$ denote the two independent integrals of equation (27) where ν is any positive number. The particular solutions of equation (22) are then:

$$q^{\nu}\underline{F}_\nu(\tau)\left[A_\nu^{(1)}\cos\nu\vartheta + A_\nu^{(2)}\sin\nu\vartheta\right], \quad q^{-\nu}\underline{G}_\nu(\tau)\left[B_\nu^{(1)}\cos\nu\vartheta + B_\nu^{(2)}\sin\nu\vartheta\right] \tag{40}$$

where $A_\nu^{(1)}$, $A_\nu^{(2)}$, $B_\nu^{(1)}$, and $B_\nu^{(2)}$ are constants. Similarly, those of equation (23) are

$$q^{\nu}\underline{\tilde{F}}_\nu(\tau)\left[\tilde{A}_\nu^{(1)}\cos\nu\vartheta + \tilde{A}_\nu^{(2)}\sin\nu\vartheta\right], \quad q^{-\nu}\underline{\tilde{G}}_\nu(\tau)\left[\tilde{B}_\nu^{(1)}\cos\nu\vartheta + \tilde{B}_\nu^{(2)}\sin\nu\vartheta\right] \tag{41}$$

where $\underline{\tilde{F}}_\nu(\tau)$ and $q^{-2\nu}\underline{\tilde{G}}_\nu(\tau)$ are the two independent integrals of equation (28) and $\tilde{A}_\nu^{(1)}$, $\tilde{A}_\nu^{(2)}$, $\tilde{B}_\nu^{(1)}$, and $\tilde{B}_\nu^{(2)}$ are constants.

In addition to these solutions, there are two other integrals each of which is a function of one variable only. Assuming $\psi = \psi(q)$ or $\psi(\vartheta)$, then equations (22) and (23) yield respectively:

$$c_1\vartheta \quad \text{and} \quad c_2\int(1-\tau)^{\beta}\frac{d\tau}{\tau} \tag{42}$$

$$\tilde{c}_1\vartheta \quad \text{and} \quad \tilde{c}_2\int(1-\tau)^{-\beta}\frac{d\tau}{\tau} \tag{43}$$

which correspond to the fundamental solution of the Laplace equation.

As c_0 approaches infinity, all these particular solutions reduce to the familiar harmonic functions: namely,

$$q^{\nu}\left[A_\nu^{(1)}\cos\nu\vartheta + A_\nu^{(2)}\sin\nu\vartheta\right], \quad q^{-\nu}\left[B_\nu^{(1)}\cos\nu\vartheta + B_\nu^{(2)}\sin\nu\vartheta\right] \tag{44}$$

and

$$c_1\vartheta, \qquad c_2 \log q \tag{45}$$

This property which is the consequence of (30) and (36) is essential in the method presented in this report for connecting a compressible flow with the incompressible flow of similar configuration.

In the subsequent calculations, another integral will be encountered for the function $\chi(q,\vartheta)$ which corresponds to the imaginary part of $w \log w\, e^{i\pi}$ or $q \log q \sin\vartheta - q(\pi - \vartheta)\cos\vartheta$ of the incompressible flow. Suppose the solution possesses the form:

$$\chi(q,\vartheta) = \chi_1(q) \sin\vartheta - \chi_2(q)(\pi - \vartheta)\cos\vartheta \tag{46}$$

By substituting the expression in equation (23), χ_1 and χ_2 are found to satisfy simultaneously the following differential equations:

$$q^2\chi_1''(q) + \left(1 - \frac{q^2}{c^2}\right)\left(q\chi_1' - \chi_1\right) = 2\left(1 - \frac{q^2}{c^2}\right)\chi_2 \tag{47}$$

$$q^2\chi_2'' + \left(1 - \frac{q^2}{c^2}\right)\left(q\chi_2' - \chi_2\right) = 0 \tag{48}$$

Equation (48) can be easily integrated by putting $\chi_2 = qk_2(q)$. The condition that $\chi_2 \rightarrow q$ as $c_0 \rightarrow \infty$ requires $k_2(q)$ to be a constant. The second integral of equation (48) is just the second of (43) which, in the limit, tends to $\log q$. Thus $\chi_2 = q$ is the appropriate solution. With this solution, it is possible to proceed to solve equation (47) by assuming $\chi_1 = q\,k_1(q)$. The equation for $k_1(q)$ is again integrable by quadrature, and the result is

$$k_1(q) = \frac{1}{2(\beta + 1)}\left[(2\beta + 1)\log\tau - \frac{1}{\tau} + K_1\int^{\tau}(1 - \tau)^{-\beta}\frac{d\tau}{\tau^2}\right] + K_2 \tag{49}$$

where K_1 and K_2 are the constants of integration. Hence, the desired particular integral is

$$\chi(q,\vartheta) = qk_1(\tau) \sin \vartheta - q(\pi - \vartheta) \cos \vartheta \tag{50}$$

The correspondence between solutions for compressible flow and for incompressible flow is summarized in table 1.

4. The Properties of the Hypergeometric Functions of Large Order

The behavior of $F(a_\nu, b_\nu; c_\nu; \tau)$ for large positive values of ν has been discussed by Chaplygin in connection with the question of convergence of his series solution for the flow of a gas jet. However, his discussions are limited to the subsonic flow and, for this reason, the value of τ is restricted to the interval $0 \leq \tau \leq \frac{1}{2\beta + 1}$. In the more general problem where both subsonic and supersonic flow may exist, the whole interval $0 \leq \tau \leq 1$ has to be considered. Furthermore, both integrals of the hypergeometric equation are involved, as will be shown in part II. As a preparation for the proof of the convergence of the solutions, the properties of the hypergeometric functions of large order in the extended interval will be discussed presently.

Chaplygin (reference 6) introduced a new function $\frac{\nu}{2\tau} \xi_\nu(\tau)$ defined as the logarithmic derivative of $\tau^{\frac{\nu}{2}} \underline{F}_\nu(\tau)$: namely,

$$\nu\xi_\nu(\tau) = 2\tau \frac{d}{d\tau} \log \tau^{\frac{\nu}{2}} \underline{F}_\nu(\tau), \quad \nu \neq 0 \tag{51}$$

where $\underline{F}_\nu(\tau)$ denotes the first integral of the hypergeometric equation (27) or (28) and ν can be either an integer or not an integer. Then in the place of equation (27) or (28), the corresponding differential equation for ξ_ν is a Riccati equation:

$$X(\xi_\nu) \equiv \xi_\nu' \pm \frac{\beta}{1 - \tau}\xi_\nu + \frac{\nu}{2\tau}\left[\xi_\nu^2 - \frac{1 - (2\beta + 1)}{1 - \tau}\right] = 0 \tag{52}$$

where the lower sign corresponds to equation (28). As shown by Chaplygin, $\underline{F}_\nu(\tau)$, although an oscillatory function,

can have no root in $0 \leq \tau \leq \frac{1}{2\beta + 1}$ and, consequently, $\xi_\nu(\tau)$ is finite and continuous in the same interval. Moreover, it can be deduced also that $\xi_\nu(0) = 1$ and $\xi_\nu'(0) = -\beta$. Since $\xi_\nu'(\tau)$ does not change sign in $0 \leq \tau \leq \frac{1}{2\beta + 1}$; $\xi_\nu(\tau)$ is monotonic decreasing and eventually vanishes at $\tau_0 \leq \tau^*$, τ^* being the first root of the hypergeometric function for $\nu > 0$. Since τ^* is a decreasing function of ν, when ν becomes large, τ^* and consequently τ_0 will differ from $\frac{1}{2\beta + 1}$ by a small quantity.

<u>Chaplygin's theorem</u>.- In $0 \leq \tau \leq \frac{1}{2\beta + 1}$, if a monotonic continuous function $\eta_\nu(\tau)$ satisfies (i) $\eta_\nu(0) = 1$ and (ii) $X(\eta_\nu) \gtrless 0$, then

$$\eta_\nu(\tau) \gtrless \xi_\nu(\tau), \quad \nu > 1 \tag{53}$$

The proof is given in Chaplygin's paper (reference 6.) In the case of the second integral $F_\nu(\tau)$, the theorem is still true with the signs of inequalities reversed because it can be verified that $X(\xi_{-\nu}) = 0$, where $\xi_{-\nu}(\tau)$ corresponds to the case of $F_\nu(\tau)$ instead of $\underline{F}_\nu(\tau)$ in (51), and $\xi_{-\nu}(0) = -1$; therefore $\xi_{-\nu}(\tau)$ is negative in $0 \leq \tau \leq \frac{1}{2\beta + 1}$.

<u>Corollary (51)</u>.- In $0 \leq \tau \leq \frac{1}{2\beta + 1}$, the functions $\underline{F}_\nu(\tau)$ and $\underline{G}_\nu(\tau)$ fall respectively between the limits:

$$\text{(i)} \qquad T_1^{\nu}(\tau) < \underline{F}_\nu(\tau) < T_2^{\nu}(\tau) \tag{54}$$

$$\text{(ii)} \qquad T_1^{-\nu}(\tau) > \underline{G}_\nu(\tau) > T_2^{-\nu}(\tau), \quad \nu > 1 \tag{55}$$

where

$$T_1(\tau) = \exp\left\{-\int_0^\tau \left[1 - \sqrt{\frac{1 - (2\beta + 1)\tau}{1 - \tau}}\right] \frac{d\tau}{2\tau}\right\} \tag{56}$$

$$T_2(\tau) = \exp\left\{-\int_0^\tau \left[1 - (1 - \tau)^\beta\right] \frac{d\tau}{2\tau}\right\} \tag{57}$$

This can be verified easily by choosing η_ν to be $\sqrt{\frac{1 - (2\beta + 1)\tau}{1 - \tau}}$ or $(1 - \tau)^\beta$. As, evidently, in $0 \leq \tau \leq \frac{1}{2\beta+1}$ when

$$\nu > 1, \quad \sqrt{\frac{1 - (2\beta + 1)\tau}{1 - \tau}} < \xi_\nu < (1 - \tau)^\beta \tag{58}$$

and

$$-\left(1+O\left(\frac{1}{\nu}\right)\right)\sqrt{\frac{1 - (2\beta + 1)\tau}{1 - \tau}} > \xi_{-\nu} > -(1 - \tau)^\beta \tag{59}$$

and furthermore, $X(\eta_\nu) \gtrless 0$ are satisfied, consequently, it follows the results.

<u>Corollary (52)</u>.- In $0 \leq \tau \leq \frac{1}{2\beta + 1}$, the absolute value of the logarithmic derivative of $F(a_\nu, b_\nu; c_\nu; \tau)$ divided by ν, is bounded both above and below - that is,

$$M_1(\tau) \leq \frac{F(a_\nu + 1, b_\nu + 1; c_\nu + 1; \tau)}{F(a_\nu, b_\nu; c_\nu; \tau)} \leq M_2(\tau) \tag{60}$$

where $M_1(\tau)$ and $M_2(\tau)$ are independent of ν. This really is a consequence of (58) and (59).

It shall be noted that the results established in the foregoing are applicable to $\underline{F}_\nu(\tau) = F(a_\nu + \beta, b_\nu + \beta; c_\nu; \tau)$, provided ν is large, because then the two equations (27) and (28) tend to be the same.

Obviously, Chaplygin's theorem ceases to be true when $\tau > \frac{1}{2\beta + 1}$. For in the interval $\frac{1}{2\beta + 1} < \tau < 1$, the solutions of the hypergeometric equation are oscillatory and, hence, within any closed interval in $\frac{1}{2\beta + 1} < \tau < 1$ the number of roots of $\underline{F}_\nu(\tau)$ will be proportional to ν. (See reference 17.) When ν is large, there will be a large number of roots in the interval. As a consequence, the function

$\xi_\nu(\tau)$ will have there an ever increasing number of simple poles, and a finite interval in which $\xi_\nu(\tau)$ remains finite for all ν does not exist.

To carry the investigation over into $\frac{1}{2\beta + 1} < \tau < 1$, the method is modified. Let $\underline{F}_\nu(\tau)$ and $F_\nu(\tau)$ be two independent solutions of equation (27) or (28); and let the linear combination be denoted by

$$F_\nu^*(\tau) = \underline{F}_\nu(\tau) + i\, F_\nu(\tau) \tag{61}$$

The complex function is, of course, a solution of the same differential equation. In terms of its modulus $R_\nu(\tau)$ and argument $\Phi_\nu(\tau)$, the function may also be expressed as

$$F_\nu^*(\tau) = R_\nu(\tau)\, e^{i\,\Phi_\nu(\tau)} \tag{62}$$

where both $R_\nu(\tau)$ and $\Phi_\nu(\tau)$ are continuous functions with continuous derivatives. By comparing with (61), it is necessary to have

$$\underline{F}_\nu(\tau) = R_\nu(\tau) \cos \Phi_\nu(\tau) \tag{63}$$

$$F_\nu(\tau) = R_\nu(\tau) \sin \Phi_\nu(\tau) \tag{64}$$

According to the Sturm separation theorem, $\underline{F}_\nu(\tau)$ and $F_\nu(\tau)$ never vanish simultaneously in any closed interval and $R_\nu(\tau)$ never vanishes in $\frac{1}{2\beta + 1} < \tau < 1$ and remains positive in the whole interval. Then corresponding to (51), a complex function $\xi_\nu^*(\tau)$ can be defined as follows:

$$\nu\, \xi_\nu^*(\tau) = 2\,\tau \frac{d}{d\tau} \log \tau^{\frac{\nu}{2}} F_\nu^*(\tau) \tag{65}$$

which satisfies the same equation (52). On separating into real and imaginary parts, the Riccati equation for $\xi_\nu^*(\tau)$ becomes

$$X_1\left(\xi_\nu^{(1)},\xi_\nu^{(2)}\right) \equiv \xi_\nu'^{(1)} + \frac{\beta}{1-\tau}\,\xi_\nu^{(1)} + \frac{\nu}{2\tau}\left[\xi_\nu^{2(1)} - \xi_\nu^{2(2)} + \frac{(2\beta+1)\tau-1}{1-\tau}\right] = 0 \qquad (66)$$

$$X_2\left(\xi_\nu^{(2)},\xi_\nu^{(1)}\right) \equiv \xi_\nu'^{(2)} + \frac{\beta}{1-\tau}\,\xi_\nu^{(2)} + \frac{\nu}{\tau}\,\xi_\nu^{(2)}\xi_\nu^{(1)} = 0 \qquad (67)$$

where $\xi_\nu^{(1)}$ and $\xi_\nu^{(2)}$ are real continuous functions of τ defined as

$$\xi_\nu^{*}(\tau) = \xi_\nu^{(1)}(\tau) + i\,\xi_\nu^{(2)}(\tau) \qquad (68)$$

Their connection with $R_\nu(\tau)$ and $\Phi_\nu(\tau)$ separately are given by means of (65): namely,

$$\nu\,\xi_\nu^{(1)}(\tau) = 2\tau\frac{d}{d\tau}\log \tau^{\frac{\nu}{2}} R_\nu(\tau) \qquad (69)$$

$$\nu\,\xi_\nu^{(2)}(\tau) = 2\tau\frac{d}{d\tau}\,\Phi_\nu(\tau) \qquad (70)$$

Now equation (67) can be integrated in terms of $\xi_\nu^{(1)}(\tau)$ and whence $\xi_\nu^{(2)}(\tau)$ can be eliminated from equation (66). Then the equations for $\xi_\nu^{(1)}$ and $\xi_\nu^{(2)}$ are

$$X_1\left(\xi_\nu^{(1)}\right) \equiv \xi_\nu'^{(1)} + \frac{\beta}{1-\tau}\,\xi_\nu^{(1)} + \frac{\nu}{2\tau}\left[\xi_\nu^{2(1)} - \xi_0^{2}(1-\tau)^{2\beta}e^{-2\nu\int_{\tau_0}^{\tau}\xi_\nu^{(1)}\frac{d\tau}{\tau}} + \frac{(2\beta+1)\tau-1}{1-\tau}\right] = 0 \qquad (71)$$

$$\xi_\nu^{(2)}(\tau) = -\xi_0(1-\tau)^{\beta}\,e^{-\nu\int_{\tau_0}^{\tau}\xi_\nu^{(1)}\frac{d\tau}{\tau}}, \quad \xi_0 = \frac{2}{\left(\tau_0^{\frac{1}{2}}\,R_\nu(\tau_0)\right)^2} \qquad (72)$$

Equation (71) together with the condition $\xi_\nu^{(1)}(0) = -1$ determines uniquely the solution $\xi_\nu^{(1)}(\tau)$. The actual value of $\xi_\nu^{(1)}(\tau)$ can be expressed, of course, in terms of the known hypergeometric functions. But the problem on hand is to determine the properties of $\xi_\nu^{(1)}(\tau)$ for large ν which are given by the following theorem.

Theorem (52).- If $\eta_\nu^{(1)}(\tau)$ is continuous and monotonic in $\tau_0 < \tau < 1$ and satisfies $X_1(\eta_\nu^{(1)}) \gtrless 0$, then for all $\nu > N$

$$\eta_\nu^{(1)}(\tau) \gtrless \xi_\nu^{(1)}(\tau) \tag{73}$$

The proof is given in appendix A.

Corollary (53).- In $\tau_0 < \tau < 1$, the following inequality holds for the modulus of $\underline{F}_\nu^*(\tau)$:

$$R_\nu(\tau)/R_\nu(\tau_0) < \left(\frac{\tau_0}{\tau}\right)^{\nu/2}, \quad \nu > N \tag{74}$$

where

$$(2\beta + 1)\tau_0 - 1 \geq 0$$

For in $\tau_0 < \tau < 1$, $\xi_\nu^{(1)} < 0$; and hence $\eta_\nu^{(1)}(\tau) = 0$ satisfies the condition $0 > \xi_\nu^{(1)}(\tau)$, which gives (74) by integration.

Now, since $\xi_\nu^{(1)}(\tau)$ is bounded by zero for all $\nu \neq 0$ in $\tau_0 < \tau < 1$, it is implied also that

$$R_\nu(\tau) < T_3^\nu(\tau) \tag{75}$$

where $T_3(\tau) = \frac{t_0}{\tau^{1/2}}$. Here the constant t_0 can be determined by joining T_3 at $\tau = \tau_0$ with T_1 or T_2 defined by equations (56) and (57). Then from equations (63) and (64) it follows that for $\nu > N$

$$\left|\underline{F}_\nu(\tau)\right| < T_3^\nu(\tau) \tag{76}$$

$$\left|\underline{G}_\nu(\tau)\right| < T_3^{-\nu}(\tau), \quad \tau_0 < \tau < 1 \tag{77}$$

PART II

CONSTRUCTION OF THE SOLUTIONS FOR COMPRESSIBLE FLOW AROUND A BODY

5. Chaplygin's Procedure

In the previous sections, the particular solutions of the differential equations in the hodograph plane are obtained. Since the differential equations in the hodograph plane are linear, superposition of solutions is allowed. In other words, if these particular solutions are multiplied by different constants and then added together, the sum is again a solution of the differential equations. By this procedure, general solutions can be constructed from the particular solutions.

However, there is a difficulty in such a method of constructing the general solution — the difficulty of making a proper choice of the multiplying constants for the particular solutions so that the resultant solution will give a flow satisfying the boundary conditions specified in the physical plane. This can be seen from the fact that the space coordinates x and y are obtained from χ which is not explicitly connected with ψ, the stream function. In fact, to obtain the coordinate x and y directly from ψ would involve an integration in the hodograph plane, Thus the linearization of differential equations in the hodograph plane is obtained at the expense of the simplicity in boundary value problem. To guarantee that ψ and χ do actually belong to the same flow in the physical plane, an additional condition besides the differential equations for ψ and χ has to be satisfied. This condition will be discussed in section 11.

Chaplygin (reference 6) suggested an ingenious method of solving this difficulty by using the well-known solutions of the incompressible flow. The first step in this method is to find the incompressible flow around a body "similar" to the body concerned. (The meaning of the word "similar" will be made clear in the following paragraph.)

The stream function ψ_0, for instance, is then expressed in terms of the speed q and the inclination θ. The function

ψ_0 (q, θ) can be expanded into an infinite series each term of which is of the form $q^n \cos n\vartheta$ or $q^n \sin n\vartheta$. For the flow around a body with constant velocity U at infinity, the function $\psi_0(q, \vartheta)$ has a singularity at the point $q = U$, $\theta = 0$ in the hodograph plane, since there all the streamlines, or lines of constant ψ_0 originate. Thus, there are two forms of the series expansion of ψ_0: One is convergent within the circle $q = U$; while the other is convergent outside of the circle $q = U$. The first, or "inside," series must be regular at the origin of the hodograph plane. Therefore, only positive values of the integers n can occur. The second, or "outside," series can have both positive and negative ν. Chaplygin's method is to use the inside series for ψ_0 as the starting point for obtaining the desired solution ψ for the compressible fluid. He suggested choosing the multiplying coefficient of the particular solutions for the compressible flow by the condition that for the limiting case of infinite sonic speed, or incompressible fluid, the series will degenerate to the inside series of the incompressible flow already obtained. The series for the compressible stream function ψ so constructed can be called as the inside series of ψ. The outside series for ψ then can be obtained by the method of analytical continuation with the aid of the "outside series" of the incompressible flow.

These solutions so constructed for the compressible flow contain the Mach number of the undisturbed flow as a parameter. They constitute a family of singly infinite solutions. Included in this family of solutions is the limiting case of zero Mach number of the free stream. This limiting case will give the incompressible flow around a body used as the starting point of this method. For other values of the free-stream Mach number, the body contour is generally different from that corresponding to zero Mach number. Thus, if the compressible flow around a given body is desired, the body shape for the initial incompressible flow must be slightly different from the given body shape. However, if a geometric parameter is included in the solution, such an adjustment is not difficult to make.

It may be stated here that owing to the regularity of the solution at the origin of the hodograph plane, only the first solution of the hypergeometric differential equation appears in the inside series. For the outside series, both the first and the second solution of the hypergeometric differential equation are necessary. This is in direct analogy with the appearance of both positive and negative exponents of q in

the incompressible outside series. This fact is particularly important, since previous investigators seem to be unaware of it. Chaplygin himself did not use the second solution of the hypergeometric differential equation, but that is simply because, for his problem, there is no singularity in the hodograph plane and hence only the inside series is needed.

6. The Functions for Incompressible Flow

Following the procedure outlined in the previous section, the analysis starts with the functions required in defining an irrotational incompressible flow. For this case, the sonic speed c_o tends to infinity, and the equations for the velocity potential $\varphi_o(x,y)$ and the stream function $\psi_o(x,y)$ all became harmonic:

$$\Delta\varphi_o = 0 \tag{78}$$

$$\Delta\psi_o = 0 \tag{79}$$

where Δ stands for the Laplacian operator. If $W_o(z)$ is the complex potential, it can be shown that

$$W_o(z) = \varphi_o + i\,\psi_o \tag{80}$$

where

$$z = x + i\,y$$

If w denotes the complex velocity $u-i\,v$, it is connected with $W_o(z)$ by

$$w = \frac{dW_o}{dz} \equiv w(z) \tag{81}$$

If $w'(z)\neq 0$, it always is possible to solve for z in terms of w; namely,

$$z = z_o(w) \tag{82}$$

In general, this solution is not single-valued and will be discussed later. By introducing this relation into equation (80), the complex potential function in the hodograph plane can be obtained

$$W_o(w) = \varphi_o(u,v) + i\,\psi_o(u,v) \tag{83}$$

In case equation (82) is many-valued, this would correspond to one branch of the function.

It is clear that in this case $\chi_o(u,v)$ is also a harmonic function. Let $\sigma_o(u,v)$ be the conjugate function defined by

$$\frac{\partial \chi_o}{\partial u} = -\frac{\partial \sigma_o}{\partial v} \tag{84}$$

$$\frac{\partial \chi_o}{\partial v} = \frac{\partial \sigma_o}{\partial u} \tag{85}$$

Hence

$$\Lambda_o(w) = \chi_o - i\,\sigma_o \tag{86}$$

where

$$w = u - i\,v$$

Thus $\Lambda_o(w)$ is an analytic function of w. From equation (13) the derivative of $\Lambda_o(w)$ with respect to w must be z. That is,

$$\frac{d\Lambda_o}{dw} = z_o(w)$$

But $z_o(w)$ already has been found from equation (82). Therefore,

$$\Lambda_o(w) = \int z_o(w)dw + \text{constant} \tag{87}$$

The real part of $\Lambda_0(w)$ gives $\chi_0(u,v)$ as required, according to (86).

7. Conformal Mapping of Incompressible Flow on the Hodograph Plane

Before the construction of solutions for the compressible flow, the general character of the solutions in the hodograph plane should be examined. This can be done easily by investigating the behavior of the transition function $z_0(w)$ for an incompressible fluid. To start with the simplest case first, consider a steady irrotational flow in an infinite, simply connected domain D bounded by a curve C in the z-plane, with a parallel flow at infinity (fig. 1). At every point z of D there is one, and only one, velocity vector $\vec{q}$. If the curve C is mapped into $\underline{C}$ and infinity corresponds to a point $\underline{P}$ on the axis of reals of w within $\underline{C}$, then the domain D is mapped into $\underline{D}$ by a mapping function

$$w = w(z)$$

defined in (81), where $w(z)$ is an analytic function of z. The inverse function

$$z = z_0(w)$$

will set up a continuous one-to-one correspondence between w- and z-plane, provided the mapping is conformal. This requires that $w(z)$ is analytic, simple within D, and $w'(z) \neq 0$.

However, for most problems these conditions cannot be satisfied throughout the field of flow. In the first place, the function $w(z)$ is generally nonsimple, for example, in the case of a uniform flow, $w(z) = \text{constant}$, thus $w'(z) \equiv 0$ and the whole z-plane corresponds only to a point in the w-plane. Furthermore, the complex velocity for a two-dimensional boundary-value problem generally can be put in the following form:

$$w = w_\infty + w^*(z)$$

where w_∞ is a constant. The boundary condition requires that $w^*(z) = 0$ and, as a consequence, $w'^*(z) = 0$ as z becomes infinite. Therefore, in all cases, the point P in the w-plane, is a singular point. It is a branch point at w_∞ if $z_o(w)$ is many-valued; or a pole, if otherwise. In practice, there are two kinds of singularities that play a dominant role in the problem of two-dimensional flow. These singularities will be investigated presently.

Branch point of order 1.[1]- It may be recalled that, when a closed body is present in a uniform flow, there always exist two stagnation points both of which correspond to the origin of the w-plane. If a streamline PS is followed, for instance, (see fig. 2) from $+\infty$ to S, the portion SMS' and then to $-\infty$, a curve PS in w-plane would be described twice. This indicates that the function $z_o(w)$ possesses two branches of Riemann surfaces joining together about the branch point P. In order to make the domain D single-valued, a cut is put along the axis of reals from the branch point to $+\infty$. Then one portion of the z-plane is mapped into a definite branch of the Riemann surfaces in the w-plane, and this will be defined as the domain D. If the body is symmetrical with respect to the coordinate axes with parallel flow at infinity, then the domain $D:R\ell z \leq 0$ will be mapped conformally into D on one branch of the Riemann surfaces and $D':R\ell z > 0$ on the other, where the region within C is excluded.

Logarithmic singularity.- The flow over a wavy surface, for instance, placed parallel to a uniform stream has a periodic nature. For such flows there are infinitely many points in the physical plane that have the same velocity. Hence, there are an infinite number of branches in the w-plane, each of which corresponds to a definite portion of the z-plane. The function $z_o(w)$ must have a term $\log\left(1 - \frac{w}{U}\right)$ and the point P now is a logarithmic singularity. If, however, a cut is introduced from the branch point to $+\infty$ and $-\pi < \arg\left(1 - \frac{w}{U}\right) < \pi$, then the domain D is again made single-valued.

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[1]The function $z_o(w)$ is said to have a branch point of order k at $w = w_\infty$ if its inverse $w(z)$ contains the part w^* which has a zero of order $k + 1$ at $z = \infty$.

8. Construction of a Solution about the Origin

Stream function.– From the considerations of the last section, the domain within a circle with radius $|w| = \underline{PS} = U$, where U is the absolute value of w at infinity in z-plane, is in all cases single-valued. If a function $W_0(w)$ is associated with a definite flow in z-plane, from section 6 it is an analytic function of w and regular within the circle $|w| = U$. Consequently, the following Taylor expansion exists:

$$W_0(w) = \sum_{n=0}^{\infty} A_n w^n, \quad |w| < U \tag{88}$$

where A_n's are, in general, complex. Since $w = qe^{-i\vartheta}$ and by (80) the imaginary part of $W_0(w)$ is equal to incompressible stream function ψ_0, it can be written as

$$\psi_0(q,\vartheta) = \mathrm{Im}\left\{W_0(w)\right\} = \sum_{n=0}^{\infty} q^n \left\{A_n^{(1)} \cos n\vartheta + A_n^{(2)} \sin n\vartheta\right\} \tag{89}$$

According to Chaplygin's procedure, the corresponding compressible solution can be obtained by simply replacing the function q^n in equation (89) by the corresponding $q^n \underline{F}_n^{(r)}(\tau)$ as shown by (40). The second integral is excluded by the regularity requirement at $q = 0$. However, in order to preserve the proper singularity at the point $(U,0)$ in the hodograph plane, the compressible stream function ψ is written as

$$\psi(q,\vartheta) = \sum_{n=0}^{\infty} q^n \underline{F}_n^{(r)}(\tau) \left\{A_n^{(1)} \cos n\vartheta + A_n^{(2)} \sin n\vartheta\right\} \tag{90}$$

where

$$\underline{F}_n^{(r)}(\tau) = \frac{\underline{F}_n(\tau)}{\underline{F}_n(\tau_1)} = \frac{F(a_n, b_n; c_n; \tau)}{F(a_n, b_n; c_n; \tau_1)}, \quad q < U \tag{91}$$

and $\tau_1 = \frac{1}{2\beta}\frac{U^2}{c_o^2}$, the value of τ_1 corresponding to the free-stream velocity U. It is seen that if $c_o \to \infty$, then $\tau = \tau_1 \to 0$, and $\underline{F}_n^{(r)}(\tau) \to 1$ due to the normalizing condition (30). Thus the solution is reduced to the incompressible form. Furthermore, if $q \to U$ the character of the solution is exactly like that of the incompressible solution. Hence, all the specified conditions are satisfied. Of course, for the mixed subsonic and supersonic flow, the free-stream Mach number is always less than unity. Thus $\tau_1 < 1/2\beta+1$.

For later analysis as given in part III, it is convenient to write ψ in a different form. Since $\underline{F}_n^{(r)}(\tau)$ is a purely real quantity, a complex function $W(w;\tau)$ can be constructed as

$$W(w;\tau) = \sum_{n=0}^{\infty} A_n \underline{F}_n^{(r)}(\tau)\, w^n, \qquad |w| < U \tag{92}$$

Then, similar to the relation between equations (88) and (89), $\psi(q,\theta)$ can be taken as the imaginary part of the new function $W(w;\tau)$. Thus,

$$\psi(q,\theta) = \operatorname{Im}\left\{W(w;\tau)\right\} \tag{93}$$

<u>Transformed potential function</u>.-- Similarly, it is possible to construct another function $\Lambda(w;\tau)$ defined by

$$\Lambda(w;\tau) = \sum_{n=0}^{\infty} \tilde{A}_n \tilde{\underline{F}}_n^{(r)}(\tau)\, w^n, \qquad q < U \tag{94}$$

In this expression, the coefficients $\tilde{A}_n$ are obtained from the expansion of $\Lambda_o(w)$ for the incompressible flow (87):

$$\Lambda_o(w) = \sum_{n=0}^{\infty} \tilde{A}_n w^n, \qquad |w| < U \tag{95}$$

and

$$\tilde{F}_n^{(r)}(\tau) = \frac{\tilde{F}_n(\tau)}{F_n(\tau_1)} \tag{96}$$

Equation (96) is the result of equation (91) and the equation of compatibility given by equation (34). Then the function $\chi(q,\vartheta)$ for the compressible flow is given by

$$\chi(q,\vartheta) = \mathrm{Rl}\left\{\Lambda(w;\tau)\right\} \tag{97}$$

The functions $W(w;\tau)$ and $\Lambda(w;\tau)$ are actually regular at the origin and satisfy the imposed conditions. However, the following question may be raised: Do the series (92) and (94) converge and represent the functions $\psi(q,\vartheta)$ and $\chi(q,\vartheta)$ in the domain of validity? To settle this question, it is necessary to prove the following theorem:

Theorem (88). If the constants A_n and $\tilde{A}_n$ are given in equations (88) and (95), while $F_n^{(r)}(\tau)$ and $\tilde{F}_n^{(r)}(\tau)$ are defined respectively by equations (91) and (96), the series (92) and (94) are uniformly and absolutely convergent in the same domain as those of (88) and (95). The proof is given in appendix B.

9. Analytic Continuation of the Solution

Branch Point of Order 1

Stream function.— As proved in the appendix B, the series (92) is absolutely and uniformly convergent and does represent a regular function $W(w;\tau)$ for every τ in $0 \leq \tau \leq \tau_1$ and on the circle of convergence it agrees with $W_o(Ue^{-i\theta})$, of which the Fourier expansion exists:

$$W_o(Ue^{-i\theta}) = \sum_{n=o}^{\infty} A_n U^n e^{-in\theta} \tag{98}$$

In the present section, it is proposed to continue the solution (92) analytically outside the domain $|w| \leq U$ with the initial value given by equation (98). The domain outside $|w| \leq U$ is generally many-valued. To fix ideas, discuss first the case of a branch point of order 1. Generally, the function $W_o(w)$ has other singularities besides the one at $w = U$.[1] However, such singularities lie outside the region of interest and thus need not be investigated. Let the nearest singularity be given by $|w| = V > U$. Then, the domain to be considered outside $|w| = U$ is an annulus with a cut joining the two singularities. The proper representation of $W_o(w)$ in such a region which has a branch point of order 1 at $w = U$, is

$$W_o(w) = iw^{-\frac{1}{2}} W_o^*(w) \tag{99}$$

where $W_o^*(w)$ is single-valued and regular within the open annulus $U < |w| < V$. Hence, in any closed domain

$$U + \delta \leq |w| \leq V - \delta, \delta$$

being a small number, there exists a uniformly and absolutely convergent series:

$$W_o^*(w) = \sum_{n=o}^{\infty} \left[B_n w^n + C_n w^{-n}\right] \tag{100}$$

which, on substituting in (99), will give the continuation of the Taylor series (88).

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[1]For instance, in the problem of the flow around an elliptic cylinder treated in part V, there are two singularities of the W_o function given by equation (280): namely, $w = 1$ and $w = 1/\epsilon^2$. The first singularity corresponds to the flow at infinity and is the singularity under discussion. The second singularity corresponds to a point inside the circle of the ζ-plane, the plane of the circular section. Since only the flow outside the circle of the ζ-plane is of interest here, the singularity $w = 1/\epsilon^2$ need not be investigated. In other words, it is necessary only to expand the W_o function in the annular region $1 < \left|\frac{w}{U}\right| < \frac{1}{\epsilon^2}$.

The solution for a compressible fluid, which has the same character of singularities as $W_o(w)$ and is valid in the annulus $U < |w| < V$, can be obtained from (100) by introducing the proper hypergeometric functions corresponding to each exponent of w. That is:

$$W^{(o)}(w;\tau) = i \sum_{n=o}^{\infty} \left[B_n^* \, \underline{F}_\nu(\tau) w^\nu + C_n^* \, \underline{G}_\nu(\tau) w^{-\nu} \right] \qquad (101)$$

which is the continuation of $W^{(i)}(w;\tau)$. Here $\nu = n + \frac{1}{2}$, n being a positive integer; $\underline{F}_\nu(\tau)$ and $q^{-2\nu} \, \underline{G}_\nu(\tau)$ are the first and second integrals of the hypergeometric equation; and B_n^* and C_n^* are constants. It should be noticed that the coefficients B_n^* and C_n^* in the outside series for the compressible flow are not equal to B_n and C_n in equation (100) for the outside series of the incompressible flow. The outside series of the incompressible flow is used only to give the proper form of $W^{(o)}(w;\tau)$ for the desired branch point characteristics; while the exact determination of $W^{(o)}(w;\tau)$ has to be made by the conditions of continuity, which will be discussed presently.

Since the partial differential equation considered here is of the second order, to ensure that $W^{(o)}(w;\tau)$ is the analytic continuation of $W^{(i)}(w;\tau)$, two conditions have to be imposed at the boundary of the respective regions of convergence; that is, the circle $q = U$. These two conditions are the following:

$$W^{(i)}(Ue^{-i\theta}; \tau_1) = W^{(o)}(Ue^{-i\theta}; \tau_1) \qquad (102)$$

$$\left[\frac{\partial}{\partial q} W^{(i)}(w;\tau)\right]_{\tau=\tau_1} = \left[\frac{\partial}{\partial q} W^{(o)}(w;\tau)\right]_{\tau=\tau_1} \qquad (103)$$

On account of equations (102) and (103), there are two relations which have the imaginary parts:

$$\sum_{n=0}^{\infty}\left[B_n^* \underline{F}_\nu(\tau_1)\, U^\nu + C_n^* \underline{G}_\nu(\tau_1)\, U^{-\nu}\right]\cos\nu\vartheta = -\sum_{n=0}^{\infty} A_n\, U^n \sin n\vartheta$$

$$0 \leq \vartheta < 2\pi$$

$$\sum_{n=0}^{\infty}\left[B_n^* U^\nu\left(\nu\underline{F}_\nu(\tau) + 2\tau_1\underline{F}_\nu'(\tau_1)\right) + C_n^* U^{-\nu}\left(-\nu\underline{G}_\nu(\tau_1) + 2\tau_1\underline{G}_\nu'(\tau_1)\right)\right]\cos\nu\vartheta$$

$$= -\sum_{n=0}^{\infty} A_n\, U^n\left(n + 2\tau_1\frac{\underline{F}_n'(\tau_1)}{\underline{F}_n(\tau_1)}\right)\sin n\vartheta$$

Here the prime denotes differentiation with respect to τ. Evidently, the coefficients on the left-hand side can be solved for in terms of the known constants A_n. They are:

$$B_n^* \underline{F}_\nu(\tau_1)\, U^\nu + C_n^* \underline{G}_\nu(\tau_1)\, U^{-\nu} = -\frac{1}{\pi}\sum_{m=0}^{\infty} A_m\, U^m\left(\frac{1}{m+\nu} + \frac{1}{m-\nu}\right) \tag{104}$$

$$B_n^* U^\nu\left(\nu\underline{F}_\nu(\tau_1) + 2\tau_1\underline{F}_\nu'(\tau_1)\right) + C_n^* U^{-\nu}\left(-\nu\underline{G}_\nu(\tau_1) + 2\tau_1\underline{G}_\nu'(\tau_1)\right)$$

$$= -\frac{1}{\pi}\sum_{m=0}^{\infty} m\, A_m\, U^m \xi_m(\tau_1)\left(\frac{1}{m+\nu} + \frac{1}{m-\nu}\right) \tag{105}$$

From these two equations, the constants B_n^* and C_n^* can be uniquely determined, provided the determinant $\Delta(\underline{F}_\nu, F_\nu)$ does not vanish. These results are:

$$B_n^* U^{\nu} = -\frac{\underline{G}_{\nu}(\tau_1)}{2\nu\pi(1-\tau_1)^{\beta}} \sum_{m=0}^{\infty} A_m U^m \left(\frac{1}{m+\nu}+\frac{1}{m-\nu}\right)\left(m\xi_m(\tau_1)-\nu\xi_{-\nu}(\tau_1)\right) \quad (106)$$

$$C_n^* U^{-\nu} = +\frac{\underline{F}_{\nu}(\tau_1)}{2\nu\pi(1-\tau_1)^{\beta}} \sum_{m=0}^{\infty} A_m U^m \left(\frac{1}{m+\nu}+\frac{1}{m-\nu}\right)\left(m\xi_m(\tau_1)-\nu\xi_{\nu}(\tau_1)\right) \quad (107)$$

as the Wronskian $\Delta(\underline{F}_{\nu}, F_{\nu}) = -\frac{\nu}{\tau} q^{-2\nu} (1-\tau)^{\beta} \neq 0$ and $\xi_{\nu}(\tau)$ is defined in (51).

The solution is again formal. To prove that the function $W(w;\tau)$ is a regular function in the annulus region, the truth of the following theorem must be first demonstrated. (See appendix C.)

Theorem (98). If the constants B_n^* and C_n^* are determined according to (102) and (103) and if the series (100) converges uniformly and absolutely in a closed domain $U + \delta \leq |w| \leq V - \delta$, then the series (101) will converge uniformly and absolutely in the domain $U + \delta \leq |w| \leq V - \delta$, $\delta > 0$.

Transformed potential function.– By a similar procedure, the continuation of (94) is

$$\Lambda^{(0)}(w;\tau) = i \sum_{n=0}^{\infty} \left[\tilde{B}_n^* \, \tilde{\underline{F}}_{\nu}(\tau) w^{\nu} + \tilde{C}_n^* \, \tilde{\underline{G}}_{\nu}(\tau) \, w^{-\nu}\right] \quad (108)$$

where $\tilde{\underline{F}}_{\nu}(\tau)$ and $\tilde{\underline{G}}_{\nu}(\tau)$ are the first and second integrals of equation (28) and the constants $\tilde{B}_n^*$ and $\tilde{C}_n^*$ can be similarly determined; namely,

$$\tilde{B}_n^* U^{\nu} = \frac{\tilde{\underline{G}}_{\nu}(\tau_1)}{2\pi\nu(1-\tau_1)^{-\beta}} \sum_{n=0}^{\infty} \tilde{A}_m U^m \left(\frac{1}{m+\nu}-\frac{1}{m-\nu}\right)\left(m\tilde{\xi}_m(\tau_1)-\nu\tilde{\xi}_{-\nu}(\tau_1)\right)\tilde{\underline{F}}_m^{(r)}(\tau_1) \quad (109)$$

$$\tilde{C}_n^* U^{-\nu}$$

$$= \frac{\tilde{\underline{F}}_\nu(\tau_1)}{2\pi\nu(1-\tau_1)^{-\beta}} \sum_{n=0}^{\infty} A_m U^m \left(\frac{1}{m+\nu} - \frac{1}{m-\nu}\right)\left(m\tilde{\xi}_m(\tau_1) - \nu\tilde{\xi}_\nu(\tau_1)\right)\tilde{\underline{F}}_m^{(r)}(\tau_1) \tag{110}$$

The solution determined so far is understood to be the principal branch of the function $W(w;\tau)$. It was assumed that the flow at infinity is parallel to the x-axis. If, in addition, the body is symmetrical with respect to the coordinate axes, the expression for the second branch of $W^{(o)}(w;\tau)$ will be identical. However, in a more general case where asymmetry exists, the two branches will require separate consideration.

10. Continuation – Logarithmic Singularity

Stream function.– Consider now the second important type of singularity: it is assumed here that the only singularity possessed by the function $W_o(w)$ in the finite part of the w-plane is a logarithmic branch point at $w = U$ about which infinitely many Riemann surfaces are joined. By analogy with (99), $W_o(w)$ now can be conveniently written as

$$W_o(w) = W_o^*(w) + \tilde{W}_o(w) \tag{111}$$

where $W_o^*(w)$ is a regular function in the entire domain with possibly an essential singularity at infinity, and hence generally is given by a Taylor series or a polynomial in w, and $\tilde{W}_o(w) = \tilde{\varphi}_o(q,\vartheta) + i\tilde{\Psi}_o(q,\vartheta)$ is an analytic function which characterizes the singularity of $W_o(w)$. Thus, aside from a constant factor,

$$\tilde{W}_o(w) = \frac{1}{i}\log\left(1 - \frac{w}{U}\right) \tag{112}$$

If a cut is laid from $+U$ to $+\infty$ and the argument of $\left(1 - \frac{w}{U}\right)$ is restricted in $-\pi < \arg\left(1 - \frac{w}{U}\right) < \pi$, then the function $\tilde{W}_o(w)$ will be single-valued in the whole cut plane.

The question of constructing a solution for the compressible fluid consists, therefore, of two parts: $W_o^*(w)$ and $\tilde{W}_o(w)$. However, the construction for $W_o^*(w)$ is, in principle, exactly the same as that of (92) and hence only $\tilde{W}_o(w)$ will be considered. First, let $\tilde{W}_o(w)$ be developed into power series in the respective domains of validity. The imaginary parts are:

$$\tilde{\psi}_o^{(i)}(q,\vartheta) = \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{q}{U}\right)^n \cos n\vartheta, \quad q < U \tag{113}$$

$$\tilde{\psi}_o^{(o)}(q,\vartheta) = -\log\frac{q}{U} + \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{q}{U}\right)^{-n} \cos n\vartheta, \quad q > U \tag{114}$$

The corresponding expression for $\tilde{\psi}(q,\vartheta)$, accordingly, will be:

$$\tilde{\psi}^{(i)}(q,\vartheta) = \sum_{n=1}^{\infty} A_n \underline{F}_n(\tau)\left(\frac{q}{U}\right)^n \cos n\vartheta, \quad q < U \tag{115}$$

$$\tilde{\psi}^{(o)}(q,\vartheta) = -B\int_{\tau_1}^{\tau} (1-\tau)^{\beta}\frac{d\tau}{\tau} + \sum_{n=1}^{\infty} C_n \underline{G}_n(\tau)\left(\frac{q}{U}\right)^{-n} \cos n\vartheta, \quad q > U \tag{116}$$

where $\underline{F}_n(\tau)$ stands for $F(a_n, b_n; c_n; \tau)$ and $\underline{G}_n(\tau)$ is defined by (39).

The function $\tilde{W}_o(w)$ may be regarded as the complex potential of a complex source situated at $w = U$. It is known that in this case the normal derivative of $\tilde{\psi}_o(q,\vartheta)$

on $|w| = U$ is a constant, except at $w = U$, where it becomes infinite. This boundary value can be expanded uniquely:

$$-\sum_{n=1}^{\infty} \cos n\vartheta = \frac{1}{2}, \quad \vartheta \neq 0 \tag{117}$$

The corresponding problem in the case of compressible flow can be put in an analogous manner: to find a function $\tilde{\psi}(q,\vartheta)$ which is continuous together with continuous partial derivatives and the normal derivative of which on $|w| = U$ is constant. Thus, the conditions (102) and (103) in conjuction with equation (117) demand:

$$\underline{F}_n(\tau_1) A_n - \underline{G}_n(\tau_1) C_n = 0 \tag{118}$$

$$\left[n\,\underline{F}_n(\tau_1) + 2\tau_1\,\underline{F}_n'(\tau_1)\right] A_n + \left[n\,\underline{G}_n(\tau_1) - 2\tau_1\,\underline{G}_n'(\tau_1)\right] C_n = 4B(1-\tau_1)^{\beta} \tag{119}$$

where the constant B can be determined when the normal derivative $\psi_q(q,\theta)$ on $|w| = U$ is assigned. By solving equations (118) and (119) and using the relation of the Wronskian of the two independent integrals of equation (27), there is obtained

$$A_n = \frac{2}{n} B\,\underline{G}_n(\tau_1) \tag{120}$$

$$C_n = \frac{2}{n} B\,\underline{F}_n(\tau_1) \tag{121}$$

Thus the function $\tilde{\psi}(q,\theta)$ is completely determined.

<u>Transformed potential function</u>.– The associated function $\chi(q,\theta)$ can be similarly constructed. As $\Lambda_0(w)$ is derived

from (87) by integration of the inverse mapping function, it must involve a term $\left(1 - \frac{w}{U}\right) \log \left(1 - \frac{w}{U}\right)$ which represents the singularity of the function $\Lambda_o(w)$. As in equation (111), $\Lambda_o(w)$ is split again into two parts:

$$\Lambda_o(w) = \Lambda_o^*(w) + \tilde{\Lambda}_o(w) \tag{122}$$

where $\Lambda_o^*(w)$ is an entire function and $\tilde{\Lambda}_o(w)$ is

$$\tilde{\Lambda}_o(w) = \frac{1}{i}\left(1 - \frac{w}{U}\right) \log \left(1 - \frac{w}{U}\right) \tag{123}$$

Now the solution corresponding to $\log \left(1 - \frac{w}{U}\right)$ can be determined in exactly the same manner except that the hypergeometric functions involved are $\tilde{\underline{F}}_n(\tau)$ and $\tilde{\underline{G}}_n(\tau)$ instead of $\underline{F}_n(\tau)$ and $\underline{G}_n(\tau)$. The part that will require special consideration is the term $\frac{w}{U} \log \left(1 - \frac{w}{U}\right)$. Let it be denoted by $\tilde{\lambda}_o(w) = \tilde{\chi}_o - i\tilde{\sigma}_o$:

$$\tilde{\lambda}_o(w) = -\frac{1}{i}\frac{w}{U} \log \left(1 - \frac{w}{U}\right) \tag{124}$$

This function is also multiple-valued. Let the argument of $\left(1 - \frac{w}{U}\right)$ again be restricted in $-\pi < \arg \left(1 - \frac{w}{U}\right) < \pi$; then in the cut plane the result will be

$$\tilde{\lambda}_o^{(i)} = \frac{1}{i} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{w}{U}\right)^{n+1}, \quad |w| < U \tag{125}$$

$$\tilde{\lambda}_o^{(o)} = \frac{1}{i}\left[-\frac{w}{U} \log \frac{w}{U}\, e^{i\pi} + \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{w}{U}\right)^{-n+1}\right], \quad |w| > U \tag{126}$$

According to equation (86), the function $\tilde{\chi}_o(q,\vartheta)$ is defined as the real part of $A_o(w)$. That part represented by equations (125) and (126) is then

$$\tilde{\chi}_o^{(i)}(q,\vartheta) = -\sum_{n=2}^{\infty} \frac{1}{n-1}\left(\frac{q}{U}\right)^n \sin n\vartheta \tag{127}$$

$$\tilde{\chi}_o^{(o)}(q,\vartheta) = \frac{q}{U}\log\frac{q}{U}\sin\vartheta - \frac{q}{U}(\pi-\vartheta)\cos\vartheta + \sum_{n=1}^{\infty}\frac{1}{n+1}\left(\frac{q}{U}\right)^{-n}\sin n\vartheta \tag{128}$$

The particular solution corresponding to

$$\frac{q}{U}\log\frac{q}{U}\sin\vartheta - \frac{q}{U}(\pi-\vartheta)\cos\vartheta$$

already has been given in equation (50). Hence the solution for the compressible flow is

$$\tilde{\chi}^{(i)}(q,\vartheta) = -\sum_{n=2}^{\infty}\tilde{A}_n\,\tilde{F}_n(\tau)\left(\frac{q}{U}\right)^n \sin n\vartheta \tag{129}$$

$$\tilde{\chi}^{(o)}(q,\vartheta) = \frac{q}{U}k(\tau)\sin\vartheta - \frac{q}{U}(\pi-\vartheta)\cos\vartheta + \sum_{n=1}^{\infty}\tilde{C}_n\,\tilde{G}_n(\tau)\left(\frac{q}{U}\right)^{-n}\sin n\vartheta \tag{130}$$

where

$$k(\tau) = \frac{1}{2(\beta+1)}\left[(2\beta+1)\log\frac{\tau}{\tau_1} - \left(\frac{1}{\tau}-\frac{1}{\tau_1}\right) + K_1\int_{\tau_1}^{\tau}(1-\tau)^{-\beta}\frac{d\tau}{\tau^2}\right] \tag{131}$$

The conditions (102) and (103) together with an expansion

$$\frac{1}{2}\sin\vartheta + \sum_{n=2}^{\infty}\left(\frac{1}{n+1}+\frac{1}{n-1}\right)\sin n\,\vartheta = (\pi-\vartheta)\cos\vartheta,\quad 0<\vartheta<2\pi$$

require that:

$$\underline{\tilde{F}}_n(\tau_1)\,\tilde{A}_n + \underline{G}_n(\tau_1)\,\tilde{C}_n = \frac{1}{n+1}+\frac{1}{n-1} \tag{132}$$

$$\left[n\,\underline{\tilde{F}}_n(\tau_1) + 2\tau_1\,\underline{\tilde{F}}'_n(\tau_1)\right]\tilde{A}_n$$

$$+\left[-n\,\underline{\tilde{G}}_n(\tau_1) + 2\tau_1\,\underline{\tilde{G}}'_n(\tau_1)\right]\tilde{C}_n = \frac{1}{n+1}+\frac{1}{n-1};\quad n\neq 1 \tag{133}$$

and

$$\underline{\tilde{G}}_1(\tau_1)\,\tilde{C}_1 = \frac{1}{2} \tag{134}$$

$$\left[-\underline{\tilde{G}}_1(\tau_1) + 2\tau_1\,\underline{\tilde{G}}'_1(\tau_1)\right]\tilde{C}_1 + 2\tau_1\,k'(\tau_1) = \frac{1}{2},\quad n=1 \tag{135}$$

By solving (132) and (133) for $\tilde{A}_n$ and $\tilde{C}_n$, there is obtained:

$$\tilde{A}_n = \frac{(1-\tau_1)^{\beta}}{n^2-1}\left(1-n\,\tilde{\xi}_{-n}(\tau_1)\right)\underline{\tilde{G}}_n(\tau_1) \tag{136}$$

$$\tilde{C}_n = -\frac{(1-\tau_1)^{\beta}}{n^2-1}\left(1-n\,\tilde{\xi}_n(\tau_1)\right)\underline{\tilde{F}}_n(\tau_1),\quad n\neq 1 \tag{137}$$

by using the Wronskian of the independent integrals of equation (28). With C_1 given by (134), the constant K_1 can be solved for from (135); it is

$$K_1 = -(1-\tau_1)^{\beta}\left[1 + \beta\tau_1 + (\beta+1)\tau_1^{2}\frac{\tilde{G}_1'(\tau_1)}{\tilde{G}_1(\tau_1)}\right] \qquad (138)$$

The solutions $\tilde{\psi}(q,\vartheta)$ and $\tilde{\chi}(q,\vartheta)$ in the whole domain under consideration are uniquely determined. Since the dominant properties of the hypergeometric functions discussed in section 4 hold, in general, the equation of convergence can be similarly settled.

11. Transition to Physical Plane

In the previous sections, it has been proved that, for a given incompressible flow for which two associated functions $\psi_0(q,\vartheta)$ and $\chi(q,\vartheta)$ are defined, there exist two associated functions $\psi(q,\vartheta)$ and $\chi(q,\vartheta)$ for the corresponding compressible flow, depending upon two parameters γ and τ_1. The question is whether the associated functions $\psi(q,\vartheta)$ and $\chi(q,\vartheta)$ belong to the same flow pattern in the physical plane. To answer this question it is necessary to fall back once more on the equation of compatibility (24); since when $\psi(q,\vartheta)$ is given, $\varphi(q,\vartheta)$ is known by solving equations (20) and (21). Hence, if $\chi(q,\vartheta)$, satisfying equation (23) and approaching χ_0 as $C_0 \longrightarrow \infty$ is to be associated with $\psi(q,\vartheta)$ for the same flow, then it is necessary that the equation of compatibility be satisfied. Except in the case of logarithmic singularity in section 10 where the complete $\psi(q,\vartheta)$ function was not discussed, this condition has been properly considered.

Once the relationship between $\psi(q,\vartheta)$ and $\chi(q,\vartheta)$ is established, the next object is to calculate the flow pattern $\psi(x,y)$ = constant in the physical plane corresponding to $\psi(q,\vartheta)$ and $\chi(q,\vartheta)$. In the first place, the fact that the transformation defined by equations (9) and (10) is generally one-to-one must be recalled. Suppose that in the hodograph plane there is a line defined by

$$\psi(q,\vartheta) = \text{constant} = K \qquad (139)$$

which will correspond to a definite streamline in the physical plane or a definite part of it. The streamline can be obtained by eliminating one of the two variables in $x(q,\theta)$ and $y(q,\theta)$.

To do this, first the equation (139) is solved for ϑ; namely,

$$\vartheta = \vartheta(q,K) \tag{140}$$

provided $\psi_\vartheta(q,\vartheta) \neq 0$. Introducing this relation into equation (13) which, when transformed into polar coordinates, are

$$x = \cos\vartheta \frac{\partial\chi}{\partial q} - \frac{\sin\vartheta}{q}\frac{\partial\chi}{\partial\vartheta} \tag{141}$$

$$y = \sin\vartheta \frac{\partial\chi}{\partial q} + \frac{\cos\vartheta}{q}\frac{\partial\chi}{\partial\vartheta} \tag{142}$$

gives a parametric representation of this particular streamline corresponding to $\psi(q,\vartheta) = K$ in the hodograph plane.

PART III

IMPROVEMENT OF THE CONVERGENCE OF SOLUTION BY THE ASYMPTOTIC PROPERTIES OF HYPERGEOMETRIC FUNCTIONS

12. General Concepts

The significance of the general solutions constructed in part II of the present report when viewed from the practical point, rests in the fact that they constitute an existence theorem. It shows that an irrotational isentropic flow about a body can be obtained from the corresponding problem of an incompressible fluid, if the free-stream Mach number is not too high. However, the solution in the form of a slowly convergent infinite series cannot be conveniently used to obtain numerical values, as the labor of computation would be prohibitive.

By examining the infinite series obtained in part II, the essential difference between the compressible flow solution and the incompressible flow solution is seen to be associated with the fact that, while in incompressible flow solution the individual terms of the series are of the forms

$$q^{\nu} \begin{matrix} \cos \nu\vartheta \\ \sin \nu\vartheta \end{matrix} \qquad q^{-\nu} \begin{matrix} \cos \nu\vartheta \\ \sin \nu\vartheta \end{matrix}$$

in compressible flow solution the individual terms of the series are of the forms

$$q^{\nu}\, \underline{F}_{\nu}(\tau) \begin{matrix} \cos \nu\vartheta \\ \sin \nu\vartheta \end{matrix} \qquad q^{-\nu}\, \underline{G}_{\nu}(\tau) \begin{matrix} \cos \nu\vartheta \\ \sin \nu\vartheta \end{matrix}$$

If it were possible to write

$$q^{\nu}\, \underline{F}_{\nu}(\tau) = \left[Q(q)\right]^{\nu}, \quad q^{-\nu}\, \underline{G}_{\nu}(\tau) = \left[Q(q)\right]^{-\nu}$$

there would be no difference between the incompressible flow

solution and the compressible flow solution except the "distortion of the speed" q by the new scale Q. In fact, this possibility is realized under the special condition of $\gamma = -1$ as shown by von Kármán (reference 1) and Tsien (reference 9).

For the case of isentropic flow with the general exponent γ there is no such scale factor Q. However, if ν is assumed to be very large, then there is such a function Q, at least to a first approximation. In other words, the leading term in the asymptotic representations of $\underline{F}_\nu(\tau)$ and $\underline{G}_\nu(\tau)$ does give the desired form. On the other hand, the use of asymptotic representation necessarily implies an approximation. But this defect is not difficult to remedy as the difference between an exact hypergeometric function and its asymptotic form can be added to the approximate solution as a correction term. Since there are an infinite number of terms in the series form of the solution and each gives a correction term, the correction terms also constitute an infinite series. Therefore, the original infinite series is now transformed into a closed function plus another infinite series of correction terms. At first sight, such a transformation seems unable to give a result that will avoid the difficulty of prohibitive computational work. But actually, owing to the good approximation given by the asymptotic representation even for moderate values of ν, the correction series converges very rapidly. A few terms seem to be all that are necessary. Thus, for all practical purposes, the original infinite series is now converted into a closed function with "speed distortion" plus a few correction terms. The fundamentally interesting point is that for the case of a general exponent γ, the simple method of speed distortion will not give an accurate enough solution. (Cf. reference 8.)

The change in type of the differential equation at the sonic speed also introduces a singularity in the speed distortion function Q. However, by using the correction terms, the effect of the singularity can be limited to a very narrow range in the neighborhood of sonic speed, and no practical difficulty is experienced. This will be made clear by the numerical example given in part V of this report.

13. Asymptotic Solutions of the Hypergeometric Equations

Let $U_\nu(\tau)$ and $V_\nu(\tau)$ be two new dependent variables defined by

$$\psi_\nu(\tau) = \tau^{-\frac{\nu+1}{2}} (1 - \tau)^{\frac{\beta}{2}} U_\nu(\tau) \tag{143}$$

$$\chi_\nu(\tau) = \tau^{-\frac{\nu+1}{2}} (1 - \tau)^{-\frac{\beta}{2}} V_\nu(\tau) \tag{144}$$

The differential equations (27) and (28) reduce respectively to

$$U_\nu''(\tau) - \left[\nu^2 \varphi(\tau) + \rho_\beta(\tau)\right] U_\nu(\tau) = 0 \tag{145}$$

$$V_\nu''(\tau) - \left[\nu^2 \varphi(\tau) + \rho_{-\beta}(\tau)\right] V_\nu(\tau) = 0 \tag{146}$$

where

$$\varphi(\tau) = \frac{1 - (2\beta + 1)\tau}{4\tau^2(1 - \tau)}$$

$$\rho_{\pm\beta}(\tau) = \frac{\beta\tau(\beta\tau \pm 2) - (1 \pm \tau)^2}{4\tau^2(1 - \tau)^2}$$

Both equations (145) and (146) involve a constant parameter ν which is real and positive but otherwise arbitrary for any fixed constant β. In the interval $0 < \tau < 1$ in which the flow takes place, the functions $\varphi(\tau)$ and $\rho_{\pm\beta}(\tau)$ are finite and continuous except at the extremities $\tau = 0$ and $\tau = 1$. To avoid the repetition, let equations (145) and (146) be replaced by

$$U_{\alpha,\nu}''(\tau) - \left[\nu^2 \varphi(\tau) + \rho_\alpha(\tau)\right] U_{\alpha,\nu}(\tau) = 0 \tag{147}$$

where $U_{\beta,\nu}(\tau) = U_\nu(\tau)$ when $\alpha = \beta$; and $U_{-\beta,\nu}(\tau) = V_\nu(\tau)$ when $\alpha = -\beta$. In the interval $\delta \leq \tau \leq \frac{1}{2\beta + 1} - \delta$, $\delta > 0$, $\varphi(\tau)$ is bounded from zero and is positive. F. Horn

(reference 18) showed that when ν is a large positive number, a pair of solutions of the following forms exist in the interval concerned:

$$U_{\alpha,\nu}^{(1)}(\tau) \sim e^{\nu K}\left[\varphi^{-\frac{1}{4}} + \frac{f_{11}(\tau)}{\nu} + \frac{f_{12}(\tau)}{\nu^2} + \ldots + \frac{f_{1s}(\tau)}{\nu^s}\right] \quad (148)$$

$$U_{\alpha,\nu}^{(2)}(\tau) \sim e^{-\nu K}\left[\varphi^{-\frac{1}{4}} + \frac{f_{21}(\tau)}{\nu} + \frac{f_{22}(\tau)}{\nu^2} + \ldots + \frac{f_{2s}(\tau)}{\nu^s}\right] \quad (149)$$

where

$$K(\tau) = \int^{\tau} \varphi^{\frac{1}{2}}(\tau)\, d\tau, \qquad 0 < \tau < \frac{1}{2\beta + 1} \quad (150)$$

A constant in equation (150) was left out, as it can be absorbed in the constant factor in equations (148) and (149). This representation can be shown to be unique as long as ν remains greater than a large positive number N. By substituting $U_{\alpha,\nu}^{(1)}(\tau)$ and $U_{\alpha,\nu}^{(2)}(\tau)$ in equation (147) and choosing the coefficients $f_{r,s}(\tau)$ ($r = 1$ and 2; and $s = 1, 2, 3, \ldots$) to make the individual terms vanish, equation (147) reduces to

$$2K' f'_{1,s+1} + K'' f_{1,s+1} = \rho_\alpha f_{1,s} - f''_{1,s} \quad (151)$$

$$2K' f'_{2,s+1} + K'' f_{2,s+1} = -\rho_\alpha f_{2,s} + f''_{2,s}, \quad s = 0, 1, 2, \ldots \quad (152)$$

where $f_{1,0}(\tau) = f_{2,0}(\tau) = \varphi^{-\frac{1}{4}}$. The coefficients $f_{r,s}(\tau)$ then are given successively by a first order ordinary differential equation and their determination does not involve any difficulty. The problem is thus formally solved.

Obviously, the solution is of the exponential type when $\varphi(\tau)$ is positive in the range concerned and of an oscillatory type when $\varphi(\tau)$ is negative. Now in the interval $\delta \leq \tau \leq 1 - \delta$, $\delta > 0$ where $\varphi(\tau) \gtrless 0$ when $\tau \lessgtr \frac{1}{2\beta + 1}$, both types of solution exist. It is evident that in the neighborhood of $\tau = \frac{1}{2\beta + 1}$ a change of character of the solutions

must take place, but the manner in which the transition occurs cannot be deduced from equations (148) and (149) because of the failure of the representation of the solutions in the neighborhood $\tau = \frac{1}{2\beta + 1}$. This is closely related to the Stokes phenomenon.

The method was extended by Jeffreys (reference 19) to include the case where $\varphi(\tau)$ has a simple root in an interval under consideration and can be applied suitably to the first order of approximation. The general problem has been treated by Langer (reference 20) in a series of papers, considering both the case where ν and τ are real and that where ν and τ are complex. Attention was given especially to the Stokes phenomenon, and a law of connection of the solution valid on each side of the critical point was explicitly stated. In the present case, however, only the first approximation is used and Jeffreys' method is adopted for convenience.

It is seen from equations (148) and (149) that the first approximation depends only on $\varphi(\tau)$, and the effect of $\rho_\alpha(\tau)$ is felt only by the higher order terms. Hence, for the first approximation only, equation (147) can be written as

$$U_\nu''(\tau) - \nu^2 \varphi(\tau) U_\nu(\tau) = 0 \tag{153}$$

where $U_{\beta,\nu} = U_{-\beta,\nu} = U_\nu$. Thus, when $\nu > N$, the dominant terms of the asymptotic solutions are

$$U_\nu^{(1)}(\tau) \sim \varphi^{-\frac{1}{4}} e^{\nu K} \left[1 + O\left(\frac{1}{\nu}\right)\right] \tag{154}$$

$$U_\nu^{(2)}(\tau) \sim \varphi^{-\frac{1}{4}} e^{-\nu K} \left[1 + O\left(\frac{1}{\nu}\right)\right] \qquad 0 < \tau < \frac{1}{2\beta+1} \tag{155}$$

Here $O\left(\frac{1}{\nu}\right)$, in each case, denotes the fact that the term is uniformly of the order ν^{-1} when ν is sufficiently large in an interval $\delta \leq \tau \leq \frac{1}{2\beta + 1} - \delta$, $\delta > 0$ and is a function of ν^{-1}.

On the other hand, in the interval $\frac{1}{2\beta+1} + \delta \leq \tau \leq 1 - \delta$, where $\varphi(\tau) < 0$ and K is a pure imaginary quantity $i\omega$ where ω is real, the dominant terms of the asymptotic solutions must be a linear combination of equations (148) and (149) and must be of the forms:

$$U_\nu^{(1)}(\tau) \sim \frac{c_1}{\varphi^{\frac{1}{4}}} \cos\,(\nu\omega + \epsilon_\nu) \tag{156}$$

$$U_\nu^{(2)}(\tau) \sim \frac{c_2}{\varphi^{\frac{1}{4}}} \sin\,(\nu\omega + \epsilon_\nu); \quad \frac{1}{2\beta+1} < \tau < 1 \tag{157}$$

where c_1, c_2, and ϵ_ν are constants to be determined.

The question of determination of these constants is actually the same as that of determining the mode of continuation of the asymptotic representation of the solutions in the range $\frac{1}{2\beta+1} + \delta \leq \tau \leq 1 - \delta$. This can be done, according to Jeffreys, by considering the solutions valid in the neighborhood of $\tau = \frac{1}{2\beta+1}$. Let $\xi = \tau - \frac{1}{2\beta+1}$. When ξ is sufficiently small and ν is large, equation (153) can be written approximately as

$$U_\nu''(\xi) + \nu^2\, \varphi'(0) \cdot \xi\, U_\nu(\xi) = 0 \tag{158}$$

provided $\frac{\varphi^{(n)}(0)}{n!\,\varphi'(0)} \sim 1$. This is known as Stokes equation.

The independent integrals are

$$\xi^{\frac{1}{2}} H_{\frac{1}{3}}^{(1)}(\zeta), \quad \xi^{\frac{1}{2}} H_{\frac{1}{3}}^{(2)}(\zeta); \quad \text{with} \quad \zeta = \frac{2}{3}\, \nu\varphi'^{\frac{1}{2}}(0)\, \xi^{\frac{3}{2}} \tag{159}$$

where $H_{\frac{1}{3}}^{(1)}(\zeta)$ and $H_{\frac{1}{3}}^{(2)}(\zeta)$ are the Hankel functions of order $\frac{1}{3}$. Consider as two independent solutions the following linear combinations:

$$U_\nu^{(1)}(\zeta) = \xi^{\frac{1}{2}} H_{\frac{1}{3}}^{(1)}(\zeta) + \xi^{\frac{1}{2}} H_{\frac{1}{3}}^{(2)}(\zeta) \tag{160}$$

$$U_\nu^{(2)}(\xi) = \xi^{\frac{1}{2}} H_{\frac{1}{3}}^{(1)}(\zeta) - \xi^{\frac{1}{2}} H_{\frac{1}{3}}^{(2)}(\zeta) \tag{161}$$

As $H_{\frac{1}{3}}^{(1)}(\zeta)$ and $H_{\frac{1}{3}}^{(2)}(\zeta)$ are analytic functions in the whole ζ-plane, this immediately provides a means of identifying the asymptotic forms that represent the same function.

Suppose first that for $\arg \xi = 0$, the solutions are given in equations (160) and (161). The same solutions for which $\arg \xi = \pi$ and $\arg \zeta = \frac{3}{2}\pi$ are

$$U_\nu^{(1)}(\xi) = \xi^{\frac{1}{2}} e^{\frac{\pi i}{2}} H_{\frac{1}{3}}^{(1)}\left(\zeta e^{\frac{3\pi i}{2}}\right) + \xi^{\frac{1}{2}} e^{\frac{\pi i}{2}} H_{\frac{1}{3}}^{(2)}\left(\zeta e^{\frac{3\pi i}{2}}\right) \tag{162}$$

$$U_\nu^{(2)}(\xi) = \xi^{\frac{1}{2}} e^{\frac{\pi i}{2}} H_{\frac{1}{3}}^{(1)}\left(\zeta e^{\frac{3\pi i}{2}}\right) - \xi^{\frac{1}{2}} e^{\frac{\pi i}{2}} H_{\frac{1}{3}}^{(2)}\left(\zeta e^{\frac{3\pi i}{2}}\right) \tag{163}$$

Now

$$H_{\frac{1}{3}}^{(1)}\left(\zeta e^{\frac{\pi i}{2}} e^{\pi i}\right) = -e^{-\frac{\pi i}{2}} H_{\frac{1}{3}}^{(2)}\left(\zeta e^{\frac{\pi i}{2}}\right)$$

$$H_{\frac{1}{3}}^{(2)}\left(\zeta e^{\frac{\pi i}{2}} e^{\pi i}\right) = 2 \cos \frac{\pi}{3} H_{\frac{1}{3}}^{(2)}\left(\zeta e^{\frac{\pi i}{2}}\right) + e^{\frac{\pi i}{3}} H_{\frac{1}{3}}^{(1)}\left(\zeta e^{\frac{\pi i}{2}}\right)$$

and when ζ is large and $-\pi < \arg \zeta e^{\frac{\pi i}{2}} < \pi$, the dominant terms of the asymptotic expansions of $H_{\frac{1}{3}}^{(1)}\left(\zeta e^{\frac{\pi i}{2}}\right)$ and $H_{\frac{1}{3}}^{(2)}\left(\zeta e^{\frac{\pi i}{2}}\right)$ are

$$H_{\frac{1}{3}}^{(1)}\left(\zeta e^{\frac{\pi i}{2}}\right) \sim \sqrt{\frac{2}{\pi \zeta}}\, e^{i\left(\zeta e^{\frac{\pi i}{2}} - \frac{\pi}{6} - \frac{\pi}{2}\right)} \left\{1 + O\left(\frac{1}{\zeta}\right)\right\}$$

$$H_{\frac{1}{3}}^{(2)}\left(\zeta e^{\frac{\pi i}{2}}\right) \sim \sqrt{\frac{2}{\pi \zeta}}\, e^{-i\left(\zeta e^{\frac{\pi i}{2}} - \frac{\pi}{6}\right)} \left\{1 + O\left(\frac{1}{\zeta}\right)\right\}$$

By substituting in equations (162) and (163) and neglecting the term of lower order in ζ, there is obtained by expanding at the same time equations (160) and (161);

$$2\xi^{-\frac{1}{4}} \cos\left(\zeta - \frac{\pi}{4}\right) \longrightarrow \xi^{-\frac{1}{4}} e^{-\zeta} \tag{164}$$

$$\xi^{-\frac{1}{4}} \sin\left(\zeta - \frac{\pi}{4}\right) \longrightarrow \xi^{-\frac{1}{4}} e^{\zeta} \tag{165}$$

Here the arrow is used to indicate the transition of the asymptotic representation of the same function from the left-hand to the right-hand member. For small ξ, $\xi^{-\frac{1}{4}} \sim \varphi^{-\frac{1}{4}}$, and $\zeta \sim -\nu\omega$; (156) and (157) finally become

$$U_\nu^{(1)}(\tau) \sim \frac{2}{\varphi^{\frac{1}{4}}} \cos\left(\nu\omega - \frac{\pi}{4}\right)\left\{1 + O\left(\frac{1}{\nu}\right)\right\} \tag{166}$$

$$U_\nu^{(2)}(\tau) \sim \frac{1}{\varphi^{\frac{1}{4}}} \cos\left(\nu\omega + \frac{\pi}{4}\right)\left\{1 + O\left(\frac{1}{\nu}\right)\right\} \quad \frac{1}{2\beta+1} < \tau < 1 \tag{167}$$

with $c_1 = 2$, $c_2 = -1$, and $\epsilon_\nu = -\frac{\pi}{4}$. Under the hypothesis just made, the pair of expressions (154), (166) and (155), (167) actually represent respectively the dominant terms of the two asymptotic expansions of the solutions $U_\nu^{(1)}(\tau)$ and $U_\nu^{(2)}(\tau)$ for a ν which may be any positive but large number.

14. The Asymptotic Representation of $F(a_\nu, b_\nu; c_\nu; \tau)$ and $F(a_\nu + \beta, b_\nu + \beta; c_\nu; \tau)$

The dominant terms of the asymptotic expansion of $U_\nu^{(1)}(\tau)$ and $U_\nu^{(2)}(\tau)$ are given respectively by (154), (166) and (155), (167). By evaluating the simple integrals in (154) and (166), the explicit expressions for the first approximation of $U_\nu^{(1)}(\tau)$ and $U_\nu^{(2)}(\tau)$ are

$$U_\nu^{(1)}(\tau) \sim (2\beta)^{\frac{-\nu(\alpha-1)}{2}} \left\{\frac{4(1-\tau)}{1-\alpha^2\tau}\right\}^{\frac{1}{4}} \tau^{\frac{\nu+1}{2}} T^{\nu *}(\tau) \tag{168}$$

$$U_\nu^{(2)}(\tau) \sim (2\beta)^{\frac{\nu(\alpha-1)}{2}} \left\{\frac{4(1-\tau)}{1-\alpha^2\tau}\right\}^{\frac{1}{4}} \tau^{-\frac{\nu+1}{2}} T^{-\nu *}(\tau) \qquad 0 < \tau < \frac{1}{2\beta+1} \tag{169}$$

$$U_\nu^{(1)}(\tau) \sim 2\left\{\frac{4(1-\tau)}{\alpha^2\tau-1}\right\}^{\frac{1}{4}} \tau^{\frac{1}{2}} \cos\left(\nu\omega - \frac{\pi}{4}\right) \tag{170}$$

$$U_\nu^{(2)}(\tau) \sim \left\{\frac{4(1-\tau)}{\alpha^2\tau-1}\right\}^{\frac{1}{4}} \tau^{\frac{1}{2}} \cos\left(\nu\omega + \frac{\pi}{4}\right) \qquad \frac{1}{2\beta+1} < \tau < 1 \tag{171}$$

where

$$T^*(\tau) = \frac{\left[\alpha(1-\tau)^{\frac{1}{2}} + (1-\alpha^2\tau)^{\frac{1}{2}}\right]^{\alpha}}{(1-\tau)^{\frac{1}{2}} + (1-\alpha^2\tau)^{\frac{1}{2}}}, \qquad \alpha = \left[\frac{\gamma+1}{\gamma-1}\right]^{\frac{1}{2}} \tag{172}$$

$$\omega(\tau) = \alpha \tan^{-1}\sqrt{\frac{\alpha^2\tau-1}{\alpha^2(1-\tau)}} - \tan^{-1}\sqrt{\frac{\alpha^2\tau-1}{1-\tau}}$$

The values of the function $\omega(\tau)$ are given in figure 3 together with the function $\mu(\tau)$, defined by $\cos\mu = 1/M$. In the respective ranges of validity, each pair of expressions differs from the exact solution only by a constant factor which can be determined to satisfy the normalization conditions (30) and (36). By substituting equation (168) into equation (143), these were found to be

$$c_{\pm\nu} = \frac{1}{\sqrt{2}} (2\beta)^{\pm\frac{\alpha-1}{2}\nu} \left\{\frac{2}{(1+\alpha)^2}\right\}^{\pm\nu}$$

Thus, the expressions for the desired asymptotic forms, when $\nu > N$, are, for the interval $0 \leq \tau < \frac{1}{2\beta+1}$,

$$\underline{F}_\nu(\tau) \sim f(\tau)\, T^{\nu}(\tau) \tag{173}$$

$$\underline{G}_\nu(\tau) \sim f(\tau)\, T^{-\nu}(\tau) \tag{174}$$

where

$$f(\tau) = \frac{(1-\tau)^{\frac{\beta}{2}+\frac{1}{4}}}{(1-\alpha^2\tau)^{\frac{1}{4}}}, \quad T(\tau) = \frac{2}{(1+\alpha)^{\alpha}} \frac{\left[\alpha(1-\tau)^{\frac{1}{2}} + (1-\alpha^2\tau)^{\frac{1}{2}}\right]^{\alpha}}{(1-\tau)^{\frac{1}{2}} + (1-\alpha^2\tau)^{\frac{1}{2}}} \tag{175}$$

For the interval $\frac{1}{2\beta+1} < \tau < 1$, they are

$$\underline{F}_\nu(\tau) \sim f(\tau)\, T^{\nu}(\tau) \cos\left(\nu\omega - \frac{\pi}{4}\right) \tag{176}$$

$$\underline{G}_\nu(\tau) \sim \frac{1}{2} f(\tau)\, T^{-\nu}(\tau) \cos\left(\nu\omega + \frac{\pi}{4}\right) \tag{177}$$

where

$$f(\tau) = 2\,\frac{(1-\tau)^{\frac{\beta}{2}+\frac{1}{4}}}{(\alpha^2\tau-1)^{\frac{1}{4}}}, \quad T(\tau) = 2\,\frac{(2\beta)^{\frac{\alpha}{2}}}{(1+\alpha)^{\alpha}}\,\frac{1}{\sqrt{2\beta\tau}} \tag{178}$$

The values of $T(\tau)$ are given (fig. 4) as a function of τ together with the local Mach number M.

Similarly, as from (153) $U_\nu(\tau) \sim V_\nu(\tau)$, corresponding expressions for $F(a_\nu + \beta, b_\nu + \beta; c_\nu; \tau)$ are:

$$\tilde{\underline{F}}_\nu(\tau) \sim g(\tau)\, T^{\nu}(\tau) \tag{179}$$

$$0 \leq \tau < \frac{1}{2\beta+1}$$

$$\tilde{\underline{G}}_\nu(\tau) \sim g(\tau)\, T^{-\nu}(\tau) \tag{180}$$

where

$$g(\tau) = \frac{(1-\tau)^{-\frac{\beta}{2}+\frac{1}{4}}}{(1-\alpha^2\tau)^{\frac{1}{4}}} \tag{181}$$

and

$$\tilde{\underline{F}}_{\nu}(\tau) \sim g(\tau)\ T^{\nu}(\tau)\ \cos\left(\nu\omega - \frac{\pi}{4}\right) \tag{182}$$

$$\frac{1}{2\beta + 1} < \tau < 1$$

$$\tilde{\underline{G}}_{\nu}(\tau) \sim \frac{1}{2}\ g(\tau)\ T^{-\nu}(\tau)\ \cos\left(\nu\omega + \frac{\pi}{4}\right) \tag{183}$$

where

$$g(\tau) = 2\ \frac{(1 - \tau)^{-\frac{\beta}{2} + \frac{1}{4}}}{(\alpha^2\tau - 1)^{\frac{1}{4}}} \tag{184}$$

Here $\underline{F}_{\nu}(\tau)$ denotes invariably the first integral $F(a_{\nu}, b_{\nu}; c_{\nu}; \tau)$ while $\underline{G}_{\nu}(\tau)$, when multiplied by $q^{-2\nu}$, denotes the second integral $F_{\nu}(\tau)$, defined by equation (37) when ν is not an integer or by equation (39) when ν is an integer, since the asymptotic expansions are valid for both integral and nonintegral values of ν, provided $\nu > N$.

In the domains of validity, the asymptotic expansions may be differentiated with respect to τ with the same order of approximation. Hence, for $\nu > N$, it can be shown that for $0 \leq \tau < \frac{1}{2\beta + 1}$

$$\tilde{\underline{F}}_{\nu,1}(\tau) \sim h(\tau)\ T^{\nu}(\tau) \left\{1 + 0\left(\frac{1}{\nu}\right)\right\} \tag{185}$$

$$\tilde{\underline{G}}_{\nu,1}(\tau) \sim h(\tau)\ T^{-\nu}(\tau) \left\{1 + 0\left(\frac{1}{\nu}\right)\right\} \tag{186}$$

where

$$h(\tau) = 2\ (1 - \tau)^{-\frac{\alpha^2}{4}}\ (1 - \alpha^2\tau)^{-\frac{1}{4}} \left[(1 - \tau)^{\frac{1}{2}} + (1 - \alpha^2\tau)^{\frac{1}{2}}\right]^{-1} \tag{187}$$

and for $\frac{1}{2\beta + 1} < \tau < 1$

$$\tilde{\underline{F}}_{\nu,1}(\tau) \sim h(\tau)\ T^{2}(\tau)\ \cos\left(\nu\omega - \mu - \frac{\pi}{4}\right) \left\{1 + 0\left(\frac{1}{\nu}\right)\right\} \tag{188}$$

$$\tilde{\underline{G}}_{\nu,1}(\tau) \sim \frac{1}{2} h(\tau)\ T^{-\nu}(\tau) \cos\left(\nu\omega + \mu + \frac{\pi}{4}\right)\left\{1 + O\left(\frac{1}{\nu}\right)\right\} \tag{189}$$

where

$$h(\tau) = 4(1-\tau)^{-\frac{\alpha^2}{4}}\ (\alpha^2\tau - 1)^{-\frac{1}{4}}\ (2\beta\tau)^{-\frac{1}{2}},\quad \mu(\tau) = \cos^{-1}\sqrt{\frac{1-\tau}{2\beta\tau}} \tag{190}$$

The values of the functions $g(\tau)$ and $h(\tau)$ are given in figure 5.

It is interesting to note that when $\gamma = -1$ the constant α vanishes and only the exponential type of solutions exist. In the case of $\psi_\nu(\tau)$ the solution is exact, namely, for $\beta = -\frac{1}{2}$

$$\underline{F}_\nu(\tau) = \left[\frac{2}{1 + \sqrt{1 + \frac{q^2}{c_o^2}}}\right]^\nu \tag{191}$$

$$\underline{F}_{-\nu}(\tau) = \left[\frac{2}{1 + \sqrt{1 + \frac{q^2}{c_o^2}}}\right]^{-\nu} \tag{192}$$

of which the first is in agreement with the result obtained by Tsien (reference 9), while for $\chi_\nu(\tau)$ the solutions which are not exact reduce to

$$\tilde{\underline{F}}_\nu(\tau) \sim \left[1 + \frac{q^2}{c_o^2}\right]^{\frac{1}{2}} \left[\frac{2}{1 + \left(1 + \frac{q^2}{c_o^2}\right)^{\frac{1}{2}}}\right]^\nu \left\{1 + O\left(\frac{1}{\nu}\right)\right\} \tag{193}$$

$$\nu \geq N$$

$$\tilde{\underline{G}}_\nu(\tau) \sim \left[1 + \frac{q^2}{c_o^2}\right]^{\frac{1}{2}} \left[\frac{2}{1 + \left(1 + \frac{q^2}{c_o^2}\right)^{\frac{1}{2}}}\right]^{-\nu} \left\{1 + O\left(\frac{1}{\nu}\right)\right\} \tag{194}$$

This may be the cause that destroys the analogy between the coordinates of the corresponding compressible flows and the incompressible flows.

For $\gamma = 1.405$ and $\nu = n + \frac{1}{2}$, n being a positive integer, the three groups of functions $\underline{F}_{\nu}(\tau)$, $\underline{F}_{-\nu}(\tau)$; $\tilde{\underline{F}}_{\nu}(\tau)$, $\tilde{\underline{F}}_{-\nu}(\tau)$; and $\tilde{\underline{F}}_{\nu,1}(\tau)$, $\tilde{\underline{F}}_{-\nu,1}(\tau)$, together with their asymptotic expressions were calculated for τ varying from 0 to 0.5 and n from 0 to 10. The results are presented in tables 2 to 13. The behavior of the approximation is illustrated in figures 6 to 11. It can be observed that the degree of approximation of the functions increases, on the one hand, with ν for any fixed τ. On the other hand, for any fixed n, the approximation becomes worse as τ approaches the critical point $\tau = \frac{1}{2\beta + 1}$, corresponding to the local sonic speed. On the whole, if the critical point $\tau = \frac{1}{2\beta + 1}$ is not reached, the agreement can generally be regarded as excellent, especially for larger values of n.

15. Transformation of the Function $W(w;\tau)$

Branch Point of Order 1

The function $W(w;\tau)$ for a flow that possesses a branch point of order 1 was given in sections 8 and 9. The forms of representation, as can be seen, are not, in general, suitable for practical calculation. The difficulty is twofold: First, the series involves infinitely many hypergeometric functions which themselves are, in turn, defined as infinite series. The convergence of hypergeometric series generally decreases with an increase of the parameter ν. This means that it is very difficult to compute the value of the later terms of the series for $W(w;\tau)$. Secondly, the convergence of the power series defining the function $W(w;\tau)$ itself is, as expected, very slow in the neighborhood of the singularity. To increase the convergence, the following method is used:

Observe that the corresponding function for the incompressible flow that has the same character of singularity is

$$W_o^{(i)}(w) = \sum_{n=o}^{\infty} A_n w^n, \qquad |w| < U$$

which is absolutely and uniformly convergent in any closed domain in $|w| < U$. Now, if in (92) $\underset{\sim}{F}_n^{(r)}(\tau)$ is replaced by

$$\underline{F}_n^{(r)}(\tau) \cong \frac{f(\tau)}{f(\tau_1)} t^n(\tau), \qquad 0 \leq \tau < \frac{1}{2\beta + 1} \tag{195}$$

where $t(\tau) = \frac{T(\tau)}{T(\tau_1)}$, as by hypothesis, $0 < \tau_1 << \frac{1}{2\beta + 1}$; then it is clear that

$$W^{(i)}(w;\tau) \cong \frac{f(\tau)}{f(\tau_1)} \sum_{n=o}^{\infty} A_n (tw)^n, \qquad |tw| < U \tag{196}$$

which is also absolutely and uniformly convergent in the same domain as $W_o(w)$ and, consequently, (196) will be denoted by $\frac{f(\tau)}{f(\tau_1)} W_o(tw)$. In doing this, however, the restriction that (195) holds only when n is greater than a large number N is violated. The error can be removed by adding to and subtracting from (91) the quantity given in (196); then it follows immediately that

$$W^{(i)}(w;\tau) = W_1(w;\tau) + W_2^{(i)}(w;\tau) \tag{197}$$

where

$$W_1(w;\tau) = \frac{f(\tau)}{f(\tau_1)} W_o(tw) \tag{198}$$

$$W_2^{(i)}(w;\tau) = \sum_{n=o}^{\infty} A_n G_n(\tau) w^n, \qquad |w| < U \tag{199}$$

with

$$G_n(\tau) = \underline{F}_n^{(r)}(\tau) - \frac{f(\tau)}{f(\tau_1)}\, t^n(\tau)$$

Here n is a positive integer. The function $W(w;\tau)$ then is represented by the sum of two functions $W_1(w;\tau)$, which is of closed form, and $W_2^{(i)}(w;\tau)$, which is the difference of two convergent power series and hence is also convergent. But, according to the theory of asymptotic expansion, $G_n(\tau)$ tends to zero as n approaches infinity. In fact, $G_n(\tau)$ is of order n^{-1}; therefore, the convergence of $W(w;\tau)$ is increased by the order of n^{-1}. This actually is the gist of the whole problem.

As the form of the representation of the hypergeometric function given in equation (195) is valid for all τ in $0 \leq \tau < \frac{1}{2\beta + 1}$, $W_1(w;\tau)$ given by equation (198) holds automatically even outside the circle $|w| = U$. For this reason, $W_1(w;\tau)$ should be identical in form with that derived from equation (101). That this is the case can be seen from the following consideration. For, in addition to equation (195), if it is assumed that

$$\underline{G}_\nu(\tau) \cong f(\tau)\, T^{-\nu}(\tau) \tag{200}$$

it follows that

$$\xi_\nu(\tau_1) = -\xi_{-\nu}(\tau_1) = \sqrt{\frac{1 - \alpha^2\tau_1}{1 - \tau_1}} \tag{201}$$

then equations (106) and (107) yield, by equations (108) and (109),

$$B_n^* \cong \frac{B_n}{f(\tau_1)}\, T^{-\nu}(\tau_1), \qquad C_n^* \cong \frac{C_n}{f(\tau_1)}\, T^{\nu}(\tau_1) \tag{202}$$

By using these sets of approximate coefficients and replacing $\underline{F}_\nu(\tau)$ and $\underline{G}_\nu(\tau)$ by their respective asymptotic expression, the following relation is obtained with the aid of equation (100)

$$W^{(o)}(w;\tau) = W_1(w;\tau) + W_2^{(o)}(w;\tau) \tag{203}$$

where

$$W_2^{(o)}(w;\tau) = i \sum_{n=0}^{\infty} \left\{ G_\nu^{(1)}(\tau)\ w^\nu + G_\nu^{(2)}(\tau)\ w^{-\nu} \right\} \tag{204}$$

In this case the coefficients B_n^* and C_n^*, as well as the functions $\underline{F}_\nu(\tau)$ and $\underline{G}_\nu(\tau)$ used in deriving $W_1(w;\tau)$, are approximate. Hence, if both are corrected, $G_\nu^{(1)}(\tau)$ and $G_\nu^{(2)}(\tau)$ should be of the forms

$$\left.\begin{aligned} G_\nu^{(1)}(\tau) &= \Delta B_n^*\ \underline{F}_\nu(\tau) + \frac{B_n}{f(\tau_1)}\ T^{-\nu}(\tau_1)\ \Delta\underline{F}_\nu(\tau) \\ G_\nu^{(2)}(\tau) &= \Delta C_n^*\ \underline{G}_\nu(\tau) + \frac{C_n}{f(\tau_1)}\ T^{\nu}(\tau_1)\ \Delta\underline{G}_\nu(\tau) \end{aligned}\right\} \tag{205}$$

where

$$\left.\begin{aligned} \Delta B_n^* &= B_n^* - \frac{B_n}{f(\tau_1)}\ T^{-\nu}(\tau_1), \quad & \Delta\underline{F}_\nu(\tau) &= \underline{F}_\nu(\tau) - f(\tau)\ T^{\nu}(\tau) \\ \Delta C_n^* &= C_n^* - \frac{C_n}{f(\tau_1)}\ T^{\nu}(\tau_1)\ , \quad & \Delta\underline{G}_\nu(\tau) &= \underline{G}_\nu(\tau) - f(\tau)\ T^{-\nu}(\tau) \end{aligned}\right\} \tag{206}$$

Here the differences ΔB_n^* and ΔC_n^* depend upon the condition at infinity for any sets of constants B_n and C_n, while those of $\Delta\underline{F}_\nu(\tau)$ and $\Delta\underline{G}_\nu(\tau)$ are functions of τ only and, for this reason, can be tabulated once for all. It also can be shown that the order of ΔB_n^* is at least of n^{-1} and therefore the convergence of (204) is again increased by n^{-1}.

Consequently, if $\psi(q,\theta) = \psi_1(q,\theta) + \psi_2^{(l)}(q,\theta)$ where the superscript (l) denotes either (i) or (o), and if the coefficients are real, the stream function for the subsonic flow is according to (93) given by

$$\psi_1(q,\theta) = \frac{f(\tau)}{f(\tau_1)}\,\psi_o(tq,\theta), \qquad 0 \leq \tau \leq \frac{1}{2\beta+1} \tag{207}$$

$$\psi_2^{(i)}(q,\theta) = -\sum_{n=o}^{\infty} A_n G_n(\tau)\, q^n \sin n\theta, \qquad q < U \tag{208}$$

and in $U < q < V$

$$\psi_2^{(o)}(q,\theta) = \sum_{n=o}^{\infty}\left[G_\nu^{(1)}(\tau)\quad q^{\nu} + G_\nu^{(2)}(\tau)\quad q^{-\nu}\right]\cos \nu\theta \tag{209}$$

with θ restricted by $0 \leq \theta < 2\pi$. This result is striking in that for $\tau = \tau_1$, $\psi(U,\theta) \equiv \psi_1(U,\theta)$ as $G_\nu(\tau_1) = 0$; that is, the function $\psi_1(q,\theta)$ represents the correct singularity of the exact function. Far away from the singularity the term $\psi_2^{(l)}(q,\theta)$ ($l = i$ or o) gradually comes into prominence, especially near $\tau = \frac{1}{2\beta+1}$; but the convergence there is already so rapid that a small number of terms is enough to secure a high accuracy in $\psi(q,\theta)$.

It is interesting to estimate the magnitude of the second part of the stream function. By noting the fact that $G_n(\tau_1) = 0$, $G_\nu(\tau_1) = 0$, the expansions of the $G_n(\tau)$ and $G_\nu(\tau)$ are

$$G_n(\tau) = G_n'(\tau_1)\,(\tau - \tau_1) + \ldots, \qquad 0 < \tau < \tau_1$$

$$G_\nu(\tau) = G_\nu'(\tau_1)\,(\tau - \tau_1) + \ldots, \qquad \tau_1 < \tau < \frac{1}{2\beta+1}$$

Then from corollary (52), it is shown that for $\vartheta \neq 0$

$$\psi_2^{(l)}(q,\theta) \sim \left(\frac{\partial\varphi_o}{\partial\theta}\right)_{q=U}(\tau - \tau_1) + \ldots$$

In other words, the second part of the solution is of the order of magnitude of $(\tau - \tau_1)$. However, the magnitude of

$(\tau - \tau_1)$ depends essentially upon τ_1 for a given incompressible flow. If τ_1 is not small, then when $\tau = 0$, $|\tau - \tau_1|$ will be large. Thus for large free-stream Mach numbers, the second part of the solution ψ_2 cannot be neglected. This means that for high free-stream Mach numbers the correct solution for compressible flow is considerably more complicated than the usually assumed simple speed distortion rule would lead one to believe. Thus, any theory based upon such a simple rule cannot be accurate enough for transonic flows.

On the other hand, if τ_1 is small, or $\tau_1 << \frac{1}{2p + 1}$, then the value of $|\tau - \tau_1|$ for $\tau = 0$ is small. Furthermore, if the maximum velocity of the flow is well below the sonic velocity, then the maximum value of τ also is small, thus $|\tau - \tau_1|$ for the whole flow is small. Then the second part of the solution ψ_2 is negligible. However, even then the solution for the compressible flow cannot be expressed as the solution of the incompressible solution by a simple distortion of the velocity scale, as is generally assumed in the so-called hodograph method. For this would be the case only if the multiplying factor $f(\tau)/f(\tau_1)$ is identically equal to 1. Since the multiplying factor is a function of the magnitude of velocity, the streamlines of the compressible flow and the streamlines of the incompressible flow cannot be made to correspond to each other. On the other hand, equation (207) shows that if ψ_0 is zero, then ψ_1 is also zero. Thus there is this <u>one</u> streamline, the zero streamline, in both flows satisfying the requirement of direct mapping. Since the zero streamline generally is chosen to represent the contour of the body, on the surface of the body in purely subsonic flows, the velocity of the compressible flow can be calculated from the incompressible flow by a simple "correction formula." This formula is given by equating the velocity q of the incompressible fluid to the velocity function tq of the compressible flow. Thus

$$\left(\frac{q}{U}\right)_0 = \sqrt{\frac{\tau}{\tau_1}}\, t = \sqrt{\frac{\tau}{\tau_1}}\, \frac{T(\tau)}{T(\tau_1)}$$

where the subscript o denotes the quantity for incompressible flow and $T(\tau)$ is given by equation (175). This formula is the same as that suggested by G. Temple and J. Yarwood (reference 11). This coincidence of Temple's theory with the

present investigation can be considered as a further substantiation of the method.

For the supersonic regions, $\underline{F}_\nu(\tau)$ and $\underline{G}_\nu(\tau)$ in (101) should be replaced by

$$\left.\begin{aligned} \underline{F}_\nu(\tau) &\cong f(\tau)\, T^{\nu}(\tau) \cos\left(\nu\omega - \frac{\pi}{4}\right) && (210) \\ \underline{G}_\nu(\tau) &\cong \frac{1}{2} f(\tau)\, T^{-\nu}(\tau) \cos\left(\nu\omega + \frac{\pi}{4}\right) && (211) \end{aligned}\right\} \quad \frac{1}{2\beta+1} < \tau < 1$$

where $f(\tau)$, $T(\tau)$ and $\omega(\tau)$ are given in (178) and (172); then by writing

$$\underline{F}_\nu(\tau) \cong \frac{1}{2} f(\tau) \left\{ e^{i\left(\nu\omega - \frac{\pi}{4}\right)} + e^{-i\left(\nu\omega - \frac{\pi}{4}\right)} \right\}$$

and substituting as before in equation (101), it leads again to the sum of $W_1(w;\tau)$ and $W_2^{(o)}(w;\tau)$, where

$$W_1(w;\tau) = \frac{f(\tau)}{4f(\tau_1)} \left[e^{-\frac{\pi i}{4}}\, i \sum_{n=o}^{\infty} \left\{ B_n\, (twe^{i\omega})^{\nu} + C_n\, (twe^{i\omega})^{-\nu} \right\} \right.$$

$$\left. + e^{\frac{\pi i}{4}}\, i \sum_{n=o}^{\infty} \left\{ B_n\, (twe^{-i\omega})^{\nu} + C_n\, (twe^{-i\omega})^{-\nu} \right\} \right]$$

and

$$W_2(w;\tau) = i \sum_{n=o}^{\infty} \left\{ G_\nu^{(1)}(\tau)\, w^{\nu} + G_\nu^{(2)}(\tau)\, w^{-\nu} \right\}, \qquad \frac{1}{2\beta+1} < \tau < 1$$

According to equation (100), $W_1(w;\tau)$ also can be summed:

$$W_1(w;\tau) = \frac{1}{4} \frac{f(\tau)}{f(\tau_1)} \left[e^{-\frac{\pi i}{4}}\, W_o(twe^{\omega i}) + e^{\frac{\pi i}{4}}\, W_o(twe^{-\omega i}) \right] \quad (212)$$

Furthermore, from (178) it can be seen that $|tw| = \lambda U$, λ being a constant given by

$$\lambda = \frac{2(2\beta)^{\frac{\alpha}{2}}}{(1+\alpha)^{\alpha}\,(2\beta\tau_1)^{\frac{1}{2}}}\,\frac{1}{T(\tau_1)} > 1 \tag{213}$$

which is a function of the Mach number and the characteristic constant β of the gas but independent of the shape of the boundary. The value of this function λ is given in table 14 and figure 12 for $\gamma = 1.405$. As a consequence, the functions constituting the stream function for the supersonic flow are

$\psi_1(q,\vartheta)$

$$= 2^{-\frac{5}{2}}\,\frac{f(\tau)}{f(\tau_1)}\left[\Psi_0(\vartheta+\omega) + \Psi_0(\vartheta-\omega) + \Phi_0(\vartheta+\omega) - \Phi_0(\vartheta-\omega)\right] \tag{214}$$

$$\vartheta - \omega \geq 0$$

$\psi_2(q,\vartheta)$

$$= \sum_{n=0}^{\infty}\left\{G_\nu^{(1)}(\tau)\,q^{\nu} + G_\nu^{(2)}(\tau)\,q^{-\nu}\right\}\cos\nu\vartheta,\quad U < q < V \tag{215}$$

Here the functions Ψ_0 and Φ_0 are defined, on account of (213), by

$$\Psi_0(\vartheta\pm\omega) \equiv \psi_0(\lambda U, \vartheta\pm\omega), \qquad \Phi_0(\vartheta\pm\omega) \equiv \varphi_0(\lambda U, \vartheta\pm\omega) \tag{216}$$

where φ_0 and ψ_0 are the velocity potential and the stream function, respectively, of the corresponding incompressible flow. The functions $G_\nu^{(1)}(\tau)$ and $G_\nu^{(2)}(\tau)$ are the same as defined in (205) except that $\Delta\underline{F}_\nu(\tau)$ and $\Delta\underline{G}_\nu(\tau)$ now are given by

$$\left.\begin{aligned} \Delta \underline{F}_\nu(\tau) &= \underline{F}_\nu(\tau) - \frac{f(\tau)}{2} T^\nu \cos\left(\nu\omega - \frac{\pi}{4}\right) \\ \Delta \underline{G}_\nu(\tau) &= \underline{G}_\nu(\tau) - \frac{f(\tau)}{2} T^{-\nu} \cos\left(\nu\omega + \frac{\pi}{4}\right) \end{aligned}\right\} \quad (217)$$

Unlike the previous calculations, $G_\nu^{(1)}(\tau)$ in (211) is not of the order of ν^{-1} due to the presence of 1/2 in front of $f(\tau)\, T^\nu \cos\left(\nu\omega - \frac{\pi}{4}\right)$. This, however, does not offer a serious objection, as the series in which it appears already converges with $(tq)^\nu$, t being less than unity.

It is worth noting, moreover, that in the hyperbolic domain the function $\psi_1(q,\vartheta)$ depends, aside from a factor $f(\tau)$, only on the two independent families of characteristics defined by

$$\xi = \vartheta + \omega, \quad \eta = \vartheta - \omega \qquad (218)$$

This result is most striking, as it shows that the main part of the solution satisfies the simple wave equation and thus clearly demonstrates its hyperbolic character. With both the incompressible stream function ψ_0 and the incompressible potential function φ_0 appearing in the solution, it is impossible to establish a simple relation between the incompressible streamlines and the compressible streamlines. Since such a simple relation is the foundation of the so-called speed correction formula for a quick estimation of velocity distribution in compressible flow from that of incompressible flow over the same body, this idea cannot be extended to supersonic regions. On the other hand, this also indicates that although the differential equation for $\psi(q,\vartheta)$ is hyperbolic in the supersonic range, it cannot be reduced to the simple wave equation by a mere distortion of the speed scale as given by the function $\omega(\tau)$. For if this were the case, then $\psi_1(q,\vartheta)$ would constitute an exact solution without the additional $\psi_2^{(0)}(q,\vartheta)$. This fact is all the more important as the additional $\psi_2^{(0)}(q,\vartheta)$ is not small in comparison with $\psi_1(q,\vartheta)$ for the mixed subsonic and supersonic flows, especially for the transitional region near sonic velocity. However, in the case of pure supersonic flow, $\psi_2^{(0)}(q,\vartheta)$ might be small; then $\psi_1(q,\vartheta)$ alone may

be used as a satisfactory approximation. Of course, when $\gamma = -1$, then, as in the corresponding case in subsonic flow, the exact differential equation for the stream function can be reduced to the simple wave equation. In this case, the appropriate form for the speed function ω is

$$\omega(q) = -\tan^{-1}\sqrt{\frac{1}{\frac{q^2}{q_1^2 - c_1^2} - 1}} \tag{219}$$

where the subscript 1 denotes the conditions at the point of tangency of the true isentropic curve and the approximating tangent. This agrees with the result obtained by N. Coburn. (See reference 21.)

16. Continuation: Logarithmic Singularity

In the case of the logarithmic singularity the function $W(w;\tau)$ was broken up into two parts of which only the one that characterizes the singularity was given in equations (115) and (116). As an example, it is proposed to show that this problem can be treated by the same method. If the same approximation is introduced as in equations (195) and (201), then the coefficients defined in equations (121) and (122) become approximately:

$$A_n \cong \frac{1}{n}\frac{T^{-n}(\tau_1)}{f(\tau_1)}, \qquad C_n \cong \frac{1}{n}\frac{T^{n}(\tau_1)}{f(\tau_1)} \tag{220}$$

with $B\,f^2(\tau_1) = \frac{1}{2}$, so chosen that the form of equation (207) is again preserved. With these coefficients and if there is written for the function inside the circle $q = U$:

$$\tilde{\psi}^{(i)}(q,\vartheta) = \tilde{\psi}_1(q,\vartheta) + \tilde{\psi}_2^{(i)}(q,\vartheta)$$

Equation (115) reduces to the sum of

$$\tilde{\psi}_1(q,\vartheta) = \frac{f(\tau)}{f(\tau_1)}\,\tilde{\psi}_0(tq,\vartheta), \qquad 0 \le \tau < \frac{1}{2\beta+1} \tag{221}$$

$$\tilde{\psi}_2^{(i)}(q,\vartheta) = \sum_{n=1}^{\infty} \frac{1}{n} \tilde{G}_n(\tau) \left(\frac{q}{U}\right)^n \cos n\vartheta, \qquad q < U \tag{222}$$

where

$$\tilde{G}_n(\tau) = \underline{F}_n(\tau)\, \Delta\underline{G}_n(\tau_1) + \frac{\Delta\underline{F}_n(\tau)}{f(\tau_1)\, T^n(\tau_1)} \tag{223}$$

with

$$\left.\begin{aligned} \Delta\underline{F}_n(\tau) &= \underline{F}_n(\tau) - f(\tau)\, T^n(\tau) \\ \Delta\underline{G}_n(\tau_1) &= \frac{\underline{G}_n(\tau_1)}{f^2(\tau_1)} - f^{-1}(\tau_1)\, T^{-n}(\tau_1) \end{aligned}\right\} \tag{224}$$

Similarly, in the case of equation (116) it reduces to

$$\tilde{\psi}^{(o)}(q,\vartheta) = \tilde{\psi}_1(q,\vartheta) + \tilde{\psi}_2^{(o)}(q,\vartheta)$$

Here $\tilde{\psi}_1(q,\vartheta)$ is again the same as (221); while $\tilde{\psi}_2^{(o)}(q,\vartheta)$ is

$$\tilde{\psi}_2^{(o)}(q,\vartheta) = -\frac{1}{2f^2(\tau_1)} \int_{\tau_1}^{\tau} (1-\tau)^{\beta} \frac{d\tau}{\tau} + \frac{f(\tau)}{f(\tau_1)} \log \frac{tq}{U}$$

$$+ \sum_{n=1}^{\infty} \frac{1}{n} \tilde{G}_n^{(o)} \left(\frac{q}{U}\right)^{-n} \cos n\vartheta \tag{225}$$

where

$$\tilde{G}_n^{(o)}(\tau) = \underline{G}_n(\tau)\, \Delta\underline{F}_n(\tau_1) + f^{-1}(\tau_1)\, T^n(\tau_1)\, \Delta\underline{G}_n(\tau) \tag{226}$$

with

$$\Delta\underline{F}_n(\tau_1) = \frac{\underline{F}_n(\tau_1)}{f^2(\tau_1)} - \frac{T^n(\tau_1)}{f(\tau_1)}, \quad \Delta\underline{G}_n(\tau) = \underline{G}_n(\tau) - f(\tau)\, T^{-n}(\tau) \tag{227}$$

Unlike the previous case, $\tilde{\Psi}(q,\vartheta) = \tilde{\Psi}_0(q,\vartheta)$ when, and only when, c_0 tends to infinity. Because of (221), however, the singularity of $\tilde{\Psi}(q,\vartheta)$ remains unchanged.

Again, if in (116)

$$\underline{G}_n(\tau) \cong \frac{1}{2} f(\tau)\, T^{-n}(\tau) \cos\left(n\omega + \frac{\pi}{4}\right)$$

is substituted for $\underline{G}_n(\tau)$, it can similarly be shown that

$$\tilde{\Psi}_1(q,\vartheta) = 2^{-\frac{5}{2}} \frac{f(\tau)}{f(\tau_1)} \left[\tilde{\Psi}_0(\vartheta+\omega) + \tilde{\Psi}_0(\vartheta-\omega) - \tilde{\Phi}_0(\vartheta+\omega) + \tilde{\Phi}_0(\vartheta-\omega)\right] \quad (228)$$

$$\vartheta - \omega \geq 0$$

$$\tilde{\Psi}_2^{(o)}(q,\vartheta) = -\frac{1}{2f^2(\tau_1)} \int_{\tau_1}^{\tau} (1-\tau)^{\beta} \frac{d\tau}{\tau} + 2^{-\frac{3}{2}} \frac{f(\tau)}{f(\tau_1)} (\log \lambda - \omega)$$

$$+ \sum_{n\pm 1}^{\infty} \frac{\tilde{G}_n^{(o)}}{n} \left(\frac{q}{U}\right)^{-n} \cos n\vartheta \quad (229)$$

where $\tilde{\Psi}_0(\vartheta \pm \omega)$ and $\tilde{\Phi}_0(\vartheta \pm \omega)$ are defined analogously to (216), and $\Delta\underline{G}_n(\tau)$ in $\tilde{G}_n^{(o)}(\tau)$ is now given by

$$\Delta\underline{G}_n(\tau) = \underline{G}_n(\tau) - \frac{1}{2} f(\tau)\, T^{-n}(\tau) \cos\left(n\omega + \frac{\pi}{4}\right) \quad (230)$$

This seems to indicate that the results obtained so far for $\psi_1(q,\vartheta)$ are quite general: It may differ for different cases, at most, by a constant factor. The general property, however, is not shared by $\psi_2(q,\vartheta)$, the character of which changes radically with the nature of the singularity and the shape of the boundary. Hence, its importance in the present problem is evident.

17. The Coordinate Functions $x(q,\vartheta)$ and $y(q,\vartheta)$

Whenever the function $\chi(q,\vartheta)$ for a boundary problem is determined, the coordinate functions $x(q,\vartheta)$ and $y(q,\vartheta)$ can be calculated according to equations (141) and (142). Suppose, for instance, a boundary is assigned with the property that the function $\Lambda(w;\tau)$ is truly described by (94) and (110), of which the real part $\chi(q,\vartheta)$, defined within the circle $|w| = U$, is

$$\chi(q,\vartheta) = \sum_{n=0}^{\infty} \tilde{A}_n \, \tilde{\underline{F}}_n^{(r)}(\tau) \, q^n \cos n\vartheta, \qquad q < U \tag{231}$$

where the constants $\tilde{A}_n$ in (94) are again real and are regarded as known, and $\tilde{\underline{F}}_n^{(r)}(\tau) = \tilde{\underline{F}}_n(\tau)/\underline{F}_n(\tau_1)$.

As the series is absolutely and uniformly convergent in $q < U$, it can be differentiated partially term by term with respect to q and ϑ. When the differential coefficients $\chi_q(q,\vartheta)$ and $\chi_\vartheta(q,\vartheta)$ are calculated and are substituted in equations (141) and (142), there results:

$$\begin{aligned} x(q,\vartheta) = &\sum_{n=1}^{\infty} n \, \tilde{A}_n \, \tilde{\underline{F}}_n^{(r)} \, q^{n-1} \cos (n-1)\vartheta \\ &-\beta\tau \sum_{n=1}^{\infty} n \, \tilde{A}_n \, \frac{n-1}{n+1} \, \tilde{\underline{F}}_{n,1}^{(r)}(\tau) \, q^{n-1} \cos n\vartheta \cos \vartheta \end{aligned} \tag{232}$$

$$q < U$$

$$\begin{aligned} y(q,\vartheta) = &-\sum_{n=1}^{\infty} n \, \tilde{A}_n \, \tilde{\underline{F}}_n^{(r)} \, q^{n-1} \sin (n-1)\vartheta \\ &-\beta\tau \sum_{n=1}^{\infty} n \, \tilde{A}_n \, \frac{n-1}{n+1} \, \tilde{\underline{F}}_{n,1}^{(r)}(\tau) \, q^{n-1} \cos n\vartheta \sin \vartheta \end{aligned} \tag{233}$$

where

$$\underline{F}_{n,1}^{(r)}(\tau) = \frac{F(a_n + \beta + 1,\ b_n + \beta + 1;\ c_n + 1;\ \tau)}{F(a_n,\ b_n;\ c_n;\ \tau_1)} \tag{234}$$

Now, since

$$x_o(q,\vartheta) = \sum_{n=1}^{\infty} n\,\tilde{A}_n\,q^{n-1}\cos(n-1)\vartheta$$

and

$$y_o(q,\vartheta) = -\sum_{n=1}^{\infty} n\,\tilde{A}_n\,q^{n-1}\sin(n-1)\vartheta$$

$$\sigma_o(q,\vartheta) = \sum_{n=1}^{\infty} \tilde{A}_n\,q^{n}\sin n\vartheta$$

by introducing the approximation given by equations (179) and (185), that is

$$\tilde{F}_n^{(r)}(\tau) \cong \frac{g(\tau)}{f(\tau_1)}\,t^n(\tau)$$

$$0 \leq \tau < \frac{1}{2\beta+1}$$

$$\tilde{F}_{n,1}^{(r)}(\tau) \cong \frac{h(\tau)}{f(\tau_1)}\,t^n(\tau)$$

by defining

$$x(q,\vartheta) = x_1(q,\vartheta) + x_2^{(l)}(q,\vartheta) \tag{235}$$

$$y(q,\vartheta) = y_1(q,\vartheta) + y_2^{(l)}(q,\vartheta) \tag{236}$$

it can be shown by the same manner that

$$x_1(q,\vartheta) = \frac{g(\tau)}{f(\tau_1)}\,t(\tau)\,x_o(tq,\vartheta) - \frac{\beta\tau}{q}\,\frac{h(\tau)}{f(\tau_1)}\,\Omega_o(tq,\vartheta)\cos\vartheta \tag{237}$$

$$0 \leq \tau < \frac{1}{2\beta+1}$$

$$y_1(q,\vartheta) = \frac{g(\tau)}{f(\tau_1)}\,t(\tau)\,y_o(tq,\vartheta) - \frac{\beta\tau}{q}\,\frac{h(\tau)}{f(\tau_1)}\,\Omega_o(tq,\vartheta)\sin\vartheta \tag{238}$$

and

$$x_2^{(i)}(q,\vartheta) = \sum_{n=1}^{\infty} n\,\tilde{A}_n\,\tilde{G}_n(\tau)\,q^{n-1}\cos(n-1)\vartheta$$

$$- \beta\tau \sum_{n=1}^{\infty} n\,\tilde{A}_n\,\tilde{G}_{n,1}(\tau)\,q^{n-1}\cos n\vartheta\cos\vartheta \qquad (239)$$

$$q < U$$

$$y_2^{(i)}(q,\vartheta) = -\sum_{n=1}^{\infty} n\,\tilde{A}_n\,\tilde{G}_n(\tau)\,q^{n-1}\sin(n-1)\vartheta$$

$$- \beta\tau \sum_{n=1}^{\infty} n\,\tilde{A}_n\,\tilde{G}_{n,1}(\tau)\,q^{n-1}\cos n\vartheta\sin\vartheta \qquad (240)$$

where

$$\tilde{G}_n(\tau) = \frac{F(a_n+\beta,\ b_n+\beta;\ c_n;\ \tau)}{F(a_n,\ b_n;\ c_n;\ \tau_1)} - \frac{g(\tau)}{f(\tau_1)}\,t^n(\tau) \qquad (241)$$

$$\tilde{G}_{n,1}(\tau) = \frac{n-1}{n+1}\,\frac{F(a_n+\beta+1,\ b_n+\beta+1;\ c_n+1;\ \tau)}{F(a_n,\ b_n;\ c_n;\ \tau_1)} - \frac{h(\tau)}{f(\tau_1)}\,t^n(\tau) \qquad (242)$$

$$\Omega_o(q,\vartheta) = \frac{\partial\sigma_o}{\partial\vartheta} \qquad (243)$$

On the other hand, the expression for $\chi(q,\vartheta)$ valid outside the circle of convergence is

$$\chi(q,\vartheta) = \sum_{n=0}^{\infty}\left[\tilde{B}_n^*\,\underline{\tilde{F}}_\nu(\tau)\,q^\nu - \tilde{C}_n^*\,\underline{\tilde{G}}_\nu(\tau)\,q^{-\nu}\right]\sin\nu\vartheta \qquad (244)$$

provided the coefficients $\tilde{B}_n^*$ and $\tilde{C}_n^*$ in (110) are real. The functions $x(q,\vartheta)$ and $y(q,\vartheta)$ corresponding to (244) can be found similarly. These are:

$$x(q,\vartheta) = \sum_{n=0}^{\infty} \left\{ \nu\tilde{B}_n^*\tilde{\underline{F}}_\nu(\tau)\ q^{\nu-1}\sin(\nu-1)\vartheta + \nu\tilde{C}_n^*\tilde{\underline{G}}_\nu(\tau)\ q^{-\nu-1}\sin(\nu+1)\vartheta \right\}$$

$$- \beta\tau \sum_{n=0}^{\infty} \left\{ \nu\tilde{B}_n^* \frac{\nu-1}{\nu+1} \tilde{\underline{F}}_{\nu,1}(\tau)\ q^{\nu-1} \right.$$

$$\left. + \nu\tilde{C}_n^* \frac{\nu+1}{\nu-1} \tilde{\underline{G}}_{\nu,1}(\tau)\ q^{-\nu-1} \right\} \sin \nu\vartheta \cos \vartheta \qquad (245)$$

$$U < q < V$$

$$y(q,\vartheta) = \sum_{n=0}^{\infty} \left\{ \nu\tilde{B}_n^*\tilde{\underline{F}}_\nu(\tau)\ q^{\nu-1}\cos(\nu-1)\vartheta - \nu\tilde{C}_n^*\tilde{\underline{G}}_\nu(\tau)\ q^{-\nu-1}\cos(\nu+1)\vartheta \right\}$$

$$- \beta\tau \sum_{n=0}^{\infty} \left\{ \nu\tilde{B}_n^* \frac{\nu-1}{\nu+1} \tilde{\underline{F}}_{\nu,1}(\tau)\ q^{\nu-1} \right.$$

$$\left. + \nu\tilde{C}_n^* \frac{\nu+1}{\nu-1} \tilde{\underline{G}}_{\nu,1}(\tau)\ q^{-\nu-1} \right\} \sin \nu\vartheta \sin \vartheta \qquad (246)$$

Here the constants $\tilde{B}_n^*$ and $\tilde{C}_n^*$ satisfy the relations (109) and (110) and can be reduced to

$$\tilde{B}_n^* \cong \frac{\tilde{B}_n}{f(\tau_1)}\ T^{-\nu}(\tau_1), \qquad \tilde{C}_n^* \cong \frac{\tilde{C}_n}{f(\tau_1)}\ T^{\nu}(\tau_1) \qquad (247)$$

provided the same approximation is made as in (202). Furthermore,

$$x_0(q,\vartheta) = \sum_{n=0}^{\infty} \left\{ \nu\tilde{B}_n\ q^{\nu-1} \sin(\nu-1)\vartheta + \nu\tilde{C}_n\ q^{-\nu-1} \sin(\nu+1)\vartheta \right\}$$

$$y_0(q,\vartheta) = \sum_{n=0}^{\infty} \left\{\nu\tilde{B}_n\, q^{\nu-1} \cos\,(\nu-1)\,\vartheta - \nu\tilde{C}_n\, q^{-\nu-1} \cos\,(\nu+1)\vartheta\right\}$$

and if $\tilde{\underline{F}}_\nu(\tau)$ and $\tilde{\underline{F}}_{\nu,1}(\tau)$ for the high-order terms are substituted by the asymptotic forms: namely,

$$\tilde{\underline{F}}_\nu(\tau) \cong g(\tau)\, T^{\nu}(\tau), \quad \tilde{\underline{F}}_{\nu,1}(\tau) \cong h(\tau)\, T^{-\nu}(\tau); \quad 0 \leq \tau < \frac{1}{2\beta+1}$$

then in like manner (245) and (246) can be transformed and can each be represented by the sum of two functions $x_1(q,\vartheta)$, $y_1(q,\vartheta)$, and $x_2(q,\vartheta)$, $y_2(q,\vartheta)$, where x_1 and y_1 are the same as (237) and (238); while x_2 and y_2 are:

$$x_2^{(0)}(q,\vartheta) = \sum_{n=0}^{\infty} \nu \left\{\tilde{G}_\nu^{(1)}(\tau)q^{\nu-1}\sin(\nu-1)\vartheta + \tilde{G}_\nu^{(2)}(\tau)q^{-\nu-1}\sin(\nu+1)\vartheta\right\}$$

$$- \beta\tau \sum_{n=0}^{\infty} \nu \left\{\tilde{G}_{\nu,1}^{(1)}(\tau)q^{\nu-1} + \tilde{G}_{\nu,1}^{(2)}(\tau)q^{-\nu-1}\right\} \sin\,\nu\vartheta\,\cos\vartheta \quad (248)$$

$$\tau_1 \leq \tau < \frac{1}{2\beta+1}$$

$$y_2^{(0)}(q,\vartheta) = \sum_{n=0}^{\infty} \nu\left\{\tilde{G}_\nu^{(1)}(\tau)q^{\nu-1}\cos(\nu-1)\vartheta - \tilde{G}_\nu^{(2)}(\tau)q^{-\nu-1}\cos(\nu+1)\vartheta\right\}$$

$$- \beta\tau \sum_{n=0}^{\infty} \nu \left\{\tilde{G}_{\nu,1}^{(1)}(\tau)q^{\nu-1} + \tilde{G}_{\nu,1}^{(2)}(\tau)q^{-\nu-1}\right\} \sin\,\nu\vartheta\,\sin\vartheta \quad (249)$$

where $\tilde{G}_\nu^{(\alpha)}$ and $\tilde{G}_{\nu,1}^{(\alpha)}$ are defined by:

$$\left.\begin{aligned}
\tilde{G}_\nu^{(1)}(\tau) &= \Delta\tilde{B}_n^*\, \tilde{\underline{F}}_\nu(\tau) + \frac{\tilde{B}_n}{f(\tau_1)}\, T^{-\nu}(\tau_1)\, \Delta\tilde{\underline{F}}_\nu(\tau) \\
\tilde{G}_\nu^{(2)}(\tau) &= \Delta C_n^*\, \tilde{\underline{G}}_\nu(\tau) + \frac{\tilde{C}_n}{f(\tau_1)}\, T^{\nu}(\tau_1)\, \Delta\tilde{\underline{G}}_\nu
\end{aligned}\right\} \quad (250)$$

$$\left.\begin{aligned}
\tilde{G}^{(1)}_{\nu,1}(\tau) &= \Delta\tilde{B}^{*}_{n}\,\frac{\nu-1}{\nu+1}\,\tilde{\underline{F}}_{\nu,1}(\tau) + \frac{\tilde{B}_n}{f(\tau_1)}\,T^{-\nu}(\tau_1)\,\Delta\tilde{\underline{F}}_{\nu,1}(\tau) \\
\tilde{G}^{(2)}_{\nu,1}(\tau) &= \Delta\tilde{C}^{*}_{n}\,\frac{\nu-1}{\nu+1}\,\tilde{\underline{G}}_{\nu,1} + \frac{\tilde{C}_n}{f(\tau_1)}\,T^{\nu}(\tau_1)\,\Delta\tilde{\underline{G}}_{\nu,1}
\end{aligned}\right\} \quad (251)$$

with

$$\left.\begin{aligned}
\Delta\tilde{\underline{F}}_{\nu,1}(\tau) &= \frac{\nu-1}{\nu+1}\,\tilde{\underline{F}}_{\nu,1}(\tau) - h(\tau)\,T^{\nu}(\tau) \\
\Delta\tilde{\underline{G}}_{\nu,1}(\tau) &= \frac{\nu+1}{\nu-1}\,\tilde{\underline{G}}_{\nu,1}(\tau) - h(\tau)\,T^{-\nu}(\tau)
\end{aligned}\right\} \quad (252)$$

while $\Delta\tilde{B}^{*}_{n}$ and $\Delta\tilde{\underline{F}}_{\nu}(\tau)$ are defined just the same as those given in equation (206).

Similarly, if the hypergeometric functions involved in the high-order terms are substituted by

$$\tilde{\underline{F}}_{\nu}(\tau) \cong g(\tau)\,T^{\nu}\cos\left(\nu\omega - \frac{\pi}{4}\right), \quad \tilde{\underline{F}}_{\nu,1}(\tau) \cong h(\tau)\,T^{\nu}\cos\left(\nu\omega - \mu - \frac{\pi}{4}\right)$$

$$\tilde{\underline{G}}_{\nu}(\tau) \cong \frac{1}{2}\,g(\tau)\,T^{-\nu}\cos\left(\nu\omega + \frac{\pi}{4}\right), \quad \tilde{\underline{G}}_{\nu,1}(\tau) \cong \frac{1}{2}\,h(\tau)\,T^{-\nu}\cos\left(\nu\omega + \mu + \frac{\pi}{4}\right)$$

and by resolving the products of the trigonometric functions into sums: for instance,

$$\begin{aligned}
2\sin(\nu-1)\vartheta\cos\left(\nu\omega - \frac{\pi}{4}\right) &= \sin\left[(\nu-1)(\vartheta+\omega) + \left(\omega - \frac{\pi}{4}\right)\right] \\
&\quad + \sin\left[(\nu-1)(\vartheta-\omega) - \left(\omega - \frac{\pi}{4}\right)\right] \\
2\sin(\nu+1)\vartheta\cos\left(\nu\omega + \frac{\pi}{4}\right) &= \sin\left[(\nu+1)(\vartheta+\omega) - \left(\omega - \frac{\pi}{4}\right)\right] \\
&\quad + \sin\left[(\nu+1)(\vartheta-\omega) + \left(\omega - \frac{\pi}{4}\right)\right]
\end{aligned}$$

a brief reduction gives when $\frac{1}{2\beta + 1} < \tau < 1$,

$$x_1(q,\vartheta) = \frac{t(\tau)}{4}\frac{g(\tau)}{f(\tau_1)}\left\{\left[X_0(\vartheta + \omega) + X_0(\vartheta - \omega)\right]\cos\left(\frac{\pi}{4} - \omega\right)\right.$$

$$\left. - \left[Y_0(\vartheta + \omega) - Y_0(\vartheta - \omega)\right]\sin\left(\frac{\pi}{4} - \omega\right)\right\}$$

$$- \frac{\beta\tau}{4q}\frac{h(\tau)}{f(\tau_1)}\left\{\left[\Omega_0(\vartheta + \omega) + \Omega_0(\vartheta - \omega)\right]\cos\left(\mu + \frac{\pi}{4}\right)\right.$$

$$\left. - \left[\Theta_0(\vartheta + \omega) - \Theta_0(\vartheta - \omega)\right]\sin\left(\frac{\pi}{4} + \mu\right)\right\}\cos\vartheta \qquad (253)$$

$$y_1(q,\vartheta) = \frac{t(\tau)}{4}\frac{g(\tau)}{f(\tau_1)}\left\{\left[Y_0(\vartheta + \omega) + Y_0(\vartheta - \omega)\right]\cos\left(\frac{\pi}{4} - \omega\right)\right.$$

$$\left. + \left[X_0(\vartheta + \omega) - X_0(\vartheta - \omega)\right]\sin\left(\frac{\pi}{4} - \omega\right)\right\}$$

$$- \frac{\beta\tau}{4q}\frac{h(\tau)}{f(\tau_1)}\left\{\left[\Omega_0(\vartheta + \omega) + \Omega_0(\vartheta - \omega)\right]\cos\left(\mu + \frac{\pi}{4}\right)\right.$$

$$\left. - \left[\Theta_0(\vartheta + \omega) - \Theta_0(\vartheta - \omega)\right]\sin\left(\frac{\pi}{4} - \omega\right)\right\}\sin\vartheta \qquad (254)$$

by the fact that $qt = \lambda U$ in the interval under consideration. Here

$$X_0(\vartheta \pm \omega) = x_0(\lambda U, \vartheta \pm \omega), \qquad Y_0(\vartheta \pm \omega) = y_0(\lambda U, \vartheta \pm \omega)$$

$$\Theta_0(\vartheta \pm \omega) \equiv \Theta_0(\lambda U, \vartheta \pm \omega), \qquad \Omega_0(\vartheta \pm \omega) \equiv \Omega_0(\lambda U, \vartheta \pm \omega)$$

where

$$\Theta_o(q,\vartheta) = \frac{\partial X_o}{\partial \vartheta}$$

and

$$x_2^{(o)}(q,\vartheta)$$

$$= \sum_{n=o}^{\infty} \nu \left\{ \tilde{G}_\nu^{(1)}(\tau) q^{\nu-1} \sin (\nu-1)\vartheta + \tilde{G}_\nu^{(2)}(\tau) q^{-\nu-1} \sin (\nu+1)\vartheta \right\}$$

$$-\beta\tau \sum_{n=o}^{\infty} \nu \left\{ \tilde{G}_{\nu,1}^{(1)}(\tau) q^{\nu-1} + \tilde{G}_{\nu,1}^{(2)}(\tau) q^{-\nu-1} \right\} \sin \nu\vartheta \cos \vartheta \qquad (255)$$

$$\frac{1}{2\beta+1} < \tau < 1$$

$$y_2^{(o)}(q,\vartheta)$$

$$= \sum_{n=o}^{\infty} \nu \left\{ \tilde{G}_\nu^{(1)}(\tau) q^{\nu-1} \cos (\nu-1) - \tilde{G}_\nu^{(2)}(\tau) q^{-\nu-1} \cos (\nu+1)\vartheta \right\}$$

$$-\beta\tau \sum_{n=o}^{\infty} \nu \left\{ \tilde{G}_{\nu,1}^{(1)}(\tau) q^{\nu-1} + \tilde{G}_{\nu,1}^{(2)}(\tau) q^{-\nu-1} \right\} \sin \nu\vartheta \sin \vartheta \qquad (256)$$

where $\tilde{G}_\nu^{(\alpha)}(\tau)$ and $\tilde{G}_{\nu,1}^{(\alpha)}(\tau)$ retain the definitions given in (250) and (251) except that $\Delta\tilde{\underline{F}}_\nu(\tau)$, $\Delta\tilde{\underline{F}}_{\nu,1}(\tau)$, $\Delta\tilde{\underline{G}}_\nu(\tau)$, and $\Delta\tilde{\underline{G}}_{\nu,1}(\tau)$ are replaced by

$$\left.\begin{aligned}
\Delta\tilde{\underline{F}}_\nu(\tau) &= \tilde{\underline{F}}_\nu(\tau) - \frac{1}{2} g(\tau)\, T^\nu \cos\left(\nu\omega - \frac{\pi}{4}\right) \\
\Delta\tilde{\underline{F}}_{\nu,1}(\tau) &= \frac{\nu-1}{\nu+1} \tilde{\underline{F}}_{\nu,1} - \frac{h(\tau)}{2} T^\nu \cos\left(\nu\omega - \mu - \frac{\pi}{4}\right) \\
\Delta\tilde{\underline{G}}_\nu(\tau) &= \tilde{\underline{G}}_\nu(\tau) - \frac{g(\tau)}{2} T^{-\nu} \cos\left(\nu\omega + \frac{\pi}{4}\right) \\
\Delta\tilde{\underline{G}}_{\nu,1}(\tau) &= \frac{\nu+1}{\nu-1} \tilde{\underline{G}}_{\nu,1} - \frac{h(\tau)}{2} T^{-\nu} \cos\left(\nu\omega + \mu + \frac{\pi}{4}\right)
\end{aligned}\right\} (257)$$

respectively. It must be noted again that the orders of $\tilde{G}_{\nu}^{(1)}(\tau)$ and $\tilde{G}_{\nu,1}^{(1)}(\tau)$ are the same as those of $\Delta\tilde{\underline{F}}_{\nu}(\tau)$ and $\Delta\tilde{\underline{F}}_{\nu,1}(\tau)$, respectively, because of the way they are defined in (257). For the same reason as stated in section 15, this again cannot jeopardize the basic assumption of convergence of the series.

PART IV

CRITERIA FOR THE UPPER CRITICAL MACH NUMBER

18. Limiting Line and the Breakdown of Isentropic Flow

The solutions constructed in the previous sections are known to be regular in the hodograph plane except at a few singular points. It is also known that for the limiting case of infinite sonic speed, or $c_0 \rightarrow \infty$, the solution will give the desired flow pattern in the physical plane. When the sonic speed is finite or when the Mach number of the free stream is different from zero, there is no guarantee as to the behavior of the solution in the physical plane except the probable continuity of the flow pattern with respect to the free-stream Mach number. It is found that such continuity in the flow pattern actually exists up to a certain Mach number. In other words, the pattern of the compressible flow is only slightly different from that of the incompressible flow up to a certain Mach number at which the so-called limiting lines appear. At the limiting line, the acceleration of the flow is infinite and the flow is reversed. It was shown by Tollmein (reference 13) and Tsien (reference 2) that, without considering viscosity, the flow cannot be continued across the limiting lines, and a forbidden region is created in the space where no fluid can enter. In other words, continuity of flow pattern exists up to a critical Mach number beyond which no isentropic flow is possible with the imposed physical boundary conditions.

The breakdown of isentropic flow, or the compressibility burble, can be effected in two ways. First of all, the acceleration in the neighborhood of the limiting line is very large. Thus each one of the following factors gives appreciable alterations in the dynamic relations:

(a) Viscous stress due to ordinary viscosity of the fluid (reference 22)

(b) Stress due to expansion or compression of the fluid, or viscous stress due to the second viscosity coefficient (reference 23, pp. 351 and 358)

(c) Small but appreciable relaxation time required for the vibrational modes of the molecules to reach equilibrium state (reference 24)

(d) Heat conduction from fluid element to fluid element

Secondly, the isentropic flow also can break down through the appearance of shock waves. The breakdown of isentropic flow is associated with the introduction of vorticity to the flow. Thus the flow becomes rotational with part of the mechanical energy of the fluid converted into heat energy. All these factors tend to increase the entropy of the fluid and finally to increase the drag of the body. Thus the critical Mach number so defined is of great physical importance to the aerodynamic characteristics of the body concerned.

Of course, the isentropic flow might break down due to the instability of flow fluid with the final appearance of shock waves. Furthermore, the action of boundary layer and possible condensation of one component of the fluid[1] on the flow might lead also to the premature destruction of the isentropic flow. On the other hand, shock waves can appear only in supersonic flow; thus, if the speed of the fluid is everywhere subsonic, there is no danger of the compressibility burble. Hence, the free-stream Mach number for the first appearance of sonic speed in the field is called the "lower critical Mach number"; while the free-stream Mach number for the first appearance of limiting lines is called the "upper critical Mach number." (See reference 2.) The latter is always higher than the former, due to the fact that limiting lines appear only in supersonic flow. The actual critical Mach number for the compressibility burble must lie between these two limits and depends, among other parameters, upon the Reynolds number of the flow.

19. The Condition for the Limiting Line

At the limiting hodograph, or the hodograph of the limiting line, it was shown (references 1, 2, 12, and 13) that

$$\frac{\partial(x,y)}{\partial(u,v)} \equiv -\left(\frac{\rho_o}{\rho q}\right)^2 \left[\psi_q^2 - \left(\frac{1}{c^2} - \frac{1}{q^2}\right)\psi_\vartheta^2\right] = 0 \qquad (258)$$

[1]The phenomenon of condensation shocks due to water vapor in the air flow around an airfoil was first brought to the attention of the authors by Kate Liepmann, who observed them in wind-tunnel experiments.

Since the factor before the term ψ_ϑ^2 is positive for supersonic regions only, $c < q$, where $\rho \neq 0$, the limiting line can appear only when the local speed exceeds that of sound. It should be noted that the vanishing of the Jacobian is the condition for the failure of the hodograph method, as the transformation (9) and (10) would no longer be one-to-one and continuous. Thus, the appearance of the limiting lines is then the physical counterpart of the singularity of the transformation.

As $\psi(\tau,\vartheta)$ is known, equation (258) defines two lines in the hodograph plane:

$$2\tau\left[\frac{1-\tau}{\alpha^2\tau-1}\right]^{\frac{1}{2}}\psi_\tau - \psi_\vartheta = 0 \tag{259}$$

$$2\tau\left[\frac{1-\tau}{\alpha^2\tau-1}\right]^{\frac{1}{2}}\psi_\tau + \psi_\vartheta = 0, \quad \tau \geq \frac{1}{2\beta+1} \tag{260}$$

Geometrically, this expresses the fact that the streamline $\psi(q,\vartheta)$ = constant and a characteristic curve belonging to either family has a common tangent (reference 1). The problem can then be formulated based on this property: the necessary and sufficient condition for the existence of a limiting line is that there exists a solution between the two simultaneous equations

$$2\tau\left[\frac{1-\tau}{\alpha^2\tau-1}\right]^{\frac{1}{2}}\psi_\tau - \psi_\vartheta = 0 \tag{261}$$

$$\psi = 0 \tag{262}$$

or

$$2\tau\left[\frac{1-\tau}{\alpha^2\tau-1}\right]^{\frac{1}{2}}\psi_\tau + \psi_\vartheta = 0 \tag{263}$$

$$\psi = 0 \tag{264}$$

where $\psi(\tau,\vartheta)$ is a definite branch associated with the largest possible τ for a given boundary and a free-stream

Mach number. The zero streamline is chosen, as it generally gives the highest velocity and is the place for the earliest appearance of the limiting line.

Generally, these equations may not possess a solution for a known function $\psi(\tau,\vartheta)$ when the parameter M_1 is assigned. This means that there will be a system of boundaries corresponding to a sequence of values of M_1, for which the limiting line does not occur. The first Mach number for which equations (261) and (262) have a solution will be defined as the upper critical Mach number and the corresponding boundary as the critical boundary.

The actual solution of the equation is, in general, difficult owing to the fact that $\psi(\tau,\vartheta)$ is, in most cases, represented by an infinite series. However, if the streamlines are determined in the hodograph plane for the calculation of the shape of the body, a simple graphical test of whether there is a point of tangency between the zero streamline and the characteristic can be easily made. On the other hand, if the form (214) and (215), for instance, is used, an approximate analytic solution can be obtained without involving much labor.

20. The Approximate Determination of the Upper Critical Mach Number

As can be seen from section 15, the importance of $\psi_2^{(o)}(\tau,\vartheta)$ relative to $\psi_1(\tau,\vartheta)$ will decrease as τ recedes from the critical circle $\tau = \frac{1}{2\beta+1}$ toward the supersonic region. For the first appearance of the limiting line, τ is almost always high, especially when the boundary is a slender closed body. Let this be the case; then $\psi_2^{(o)}(\tau,\vartheta)$ can be neglected in comparison with $\psi_1(\tau,\vartheta)$ and a great simplification is possible. The zero streamline then can be represented approximately by

$$\psi(\tau,\vartheta) \equiv \psi_1(\tau,\vartheta) = 0$$

Furthermore, a simple reduction shows that the two pairs of equations, (261), (262) and (263), (264) reduce respectively to

$$\Phi_o'(\eta) + \Psi_o'(\eta) = 0 \tag{265}$$

$$\Phi_o(\xi) + \Psi_o(\xi) = \Phi_o(\eta) - \Psi_o(\eta) \tag{266}$$

or

$$\Phi_o'(\xi) + \Psi_o'(\xi) = 0 \tag{267}$$

$$\Phi_o(\eta) - \Psi_o(\eta) = \Phi_o(\xi) + \Psi_o(\xi) \tag{268}$$

where ξ and η are the characteristic parameters defined in equation (218). This reduction is made possible by the fact that $f(\tau)$ never vanishes in the interval $\frac{1}{2\beta+1} < \tau < 1$.

Whenever the stream function ψ_o and the potential function φ_o of the incompressible flow are given, the functions Ψ_o and Φ_o can be easily obtained by substituting λU for q according to equation (216). Then, since λ decreases with an increase in the free-stream Mach number M, as shown in table 14 and figure 12, the upper critical Mach number will be given by the largest value of λ that gives a solution either of equations (265) and (266) or equations (267) and (268). An analytical solution can be made, as the functions Ψ_o and Φ_o are quite simple.

There is, however, an interesting direct geometrical interpretation of these sets of equations in the physical plane of the incompressible flow as shown by figure 13. According to equations (216), the functions Ψ_o and Φ_o are the stream function ψ_o and the potential function φ_o at the constant value of the speed λU. Since $\lambda > 1$, for the body shown in figure 13, the constant speed λU curve C_λ forms a loop symmetrical with respect to the y-axis.

The variables are really the angle of inclination of the incompressible velocity vector. Along the constant speed curve C_λ from the point S_2 to P, the angle of inclination of the velocity vector is monotonically decreasing. Therefore, the parameter of the angle of inclination can be replaced by the distances along the curve C_λ. Let equation (267) be satisfied at the point $S = S_2$; then

$$\Phi_0'(S_2) = -\Psi_0'(S_2) \tag{269}$$

This means that, at the point $S = S_2$, the rate of change of the potential function φ_0 along C_λ is equal to the negative of the rate of change of the stream function ψ_0. Since potential lines and streamlines in incompressible flow form an infinitesimal square mesh, this condition requires that the angle between the tangent to the curve C_λ at $S = S_2$ be 45°, as shown in figure 13. This is easily seen by remembering that from S_2 to P, the value of the stream function increases while the value of the potential function decreases, because of the indicated flow direction. Thus the point S_2 can be easily determined by this graphical condition. Equation (268) can then be written as

$$\Phi_0(s) - \Psi_0(S) = \Phi_0(S_2) + \Psi_0(S_2) \tag{270}$$

If this condition is satisfied at a point S_1, then the condition for a limiting line is completely satisfied. A similar graphical interpretation for the equations (265) and (266) can be worked out for the side of the constant speed curve lying to the right of the y-axis. From these considerations, it is clear that the upper critical Mach number is the lowest free-stream Mach number which gives a constant speed C_λ containing two points, S_1 and S_2, defined by equations (269) and (270).

PART V

APPLICATION — ELLIPTIC CYLINDERS

21. Preliminary Discussions

This part of the report is devoted to the application of the general method, developed in part III, to the study of the flow of a compressible fluid around an elliptic cylinder. According to sections 8 and 9, if a solution were constructed about the stagnation point, the continuation of this solution would require that conditions (102) and (103) and, hence, (106) and (107) be satisfied. These equations involve two sets of hypergeometric functions with parameters m and $m + 1/2$, as well as their derivatives. To shorten the lengthy calculations, in view of the limited amount of time available, the following approximate procedure was adopted.

Given the domain $\underline{D}$, the solution valid in the annulus region, rather than that about the stagnation point, was first constructed. The constants which determine the Laurent expansion of the solution, B_n^* and C_n^*, for example, are now assigned and, consequently, the set of hypergeometric functions with integral parameters is not immediately required. The difficulty, however, is the question of whether it is possible to continue the solution within the circle of convergence. This continuation may not be possible owing to the stringent continuity conditions given by equations (102) and (103), and to the requirement that the function must be regular within the circle $q = U$.

This, however, does not offer a serious objection from the practical point of view. In the first place, the summed function ($\psi_1(q,\vartheta)$, for instance) actually holds even within the circle of convergence $q < U$, and the correction function $\psi_2(q,\vartheta)$, is generally small compared with $\psi_1(q,\vartheta)$ due to the close asymptotic approximation of the hypergeometric functions in the elliptic domain. In other words, although the solution within the circle of convergence strictly represents a different flow, numerically it approximates very closely that defined in the annulus region. In the second place, since this region $q < U$ is relatively unimportant in the case of mixed flow, where τ_1 is very much less than $\frac{1}{2\beta+1}$ — that is, for free-stream Mach number considerably less

than unity - the inaccuracy of the solution is limited to a small region in the hodograph plane. Furthermore, the most interesting phenomena of such a flow, such as the appearance of limiting lines, always take place in the annulus region. Therefore, this modified procedure, although unsatisfactory from the general view point, is an expedient capable of yielding an interesting result and furnishing a test of the practicability of the proposed solution.

The situation also may be considered from another angle. The procedure used in this section can be derived by replacing the functions $\xi_{\nu}(\tau)$ and $\xi_{-\nu}(\tau)$ with the approximate values given in equation (201) in the expressions for the coefficients involved in the solution within the annulus region, that is, (106) and (107). Thus the procedure may be regarded as an appropriate method of approximation. The error introduced is generally negligible if $\tau_1 << \frac{1}{2\beta+1}$. This is indicated by the fact that the correction function $\psi_2(q,\tau)$, for instance, is very small in comparison with $\psi_1(q,\vartheta)$ when $q \leq U$.

Another simplification is made by using the elementary integral $q^{-2\nu}\,\mathfrak{F}_{-\nu}(\tau)$ instead of $q^{-2\nu}\,\mathfrak{G}_{\nu}(\tau)$ in the continued solution, as, in this case, $\mathfrak{F}_{-\nu}(\tau)$ is a well-defined function. In doing so, the asymptotic behavior of the second solution remains unchanged because the first term in $\mathfrak{G}_{\nu}(\tau)$ is always small in comparison with the second.

If, however, all the required hypergeometric functions are computed, there is no difficulty in carrying out the exact method developed in part III of the report for any accurate study of two-dimensional flow. For this reason, the expressions for the hydrodynamic functions derived for both the exact and approximate procedures for the problem at hand are given.

In the numerical example, detailed calculations are made for the flow of air about a cylindrical body derived from the incompressible flow about an elliptic section with a ratio of the minor and major axes equal to 0.6. The calculations were carried out for two different free-stream Mach numbers, 0.6 and 0.7.

22. The Functions $z_0(w)$, $W_0(w)$ and $\Lambda_0(w)$

The irrotational flow of an incompressible fluid about

an elliptic cylinder with the velocity at infinity parallel to the major axis is represented by the complex potential $W_o(z_o)$:

$$W_o(z_o) = \zeta + \frac{1}{\zeta} \tag{271}$$

with

$$z_o = \zeta + \frac{\epsilon^2}{\zeta} \tag{272}$$

For convenience in practical calculation, all the physical quantities z_o, q, and ρ, will be normalized consistently throughout the present part. The major and minor axes of the section are respectively $1 + \epsilon^2$ and $1 - \epsilon^2$, where $\epsilon < 1$; $q = 1$ at infinity and $\rho = 1$ when $q = 0$. This will automatically render the hydrodynamic functions dimensionless and the constants U and ρ_0 will be eliminated from the formulas in the succeeding sections.

By differentiating (271) with respect to z_o, the dimensionless complex velocity of the flow is

$$w = \frac{\zeta^2 - 1}{\zeta^2 - \epsilon^2}$$

Thus

$$\zeta = -\left[\frac{1 - \epsilon^2 w}{1 - w}\right]^{1/2}, \qquad |1 - \epsilon^2 w| \neq 0 \tag{273}$$

This function is two-valued with two branch points at $w = 1$ and $w = \epsilon^{-2}$. In order to make $z_o(w)$ a single-valued function of w, the expression (273) is supposed to be the principal value so that $|\arg(1 - w)| < \pi$ and $1 < |w| < \epsilon^{-2}$. The condition $|\epsilon^2 w| < 1$ must be satisfied, for $w = \epsilon^{-2}$ corresponds to $\zeta = 0$, which is another singularity. With the principal value so defined, if the negative sign in (273) is taken, then the domain $\underline{D}$ corresponds to the half plane $Rl\ \zeta \leq 0$ and $|\zeta| \geq 1$. On the other hand, since the trans-

formation (272) is one-to-one when $|\zeta| \geq 1$, then the domain D, which is $R\ell\ z_0 \leq 0$ with the region inside the section excluded, corresponds uniquely to $\underline{D}$.

Consequently, the inverse mapping function $z_0(w)$ is

$$z_0(w) = -\left\{\left[\frac{1-\epsilon^2 w}{1-w}\right]^{1/2} + \epsilon^2\left[\frac{1-w}{1-\epsilon^2 w}\right]^{1/2}\right\} \tag{274}$$

which will be single-valued, provided a cut is introduced to join the branch points in such a way that the argument of $(1-w)$ is restricted to $-\pi < \arg(1-w) < \pi$ and $|\epsilon^2 w| < 1$. On separating into real and imaginary parts, it is found that as $0 \leq \vartheta < 2\pi$

$$x_0(q,\vartheta) = -\frac{1}{2^{\frac{1}{2}}}\left[\left\{I(q,\vartheta) + J(q,\vartheta)\right\}^{1/2} + \epsilon^2\left\{I_\epsilon(q,\vartheta) + J^{-1}(q,\vartheta)\right\}^{1/2}\right] \tag{275}$$

$$y_0(q,\vartheta) = \frac{1}{2^{\frac{1}{2}}}\left[\left\{-I(q,\vartheta) + J(q,\vartheta)\right\}^{1/2} - \epsilon^2\left\{-I_\epsilon(q,\vartheta) + J^{-1}(q,\vartheta)\right\}^{1/2}\right] \tag{276}$$

with $w = q\ e^{-i\vartheta}$, where the functions $I(q,\vartheta)$, $I_\epsilon(q,\vartheta)$, and $J(q,\vartheta)$ stand for:

$$I(q,\vartheta) = \frac{1-(1+\epsilon^2)q\cos\vartheta + \epsilon^2 q^2}{1 - 2q\cos\vartheta + q^2} \tag{277}$$

$$I_\epsilon(q,\vartheta) = \frac{1-(1+\epsilon^2)q\cos\vartheta + \epsilon^2 q^2}{1 - 2\epsilon^2 q\cos\vartheta + \epsilon^4 q^2} \tag{278}$$

$$J(q,\vartheta) = \left[\frac{1-2\,\epsilon^2\,q\cos\vartheta + \epsilon^4 q^2}{1-2\,q\cos\vartheta + q^2}\right]^{1/2} \tag{279}$$

On the other hand, substituting equation (273) in equation (271), the function $W_0(z_0)$ is carried over into D; namely,

$$W_0(w) = -\left\{\left[\frac{1-\epsilon^2 w}{1-w}\right]^{\frac{1}{2}} + \left[\frac{1-w}{1-\epsilon^2 w}\right]^{\frac{1}{2}}\right\} \tag{280}$$

Now $W_0(w) = \varphi_0(q,\vartheta) + i\psi_0(q,\vartheta)$, and similarly

$$\varphi_0(q,\vartheta) = \frac{1}{2^{\frac{1}{2}}}\left[\left\{I(q,\vartheta) + J(q,\vartheta)\right\}^{\frac{1}{2}} + \left\{I_\epsilon(q,\vartheta) + J^{-1}(q,\vartheta)\right\}^{\frac{1}{2}}\right] \tag{281}$$

$$\psi_0(q,\vartheta) = \frac{1}{2^{\frac{1}{2}}}\left[\left\{-I(q,\vartheta) + J(q,\vartheta)\right\}^{\frac{1}{2}} - \left\{-I_\epsilon(q,\vartheta) + J^{-1}(q,\vartheta)\right\}^{\frac{1}{2}}\right] \tag{282}$$

By integrating $z_0(w)$, according to (87), the transformed potential function $\Lambda_0(w)$, aside from a constant, takes the form:

$$\Lambda_0(w) = 2(1-w)^{\frac{1}{2}}\,(1-\epsilon^2 w)^{\frac{1}{2}} \tag{283}$$

The principal value of this function is again defined by restricting the argument of $(1-w)$ to $-\pi < \arg\,(1-w) < \pi$ and $|w| < \epsilon^{-2}$. Within this domain D, the real and imaginary parts are:

$$\chi_0(q,\vartheta) = 2^{\frac{1}{2}}\,[K(q,\vartheta) + L(q,\vartheta)]^{\frac{1}{2}} \tag{284}$$

$$0 \leq \vartheta < 2\pi$$

$$\sigma_c(q,\vartheta) = -\,2^{\frac{1}{2}}\,[-K(q,\vartheta) + L(q,\vartheta)]^{\frac{1}{2}} \tag{285}$$

as $\Lambda_0(w) = \chi_0(q,\vartheta) - i\sigma_0(q,\vartheta)$, where the functions $K(q,\vartheta)$ and $L(q,\vartheta)$ are defined by:

$$K(q,\vartheta) = 1 - (1+\epsilon^2)q \cos \vartheta + \epsilon^2 q^2 \cos 2\vartheta \tag{286}$$

$$L(q,\vartheta) = [1-2 q \cos \vartheta + q^2]^{\frac{1}{2}} [1-2\epsilon^2 q \cos \vartheta + \epsilon^4 q^2]^{\frac{1}{2}} \tag{287}$$

23. Expansions of $W_0(w)$ and $\Lambda_0(w)$

The function $W_0(w)$ defined in (280) is single-valued and regular everywhere in $|w| < 1$ and, hence, possesses the following expansion:

$$W_0(w) = -\sum_{n=0}^{\infty} A_n w^n, \quad |w| < 1 \tag{288}$$

where the coefficients A_n are real and given by

$$A_n = 2S_n^{(i)} - (1 + \epsilon^2) S_{n-1}^{(i)} \quad n \geq 1 \tag{289}$$

$$A_0 = 2S_0^{(i)} = 2$$

with

$$S_n^{(i)}(\epsilon^2) = \frac{1}{\pi} \sum_{m=0}^{n} \frac{\Gamma(n-m+\frac{1}{2})\Gamma(m+\frac{1}{2})}{\Gamma(n-m+1)\Gamma(m+1)} \epsilon^{2m}$$

However, in the region outside $|w| < 1$ the function $W_0(w)$ is doubled-valued; and when a cut is put between the branch points $w = 1$ and $w = \epsilon^{-2}$, the principal value is discontinuous along the positive axis of reals within the annulus region. To obtain the desired expansion, the function is written in the following form

$$W_0(w) = \frac{i}{w^{\frac{1}{2}}} \frac{2 - (1+\epsilon^2)w}{(1-w^{-1})^{\frac{1}{2}} (1-\epsilon^2 w)^{\frac{1}{2}}} \tag{290}$$

Now $(1-w^{-1})^{-\frac{1}{2}}\ (1-\epsilon^2 w)^{-\frac{1}{2}}$ is single-valued and continuous within the annulus region; its Laurent expansion is

$$(1-w^{-1})^{-\frac{1}{2}}\ (1-\epsilon^2 w)^{-\frac{1}{2}}$$

$$= S_o^{(o)} + \sum_{n=1}^{\infty} S_n^{(o)} \left[\epsilon^{2n}\ w^n + w^{-n}\right], \qquad 1 < |w| < \epsilon^{-2} \tag{291}$$

where

$$S_n^{(o)}(\epsilon^2) = \frac{1}{\pi} \sum_{m=o}^{\infty} \frac{\Gamma(n+m+\frac{1}{2})\Gamma(m+\frac{1}{2})}{\Gamma(n+m+1)\Gamma(m+1)}\ \epsilon^{2m} \tag{292}$$

Substituting $(1-w)^{-\frac{1}{2}}\ (1-\epsilon^2 w)^{-\frac{1}{2}}$ from (291) in (290), the expansion for $W_o(w)$ in the annulus region is

$$W_o(w) = i \sum_{n=o}^{\infty} \left[B_n\ \epsilon^{2n}\ w^{\nu} + C_n\ w^{-\nu}\right], \qquad 1 < |w| < \epsilon^{-2} \tag{293}$$

when the constants B_n, C_n and the exponent ν are defined by:

$$\left.\begin{aligned} B_n &= 2\ \epsilon^2\ S_{n+1}^{(o)} - (1+\epsilon^2)\ S_n^{(o)} \\ C_n &= 2\ S_n^{(o)} - (1+\epsilon^2)\ S_{n+1}^{(o)} \\ \nu &= n + \tfrac{1}{2} \end{aligned}\right\} \tag{294}$$

Similarly, the transformed potential function $\Lambda_o(w)$ can be expanded and is:

$$\Lambda_o(w) = 2 \sum_{n=o}^{\infty} \tilde{A}_n\ w^n, \qquad |w| < 1 \tag{295}$$

when the constants $\tilde{A}_n$ are

$$\left.\begin{aligned} \tilde{A}_n &= S_n^{(i)} - (1+\epsilon^2)\ S_{n-1}^{(i)} + \epsilon^2\ S_{n-2}^{(i)} \\ \tilde{A}_1 &= -\frac{1}{2}\ (1+\epsilon^2), \quad \tilde{A}_o = 1 \end{aligned}\right\} \tag{296}$$

and $S_n^{(i)}$ is given in (289).

On the other hand, in the annulus region the expansion is

$$\Lambda_o(w) = -2i \sum_{n=o}^{\infty} \left[\tilde{B}_n \epsilon^{2n} w^n + \tilde{C}_n w^{-n}\right], \quad 1 < |w| < \epsilon^{-2} \tag{297}$$

with the constants $\tilde{B}_n$ and $\tilde{C}_n$ defined as

$$\left.\begin{aligned} \tilde{B}_n &= S_{n-1}^{(o)} - (1+\epsilon^2) S_n^{(o)} + \epsilon^2 S_{n+1}^{(o)}, \quad n \geq 1 \\ \tilde{B}_o &= 2\epsilon^2 S_1^{(o)} - (1+\epsilon^2) S_o^{(o)} \\ \tilde{C}_n &= S_n^{(o)} - (1+\epsilon^2) S_{n+1}^{(o)} + \epsilon^2 S_{n+2}^{(o)} \end{aligned}\right\} \tag{298}$$

where $S_n^{(o)}(\epsilon^2)$ is defined in (292).

24. The Stream Function $\psi(q,\vartheta)$

The relationship between the domain $\underline{D}$ and D is thus fully established and the functions corresponding to such domains are also given. From the general scheme developed in sections 8 and 9 the solutions for the similar motion of a compressible fluid can be constructed. First of all, the stream function $\psi(q,\vartheta)$ governing the subsonic flow is the sum of $\psi_1(q,\vartheta)$ and $\psi_2(q,\vartheta)$. According to (207), (208), and (209), for $0 \leq \tau < \frac{1}{2\beta+1}$:

$$\psi_1(q,\vartheta) = \frac{1}{2^{\frac{1}{2}}} \frac{f(\tau)}{f(\tau_1)} \left\{\left[-I(tq,\vartheta) + J(tq,\vartheta)\right]^{\frac{1}{2}} - \left[-I_\epsilon(tq,\vartheta) + J^{-1}(tq,\vartheta)\right]^{\frac{1}{2}}\right\} \tag{299}$$

where the functions $I(tq,\vartheta)$, $I_\epsilon(tq,\vartheta)$, and $J(tq,\vartheta)$ are obtained from I, I_ϵ, and J in (272) to (279) by replacing q by tq, t being defined in (195). For $q < 1$, the function $\psi_2(q,\vartheta)$ is

$$\psi_2^{(i)}(q,\vartheta) = \sum_{n=0}^{\infty} A_n\, G_n(\tau)\, q^n \sin n\,\vartheta \tag{300}$$

where A_n is defined in (289) and $G_n(\tau)$ in (199). For $q > 1$ and in subsonic region the function $\psi_2^{(o)}(q,\vartheta)$:

$$\psi_2^{(o)}(q,\vartheta) = \sum_{n=0}^{\infty}\Big[G_\nu^{(1)}(\tau)\,\epsilon^{2n}q^{\nu} + G_\nu^{(2)}(\tau)\,q^{-\nu}\Big]\cos\nu\,\vartheta, \quad 0 \le \vartheta < 2\pi \tag{301}$$

where $G_\nu^{(1)}(\tau)$ and $G_\nu^{(2)}(\tau)$ are defined by (205) with the constants B_n and C_n defined in (294).

When the motion becomes supersonic, the continuation of $\psi_1(q,\vartheta)$ defined in (299) gives

$$\begin{aligned}\psi_1(q,\vartheta) = \frac{1}{8}\,\frac{f(\tau)}{f(\tau_1)}\Big\{&\Big[-I(\lambda,\xi)+J(\lambda,\xi)\Big]^{\frac{1}{2}} - \Big[-I_\epsilon(\lambda,\xi)+J^{-1}(\lambda,\xi)\Big]^{\frac{1}{2}}\\ &+\Big[-I(\lambda,\eta)+J(\lambda,\eta)\Big]^{\frac{1}{2}} - \Big[-I_\epsilon(\lambda,\eta)+J^{-1}(\lambda,\eta)\Big]^{\frac{1}{2}}\\ &-\Big[+I(\lambda,\xi)+J(\lambda,\xi)\Big]^{\frac{1}{2}} - \Big[-I_\epsilon(\lambda,\xi)+J^{-1}(\lambda,\xi)\Big]^{\frac{1}{2}} \qquad \frac{1}{2\beta+1} < \tau < 1\\ &\pm\Big[I(\lambda,\eta)+J(\lambda,\eta)\Big]^{\frac{1}{2}} \pm \Big[I_\epsilon(\lambda,\eta)+J^{-1}(\lambda,\eta)\Big]^{\frac{1}{2}}\Big\}\end{aligned} \tag{302}$$

according to (214). Here ξ and η are the characteristic parameters defined in (218). The upper sign in the last two terms corresponds to $\eta > 0$ while the lower one, to $\eta < 0$.

The accompanying function $\psi_2^{(o)}(q,\vartheta)$ is

$$\psi_2^{(o)}(q,\vartheta) = \sum_{n=0}^{\infty} \left[G_\nu^{(1)}(\tau)\, \epsilon^{2n}\, q^{\nu} + G_\nu^{(2)}(\tau)\, q^{-\nu} \right] \cos \nu\vartheta, \quad \frac{1}{2\beta+1} < \tau < 1 \tag{303}$$

Here the functions $G_\nu^{(1)}(\tau)$ and $G_\nu^{(2)}(\tau)$ are defined by (205) in conjunction with (217) in such a way that (303) will be the continuation of (301). It also should be noticed that the variable is restricted to $\frac{1}{2\beta+1} < \tau < 1$ instead of $\frac{1}{2\beta+1} < \tau < \tau_1 \epsilon^{-4}$, as $\tau_1 \epsilon^{-4}$ is generally greater than unity, which is impossible for the actual gas.

It should be remembered that $\psi_2^{(i)}(q,\vartheta)$ is always negligible compared with $\psi_1(q,\vartheta)$ within and on the unit circle $q = 1$ when τ_1 is small in comparison with $\frac{1}{2\beta+1}$; $\psi(q,\vartheta)$ can be approximately represented by $\psi_1(q,\vartheta)$ alone throughout the interior of the unit circle. As a consequence, the calculation can be simplified considerably by constructing first a solution for the annulus region by using $\underline{F}_{-\nu}(\tau)$ instead of $\underline{G}_\nu(\tau)$ and making an approximate connection across the unit circle. In that event, the stream function will be reduced to

$$\psi(q,\vartheta) \cong \psi_1(q,\vartheta) \tag{304}$$

when $0 \leq q \leq 1$; here $\psi_1(q,\vartheta)$ is again defined in (299). On the other hand, when $\tau_1 < \tau < \frac{1}{2\beta+1}$,

$$\psi(q,\vartheta) = \psi_1(q,\vartheta) + \psi_2^{(o)}(q,\vartheta) \tag{305}$$

where the function $\psi_2^{(o)}(q,\vartheta)$ which is small on $q = 1$ is given by

$$\psi_2^{(o)}(q,\vartheta) = \sum_{n=o}^{\infty} \left[B_n G_\nu(\tau)\ \epsilon^{2n}\ q^\nu + C_n G_{-\nu}(\tau) q^{-\nu}\right] \cos \nu\ \vartheta \quad (306)$$

Here the functions $G_\nu(\tau)$ and $G_{-\nu}(\tau)$ can be shown to be

$$G_\nu(\tau) = \underline{F}_\nu^{(r)}(\tau) - \frac{f(\tau)}{f(\tau_1)}\ t^\nu, \quad G_{-\nu}(\tau) = \underline{F}_{-\nu}^{(r)}(\tau) - \frac{f(\tau)}{f(\tau_1)}\ t^{-\nu} \quad (307)$$

and the coefficients B_n and C_n are defined in (294).

The continuation of $\psi_1(q,\vartheta)$ is naturally the expression given in (302) while that of (306) differs only in the definition of $G_\nu(\tau)$ and $G_{-\nu}(\tau)$ which are

$$G_\nu(\tau) = \underline{F}_\nu^{(r)}(\tau) - \frac{1}{2}\ \frac{f(\tau)}{f(\tau_1)}\ t^\nu \cos\left(\nu\omega - \frac{\pi}{4}\right)$$

$$\frac{1}{2\beta+1} < \tau < 1 \quad (308)$$

$$G_{-\nu}(\tau) = \underline{F}_{-\nu}^{(r)}(\tau) - \frac{1}{2}\ \frac{f(\tau)}{f(\tau_1)}\ t^{-\nu} \cos\left(\nu\omega + \frac{\pi}{4}\right)$$

25. The Coordinate Functions $x(q,\vartheta)$ and $y(q,\vartheta)$

With the functions $z_o(w)$ and $\Lambda_o(w)$ defined in sections 22 and 23, the corresponding functions $\Lambda(w;\tau)$ and consequently $z(w;\tau)$ for the motion of a compressible fluid can be constructed. The coordinate functions derived from $\Lambda(w;\tau)$ are given respectively by the sum of two functions $x_1(q,\vartheta)$ and $y_1(q,\vartheta)$ which, according to equations (237) to (238), are

$$x_1(q,\vartheta) = -\frac{t(\tau)}{2^{\frac{1}{2}}}\frac{g(\tau)}{f(\tau_1)}\left\{\left[I(tq,\vartheta) + J(tq,\vartheta)\right]^{\frac{1}{2}}\right.$$

$$\left. + \epsilon^2\left[I_\epsilon(tq,\vartheta) + J^{-1}(tq,\vartheta)\right]^{\frac{1}{2}}\right\}$$

$$- \frac{\beta\tau}{2}\frac{h(\tau)}{f(\tau_1)}\frac{t\sin 2\vartheta}{\sigma_o(tq,\vartheta)}\left\{-1 + 4\epsilon^2 t q\cos\vartheta - \epsilon^2\right.$$

$$\left. + J(tq,\vartheta) + \epsilon^2 J^{-1}(tq,\vartheta)\right\} \tag{309}$$

$$y_1(q,\vartheta) = \frac{t(\tau)}{2^{\frac{1}{2}}}\frac{g(\tau)}{f(\tau_1)}\left\{\left[-I(tq,\vartheta) + J(tq,\vartheta)\right]^{\frac{1}{2}}\right.$$

$$\left. - \epsilon^2\left[-I_\epsilon(tq,\vartheta) + J^{-1}(tq,\vartheta)\right]^{\frac{1}{2}}\right\}$$

$$- \beta\tau\frac{h(\tau)}{f(\tau_1)}\frac{t\sin^2\vartheta}{\sigma_o(tq,\vartheta)}\left\{-1 + 4\epsilon^2 t q\cos\vartheta - \epsilon^2\right.$$

$$\left. + J(tq,\vartheta) + \epsilon^2 J^{-1}(tq,\vartheta)\right\} \tag{310}$$

where $\sigma_o(tq,\vartheta)$ is obtained from $\sigma_o(q,\vartheta)$ in (285) by replacing q by tq. The functions $x_2^{(i)}(q,\vartheta)$ and $y_2^{(i)}(q,\vartheta)$, according to equations (239) and (240), are

$$x_2^{(i)}(q,\vartheta) = 2\sum_{n=1}^{\infty} n\,\tilde{A}_n\,\tilde{G}_n(\tau)\,q^{n-1}\cos(n-1)\vartheta$$

$$- 2\beta\tau\sum_{n=1}^{\infty} n\,\tilde{A}_n\,\tilde{G}_{n,i}(\tau)\,q^{n-1}\cos n\vartheta\cos\vartheta \tag{311}$$

$$q < 1$$

$$y_2^{(i)}(q,\vartheta) = -2\sum_{n=1}^{\infty} n\,\tilde{A}_n\,\tilde{G}_n(\tau)\,q^{n-1}\sin(n-1)\vartheta$$

$$-2\beta\tau\sum_{n=1}^{\infty} n\,\tilde{A}_n\,\tilde{G}_{n,1}(\tau)\,q^{n-1}\cos n\vartheta\sin\vartheta \qquad (312)$$

Here the functions $\tilde{G}_n(\tau)$ and $\tilde{G}_{n,1}(\tau)$ are defined by equations (241) and (242) and the constants $\tilde{A}_n$ by (296).

The same functions valid in the annulus region are again represented by the sums $x_1(q,\vartheta) + x_2^{(o)}(q,\vartheta)$ and $y_1(q,\vartheta) + y_2^{(o)}(q,\vartheta)$, where $x_1(q,\vartheta)$ and $y_1(q,\vartheta)$ are defined by equations (309) and (310), respectively. When $\tau_1 \leq \tau < \frac{1}{2\beta+1}$, $x_2^{(o)}(q,\vartheta)$, and $y_2^{(o)}(q,\vartheta)$ are

$$x_2^{(o)}(q,\vartheta) = -2\sum_{n=0}^{\infty}\nu\left[\tilde{G}_\nu^{(1)}(\tau)\,\epsilon^{2n}\,q^{\nu-1}\sin(\nu-1)\vartheta\right.$$

$$\left.+\tilde{G}_\nu^{(2)}(\tau)q^{-\nu-1}\sin(\nu+1)\vartheta\right]+2\beta\tau\sum_{n=0}^{\infty}\nu\left[\tilde{G}_{\nu,1}^{(1)}(\tau)\epsilon^{2n}q^{\nu-1}\right.$$

$$\left.+\tilde{G}_{\nu,1}^{(2)}(\tau)\,q^{-\nu-1}\right]\sin\nu\vartheta\cos\vartheta \qquad (313)$$

$$y_2^{(o)}(q,\vartheta) = -2\sum_{n=0}^{\infty}\nu\left[\tilde{G}_\nu^{(1)}(\tau)\epsilon^{2n}q^{\nu-1}\cos(\nu-1)\vartheta\right.$$

$$\left.+\tilde{G}_\nu^{(2)}(\tau)\,q^{-\nu-1}\cos(\nu+1)\vartheta\right]$$

$$+2\beta\tau\sum_{n=0}^{\infty}\nu\left[\tilde{G}_{\nu,1}^{(1)}(\tau)\epsilon^{2n}q^{\nu-1}+\tilde{G}_{\nu,1}^{(2)}(\tau)q^{-\nu-1}\right]\sin\nu\vartheta\sin\vartheta \qquad (314)$$

The functions $\tilde{G}_\nu^{(\alpha)}(\tau)$, $\tilde{G}_{\nu,1}^{(\alpha)}(\tau)$ are defined in equations (250) and (251) together with equations (252) with the constants $\tilde{B}_n$ and $\tilde{C}_n$ defined in equations (298).

On the other hand, when $\frac{1}{2\beta+1} < \tau < 1$, the continued expressions of $x_1(q,\vartheta)$, $y_1(q,\vartheta)$ across the critical circle $\tau = \frac{1}{2\beta+1}$ are, according to equations (253) and (254),

$$
\begin{aligned}
x_1(q,\vartheta) = &-\frac{t(\tau)}{2^{5/2}}\frac{g(\tau)}{f(\tau_1)}\Bigg\{\Big[I(\lambda,\xi)+J(\lambda,\xi)\Big]^{\frac{1}{2}} \\
&+\epsilon^2\Big[I_\epsilon(\lambda,\xi)+J^{-1}(\lambda,\xi)\Big]^{\frac{1}{2}}+\Big[I(\lambda,\eta)+J(\lambda,\eta)\Big]^{\frac{1}{2}} \\
&+\epsilon^2\Big[I_\epsilon(\lambda,\eta)+J^{-1}(\lambda,\eta)\Big]^{\frac{1}{2}}\Bigg\}\cos\left(\frac{\pi}{4}-\omega\right) \\
&-\frac{t(\tau)}{2^{5/2}}\frac{g(\tau)}{f(\tau_1)}\Bigg\{\Big[-I(\lambda,\xi)+J(\lambda,\xi)\Big]^{\frac{1}{2}}-\epsilon^2\Big[-I_\epsilon(\lambda,\xi) \\
&+J^{-1}(\lambda,\xi)\Big]^{\frac{1}{2}}-\Big[-I(\lambda,\eta)+J(\lambda,\eta)\Big]^{\frac{1}{2}}+\epsilon^2\Big[-I_\epsilon(\lambda,\eta) \\
&+J^{-1}(\lambda,\eta)\Big]^{\frac{1}{2}}\Bigg\}\sin\left(\frac{\pi}{4}-\omega\right)-\frac{\beta\tau\, h(\tau)\cos\vartheta}{4\,q\,f(\tau_1)}\Bigg\{\Bigg[\frac{\lambda\sin\xi}{\sigma_0(\lambda,\xi)} \\
&\times\left(-1+4\epsilon^2\lambda\cos\xi-\epsilon^2+J(\lambda,\xi)+\epsilon^2 J^{-1}(\lambda,\xi)\right) \\
&+\frac{\lambda\sin\eta}{\sigma_0(\lambda,\eta)}\Big(-1+4\epsilon^2\lambda\cos\eta-\epsilon^2+J(\lambda,\eta) \\
&+\epsilon^2 J^{-1}(\lambda,\eta)\Big)\Bigg]\cos\left(\mu+\frac{\pi}{4}\right)-\Bigg[\frac{\lambda\sin\xi}{\chi_0(\lambda,\xi)} \\
&\times\left(1-4\epsilon^2\lambda\cos\xi+\epsilon^2+J(\lambda,\xi)+\epsilon^2 J^{-1}(\lambda,\xi)\right) \\
&-\frac{\lambda\sin\eta}{\chi_0(\lambda,\eta)}\Big(1-4\epsilon^2\lambda\cos\eta+\epsilon^2+J(\lambda,\eta) \\
&\qquad+\epsilon^2 J^{-1}(\lambda,\eta)\Big)\Bigg]\sin\left(\mu+\frac{\pi}{4}\right)\Bigg\}
\end{aligned}
\tag{315}
$$

$$y_1(q,\vartheta) = \frac{t(\tau)}{2^{5/2}}\frac{g(\tau)}{f(\tau_1)}\left\{\left[-I(\lambda,\xi) + J(\lambda,\xi)\right]^{\frac{1}{2}} - \epsilon^2\left[-I_\epsilon(\lambda,\xi)\right.\right.$$

$$\left. + J^{-1}(\lambda,\xi)\right]^{\frac{1}{2}} + \left[-I(\lambda,\eta) + J(\lambda,\eta)\right]^{\frac{1}{2}} - \epsilon^2\left[-I_\epsilon(\lambda,\eta)\right.$$

$$\left.\left. + J^{-1}(\lambda,\eta)\right]^{\frac{1}{2}}\right\}\cos\left(\frac{\pi}{4} - \omega\right) + \frac{t(\tau)}{2^{5/2}}\frac{g(\tau)}{f(\tau_1)}\left\{-\left[I(\lambda,\xi)\right.\right.$$

$$\left. + J(\lambda,\xi)\right]^{\frac{1}{2}} - \epsilon^2\left[I_\epsilon(\lambda,\xi) + J^{-1}(\lambda,\xi)\right]^{\frac{1}{2}}$$

$$\left. + \left[I(\lambda,\eta) + J(\lambda,\eta)\right]^{\frac{1}{2}} + \epsilon^2\left[I_\epsilon(\lambda,\eta) + J^{-1}(\lambda,\eta)\right]^{\frac{1}{2}}\right\}$$

$$\times \sin\left(\frac{\pi}{4} - \omega\right) - \frac{\beta\tau\, h(\tau)\sin\vartheta}{4\,q\, f(\tau_1)}\left\{\left[\frac{\lambda\sin\xi}{\sigma_o(\lambda,\xi)}\right.\right.$$

$$\times\left(-1 + 4\epsilon^2\lambda\cos\xi - \epsilon^2 + J(\lambda,\xi) + \epsilon^2 J^{-1}(\lambda,\xi)\right)$$

$$+ \frac{\lambda\sin\eta}{\sigma_o(\lambda,\eta)}\left(-1 + 4\epsilon^2\lambda\cos\eta - \epsilon^2 + J(\lambda,\eta)\right)$$

$$\left. + \epsilon^2 J^{-1}(\lambda,\eta)\right)\Bigg]\cos\left(\mu + \frac{\pi}{4}\right) - \left[\frac{\lambda\sin\xi}{\chi_o(\lambda,\xi)}\left(1 - 4\epsilon^2\lambda\cos\xi\right.\right.$$

$$\left. + \epsilon^2 + J(\lambda,\xi) + \epsilon^2 J^{-1}(\lambda,\xi)\right) - \frac{\lambda\sin\eta}{\chi_o(\lambda,\eta)}\left(1 - 4\epsilon^2\lambda\cos\eta\right.$$

$$\left.\left. + \epsilon^2 + J(\lambda,\eta) + \epsilon^2 J^{-1}(\lambda,\eta)\right)\right]\sin\left(\mu + \frac{\pi}{4}\right)\Bigg\} \tag{316}$$

While $x_2(q,\vartheta)$ and $y_2(q,\vartheta)$ remain to be defined by equations (313) and (314) except the functions $\tilde{G}_{\nu}^{(\alpha)}(\tau)$ and $\tilde{G}_{\nu,1}^{(\alpha)}(\tau)$ are replaced by those given in equations (250), (251) together with equations (257).

By the same argument as that used for the stream function, the practical calculation of $x(q,\vartheta)$ and $y(q,\vartheta)$ can be simplified by neglecting $x_2^{(i)}(q,\vartheta)$ and $y_2^{(i)}(q,\vartheta)$ when $q < 1$; namely,

$$x(q,\vartheta) \cong x_1(q,\vartheta) \tag{317}$$

$$y(q,\vartheta) \cong y_1(q,\vartheta), \quad 0 \leq q \leq 1 \tag{318}$$

where $x_1(q,\vartheta)$ and $y_1(q,\vartheta)$ are defined in equations (309) and (310); and in the annulus region

$$x(q,\vartheta) = x_1(q,\vartheta) + x_2^{(o)}(q,\vartheta) \tag{319}$$

$$\tau_1 < \tau < 1$$

$$y(q,\vartheta) = y_1(q,\vartheta) + y_2^{(o)}(q,\vartheta) \tag{320}$$

Here $x_1(q,\vartheta)$ and $y_1(q,\vartheta)$ are either given by equations (309), (310) or (315), (316). The terms $x_2^{(o)}(q,\vartheta)$ and $y_2^{(o)}(q,\vartheta)$, on the other hand, become

$$x_2^{(o)}(q,\vartheta) = -2 \sum_{n=o}^{\infty} \nu \Big[\tilde{B}_n \tilde{G}_\nu(\tau)\epsilon^{2n} \quad q^{\nu-1} \sin(\nu - 1)\vartheta$$

$$+ \tilde{C}_n \tilde{G}_{-\nu}(\tau) q^{-\nu-1} \sin(\nu + 1)\vartheta\Big] + 2\beta\tau \sum_{n=o}^{\infty} \nu \Big[\tilde{B}_n \tilde{G}_{\nu,1}(\tau)\epsilon^{2n}$$

$$\times q^{\nu-1} + \tilde{C}_n \tilde{G}_{\nu,1}(\tau) q^{-\nu-1}\Big] \sin \nu\vartheta \cos \vartheta \qquad (321)$$

$$y_2^{(o)}(q,\vartheta) = -2 \sum_{n=o}^{\infty} \nu \Big[\tilde{B}_n \tilde{G}_\nu(\tau)\epsilon^{2n} q^{\nu-1} \cos(\nu - 1)\vartheta$$

$$-\tilde{C}_n \tilde{G}_{-\nu}(\tau) q^{-\nu-1} \cos(\nu + 1)\vartheta\Big] + 2\beta\tau \sum_{n=o}^{\infty} \nu \Big[\tilde{B}_n \tilde{G}_{\nu,1}(\tau)\epsilon^{2n}$$

$$\times q^{\nu-1} + \tilde{C}_n \tilde{G}_{-\nu,1} q^{-\nu-1}\Big] \sin \nu\vartheta \sin \vartheta \qquad (322)$$

For $\tau_1 \leq \tau < \frac{1}{2\beta + 1}$, the functions $\tilde{G}_\nu(\tau)$, $\tilde{G}_{\nu,1}(\tau)$ are defined as

$$\tilde{G}_\nu(\tau) = \tilde{F}_\nu^{(r)}(\tau) - \frac{g(\tau)}{f(\tau_1)} t^\nu, \quad \tilde{G}_\nu(\tau) = \tilde{F}_{-\nu}^{(r)}(\tau) - \frac{g(\tau)}{f(\tau_1)} t^{-\nu} \qquad (323)$$

$$\left.\begin{aligned} \tilde{G}_{\nu,1}(\tau) &= \frac{\nu - 1}{\nu + 1} \tilde{F}_{\nu,1}^{(r)}(\tau) - \frac{h(\tau)}{f(\tau_1)} t^\nu \\ \tilde{G}_{-\nu,1}(\tau) &= \frac{\nu + 1}{\nu - 1} \tilde{F}_{-\nu,1}^{(r)}(\tau) - \frac{h(\tau)}{f(\tau_1)} t^{-\nu} \end{aligned}\right\} \qquad (324)$$

For $\frac{1}{2\beta + 1} < \tau < 1$

$$\left.\begin{aligned} \tilde{G}_\nu(\tau) &= \tilde{F}_\nu^{(r)}(\tau) - \frac{1}{2} \frac{g(\tau)}{f(\tau_1)} t^\nu \cos\left(\nu\omega - \frac{\pi}{4}\right) \\ \tilde{G}_{-\nu}(\tau) &= \tilde{F}_{-\nu}^{(r)}(\tau) - \frac{1}{2} \frac{g(\tau)}{f(\tau_1)} t^{-\nu} \cos\left(\nu\omega + \frac{\pi}{4}\right) \end{aligned}\right\} \qquad (325)$$

$$\left.\begin{aligned}\tilde{G}_{\nu,1}(\tau) &= \frac{\nu-1}{\nu+1}\,\tilde{F}^{(r)}_{-\nu,1}(\tau) - \frac{1}{2}\,\frac{h(\tau)}{f(\tau_1)}\,t^{\nu}\cos\left(\nu\omega - \mu - \frac{\pi}{4}\right)\\ \tilde{G}_{-\nu,1}(\tau) &= \frac{\nu+1}{\nu-1}\,\tilde{F}^{(r)}_{-\nu,1}(\tau) - \frac{h(\tau)}{2f(\tau_1)}\,t^{-\nu}\cos\left(\nu\omega + \mu + \frac{\pi}{4}\right)\end{aligned}\right\}\quad(326)$$

CONCLUSIONS

As an example, the motion of air past a cylindrical body was considered by taking $\epsilon = \frac{1}{2}$. The flow patterns in the τ,δ- plane for two free-stream Mach numbers $M_1 = 0.6$ and 0.7 have been calculated and were given in figures 14 and 15. It should be noticed that there is considerable distortion in the shape of the bodies in the compressible flow from that in the incompressible flow. If the compressible flow around a given body is desired, a series of computations should be made with various geometric parameters ϵ, so that the desired body shape at a definite Mach number M_1 could be picked out.

These computations definitely demonstrate the practicability of the proposed method. They also show that, in the case of two-dimensional motion of a compressible fluid, the mixed subsonic and supersonic flows exist within the field of an irrotational isentropic flow about a suitable body, and the transition from one to the other is continuous and reversible. Furthermore, the breakdown of the irrotational isentropic flow depends solely upon the occurrence of limiting lines which, in turn, is determined by the condition at infinity or the shape of the boundary, while the magnitude of the local speed attained is immaterial. In the case of $M_1 = 0.6$, the irrotational supersonic flow continues to exist up to the local Mach number $M = 1.25$; whereas for $M_1 = 0.7$ it breaks down as soon as $M = 1.22$ is reached. The singular behavior of the streamline is marked by the point of tangency of $\psi = 0$ with a characteristic at $M = 1.22$.

The calculation of the flow pattern in the physical plane is yet to be completed. When this is done, the pressure distribution can be compared with that over the same body of the incompressible flow.

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Pasadena, Calif., April 17, 1945.

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APPENDIX A

PROOF OF THEOREM(52)

To facilitate the discussion, equation (71) is first written in the form:

$$X_1(\xi_\nu^{(1)}) \equiv \xi_\nu'^{(1)}(\tau) + \frac{\nu}{2\tau}\,\zeta_1\,\zeta_2 = 0$$

where

$$\zeta_1(\tau) = \xi_\nu^{(1)}(\tau) + \frac{\beta\tau}{\nu(1 - \tau)} + \gamma_\nu(\tau)$$

$$\zeta_2(\tau) = \xi_\nu^{(1)}(\tau) + \frac{\beta\tau}{\nu(1 - \tau)} - \gamma_\nu(\tau)$$

and

$$\gamma_\nu(\tau) = \left\{\frac{1 - (2\beta + 1)\tau}{1 - \tau} + \frac{\beta^2\tau^2}{\nu^2(1 - \tau)^2} + \frac{4(1 - \tau)^{2\beta}}{\tau^\nu\,R_\nu^2(\tau)}\right\}^{\frac{1}{2}}$$

when ν is large, the character of the functions ζ_1 and ζ_2 can be easily studied in the $\tau, \xi_\nu^{(1)}$-plane (fig. 16) by neglecting the third term under the radical sign. This can be justified in the following manner: Consider the case when ν is positive and large but not an integer. In the interval $0 \leq \tau \leq \frac{1}{2\beta + 1}$, $\underline{F}_\nu(\tau) \ll F_\nu$ because $F_\nu(\tau) \sim \tau^{-\nu}\underline{F}_{-\nu}(\tau)$ by equations (35) and (55). Then $\tau^{\frac{\nu}{2}} R_\nu(\tau) \sim \tau^{-\frac{\nu}{2}} T_1^{-\nu}$. Therefore, $\tau^{\frac{\nu}{2}} R_\nu(\tau) \gg 1$ when ν is large. But both $\underline{F}_\nu(\tau)$ and $F_\nu(\tau)$ are continuous with respect to ν; so the foregoing result applies equally to the case of integral ν. Hence, the third term in the radical for $\gamma_\nu(\tau)$ can be neglected for large ν.

Owing to the manner in which γ_ν is defined, corresponding to each ν there is a line $\tau = \tau_0 > \frac{1}{2\beta + 1}$ such that

$\gamma_\nu^2(\tau) \gtrless 0$ when $\tau \lessgtr \tau_0$. As a consequence ζ_1 and ζ_2 are real or complex conjugate according as $\tau \lessgtr \tau_0$. In $0 \leq \tau \leq \tau_0$, $\zeta_1 = 0$ and $\zeta_2 = 0$ will give two 1-parameter families radiating from $(0, -1)$ and $(0,1)$, respectively, and joining together at a point where $\gamma_\nu^2 = 0$. If $0 \leq \tau \leq \tau_0$, the product $\zeta_1\zeta_2$ may be negative or positive according to whether the point lies to the left or the right of the curve $\zeta_1 = 0$ and $\zeta_2 = 0$. On the other hand, if $\tau > \tau_0$, $\zeta_1\zeta_2$ is always positive.

Now $\xi_\nu'^{(1)}(0) = \beta$, while the initial slope of $\zeta_1 = 0$ is $\beta\left(1 - \frac{1}{\nu}\right)$, the integral curve must lie above $\zeta_1 = 0$, and below $\zeta_2 = 0$. If it were not, the integral curve would cross the curve $\zeta_1 = 0$, $\zeta_2 = 0$, where $\xi_\nu'^{(1)}(\tau) = 0$, and $\xi_\nu'^{(1)}(\tau)$ would be negative somewhere in $0 \leq \tau \leq \frac{1}{2\beta + 1}$. This is not possible, for $\xi_\nu^{(1)} \sim \xi_{-\nu}$ by an argument similar to that used for determining the magnitude of $\tau^{\frac{\nu}{2}} R_\nu(\tau)$ and according to (55) $\sqrt{\frac{1 - (2\beta + 1)\tau}{1 - \tau}} > \xi_{-\nu} > -(1 - \tau)^\beta$ in $0 \leq \tau \leq \frac{1}{2\beta + 1}$. Hence $\xi_\nu'^{(1)} > 0$ in $0 \leq \tau \leq \frac{1}{2\beta + 1}$ and $\xi_\nu^{(1)}$ continues to increase until it intersects with $\zeta_1 = 0$. After it crosses the curve $\zeta_1 = 0$, $\xi_\nu'^{(1)} < 0$ and never changes sign as $\zeta_1\zeta_2 > 0$ in $\tau_0 < \tau < 1$. Consequently, $\xi_\nu^{(1)}(\tau)$ is monotonic and decreasing in the interval $\tau_0 < \tau < 1$. When ν is sufficiently large, τ_0 will approach very rapidly to $\frac{1}{2\beta + 1}$ and $\tau_0 = \frac{1}{2\beta + 1}$ when ν becomes infinite.

<u>Proof of theorem (52)</u>.- Form the following identity:

$$X_1(\eta_\nu^{(1)}) \equiv (\eta_\nu'^{(1)} - \xi_\nu'^{(1)}) + (\eta_\nu^{(1)} - \xi_\nu^{(1)})\left[\frac{\beta}{1 - \tau} + \frac{\nu}{2\tau}(\eta_\nu^{(1)} + \xi_\nu^{(1)})\right] + \frac{\nu}{\tau}\xi_0^2(1 - \tau)^{2\beta} e^{-\nu\int_{\tau_0}^{\tau}(\eta_\nu^{(1)} + \xi_\nu^{(1)})\frac{d\tau}{\tau}} \sinh \nu \int_{\tau_0}^{\tau}(\eta_\nu^{(1)} - \xi_\nu^{(1)})\frac{d\tau}{\tau} \gtrless 0 \qquad \text{(A1)}$$

It can be shown that the differential expression possesses an integration factor

$$(\eta_\nu^{(1)} - \xi_\nu^{(1)})\tau^{2\nu}(1 - \tau)^{-2\beta} R_\nu^2 S_\nu^2 \tag{A2}$$

where

$$R_\nu = R_\nu(\tau_0)\exp\left\{\nu\int_{\tau_0}^{\tau}(\xi_\nu^{(1)} - 1)\frac{d\tau}{2\tau}\right\}$$

$$S_\nu = S_\nu(\tau_0)\exp\left\{\nu\int_{\tau_0}^{\tau}(\eta_\nu^{(1)} - 1)\frac{d\tau}{2\tau}\right\}$$

It will be noticed that the sign of (A2) is determined by the first factor $(\eta_\nu^{(1)} - \xi_\nu^{(1)})$ only. On multiplying (A1) by (A2) and integrating the resulting total differential from τ_0 to τ, with a suitably chosen initial value $\eta_\nu^{(1)}(\tau_0) = \xi_\nu^{(1)}(\tau_0)$ it is found that

$$\frac{1}{2}(\eta_\nu^{(1)} - \xi_\nu^{(1)})^2\tau^{2\nu}(1 - \tau)^{-2\beta} R_\nu^2 S_\nu^2 + \xi_0^2 R_\nu^2(\tau_0)S_\nu^2(\tau_0)$$

$$\times\left[\cosh \nu\int_{\tau_0}^{\tau}(\eta_\nu^{(1)} - \xi_\nu^{(1)})\frac{d\tau}{\tau} - 1\right] > 0$$

which is positive if and only if $\eta_\nu^{(1)} - \xi_\nu^{(1)} \gtrless 0$ everywhere in $\tau_0 < \tau < 1$. Since both $\xi_\nu^{(1)}$ and $\eta_\nu^{(1)}$ are continuous and monotonic, the condition is both necessary and sufficient. Furthermore, it should be noticed that the condition $\eta_\nu^{(1)}(\tau_0) = \xi_\nu^{(1)}(\tau_0)$ is purely a convenience. If $\eta_\nu^{(1)}(\tau_0) \neq \xi_\nu^{(1)}(\tau_0)$, the validity of the theorem is not in the least impaired.

APPENDIX B

PROOF OF THEOREM (88)

Consider the first series: Multiplying throughout the inequality (58), namely,

$$\xi_n(\tau) > \sqrt{\frac{1-(2\beta+1)\tau}{1-\tau}}, \qquad 0 < \tau < \frac{1}{2\beta+1}$$

by $\frac{u}{2\tau}$ and integrating both sides from τ to τ_1 shows that

$$\underline{F}_n^{(r)}(\tau) < t_1^n(\tau)$$

where $t_1(\tau) = \frac{T_1(\tau)}{T_1(\tau_1)} \geq 1$. Then it follows that

$$\left| A_n \, \underline{F}_n^{(r)}(\tau) \, w^n \right| < \left| A_n (t_1 w)^n \right|$$

Now $\sum_{n=0}^{\infty} \left| A_n \, (t_1 w)^n \right|$ converges when $|t_1 w| < U$ due to equation (88). By Weirstrass's theorem, the series (92) is uniformly and absolutely convergent if $|t_1 w| = t_1 q < U$. Now $t_1(\tau_1) = 1$; thus $t_1 q$ is equal to U when $q = U$ and $= {}_1$. The term $t_1 q$ is zero if $q = 0$ and remains positive for $0 < q < U$. By the definition of $T_1(\tau)$ given by equation (56), it can be easily shown that

$$\frac{d}{dq} t_1 q > 0$$

for $0 < \tau < \tau_1$. Thus $t_1 q$ increases monotonically from zero to U in the interval $0 \leq \tau \leq \tau_1$. Therefore, the series (92) is uniformly and absolutely convergent in any closed domain in $|w| < U$.

Similarly, the convergence of the series (94) can be established.

APPENDIX C

PROOF OF THEOREM (98)

It is observed that the following identities exist among the constants involved in (98) and (99):

$$B_n U^{\nu} = -\frac{1}{2\nu\pi} \sum_{m=0}^{\infty} A_m U^m \left(\frac{1}{m+\nu} + \frac{1}{m-\nu}\right)(m+\nu)$$

$$C_n U^{-\nu} = \frac{1}{2\nu\pi} \sum_{m=0}^{\infty} A_m U^m \left(\frac{1}{m+\nu} + \frac{1}{m-\nu}\right)(m-\nu)$$

Now, by the inequalities (58) and (59), the functions $\xi_{\nu}(\tau_1)$, $\xi_{-\nu}(\tau_1)$ can be bounded both above and below for all $\nu \neq 0$, when $0 \leq \tau \leq \frac{1}{2\beta+1}$. And if a smaller value of $\Delta(\underline{F}_{\nu}, F_{\nu})$ is taken, it can be deduced that

$$\left| B_n^* \right| \leq M_1 \frac{|B_n|}{\underline{F}_{\nu}(\tau_1)}$$

$$\left| C_n^* \right| \leq M_2 \frac{|C_n|}{\underline{G}_{\nu}(\tau_1)}$$

where M_1 and M_2 are constants independent of n. On the other hand, from the inequality (58)

$$\xi_{\nu}(\tau) < (1-\tau)^{\beta}, \qquad 0 \leq \tau \leq \frac{1}{2\beta+1}$$

it follows that

$$\frac{\underline{F}_\nu(\tau)}{\underline{F}_\nu(\tau_1)} < t_2^\nu(\tau), \qquad \tau_1 \leq \tau \leq \frac{1}{2\beta+1}$$

Consequently, the first part of (101) can be dominated:

$$\left|B_n^* \underline{F}_\nu(\tau)\ w^\nu\right| < \left|B_n(t_2 w)^\nu\right|$$

where $t_2(\tau) = \frac{T_2(\tau)}{T_2(\tau_1)}$. The continuation of this inequality for $\tau > \frac{1}{2\beta+1}$ can be easily done by defining a new $t_2(\tau)$. By hypothesis, $\sum_{n=0}^{\infty} \left|B_n\ (t_2 w)^\nu\right|$ converges if $|t_2 w| < V$. Since $t_2(\tau) \leq t_2(\tau_1)$ for $\tau_1 \leq \tau < 1$, the inequality $|t_2 w| < V$ is uniformly bounded.

Similarly, it can be shown that

$$\left|C_n^* \underline{G}_\nu(\tau)\ w^{-\nu}\right| < \left|C_n\ (t_1 w)^{-\nu}\right|$$

But $\sum_{n=0}^{\infty} \left|C_n\ (t_1 w)^{-\nu}\right|$ converges if $|t_1 w| > U$. Since on $|w| = U$ $t_1(\tau_1) = 1$ and $\frac{d}{dq} \log |t_1 w| > 0$ when $0 < \tau < \frac{1}{2\beta+1}$ or $\frac{d}{dq} |t_1 w| = 0$ when $\frac{1}{2\beta+1} < \tau < 1$, the condition $|t_1 w| > U$ holds for all τ in $\tau_1 \leq \tau < 1$. Hence, by Weierstrass's theorem the series (101) converges uniformly and absolutely in $U + \delta \leq |w| \leq V - \delta$.

TABLES OF THE HYPERGEOMETRIC FUNCTIONS

The values of the hypergeometric functions given in tables 1 to 5 are calculated from power series for $\gamma = 1.405$. The function $\tilde{F}_{-\nu,1}(\tau)$ in table 6 is connected with $\tilde{F}_{\nu}(\tau)$, $\tilde{F}_{-\nu}(\tau)$, and $\tilde{F}_{\nu,1}(\tau)$ through the following equation:

$$\frac{\beta(\nu+1)}{2(\nu-1)}\,\tau\,\tilde{F}_{\nu}(\tau)\,\tilde{F}_{-\nu,1}(\tau)$$

$$= \tilde{F}_{\nu}(\tau)\,\tilde{F}_{-\nu}(\tau) - \frac{\beta(\nu-1)}{2(\nu+1)}\,\tau\,\tilde{F}_{\nu,1}(\tau)\,\tilde{F}_{-\nu}(\tau) - (1-\tau)^{-\beta}$$

This is simply the Wronskian of the two independent integrals of the hypergeometric equation and it holds everywhere except at the singularities $\tau = 0$ and $\tau = 1$. Tables 7 to 12 contain the corresponding approximate functions as indicated.

The numbers in these tables are expressed in terms of appropriate powers of 10. However, a notation was devised in which only the powers are given while the base "10" is omitted. Thus, $3.14159 \times 10^{m} \equiv 3.14159,\ m$. Here m may be either a positive or negative integer, or zero. Unless indicated by the sign † on the heading, accidental errors were detected and eliminated by the difference method.

TABLE 1.- CORRESPONDING PARTICULAR INTEGRALS FOR THE SOLUTIONS OF COMPRESSIBLE FLOW AND INCOMPRESSIBLE FLOW

	Compressible	Incompressible
$\psi(q,\theta)$	$q^{\nu}\ \underline{F}_{\nu}(\tau)\ \begin{matrix}\cos\ \nu\theta\\ \sin\ \nu\theta\end{matrix}$	$q^{\nu}\ \begin{matrix}\cos\ \nu\theta\\ \sin\ \nu\theta\end{matrix}$
	$q^{-\nu}\ \underline{G}_{\nu}(\tau)\ \begin{matrix}\cos\ \nu\theta\\ \sin\ \nu\theta\end{matrix}$	$q^{-\nu}\ \begin{matrix}\cos\ \nu\theta\\ \sin\ \nu\theta\end{matrix}$
	$\int_{\theta} (1-\tau)^{\beta}\ \frac{d\tau}{\tau}$	$\log q$
$\chi(q,\theta)$	$q^{\nu}\ \underline{\tilde{F}}_{\nu}(\tau)\ \begin{matrix}\cos\ \nu\theta\\ \sin\ \nu\theta\end{matrix}$	$q^{\nu}\ \begin{matrix}\cos\ \nu\theta\\ \sin\ \nu\theta\end{matrix}$
	$q^{-\nu}\ \underline{\tilde{G}}_{\nu}(\tau)\ \begin{matrix}\cos\ \nu\theta\\ \sin\ \nu\theta\end{matrix}$	$q^{-\nu}\ \begin{matrix}\cos\ \nu\theta\\ \sin\ \nu\theta\end{matrix}$
	$\int_{\theta} (1-\tau)^{-\beta}\ \frac{d\tau}{\tau}$	$\log q$

The functions $\underline{F}_{\nu}(\tau)$, $q^{-2\nu}\underline{G}_{\nu}(\tau)$ and $\underline{\tilde{F}}_{\nu}(\tau)$, $q^{-2\nu}\underline{\tilde{G}}_{\nu}(\tau)$ are respectively the two independent integrals of equations (27) and (28).

TABLE 14

τ	λ	M	τ	M	τ	M
0	∞	0	0.17	1.0057	0.28	1.3858
.02	2.2554	.10078	.18	1.0412	.29	1.4202
.04	1.6376	.20576	.19	1.0763	.30	1.4548
.06	1.3751	.31521	.20	1.1111	.32	1.5244
.08	1.2267	.42941	.21	1.1457	.34	1.5950
.10	1.1322	.54870	.22	1.1802	.36	1.6667
.12	1.0697	.67340	.23	1.2145	.38	1.7398
.14	1.0283	.80391	.24	1.2498	.40	1.8140
.15	1.0141		.25	1.2830	.42	1.8910
.16	1.0041	.94062	.26	1.3172	.44	1.9698
.165	1.0011		.27	1.3515	.46	2.0510

TABLE 2

τ	$F_{1/2}(\tau)$	$F_{3/2}(\tau)$	$F_{5/2}(\tau)$	$F_{7/2}(\tau)$	$F_{9/2}(\tau)$	$F_{11/2}(\tau)$	$F_{13/2}(\tau)$	$F_{15/2}(\tau)$	$F_{17/2}(\tau)$	$F_{19/2}(\tau)$	$F_{21/2}(\tau)$
0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, -0	1.00000, 0
.10	9.40592, -1	8.26748, -1	7.23508, -1	6.31726, -1	5.50840, -1	4.79886, -1	4.17817, -1	3.63614, -1	3.16339, -1	2.75137, -1	2.39625, -1
.12	9.29261, -1	7.94894, -1	6.75536, -1	5.72109, -1	4.83487, -1	4.08015, -1	3.43978, -1	2.89778, -1	2.43980, -1	2.06332, -1	1.72739, -1
.14	9.18112, -1	7.63945, -1	6.29860, -1	5.16693, -1	4.22613, -1	3.44751, -1	2.80876, -1	2.28561, -1	1.85821, -1	1.50967, -1	1.22578, -1
.15	9.12605, -1	7.48807, -1	6.07873, -1	4.90498, -1	3.94290, -1	3.16120, -1	2.52960, -1	2.02124, -1	1.61522, -1	1.28541, -1	1.02496, -1
.16	9.07143, -1	7.33892, -1	5.85410, -1	4.65282, -1	3.67498, -1	2.89341, -1	2.27271, -1	1.78195, -1	1.39515, -1	1.09110, -1	8.52437, -2
.165	9.04429, -1	7.26616, -1	5.75910, -1	4.53033, -1	3.54621, -1	2.76621, -1	2.15219, -1	1.67109, -1	1.29559, -1	1.00303, -1	7.75697, -2
.17	9.01726, -1	7.19196, -1	5.65503, -1	4.41018, -1	3.42087, -1	2.64333, -1	2.03672, -1	1.56581, -1	1.20163, -1	9.20855, -2	7.04764, -2
.175	8.99036, -1	7.11930, -1	5.55249, -1	4.29336, -1	3.29881, -1	2.52488, -1	1.92811, -1	1.46565, -1	1.13335, -1	8.44275, -2	6.39286, -2
.18	8.96365, -1	7.04719, -1	5.45130, -1	4.17682, -1	3.18004, -1	2.41012, -1	1.82027, -1	1.37102, -1	1.03035, -1	7.72969, -2	5.78904, -2
.185	8.93684, -1	6.97563, -1	5.35140, -1	4.06356, -1	3.06448, -1	2.29956, -1	1.71898, -1	1.28111, -1	9.52408, -2	7.06636 -2	5.25290, -2
.19	8.91217, -1	6.90461, -1	5.25281, -1	3.95253, -1	2.95208, -1	2.19288, -1	1.62213, -1	1.19592, -1	8.79285, -2	6.44991, -2	4.72128, -2
.195	8.88380, -1	6.83411, -1	5.15553, -1	3.84370, -1	2.84274, -1	2.09005, -1	1.52955, -1	1.11523, -1	8.10647, -2	5.96499, -2	4.25122, -2
.20	8.85743, -1	6.76417, -1	5.05952, -1	3.73707, -1	2.73642, -1	1.99085, -1	1.44109, -1	1.03890, -1	7.46368, -2	5.34693, -2	3.81990, -2
.21.	8.80715, -1	6.62589, -1	4.87834, -1	3.53020, -1	2.53258, -1	1.80315, -1	1.27601, -1	8.98454, -2	6.29905, -2	4.40049, -2	3.06307, -2
.22	8.75312, -1	6.48974, -1	4.68817, -1	3.33172, -1	2.34026, -1	1.62911, -1	1.12582, -1	7.73225, -2	5.28229, -2	3.69219, -2	2.43151, -2
.23	8.70161, -1	6.35571, -1	4.50995, -1	3.14141, -1	2.15882, -1	1.46889, -1	1.08552, -1	6.61895, -2	4.39613, -2	2.90544, -2	1.90756, -2
.24	8.65064, -1	6.22379, -1	4.33862, -1	2.95901, -1	1.98792, -1	1.31893, -1	8.85939, -2	5.63245, -2	3.63254, -2	2.32524, -2	1.47590, -2
.25	8.59991, -1	6.09398, -1	4.16807, -1	2.78456, -1	1.82708, -1	1.18145, -1	7.54363, -2	4.76140, -2	2.97302, -2	1.83780, -2	1.12807, -2
.26	8.54973, -1	5.96618, -1	4.00425, -1	2.61721, -1	1.67591, -1	1.06460, -1	6.83835, -2	3.99519, -2	2.40758, -2	1.43192, -2	8.58040, -3
.27	8.49998, -1	5.84048, -1	3.84507, -1	2.45734, -1	1.53401, -1	9.58418, -2	5.83524, -2	3.32399, -2	1.92562, -2	1.09583, -2	6.09334, -3
.28.	8.45066, -1	5.71682, -1	3.69046, -1	2.30457, -1	1.40094, -1	8.31600, -2	4.82637, -2	2.73664, -2	1.51785, -2	8.20153, -3	4.28202, -3
.29	8.40177, -1	5.59519, -1	3.54034, -1	2.15889, -1	1.27838, -1	7.33811, -2	4.10444, -2	2.23067, -2	1.17406, -2	5.96281, -3	2.88731, -3
.30	8.35332, -1	5.47857, -1	3.39463, -1	2.01951, -1	1.15989, -1	6.44493, -2	3.46231, -2	1.79223, -2	8.87599, -3	4.16879, -3	1.78066, -3
.32	8.25771, -1	5.24232, -1	3.11619, -1	1.76035, -1	9.49754, -2	4.89140, -2	2.39184, -2	1.09649, -2	4.86947, -3	1.63491, -3	3.85966, -4
.34	8.16381, -1	5.01692, -1	2.85451, -1	1.52558, -1	7.67739, -2	3.61899, -2	1.56592, -2	5.97068, -3	1.75636, -3	1.61758, -4	-3.44849, -4
.36	8.07162, -1	4.79926, -1	2.60899, -1	1.31370, -1	6.11184, -2	2.58187, -2	9.43050, -3	2.53632, -3	3.65867, -5	-5.94075, -4	-8.15158, -4
.38	7.98111, -1	4.58923, -1	2.37908, -1	1.12320, -1	4.77856, -2	1.75508, -2	4.65885, -3	2.90912, -4	-9.08482, -4	-8.86334, -4	-8.55445, -4
.40	7.89228, -1	4.38870, -1	2.16418, -1	9.52880, -2	3.64602, -2	1.10841, -2	1.62280, -3	-1.06371, -3	-1.31917, -3	-8.99307, -4	-5.31185, -4
.42	7.80512, -1	4.19814, -1	1.96359, -1	8.00764, -2	2.70013, -2	5.06927, -3	-5.54379, -4	-1.77029, -3	-1.38933, -3	-7.52517, -4	-3.82053, -4
.44	7.71962, -1	4.00862, -1	1.77706, -1	6.66119, -2	1.91178, -2	2.36621, -3	-1.90872, -3	-2.02354, -3	-1.25658, -3	-5.63157, -4	-2.54660, -4
.46	7.63577, -1	3.82284, -1	1.60388, -1	5.47462, -2	1.27938, -2	1.30069, -4	-2.53949, -3	-1.[illegible]7681, -3	-1.02127, -3	-3.56237, -4	-1.12718, -4
.48	7.55355, -1	3.64899, -1	1.44304, -1	4.43664, -2	7.87304, -3	-2.12548, -3	-2.90284, -3	-1.74815, -3	-7.82660, -4	-1.75337, -4	-2.46137, -5
.50	7.47298, -1	3.48202, -1	1.29454, -1	3.53204, -2	3.65270, -3	-4.82362, -3	-2.86456, -3	-1.42591, -3	-4.95162, -4	-2.923 ?, -5	+2.98827, -5

TABLE 3

τ	$F_{-1/2}(\tau)$	$F_{-3/2}(\tau)$	$F_{-5/2}(\tau)$	$F_{-7/2}(\tau)$	$F_{-9/2}(\tau)$	$F_{-11/2}(\tau)$	† $F_{-13/2}(\tau)$	† $F_{-15/2}(\tau)$	† $F_{-17/2}(\tau)$	† $F_{-19/2}(\tau)$	† $F_{-21/2}(\tau)$
0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0
.10	1.06839, 0	1.12055, 0	1.41771, 0	1.72608, 0	2.06119, 0	2.40570, 0	2.79348, 0	3.23017, 0	3.72368, 0	4.28471, 0	4.92493, 0
.12	1.06928, 0	1.13121, 0	1.49211, 0	1.90884, 0	2.38596, 0	2.93835, 0	3.56362, 0	4.33940, 0	5.22992, 0	6.28099, 0	7.52351, 0
.14	1.07993, 0	1.13814, 0	1.55339, 0	2.07768, 0	2.72458, 0	3.52556, 0	4.51952, 0	5.75483, 0	7.28997, 0	9.19794, 0	1.15685, 1
.15	1.08516, 0	1.14030, 0	1.57887, 0	2.15367, 0	2.88480, 0	3.81970, 0	5.01470, 0	6.54253, 0	8.49624, 0	1.09896, 1	1.41735, 1
.16	1.09032, 0	1.14163, 0	1.59998, 0	2.21820, 0	3.03264, 0	4.10061, 0	5.50301, 0	7.34428, 0	9.76040, 0	1.29287, 1	1.70796, 1
.165	1.09289, 0	1.14201, 0	1.60869, 0	2.24885, 0	3.10039, 0	4.23237, 0	5.73721, 0	7.73704, 0	1.03952, 1	1.39181, 1	1.85944, 1
.17	1.09543, 0	1.14219, 0	1.61875, 0	2.27577, 0	3.16514, 0	4.35828, 0	5.96026, 0	8.11576, 0	1.10104, 1	1.48961, 1	2.01070, 1
.175	1.09796, 0	1.14216, 0	1.62348, 0	2.29984, 0	3.23029, 0	4.47061, 0	6.16844, 0	8.47296, 0	1.15990, 1	1.58567, 1	2.15772, 1
.18	1.10048, 0	1.14199, 0	1.62910, 0	2.32087, 0	3.27119, 0	4.57366, 0	6.35782, 0	8.80060, 0	1.21425, 1	1.67116, 1	2.29839, 1
.185	1.10298, 0	1.14182, 0	1.63355, 0	2.33878, 0	3.31524, 0	4.66569, 0	6.52426, 0	9.08955, 0	1.26235, 1	1.74878, 1	2.41773, 1
.19	1.10546, 0	1.14108, 0	1.63687, 0	2.35325, 0	3.35182, 0	4.73892, 0	6.66343, 0	9.33065, 0	1.30229, 1	1.81289, 1	2.51778, 1
.195	1.10793, 0	1.14035, 0	1.63905, 0	2.36430, 0	3.38032, 0	4.79756, 0	6.77089, 0	9.51383, 0	1.33197, 1	1.85906, 1	2.58755, 1
.20	1.11039, 0	1.13949, 0	1.64000, 0	2.37172, 0	3.40014, 0	4.83784, 0	6.84198, 0	9.62282, 0	1.34816, 1	1.88307, 1	2.61858, 1
.21	1.11525, 0	1.13724, 0	1.63842, 0	2.37525, 0	3.41159, 0	4.85625, 0	6.85711, 0	9.60928, 0	1.33653, 1	1.84450, 1	2.52419, 1
.22	1.12006, 0	1.13439, 0	1.63210, 0	2.36293, 0	3.38194, 0	4.78045, 0	6.67156, 0	9.15810, 0	1.24449, 1	1.65504, 1	2.14954, 1
.23	1.12481, 0	1.13095, 0	1.62166, 0	2.33411, 0	3.30745, 0	4.59764, 0	6.24967, 0	8.28238, 0	1.05258, 1	1.27029, 1	1.40274, 1
.24	1.12950, 0	1.12695, 0	1.60636, 0	2.28816, 0	3.18493, 0	4.29634, 0	5.55798, 0	6.78110, 0	7.40995, 0	6.46595, 0	1.90668, 0
.25	1.13414, 0	1.12244, 0	1.58603, 0	2.22477, 0	3.01181, 0	3.86685, 0	4.56762, 0	4.60497, 0	2.91933, 0	-2.55113, 0	-1.58729, 1
.26	1.13871, 0	1.11743, 0	1.56018, 0	2.14373, 0	2.78819, 0	3.30152, 0	3.25817, 0	1.73352, 0	-3.08469, 0	-1.46516, 1	-3.93195, 1
.27	1.14323, 0	1.11196, 0	1.53092, 0	2.04504, 0	2.50693, 0	2.59526, 0	1.50707, 0	-1.89268, 0	-1.08908, 1	-3.00024, 1	-6.93167, 1
.28	1.14770, 0	1.10605, 0	1.49738, 0	1.92884, 0	2.17366, 0	1.74570, 0	-3.86533, -1	-6.23854, 0	-1.99169, 1	-4.85917, 1	-1.05468, 2
.29	1.15211, 0	1.09973, 0	1.45973, 0	1.79545, 0	1.78883, 0	8.53485, -1	-3.82193, 0	-1.14395, 1	-3.08982, 1	-7.01848, 1	-1.47011, 2
.30	1.15646, 0	1.09304, 0	1.41812, 0	1.64535, 0	1.34769, 0	-3.77564, -1	-6.38666, 0	-1.72970, 1	-4.28751, 1	-9.42783, 1	-1.92579, 2
.32	1.16500, 0	1.07861, 0	1.32381, 0	1.29780, 0	3.21587, -1	-3.02435, 0	-1.15868, 1	-3.07618, 1	-7.02522, 1	-1.46595, 2	-2.56312, 2
.34	1.17333, 0	1.06296, 0	1.21609, 0	8.93074, -4	-8.78444, -1	-8.09300, 0	-1.86550, 1	-4.65705, 1	-9.88607, 1	-1.96922, 2	-3.63061, 2
.36	1.18145, 0	1.04630, 0	1.09682, 0	4.40208, -1	-2.21429, 0	-9.48146, 0	-2.80688, 1	-6.00882, 1	-1.23174, 2	-2.28798, 2	-3.82623, 2
.38	1.18935, 0	1.02861, 0	9.67976, -1	-5.00266, -2	-3.63755, 0	-1.29941, 1	-3.32151, 1	-7.21257, 1	-1.37846, 2	-2.29905, 2	-3.15371, 2
.40	1.19706, 0	1.01064, 0	8.31623, -1	-5.65487, -1	-5.09175, 0	-1.83038, 1	-3.91243, 1	-7.98883, 1	-1.36028, 2	-1.85609, 2	-1.32867, 2
.42	1.20456, 0	9.91968, -1	6.89672, -1	-1.09309, 0	-6.51410, 0	-1.91385, 1	-4.32731, 1	-7.99367, 1	-1.14410, 2	-8.63367, 1	1.70387, 2
.44	1.21187, 0	9.72948, -1	5.44847, -1	-1.61923, 0	-7.84257, 0	-2.14584, 1	-4.48151, 1	-7.17273, 1	-6.87391, 1	46.63417, 1	5.67107, 2
.46	1.21899, 0	9.53707, -1	3.96639, -1	-2.15027, 0	-9.01412, 0	-2.29447, 1	-4.31290, 1	-5.39506, 1	4.00001, -1	2.59677, 2	9.94177, 2
.48	1.22591, 0	9.34406, -1	2.53286, -1	-2.61293, 0	-9.96814, 0	-2.34099, 1	-3.79201, 1	-2.67117, 1	8.74701, 1	4.67461, 2	1.36427, 3
.50	1.23268, 0	9.15143, -1	1.10727, -1	-3.05472, 0	-1.06545, 1	-2.27261, 1	-2.91157, 1	-8.7286 , 0	1.83516, 2	6.52931, 2	1.53557, 3

TABLE 4

τ	$\tilde{F}_{1/2}(\tau)$	$\tilde{F}_{3/2}(\tau)$	$\tilde{F}_{5/2}(\tau)$	$\tilde{F}_{7/2}(\tau)$	$\tilde{F}_{9/2}(\tau)$	$\tilde{F}_{11/2}(\tau)$	$\tilde{F}_{13/2}(\tau)$	$\tilde{F}_{15/2}(\tau)$	$\tilde{F}_{17/2}(\tau)$	$\tilde{F}_{19/2}(\tau)$	$\tilde{F}_{21/2}(\tau)$
0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0
.10	1.02253, 0	9.60632, -1	8.65836, -1	7.68544, -1	6.77456, -1	5.94750, -1	5.20766, -1	4.55218, -1	3.97428, -1	3.46662, -1	3.02175, -1
.12	1.02756, 0	9.52135, -1	8.38224, -1	7.24388, -1	6.20321, -1	5.28408, -1	4.48602, -1	3.79976, -1	3.21518, -1	2.71383, -1	2.28994, -1
.14	1.03281, 0	9.43406, -1	8.10618, -1	6.80974, -1	5.65617, -1	4.66675, -1	3.83576, -1	3.14005, -1	2.56617, -1	2.09374, -1	1.70610, -1
.15	1.03551, 0	9.38948, -1	7.96735, -1	6.59551, -1	5.39173, -1	4.37484, -1	3.53240, -1	2.84246, -1	2.28146, -1	1.82770, -1	1.46199, -1
.16	1.03827, 0	9.34427, -1	7.82803, -1	6.38317, -1	5.15332, -1	4.19773, -1	3.24641, -1	2.56517, -1	2.02045, -1	1.58815, -1	1.24604, -1
.165	1.03987, 0	9.32141, -1	7.75816, -1	6.27773, -1	5.00658, -1	3.95745, -1	3.10998, -1	2.43385, -1	1.89850, -1	1.47768, -1	1.14781, -1
.17	1.04109, 0	9.29839, -1	7.68817, -1	6.17278, -1	4.88093, -1	3.82370, -1	2.97682, -1	2.30725, -1	1.78196, -1	1.37309, -1	1.05568, -1
.175	1.04252, 0	9.27520, -1	7.61804, -1	6.06250, -1	4.75699, -1	3.69258, -1	2.84739, -1	2.18627, -1	1.67069, -1	1.27415, -1	9.69383, -2
.18	1.04397, 0	9.25183, -1	7.54777, -1	5.96432, -1	4.63455, -1	3.56574, -1	2.72164, -1	2.06781, -1	1.56450, -1	1.18063, -1	8.88860, -2
.185	1.04543, 0	9.22829, -1	7.47738, -1	5.86063, -1	4.51360, -1	3.45822, -1	2.59949, -1	1.96475, -1	1.46323, -1	1.09232, -1	8.15007, -2
.19	1.04691, 0	9.20457, -1	7.40681, -1	5.75784, -1	4.39418, -1	3.31491, -1	2.48089, -1	1.84898, -1	1.36673, -1	1.00900, -1	7.42442, -2
.195	1.04841, 0	9.18068, -1	7.33611, -1	5.65635, -1	4.27619, -1	3.19416, -1	2.36579, -1	1.74140, -1	1.27463, -1	9.30451, -2	6.76615, -2
.20	1.04992, 0	9.15669, -1	7.26528, -1	5.55335, -1	4.15972, -1	3.07595, -1	2.26412, -1	1.64091, -1	1.18737, -1	8.56588, -2	6.15277, -2
.21	1.05230, 0	9.10786, -1	7.12317, -1	5.35067, -1	3.93124, -1	2.84726, -1	2.04085, -1	1.46177, -1	1.02521, -1	7.21522, -2	5.05188, -2
.22	1.05514, 0	9.05835, -1	6.98047, -1	5.15041, -1	3.70870, -1	2.62803, -1	1.84065, -1	1.27778, -1	8.79060, -2	6.02599, -2	4.10322, -2
.23	1.05936, 0	9.00806, -1	6.83715, -1	4.95203, -1	3.49206, -1	2.41870, -1	1.66304, -1	1.11613, -1	7.47778, -2	4.98307, -2	3.29142, -2
.24	1.06288, 0	8.95694, -1	6.69322, -1	4.75571, -1	3.28136, -1	2.21889, -1	1.47759, -1	9.72075, -2	6.30501, -2	4.07323, -2	2.60114, -2
.25	1.06604, 0	8.90496, -1	6.54864, -1	4.56150, -1	3.07652, -1	2.02841, -1	1.31386, -1	8.38876, -2	5.25569, -2	3.28409, -2	2.01863, -2
.26	1.06950, 0	8.85210, -1	6.40342, -1	4.36942, -1	2.87755, -1	1.84708, -1	1.16140, -1	7.17813, -2	4.32651, -2	2.60403, -2	1.53124, -2
.27	1.07306, 0	8.79833, -1	6.25752, -1	4.17949, -1	2.68443, -1	1.67473, -1	1.01979, -1	6.08159, -2	3.50638, -2	2.02219, -2	1.12740, -2
.28	1.07669, 0	8.74362, -1	6.11079, -1	3.99174, -1	2.49714, -1	1.51115, -1	8.88898, -2	5.09333, -2	2.78347, -2	1.52852, -2	7.96572, -3
.29	1.08042, 0	8.68792, -1	5.96314, -1	3.80521, -1	2.31565, -1	1.35618, -1	7.67415, -2	4.20688, -2	2.15215, -2	1.11358, -2	5.29181, -3
.30	1.08425, 0	8.63120, -1	5.81562, -1	3.62289, -1	2.13997, -1	1.20963, -1	6.66817, -2	3.41321, -2	1.60320, -2	7.68655, -3	3.16544, -3
.32	1.09221, 0	8.51457, -1	5.51734, -1	3.24310, -1	1.80688, -1	9.41032, -2	4.59760, -2	2.06796, -2	7.21412, -3	2.57256, -3	2.49754, -4
.34	1.10063, 0	8.39338, -1	5.21591, -1	2.91282, -1	1.49470, -1	7.03861, -2	2.97227, -2	1.07105, -2	7.98990, -4	-6.29653, -4	-1.31670, -3
.36	1.10964, 0	8.26726, -1	4.91116, -1	2.87170, -1	1.20628, -1	4.96611, -2	1.65104, -2	3.19256, -3	-3.73923, -3	-2.40270, -3	-1.95931, -3
.38	1.11898, 0	8.13792, -1	4.60285, -1	2.24064, -1	9.40421, -2	3.17771, -2	6.03759, -3	-2.07404, -3	-6.87587, -3	-3.14716, -3	-2.00667, -3
.40	1.12902, 0	7.99850, -1	4.29064, -1	1.81964, -1	6.96946, -2	1.65772, -2	-1.98761, -3	-5.46776, -3	-9.02991, -3	-3.19032, -3	-1.71568, -3
.42	1.13972, 0	7.85822, -1	3.97480, -1	1.60943, -1	4.75856, -2	3.90922, -3	-7.84725, -3	-7.29513, -3	-1.05861, -2	-2.79384, -3	-1.26203, -3
.44	1.15114, 0	7.70425, -1	3.65450, -1	1.30978, -1	2.76346, -2	-6.38482, -3	-1.18131, -2	-7.89179, -3	-1.18892, -2	-2.16146, -3	-7.74904, -4
.46	1.16338, 0	7.54598, -1	3.32955, -1	1.02146, -1	9.87962, -3	-1.44637, -2	-1.41465, -2	-7.52273, -3	-1.32383, -2	-1.44623, -3	-3.36841, -4
.48	1.17653, 0	7.37924, -1	2.99958, -1	7.44810, -2	-5.72230, -3	-2.36344, -2	-1.60977, -2	-6.43288, -3	-1.49067, -2	-7.57447, -4	+1.00311, -5
.50	1.19071, 0	7.20312, -1	2.66446, -1	4.80326, -2	-1.91957, -2	-2.48214, -2	-1.49069, -2	-4.83756, -3	-1.71471, -2	-1.67292, -4	2.43691, -4

TABLE 5

τ	$\bar{F}_{-1/2}(\tau)$	$\bar{F}_{-3/2}(\tau)$	$\bar{F}_{-5/2}(\tau)$	$\bar{F}_{-7/2}(\tau)$	$\bar{F}_{-9/2}(\tau)$	$\bar{F}_{-11/2}(\tau)$	$\bar{F}_{-13/2}(\tau)$	$\bar{F}_{-15/2}(\tau)$	$\bar{F}_{-17/2}(\tau)$	$\bar{F}_{-19/2}(\tau)$	$\bar{F}_{-21/2}(\tau)$
0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0
.10	8.00892, -1	1.67186, 0	2.11694, 0	2.51348, 0	2.92885, 0	3.37877, 0	3.87757, 0	4.44205, 0	5.08274, 0	5.81560, 0	6.64972, 0
.12	7.57300, -1	1.74279, 0	2.56652, 0	2.95210, 0	3.42667, 0	4.40740, 0	5.31493, 0	6.37910, 0	7.63149, 0	9.10875, 0	1.06544, 1
.14	7.12257, -1	1.79153, 0	2.54679, 0	3.38068, 0	4.37911, 0	5.60413, 0	7.11764, 0	8.99243, 0	1.13169, 1	1.41997, 1	1.77743, 1
.15	6.89180, -1	1.80740, 0	2.63522, 0	3.60150, 0	4.74977, 0	6.22984, 0	8.11157, 0	1.05092, 1	1.35651, 1	1.74591, 1	2.24192, 1
.16	6.66865, -1	1.81752, 0	2.70976, 0	3.76074, 0	5.10184, 0	6.84454, 0	9.11876, 0	1.20984, 1	1.59928, 1	2.11094, 1	2.67621, 1
.165	6.53760, -1	1.82041, 0	2.74209, 0	3.84332, 0	5.26676, 0	7.72587, 0	9.61556, 0	1.28942, 1	1.72361, 1	2.29849, 1	3.06882, 1
.17	6.41751, -1	1.82182, 0	2.77080, 0	3.91985, 0	5.42181, 0	7.42084, 0	1.00938, 1	1.38735, 1	1.84660, 1	2.48836, 1	3.34724, 1
.175	6.29536, -1	1.82177, 0	2.79800, 0	3.98868, 0	5.56512, 0	7.68458, 0	1.05469, 1	1.44183, 1	1.98554, 1	2.67380, 1	3.63126, 1
.18	6.17410, -1	1.82024, 0	2.81720, 0	4.05005, 0	5.69471, 0	7.92551, 0	1.09652, 1	1.51121, 1	2.07705, 1	2.84888, 1	3.90136, 1
.185	6.050740 -1	1.81720, 0	2.83484, 0	4.10291, 0	5.80852, 0	8.13935, 0	1.13384, 1	1.57337, 1	2.17724, 1	3.00655, 1	4.14478, 1
.19	5.92625, -1	1.81267, 0	2.84724, 0	4.14655, 0	5.90445, 0	8.32074, 0	1.16552, 1	1.62601, 1	2.28169, 1	3.13854, 1	4.34694, 1
.195	5.80060, -1	1.80662, 0	2.85574, 0	4.18023, 0	5.98032, 0	8.46431, 0	1.19056, 1	1.66660, 1	2.32540, 1	3.23640, 1	4.49029, 1
.20	5.67378, -1	1.79904, 0	2.85955, 0	4.20322, 0	6.03394, 0	8.56441, 0	1.20706, 1	1.69241, 1	2.35284, 1	3.28658, 1	4.56460, 1
.21	5.41653, -1	1.77928, 0	2.85309, 0	4.21425, 0	6.06534, 0	8.61069, 0	1.21056, 1	1.68774, 1	2.33425, 1	3.20172, 1	4.35196, 1
.22	5.15436, -1	1.75350, 0	2.82624, 0	4.17383, 0	5.98032, 0	8.41095, 0	1.16441, 1	1.58525, 1	2.12176, 1	2.77520, 1	3.52512, 1
.23	4.88698, -1	1.72100, 0	2.77789, 0	4.07623, 0	5.76031, 0	7.91461, 0	1.05626, 1	1.38985, 1	1.68490, 1	1.95177, 1	1.82495, 1
.24	4.61429, -1	1.68229, 0	2.70681, 0	3.91586, 0	5.38685, 0	7.07039, 0	8.73318, 0	9.79024, 1	8.99249, 0	5.87423, 0	-1.02029, 1
.25	4.33605, -1	1.63707, 0	2.61182, 0	3.68733, 0	4.84185, 0	5.82752, 0	6.02826, 0	4.14123, 0	-2.40221, 0	-1.84421, 1	-5.28308, 1
.26	4.05202, -1	1.58625, 0	2.49182, 0	3.38543, 0	4.10792, 0	4.13714, 0	2.32552, 0	-3.63181, 0	-1.81466, 1	-4.93864, 1	-1.12090, 2
.27	3.76200, -1	1.52671, 0	2.34574, 0	3.00628, 0	3.16880, 0	1.95394, 0	-2.48709, 0	-1.57802, 1	-3.87854, 1	-8.99703, 1	-1.89792, 2
.28	3.45871, -1	1.46134, 0	2.17255, 0	2.54232, 0	2.00960, 0	-7.82541, -1	-5.50472, 0	-2.85047, 1	-6.56340, 1	-1.40799, 2	-2.87866, 2
.29	3.16291, -1	1.38903, 0	1.97129, 0	1.99236, 0	6.17218, -1	-4.04621, 0	-1.68005, 1	-4.29331, 1	-9.59151, 1	-2.01957, 2	-4.02893, 2
.30	2.85332, -1	1.30984, 0	1.74105, 0	1.35168, 0	-1.01931, 0	-7.91965, 0	-2.44135, 1	-5.00989, 1	-1.32492, 2	-2.72472, 2	-5.32552, 2
.32	2.21268, -1	1.12915, 0	1.19021, 0	-2.14314, -1	-5.05571, 0	-1.74900, 1	-4.55801, 1	-1.04167, 2	-2.19261, 2	-4.34121, 2	-9.16299, 2
.34	1.54090, -1	9.18760, -1	5.13902, -1	-2.17479, 0	-1.01306, 1	-2.94366, 1	-7.14952, 1	-1.56252, 2	-3.16041, 2	-5.97782, 2	-1.05987, 3
.36	8.35413, -2	6.77247, -1	-2.95006, -1	-4.63649, 0	-1.62171, 1	-4.34790, 1	-1.00772, 2	-2.11116, 2	-4.15818, 2	-7.14925, 2	-1.13299, 3
.38	9.28131, -3	4.03256, -1	-1.23454, 0	-7.30206, 0	-2.32206, 1	-5.94545, 1	-1.31005, 2	-2.60430, 2	-4.63663, 2	-7.18266, 2	-8.75530, 2
.40	-6.90636, -2	9.62728, -2	-2.31348, 0	-1.04453, 1	-3.09676, 1	-7.53560, 1	-1.59694, 2	-2.95013, 2	-4.57159, 2	-6.30341, 2	-1.23821, 2
.42	-1.51922, -1	-2.48582, -1	-3.53126, 0	-1.39444, 1	-3.91986, 1	-8.08802, 1	-1.79336, 2	-2.94789, 2	-3.63661, 2	-7.81747, 1	1.23152, 3
.44	-2.39785, -1	-6.29646, -1	-4.68782, 0	-1.77401, 1	-4.75605, 1	-1.04238, 2	-1.87549, 2	-2.51383, 2	-1.19912, 2	+6.79217, 2	3.15928, 3
.46	-3.33223, -1	-1.05064, 0	-6.38142, 0	-2.17661, 1	-5.56041, 1	-1.13588, 2	-1.77439, 2	-1.49208, 2	+2.63399, 2	1.72553, 3	5.42030, 3
.48	-4.32896, -1	-1.51424, 0	-8.00640, 0	-2.59318, 1	-6.31298, 1	-1.16761, 2	-1.43178, 2	+2.21192, 1	7.93197, 2	2.95775, 3	7.50837, 3
.50	-5.39675, -1	-2.02198, 0	-9.76294, 0	-3.01257, 1	-6.84533, 1	-1.11466, 2	-7.95316, 1	2.65768, 2	2.43521, 3	4.16443, 3	8.65307, 3

TABLE 6

τ	$\tilde{F}_{1/2,1}(\tau)$	$\tilde{F}_{3/2,1}(\tau)$	$\tilde{F}_{5/2,1}(\tau)$	$\tilde{F}_{7/2,1}(\tau)$	$\tilde{F}_{9/2,1}(\tau)$	$\tilde{F}_{11/2,1}(\tau)$	$\tilde{F}_{13/2,1}(\tau)$	$\tilde{F}_{15/2,1}(\tau)$	$\tilde{F}_{17/2,1}(\tau)$	$\tilde{F}_{19/2,1}(\tau)$	$\tilde{F}_{21/2,1}(\tau)$
0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0
.10	1.20005, 0	1.13180, 0	1.03256, 0	9.27334, -1	8.25307, -1	7.30312, -1	6.43762, -1	5.65890, -1	4.96434, -1	4.34827, -1	3.80406, -1
.12	1.24830, 0	1.16243, 0	1.05984, 0	9.12019, -1	7.90857, -1	6.80772, -1	5.83106, -1	4.97660, -1	4.23697, -1	3.59815, -1	3.05158, -1
.14	1.29993, 0	1.19507, 0	1.04741, 0	8.96412, -1	7.58534, -1	6.32718, -1	5.25848, -1	4.35030, -1	3.58652, -1	2.94889, -1	2.41942, -1
.15	1.32712, 0	1.21211, 0	1.05151, 0	8.88495, -1	7.39446, -1	6.09252, -1	4.98469, -1	4.06736, -1	3.28963, -1	2.65901, -1	2.14400, -1
.16	1.35528, 0	1.22967, 0	1.05530, 0	8.80499, -1	7.22402, -1	5.86159, -1	4.71913, -1	3.77747, -1	3.01038, -1	2.39071, -1	1.89326, -1
.166	1.36973, 0	1.23866, 0	1.05733, 0	8.76471, -1	7.13896, -1	5.74754, -1	4,58940, -1	3.64223, -1	2.87715, -1	2.28429, -1	1.77663, -1
.17	1.38445, 0	1.24777, 0	1.05937, 0	8.72422, -1	7.05402, -1	5.63443, -1	4.46171, -1	3.51033, -1	2.74809, -1	2.14288, -1	1.66555, -1
.176	1.39944, 0	1.25702, 0	1.06144, 0	8.68363, -1	6.96919, -1	5.52226, -1	4.33802, -1	3.38145, -1	2.62310, -1	2.02625, -1	1.56963, -1
.18	1.41470, 0	1.26642, 0	1.06353, 0	8.64262, -1	6.88447, -1	5.41104, -1	4.21256, -1	3.25563, -1	2.50211, -1	1.91436, -1	1.46927, -1
.186	1.43024, 0	1.27597, 0	1.06565, 0	8.60160, -1	6.79987, -1	5.30077, -1	4.09066, -1	3.13285, -1	2.38503, -1	1.80703, -1	1.36370, -1
.19	1.44607, 0	1.28567, 0	1.06778, 0	8.56017, -1	6.71858, -1	5.19144, -1	3.97096, -1	3.01305, -1	2.27179, -1	1.70414, -1	1.27292, -1
.196	1.46220, 0	1.29582, 0	1.06994, 0	8.51861, -1	6.63101, -1	5.08306, -1	3.85323, -1	2.89622, -1	2.18230, -1	1.60687, -1	1.18677, -1
.20	1.47863, 0	1.30555, 0	1.07215, 0	8.47684, -1	6.54675, -1	4.97563, -1	3.73747, -1	2.78230, -1	2.09650, -1	1.51116, -1	1.10607, -1
.21	1.51243, 0	1.32603, 0	1.07668, 0	8.39260, -1	6.37859, -1	4.76353, -1	3.51178, -1	2.56305, -1	1.84102, -1	1.33460, -1	9.54330, -2
.22	1.54755, 0	1.34720, 0	1.08112, 0	8.30744, -1	6.21092, -1	4.55546, -1	3.29381, -1	2.35802, -1	1.66860, -1	1.17311, -1	8.19421, -2
.23	1.58405, 0	1.36909, 0	1.08578, 0	8.22132, -1	6.04373, -1	4.35112, -1	3.08346, -1	2.15789, -1	1.49469, -1	1.02612, -1	6.99108, -2
.24	1.62200, 0	1.39171, 0	1.09055, 0	8.13421, -1	5.87704, -1	4.15063, -1	2.88067, -1	1.97136, -1	1.33329, -1	8.92618, -2	5.92227, -2
.25	1.66150, 0	1.41512, 0	1.09543, 0	8.04609, -1	5.71065, -1	3.95401, -1	2.68552, -1	1.79514, -1	1.18402, -1	7.71756, -2	4.97677, -2
.26	1.70262, 0	1.43935, 0	1.10044, 0	7.95692, -1	5.54518, -1	3.76126, -1	2.49734, -1	1.62893, -1	1.04622, -1	6.62707, -2	4.14415, -2
.27	1.74546, 0	1.46443, 0	1.10557, 0	7.86665, -1	5.38004, -1	3.57241, -1	2.31663, -1	1.47242, -1	9.19344, -2	5.64681, -2	3.41430, -2
.28	1.79011, 0	1.49043, 0	1.11084, 0	7.77529, -1	5.21543, -1	3.38747, -1	2.14310, -1	1.32553, -1	8.02849, -2	4.76217, -2	2.77898, -2
.29	1.83670, 0	1.51738, 0	1.11624, 0	7.68276, -1	5.05136, -1	3.20645, -1	1.97666, -1	1.18737, -1	6.96212, -2	3.96686, -2	2.22850, -2
.30	1.88532, 0	1.54534, 0	1.12179, 0	7.58904, -1	4.88785, -1	3.02958, -1	1.81721, -1	1.05825, -1	5.98920, -2	3.29290, -2	1.75505, -2
.32	1.98918, 0	1.60452, 0	1.13334, 0	7.39786, -1	4.56255, -1	2.68712, -1	1.51891, -1	8.25283, -2	4.30385, -2	2.14362, -2	1.00926, -2
.34	2.10288, 0	1.66845, 0	1.14556, 0	7.20138, -1	4.25961, -1	2.38082, -1	1.24744, -1	6.24199, -2	2.93387, -2	1.27144, -2	4.88552, -3
.36	2.22752, 0	1.73771, 0	1.15850, 0	6.99920, -1	3.91914, -1	2.05064, -1	1.00200, -1	4.52706, -2	1.84272, -2	6.31080, -3	1.39198, -3
.38	2.36481, 0	1.81298, 0	1.17228, 0	6.79089, -1	3.60125, -1	1.75673, -1	7.81776, -2	3.08661, -2	9.95791, -3	1.81686, -3	-7.35494, -4
.40	2.51646, 0	1.89502, 0	1.18691, 0	6.57594, -1	3.28608, -1	1.47924, -1	5.85941, -2	1.89558, -2	3.50510, -3	-1.13327, -3	-1.88216, -3
.42	2.68460, 0	1.98476, 0	1.20255, 0	6.35378, -1	2.97374, -1	1.21836, -1	4.13637, -2	9.35188, -3	-9.37188, -4	-2.86447, -3	-2.27552, -3
.44	2.87173, 0	2.08328, 0	1.21931, 0	6.12578, -1	2.66441, -1	9.74275, -2	2.63983, -2	1.82997, -3	-3.95479, -3	-3.68243, -3	-2.21080, -3
.46	3.08088, 0	2.19185, 0	1.23732, 0	5.88518, -1	2.36825, -1	7.47181, -2	1.36070, -2	-3.82048, -3	-5.71341, -3	-5.77545, -3	-1.85620, -3
.48	3.81571, 0	2.31202, 0	1.25675, 0	5.63715, -1	2.05544, -1	5.37297, -2	2.82593, -3	-7.80648, -3	-6.45857, -3	-3.41625, -3	-1.36790, -3
.50	3.58063, 0	2.44663, 0	1.27779, 0	5.37870, -1	1.76621, -1	3.44858, -2	-5.83216, -3	-1.03310, -2	-6.41547, -3	-2.76389, -3	-6.25008, -4

TABLE 7

τ	$\tilde{F}_{-1/2,1}(\tau)$	$\tilde{F}_{-3/2,1}(\tau)$	$\tilde{F}_{-5/2,1}(\tau)$	$\tilde{F}_{-7/2,1}(\tau)$	$\tilde{F}_{-9/2,1}(\tau)$	$\tilde{F}_{-11/2,1}(\tau)$	$\tilde{F}_{-13/2,1}(\tau)$	$\tilde{F}_{-15/2,1}(\tau)$	$\tilde{F}_{-17/2,1}(\tau)$	$\tilde{F}_{-19/2,1}(\tau)$	$\tilde{F}_{-21/2,1}(\tau)$
0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0	1.00000, 0
.10	1.15825, 0	4.42163, -1	1.67870, 0	2.77963, 0	3.77324, 0	4.71787, 0	5.66971, 0	6.63494, 0	7.69322, 0	8.80658, 0	7.87890, 0
.12	1.19815, 0	3.23681, -1	1.52098, 0	2.82519, 0	4.21870, 0	5.74867, 0	7.46416, 0	9.41564, 0	1.17533, 1	1.42513, 1	1.31198, 1
.14	1.23663, 0	2.02270, -1	1.27591, 0	2.84275, 0	4.30253, 0	6.34974, 0	8.88812, 0	1.20820, 1	1.63376, 1	2.09370, 1	2.09748, 1
.15	1.25790, 0	1.40631, -1	1.12111, 0	2.45530, 0	4.16385, 0	6.38661, 0	9.22060, 0	1.27240, 1	1.82964, 1	2.39492, 1	2.56353, 1
.16	1.27991, 0	7.96450, -2	9.48388, -1	2.19925, 0	3.87620, 0	6.12455, 0	9.14821, 0	1.32069, 1	1.95739, 1	2.69375, 1	3.02081, 1
.165	1.29120, 0	4.54987, -2	8.49844, -1	2.04465, 0	3.67204, 0	5.88994, 0	8.91703, 0	1.30447, 1	1.98170, 1	2.82834, 1	3.24452, 1
.17	1.30269, 0	1.47556, -2	7.49250, -1	1.87203, 0	3.42510, 0	5.57106, 0	8.53824, 0	1.26352, 1	1.96691, 1	2.60292, 1	3.43095, 1
.175	1.31439, 0	-1.71897, -2	6.43698, -1	1.68112, 0	3.13363, 0	5.16166, 0	7.99346, 0	1.19407, 1	1.96642, 1	2.50354, 1	3.57645, 1
.18	1.32628, 0	-4.93450, -2	5.33227, -1	1.47177, 0	2.79597, 0	4.65677, 0	7.25647, 0	1.09237, 1	1.81867, 1	2.31429, 1	3.66546, 1
.185	1.33840, 0	-8.17085, -2	4.17912, -1	1.24383, 0	2.41065, 0	4.04768, 0	6.34140, 0	9.54661, 0	1.66176, 1	2.01975, 1	3.68030, 1
.19	1.36073, 0	-1.14294, -1	2.97819, -1	9.97206, -1	1.97632, 0	3.33198, 0	5.20279, 0	7.77185, 0	1.44118, 1	1.60286, 1	3.60106, 1
.195	1.36328, 0	-1.47092, -1	1.75016, -1	7.31847, -1	1.49162, 0	2.50383, 0	3.83503, 0	5.56263, 0	1.14947, 1	9.38276, 0	3.40667, 1
.20	1.37606, 0	-1.80120, -1	4.35729, -2	4.47734, -1	9.56101, -1	1.55872, 0	2.22751, 0	2.85430, 0	7.79362, 0	3.53507, 0	3.07104, 1
.21	1.40234, 0	-2.46867, -1	-2.28949, -1	-1.76609, -1	-2.71791, -1	-5.97655, -1	-1.75506, 0	-4.01080, 0	-2.23285, 0	-1.62844, 1	1.87986, 1
.22	1.42961, 0	-3.14570, -1	-5.19178, -1	-5.75135, -1	-1.71154, 0	-3.46162, 0	-6.85760, 0	-1.31405, 1	-1.61473, 1	-4.40372, 1	-1.88591, 0
.23	1.45793, 0	-3.83268, -1	-8.26516, -1	-1.64650, 0	-3.38398, 0	-5.74685, 0	-1.31049, 1	-2.46641, 1	-3.62610, 1	-5.06254, 1	-3.34772, 1
.24	1.48734, 0	-4.53005, -1	-1.18035, 0	-2.48872, 0	-5.23640, 0	-1.05848, 1	-2.05260, 1	-3.86501, 1	-6.67502, 1	-1.27063, 2	-7.80258, 1
.25	1.51792, 0	-5.23825, -1	-1.49003, 0	-3.39913, 0	-7.29248, 0	-1.48729, 1	-2.90976, 1	-5.60523, 1	-8.32837, 1	-1.83472, 2	-1.37066, 2
.26	1.54972, 0	-5.95778, -1	-1.84493, 0	-4.37443, 0	-9.55201, 0	-1.96738, 1	-3.87493, 1	-7.36874, 1	-1.13664, 2	-2.45881, 2	-2.11339, 2
.27	1.58281, 0	-6.68908, -1	-2.21434, 0	-5.41065, 0	-1.19908, 1	-2.49138, 1	-5.03564, 1	-9.42190, 1	-1.46656, 2	-3.15074, 2	-3.00505, 2
.28	1.61728, 0	-7.43277, -1	-2.59756, 0	-6.50314, 0	-1.45906, 1	-3.05329, 1	-6.07380, 1	-1.16138, 2	-1.80955, 2	-3.88520, 2	-4.02693, 2
.29	1.65320, 0	-8.18940, -1	-2.99355, 0	-7.64661, 0	-1.73286, 1	-3.64828, 1	-7.26825, 1	-1.38754, 2	-2.14706, 2	-4.85810, 2	-5.14194, 2
.30	1.69065, 0	-8.95969, -1	-3.40246, 0	-8.85506, 0	-2.01788, 1	-4.25781, 1	-8.47973, 1	-1.61190, 2	-2.45493, 2	-5.16524, 2	-6.29134, 2
.32	1.77064, 0	-1.05433, 0	-4.26339, 0	-1.15198, 1	-2.60870, 1	-5.49728, 1	-1.08259, 2	-2.01111, 2	-2.86267, 2	-5.83220, 2	-8.54089, 2
.34	1.85778, 0	-1.21899, 0	-5.14357, 0	-1.38959, 1	-3.20182, 1	-6.63000, 1	-1.27820, 2	-2.25468, 2	-2.75516, 2	-5.18408, 2	-9.23627, 2
.36	1.95335, 0	-1.39057, 0	-6.05552, 0	-1.64912, 1	-3.76112, 1	-7.60901, 1	-1.38194, 2	-2.21906, 2	-1.81791, 2	-2.50924, 2	-7.73759, 2
.38	2.05837, 0	-1.56990, 0	-7.01143, 0	-1.90222, 1	-4.24450, 1	-8.18855, 1	-1.38408, 2	-1.77536, 2	+2.46745, 1	2.73928, 2	-2.52934, 2
.40	2.17418, 0	-1.75784, 0	-7.97208, 0	-2.13945, 1	-4.60461, 1	-8.22140, 1	-1.14230, 2	-8.08123, 1	3.62999, 2	1.07363, 3	+7.37065, 2
.42	2.30237, 0	-1.95548, 0	-8.95768, 0	-2.35015, 1	-4.78963, 1	-7.53855, 1	-7.02419, 1	+7.53641, 1	8.40863, 2	2.07530, 3	2.21336, 3
.44	2.44482, 0	-2.16400, 0	-9.89719, 0	-2.52286, 1	-4.74655, 1	-6.97172, 1	+2.22207, -1	2.91840, 2	1.43076, 3	3.14186, 3	4.04131, 3
.46	2.60378, 0	-2.38491, 0	-1.08382, 1	-2.64393, 1	-4.41578, 1	-3.37787, 1	9.70606, 1	5.59780, 2	2.07404, 3	4.02083, 3	5.83512, 3
.48	2.78201, 0	-2.61992, 0	-1.17468, 1	-2.69982, 1	-3.84513, 1	+3.44903, 0	2.20014, 2	8.57613, 2	2.67861, 3	4.36293, 3	7.23171, 3
.50	2.98280, 0	-2.87114, 0	-1.26069, 1	-2.67489, 1	-2.68133, 1	5.23818, 1	3.83460, 2	1.16006, 3	3.12506, 3	3.77956, 3	7.32977, 3

TABLE 8.- $\underline{F}_{\nu}^{(0)}(\tau) = f(\tau)\ T^{\nu}(\tau) \longrightarrow f(\tau)\ T^{\nu}(\tau)\ \cos\left(\nu\omega - \frac{\pi}{4}\right)$

τ	$\underline{F}_{1/2}^{(0)}(\tau)$	$\underline{F}_{3/2}^{(0)}(\tau)$	$\underline{F}_{5/2}^{(0)}(\tau)$	$\underline{F}_{7/2}^{(0)}(\tau)$	$\underline{F}_{9/2}^{(0)}(\tau)$	$\underline{F}_{11/2}^{(0)}(\tau)$	$\underline{F}_{13/2}^{(0)}(\tau)$	$\underline{F}_{15/2}^{(0)}(\tau)$	$\underline{F}_{17/2}^{(0)}(\tau)$	$\underline{F}_{19/2}^{(0)}(\tau)$	$\underline{F}_{21/2}^{(0)}(\tau)$
.02	9.89043, -1	9.64368, -1	9.40308, -1	9.16845, -1	8.93970, -1	8.71665, -1	8.49927, -1	8.28712, -1	8.08035, -1	7.87875, -1	7.68217, -1
.04	9.81477, -1	9.31947, -1	8.84917, -1	8.40269, -1	7.97856, -1	7.57592, -1	7.19360, -1	6.83058, -1	6.48587, -1	6.15856, -1	5.84777, -1
.06	9.78834, -1	9.03632, -1	8.34378, -1	7.70431, -1	7.11385, -1	6.56865, -1	6.06523, -1	5.60039, -1	5.17118, -1	4.77486, -1	4.40891, -1
.08	9.82797, -1	8.80976, -1	7.89704, -1	7.07889, -1	6.34549, -1	5.68808, -1	5.09878, -1	4.57053, -1	4.09701, -1	3.67258, -1	3.29206, -1
.10	9.98516, -1	8.67001, -1	7.52959, -1	6.53918, -1	5.67904, -1	4.93204, -1	4.28330, -1	3.71989, -1	3.23069, -1	2.80565, -1	2.43661, -1
.12	1.03492, 0	8.68573, -1	7.28264, -1	6.11796, -1	5.13480, -1	4.30850, -1	3.61665, -1	3.03534, -1	2.54746, -1	2.13800, -1	1.79435, -1
.14	1.12150, 0	9.06499, -1	7.32713, -1	5.92244, -1	4.78704, -1	3.86931, -1	3.12752, -1	2.52794, -1	2.04331, -1	1.65188, -1	1.33496, -1
.15	1.21594, 0	9.62870, -1	7.62405, -1	6.03625, -1	4.77954, -1	3.78447, -1	2.99686, -1	2.37289, -1	1.87871, -1	1.48757, -1	1.17787, -1
.16	1.43733, 0	1.11286, 0	8.61644, -1	6.67136, -1	5.16536, -1	3.99013, -1	3.09651, -1	2.39750, -1	1.85629, -1	1.43725, -1	1.11180, -1
.165	1.77527, 0	1.35777, 0	1.03845, 0	7.94235, -1	6.07450, -1	4.64692, -1	3.55531, -1	2.71756, -1	2.07853, -1	1.58971, -1	1.21585, -1
.17	2.95346, 0	2.25111, 0	1.69864, 0	1.28169, 0	9.67006, -1	7.29835, -1	5.50531, -1	4.15392, -1	3.13426, -1	2.36483, -1	1.78438, -1
.175	2.06247, 0	1.53751, 0	1.14614, 0	8.54367, -1	6.36892, -1	4.74754, -1	3.53867, -1	2.51916, -1	1.96626, -1	1.46561, -1	1.09242, -1
.18	1.76592, 0	1.30268, 0	9.60844, -1	7.08567, -1	5.22637, -1	3.85409, -1	2.84192, -1	2.09641, -1	1.54488, -1	1.13891, -1	8.39855, -2
.185	1.59244, 0	1.16351, 0	8.49924, -1	6.20721, -1	4.53232, -1	3.30968, -1	2.41491, -1	1.76225, -1	1.28569, -1	9.37840, -2	6.85972, -2
.19	1.47126, 0	1.06872, 0	7.71681, -1	5.58486, -1	4.04013, -1	2.92140, -1	2.11158, -1	1.52582, -1	1.10182, -1	7.95442, -2	5.74038, -2
.195	1.37854, 0	9.90878, -1	7.11764, -1	5.10889, -1	3.66084, -1	2.62272, -1	1.87771, -1	1.34534, -1	9.60588, -2	6.86150, -2	4.89904, -2
.20	1.30390, 0	9.30566, -1	6.62938, -1	4.71769, -1	3.35310, -1	2.38040, -1	1.68795, -1	1.19662, -1	8.48994, -2	5.98000, -2	4.22286, -2
.21	1.18753, 0	8.36116, -1	5.87025, -1	4.11061, -1	2.87115, -1	2.00077, -1	1.39114, -1	9.65218, -2	6.68536, -2	4.61870, -2	3.18557, -2
.22	1.09852, 0	7.64386, -1	5.29204, -1	3.64720, -1	2.50306, -1	1.71109, -1	1.16539, -1	7.90955, -2	5.36033, -2	3.60757, -2	2.42494, -2
.23	1.02575, 0	7.08681, -1	4.82631, -1	3.27578, -1	2.20673, -1	1.47783, -1	9.84394, -2	6.52368, -2	4.30163, -2	2.82288, -2	1.84094, -2
.24	9.66405, -1	6.58445, -1	4.43616, -1	2.95985, -1	1.95625, -1	1.28253, -1	8.34320, -2	5.38736, -2	3.45389, -2	2.19882, -2	1.39007, -2
.25	9.14389, -1	6.17021, -1	4.10028, -1	2.68944, -1	1.74159, -1	1.11574, -1	7.07345, -2	4.43938, -2	2.75871, -2	1.69730, -2	1.03585, -2
.26	8.65846, -1	5.80716, -1	3.80511, -1	2.46035, -1	1.55356, -1	9.70898, -2	5.98602, -2	3.64024, -2	2.18440, -2	1.29264, -2	7.53683, -3
.27	8.27790, -1	5.48583, -1	3.54141, -1	2.23772, -1	1.38667, -1	8.43758, -2	5.04444, -2	2.96306, -2	1.70884, -2	9.66225, -3	5.34348, -3
.28	7.90640, -1	5.19156, -1	3.30282, -1	2.04584, -1	1.23714, -1	7.31373, -2	4.22786, -2	2.38802, -2	1.31538, -2	7.04085, -3	3.64048, -3
.29	7.57068, -1	4.92479, -1	3.08461, -1	1.87110, -1	1.10237, -1	6.31635, -2	3.51781, -2	1.90052, -2	9.91462, -3	4.95455, -3	2.33874, -3
.30	7.25975, -1	4.67950, -1	2.88376, -1	1.71084, -1	9.80291, -2	5.42846, -2	2.90000, -2	1.48748, -2	7.25698, -3	3.30619, -3	1.35145, -3
.32	6.70151, -1	4.23917, -1	2.52384, -1	1.42589, -1	7.68465, -2	3.93580, -2	1.89878, -2	8.48627, -3	3.36774, -3	1.04900, -3	1.08178, -3
.34	6.20992, -1	3.86159, -1	2.20817, -1	1.18310, -1	5.92761, -2	2.75151, -2	1.15589, -2	4.07567, -3	9.27234, -4	-2.06100, -4	-4.75333, -4
.36	5.76927, -1	3.50441, -1	1.92835, -1	9.72485, -2	4.47172, -2	1.82567, -2	6.10691, -3	1.15648, -3	-4.98892, -4	-7.99940, -4	-6.61575, -4
.38	5.36904, -1	3.18976, -1	1.67529, -1	7.90176, -2	3.27109, -2	1.11127, -2	2.28204, -3	-6.50167, -4	-1.20150, -3	-9.81436, -4	-6.31046, -4
.40	5.00158, -1	2.90193, -1	1.45383, -1	6.32597, -2	2.29020, -2	5.72187, -3	-2.86008, -4	-1.64655, -3	-1.44186, -3	-9.26138, -4	-5.00405, -4
.42	4.66139, -1	2.63699, -1	1.25202, -1	4.96883, -2	1.49833, -2	1.77123, -3	-1.88852, -3	-2.48134, -3	-1.39353, -3	-7.51446, -4	-3.41763, -4
.44	4.34424, -1	2.39199, -1	1.07048, -1	3.80819, -2	8.69761, -3	-1.00215, -3	-2.67053, -3	-2.11722, -3	-1.18595, -3	-5.38797, -4	-2.00066, -4
.46	4.04696, -1	2.16473, -1	9.07389, -2	2.81887, -2	3.81561, -3	-2.82746, -3	-3.09894, -3	-1.92264, -3	-9.10079, -4	-3.34957, -4	-8.07136, -5
.48	3.76233, -1	1.95351, -1	6.74098, -2	1.98803, -2	1.32100, -4	-3.90218, -3	-3.33262, -3	-1.59628, -3	-6.26381, -4	-1.65424, -4	-2.52289, -6
.50	3.50230, -1	1.75664, -1	5.43218, -2	1.29782, -2	-2.53521, -3	-4.39783, -3	-2.78413, -3	-1.22235, -3	-3.71628, -4	-4.02791, -5	+4.18888, -5

TABLE 9.- $\underline{F}^{(0)}_{-\nu}(\tau) = f(\tau)\ \tau^{-\nu}(\tau) \longrightarrow \frac{1}{2} f(\tau)\ \tau^{-\nu}(\tau)\ \cos\left(\nu\pi + \frac{\pi}{4}\right)$

τ	$\underline{F}^{(0)}_{-1/2}(\tau)$	$\underline{F}^{(0)}_{-3/2}(\tau)$	$\underline{F}^{(0)}_{-5/2}(\tau)$	$\underline{F}^{(0)}_{-7/2}(\tau)$	$\underline{F}^{(0)}_{-9/2}(\tau)$	$\underline{F}^{(0)}_{-11/2}(\tau)$	$\underline{F}^{(0)}_{-13/2}(\tau)$	$\underline{F}^{(0)}_{-15/2}(\tau)$	$\underline{F}^{(0)}_{-17/2}(\tau)$	$\underline{F}^{(0)}_{-19/2}(\tau)$	$\underline{F}^{(0)}_{-21/2}(\tau)$
0.02	1.01436, 0	1.04031, 0	1.06693, 0	1.09423, 0	1.12223, 0	1.15094, 0	1.18039, 0	1.21060, 0	1.24157, 0	1.27334, 0	1.30593, 0
0.04	1.03364, 0	1.08857, 0	1.14643, 0	1.20736, 0	1.27155, 0	1.33910, 0	1.41027, 0	1.48523, 0	1.56416, 0	1.64729, 0	1.73484, 0
0.06	1.06986, 0	1.14783, 0	1.24310, 0	1.34626, 0	1.45802, 0	1.57904, 0	1.71010, 0	1.85206, 0	2.00577, 0	2.17225, 0	2.35255, 0
0.08	1.09639, 0	1.22310, 0	1.36443, 0	1.52217, 0	1.69809, 0	1.89435, 0	2.11330, 0	2.35765, 0	2.63003, 0	2.93400, 0	3.27310, 0
0.10	1.14952, 0	1.32362, 0	1.52410, 0	1.75493, 0	2.02073, 0	2.32879, 0	2.67920, 0	3.06499, 0	3.55224, 0	4.09025, 0	4.70975, 0
0.12	1.23322, 0	1.46925, 0	1.75068, 0	2.08598, 0	2.48545, 0	2.96146, 0	3.52862, 0	4.20441, 0	5.00962, 0	5.96905, 0	7.11221, 0
0.14	1.38750, 0	1.71659, 0	2.12373, 0	2.62744, 0	3.25062, 0	4.02161, 0	4.97546, 0	6.15553, 0	7.61553, 0	9.42179, 0	1.16565, 1
0.16	1.53556, 0	1.93944, 0	2.44939, 0	3.09343, 0	3.90680, 0	4.93446, 0	6.23139, 0	7.86986, 0	9.93913, 0	1.25525, 1	1.58530, 1
0.16	1.85639, 0	2.39764, 0	3.09669, 0	3.99955, 0	5.16565, 0	6.67175, 0	8.61692, 0	1.11293, 1	1.43741, 1	1.85648, 1	2.39777, 1
0.165	2.32115, 0	3.03488, 0	3.96807, 0	5.18822, 0	6.78356, 0	8.86942, 0	1.15967, 1	1.51626, 1	1.98249, 1	2.59209, 1	3.38913, 1
0.17	1.97704, 0	2.62022, 0	3.47696, 0	4.60237, 0	6.09965, 0	8.08400, 0	1.07140, 1	1.41996, 1	1.88189, 1	2.49412, 1	3.30551, 1
0.175	1.68331, 0	1.85658, 0	2.48904, 0	3.33870, 0	4.47831, 0	6.00686, 0	8.05696, 0	1.08066, 1	1.44944, 1	1.94403, 1	2.60757, 1
0.18	1.19687, 0	1.62226, 0	2.19684, 0	2.97957, 0	4.03752, 0	5.47065, 0	7.41186, 0	1.00410, 1	1.36014, 1	1.84229, 1	2.49508, 1
0.185	1.08951, 0	1.49049, 0	2.03856, 0	2.78751, 0	3.81068, 0	5.20810, 0	7.11615, 0	9.72065, 0	1.32748, 1	1.81236, 1	2.47362, 1
0.19	1.01502, 0	1.39982, 0	1.92961, 0	2.66822, 0	3.66014, 0	5.03680, 0	6.92715, 0	9.52115, 0	1.30782, 1	1.79621, 1	2.46253, 1
0.195	9.58226, -1	1.33082, 0	1.84661, 0	2.55947, 0	3.54596, 0	4.90170, 0	6.77165, 0	9.34346, 0	1.28762, 1	1.77174, 1	2.43445, 1
0.20	9.12386, -1	1.27494, 0	1.77869, 0	2.47727, 0	3.44396, 0	4.77857, 0	6.61655, 0	9.14090, 0	1.25974, 1	1.73152, 1	2.37306, 1
0.21	8.40756, -1	1.18656, 0	1.66788, 0	2.33714, 0	3.25928, 0	4.50231, 0	6.24245, 0	8.56135, 0	1.16569, 1	1.57371, 1	2.10311, 1
0.22	7.87016, -1	1.11580, 0	1.57419, 0	2.20397, 0	3.05775, 0	4.19690, 0	5.68605, 0	7.58025, 0	9.90410, 0	1.25725, 1	1.53538, 1
0.23	7.39396, -1	1.06431, 0	1.48579, 0	2.08459, 0	2.81962, 0	3.78601, 0	4.88319, 0	6.06990, 0	7.06475, 0	7.30595, 0	7.10740, 0
0.24	7.00080, -1	9.98920, -1	1.39893, 0	1.91322, 0	2.53530, 0	3.21218, 0	3.79353, 0	3.93931, 0	2.95415, 0	-4.99969, -1	-2.81451, 0
0.25	6.66402, -1	9.47280, -1	1.31099, 0	1.74612, 0	2.19831, 0	2.52035, 0	2.38206, 0	1.10702, 0	-2.61475, 0	-1.12276, 1	-2.91650, 1
0.26	6.34188, -1	8.98065, -1	1.22048, 0	1.56128, 0	1.80448, 0	1.68068, 0	6.23765, 0	-2.48611, 0	-9.77445, 0	-2.51542, 1	-5.56570, 1
0.27	6.05584, -1	8.50627, -1	1.12676, 0	1.35793, 0	1.35267, 0	6.89515, -1	-1.49096, 0	-6.86430, 0	-1.85750, 1	-4.23845, 1	-8.84746, 1
0.28	5.79147, -1	8.06993, -1	1.02938, 0	1.13598, 0	8.42660, -1	-4.52903, -1	-3.96352, 0	-1.20262, 1	-2.89943, 1	-6.27515, 1	-1.27225, 2
0.29	5.54475, -1	7.58697, -1	9.28625, -1	8.98665, -1	2.78121, -1	-1.73890, 0	-6.77225, 0	-1.79149, 1	-4.08778, 1	-8.58920, 1	-1.70775, 2
0.30	5.31244, -1	7.13734, -1	8.23800, -1	6.38875, -1	-3.42830, -1	-3.16672, 0	-9.90920, 0	-2.44750, 1	-5.40535, 1	-1.11232, 2	-2.17516, 2
0.32	4.88404, -1	6.26910, -1	6.04795, -1	7.78700, -2	-1.72409, 0	-6.37915, 0	-1.69445, 1	-3.90549, 1	-8.26015, 1	-1.64058, 2	-3.09189, 2
0.34	4.49410, -1	5.38827, -1	3.74912, -1	-5.34045, -1	-3.25521, 0	-9.93145, 0	-2.34783, 1	-5.43695, 1	-1.10463, 2	-2.09452, 2	-3.73929, 2
0.36	4.13518, -1	4.64018, -1	1.38204, -1	-1.17927, 0	-4.87107, 0	-1.36061, 1	-3.21358, 1	-6.50790, 1	-1.31899, 2	-2.33893, 2	-3.73154, 2
0.38	3.80199, -1	3.71165, -1	-1.01642, -1	-1.83946, 0	-6.49930, 0	-1.71478, 1	-3.87758, 1	-7.80525, 1	-1.40383, 2	-2.19833, 2	-2.72873, 2
0.40	3.49107, -1	2.90807, -1	-3.40012, -1	-2.49311, 0	-8.08630, 0	-2.02670, 1	-4.36049, 1	-8.16840, 1	-1.29698, 2	-1.54454, 2	-4.98521, 1
0.42	3.19980, -1	2.13267, -1	-5.72770, -1	-3.11942, 0	-9.45830, 0	-2.26740, 1	-4.87587, 1	-8.48765, 1	-9.82830, 1	-3.14725, 1	2.93140, 2
0.44	2.92630, -1	1.39091, -1	-7.96420, -1	-3.69668, 0	-1.06211, 1	-2.40959, 1	-4.45007, 1	-6.18020, 1	-3.58233, 0	-1.43921, 2	7.16436, 2
0.46	2.66912, -1	6.87472, -2	-1.00382, 0	-4.20472, 0	-1.14689, 1	-2.43071, 1	-3.93544, 1	-3.64254, 1	4.64826, 1	3.52680, 2	1.14146, 3
0.48	2.42713, -1	2.69075, -3	-1.53858, 0	-4.82520, 0	-1.19375, 1	-2.31534, 1	-2.09825, 1	-1.86651, 0	1.43391, 2	5.61960, 2	1.45880, 3
0.50	2.19955, -1	-5.86493, -2	-1.67510, 0	-4.94366, 0	-1.19840, 1	-2.05734, 1	-1.72586, 1	-4.00740, 1	2.43137, 2	7.29125, 2	1.67868, 3

TABLE 10.- $\tilde{F}_{\nu}^{(0)}(\tau) = g(\tau)\, T^{\nu}(\tau) \longrightarrow g(\tau)\, T^{\nu}(\tau) \cos\left(\nu\pi - \frac{\pi}{4}\right)$

τ	$\tilde{F}_{1/2}^{(0)}(\tau)$	$\tilde{F}_{3/2}^{(0)}(\tau)$	$\tilde{F}_{5/2}^{(0)}(\tau)$	$\tilde{F}_{7/2}^{(0)}(\tau)$	$\tilde{F}_{9/2}^{(0)}(\tau)$	$\tilde{F}_{11/2}^{(0)}(\tau)$	$\tilde{F}_{13/2}^{(0)}(\tau)$	$\tilde{F}_{15/2}^{(0)}(\tau)$	$\tilde{F}_{17/2}^{(0)}(\tau)$	$\tilde{F}_{19/2}^{(0)}(\tau)$	$\tilde{F}_{21/2}^{(0)}(\tau)$
.02	1.03963, 0	1.01389, 0	9.88402, -1	9.63742, -1	9.39697, -1	9.16251, -1	8.93390, -1	8.71101, -1	8.49366, -1	8.28175, -1	8.07511, -1
.04	1.08556, 0	1.03078, 0	9.78762, -1	9.29366, -1	8.82469, -1	8.37935, -1	7.95648, -1	7.55496, -1	7.17370, -1	6.81166, -1	6.46793, -1
.06	1.14018, 0	1.06279, 0	9.72109, -1	8.97606, -1	8.28813, -1	7.65294, -1	7.06642, -1	6.52485, -1	6.02479, -1	5.56306, -1	5.13669, -1
.08	1.20746, 0	1.08237, 0	9.70230, -1	8.96712, -1	7.79807, -1	6.96838, -1	6.25436, -1	5.61535, -1	5.03359, -1	4.51209, -1	4.04462, -1
.10	1.29496, 0	1.12461, 0	9.78886, -1	8.48217, -1	7.36845, -1	6.39750, -1	5.55600, -1	4.82518, -1	4.19050, -1	3.63929, -1	3.16060, -1
.12	1.41902, 0	1.19094, 0	9.99512, -1	8.38858, -1	7.04026, -1	5.90865, -1	4.95893, -1	4.16188, -1	3.49292, -1	2.93150, -1	2.46031, -1
.14	1.62754, 0	1.31553, 0	1.06333, 0	8.59476, -1	6.94705, -1	5.61522, -1	4.53872, -1	3.66860, -1	2.96529, -1	2.39681, -1	1.93732, -1
.15	1.81530, 0	1.43828, 0	1.13863, 0	9.01660, -1	7.15939, -1	5.65301, -1	4.47608, -1	3.54418, -1	2.80630, -1	2.22204, -1	1.75943, -1
.16	2.21066, 0	1.71181, 0	1.32540, 0	1.02608, 0	7.94448, -1	6.13694, -1	4.76253, -1	3.68743, -1	2.85503, -1	2.21053, -1	1.70998, -1
.165	2.77097, 0	2.11930, 0	1.62089, 0	1.23970, 0	9.48150, -1	7.25168, -1	5.54625, -1	4.24191, -1	3.24432, -1	2.48133, -1	1.89778, -1
.17	4.72537, 0	3.55034, 0	2.69081, 0	2.03030, 0	1.53193, 0	1.15587, 0	8.72144, -1	6.58217, -1	4.96553, -1	3.74629, -1	2.82683, -1
.175	3.31647, 0	2.47233, 0	1.84300, 0	1.37386, 0	1.02413, 0	7.63403, -1	5.69069, -1	4.06090, -1	3.16183, -1	2.35670, -1	1.75869, -1
.18	2.88255, 0	2.12635, 0	1.56840, 0	1.15677, 0	8.53116, -1	6.29112, -1	4.63889, -1	3.42038, -1	2.52177, -1	1.88906, -1	1.37041, -1
.185	2.63893, 0	1.92812, 0	1.40645, 0	1.02863, 0	7.51078, -1	5.48305, -1	4.00158, -1	2.92025, -1	2.13061, -1	1.55415, -1	1.13345, -1
.19	2.47544, 0	1.79311, 0	1.29637, 0	9.39669, -1	6.79759, -1	4.91534, -1	3.55283, -1	2.56667, -1	1.85381, -1	1.33835, -1	9.65840, -2
.195	2.35534, 0	1.69287, 0	1.21601, 0	8.72516, -1	6.25436, -1	4.48075, -1	3.20796, -1	2.29498, -1	1.64078, -1	1.17225, -1	8.38970, -2
.20	2.26219, 0	1.61413, 0	1.15030, 0	8.16491, -1	5.81743, -1	4.12985, -1	2.92841, -1	2.07429, -1	1.46774, -1	1.03749, -1	7.32630, -2
.21	2.12539, 0	1.49637, 0	1.05068, 0	7.38646, -1	5.13841, -1	3.58072, -1	2.48968, -1	1.72742, -1	1.19610, -1	8.28595, -2	5.70166, -2
.22	2.02900, 0	1.41171, 0	9.77366, -1	6.73587, -1	4.62278, -1	3.16014, -1	2.15231, -1	1.46078, -1	9.88131, -2	6.66258, -2	4.47852, -2
.23	1.95765, 0	1.34739, 0	9.20208, -1	6.24196, -1	4.20556, -1	2.81732, -1	1.87689, -1	1.24384, -1	8.20170, -2	5.38186, -2	3.51003, -2
.24	1.90304, 0	1.29681, 0	8.73869, -1	5.82184, -1	3.85225, -1	2.52566, -1	1.64294, -1	1.06088, -1	6.80140, -2	4.32992, -2	2.73733, -2
.25	1.86045, 0	1.25542, 0	8.34263, -1	5.47003, -1	3.54363, -1	2.27014, -1	1.43920, -1	9.03252, -2	5.61301, -2	3.46341, -2	2.10352, -2
.26	1.82595, 0	1.22137, 0	8.00298, -1	5.15563, -1	3.28748, -1	2.04201, -1	1.28878, -1	7.85623, -2	4.89427, -2	2.71871, -2	1.58516, -2
.27	1.80155, 0	1.19342, 0	7.70731, -1	4.87004, -1	3.01787, -1	1.83630, -1	1.09784, -1	6.44880, -2	3.71902, -2	2.10283, -2	1.16292, -2
.28	1.77972, 0	1.16832, 0	7.43273, -1	4.60400, -1	2.78408, -1	1.64590, -1	9.51444, -2	5.37404, -2	2.96016, -2	1.58448, -2	8.19256, -3
.29	1.70372, 0	1.10828, 0	6.94167, -1	4.21076, -1	2.46080, -1	1.42144, -1	7.91678, -2	4.27697, -2	2.33191, -2	1.11498, -2	5.25885, -3
.30	1.75144, 0	1.12896, 0	6.25719, -1	4.12747, -1	2.36499, -1	1.30984, -1	6.99637, -2	3.58860, -2	1.75078, -2	7.97832, -3	3.26045, -3
.31	1.73673, 0	1.09860, 0	6.54014, -1	3.69786, -1	1.99152, -1	1.01948, -1	4.92078, -2	2.19926, -2	8.72767, -3	2.71854, -3	2.80349, -3
.32	1.73244, 0	1.07452, 0	6.16035, -1	3.30081, -1	1.65368, -1	7.67615, -2	3.22470, -2	1.13647, -2	2.57354, -3	-5.74978, -3	-1.32608, -3
.33	1.73856, 0	1.05463, 0	5.80435, -1	2.92719, -1	1.34599, -1	5.49528, -2	1.33819, -2	3.48102, -3	-1.48351, -3	-2.40783, -3	-1.99135, -3
.34	1.74787, 0	1.03842, 0	5.46362, -1	2.57239, -1	1.06489, -1	3.61771, -2	7.42911, -3	-2.11660, -3	-3.91145, -3	-3.19604, -3	-2.05435, -3
.35	1.76558, 0	1.02438, 0	5.15202, -1	2.23307, -1	8.08441, -2	2.01980, -2	-1.00608, -3	-5.81232, -3	-5.08977, -3	-3.28574, -3	-1.76643, -3
.40	1.78914, 0	1.01213, 0	4.80563, -1	1.90705, -1	5.75092, -2	6.79857, -3	-7.24886, -3	-9.52393, -3	-5.34867, -3	-2.38421, -3	-1.31178, -3
.45	1.81833, 0	1.00119, 0	4.48061, -1	1.59312, -1	3.64048, -2	-4.19461, -3	-1.11778, -2	-8.86186, -3	-6.98392, -3	-2.28519, -3	-8.37482, -4
.55	1.85306, 0	9.91200, -1	4.15481, -1	1.29063, -1	1.74711, -2	-1.29468, -2	-1.41898, -2	-8.80360, -3	-4.16712, -3	-1.53372, -3	-5.89578, -4
.65	1.89097, 0	9.81846, -1	3.38806, -1	9.99196, -2	6.63943, -4	-1.98124, -2	-1.67499, -2	-8.03296, -3	-3.14823, -3	-8.31431, -4	-1.26787, -5
.80	1.93927, 0	9.72785, -1	3.00787, -1	7.18620, -2	-1.40878, -2	-2.43514, -2	-1.54161, -2	-6.76851, -3	-2.06775, -3	-2.23031, -4	-2.319421, -4

TABLE 11.- $\tilde{F}^{(0)}_{-\nu}(\tau) = g(\tau)\ \tau^{-\nu}(\tau) \longrightarrow \frac{1}{2}g(\tau)\ \tau^{-\nu}(\tau)\ \cos\left(\nu\omega + \frac{\pi}{4}\right)$

τ	$\tilde{F}^{(0)}_{-1/2}(\tau)$	$\tilde{F}^{(0)}_{-3/2}(\tau)$	$\tilde{F}^{(0)}_{-5/2}(\tau)$	$\tilde{F}^{(0)}_{-7/2}(\tau)$	$\tilde{F}^{(0)}_{-9/2}(\tau)$	$\tilde{F}^{(0)}_{-11/2}(\tau)$	$\tilde{F}^{(0)}_{-13/2}(\tau)$	$\tilde{F}^{(0)}_{-15/2}(\tau)$	$\tilde{F}^{(0)}_{-17/2}(\tau)$	$\tilde{F}^{(0)}_{-19/2}(\tau)$	$\tilde{F}^{(0)}_{-21/2}(\tau)$
.02	1.06623, 0	1.09352, 0	1.12150, 0	1.15020, 0	1.17963, 0	1.20981, 0	1.24077, 0	1.27252, 0	1.30508, 0	1.33847, 0	1.37273, 0
.04	1.14326, 0	1.20401, 0	1.26801, 0	1.33540, 0	1.40638, 0	1.48111, 0	1.55983, 0	1.64274, 0	1.73004, 0	1.82199, 0	1.91882, 0
.06	1.23481, 0	1.33730, 0	1.44830, 0	1.56851, 0	1.69870, 0	1.83969, 0	1.99239, 0	2.15777, 0	2.33686, 0	2.53082, 0	2.74069, 0
.08	1.34702, 0	1.50270, 0	1.67634, 0	1.87014, 0	2.08627, 0	2.32740, 0	2.59640, 0	2.89649, 0	3.23125, 0	3.60471, 0	4.02133, 0
.10	1.49108, 0	1.71691, 0	1.97696, 0	2.27837, 0	2.62118, 0	3.01815, 0	3.47527, 0	4.00163, 0	4.60772, 0	5.30659, 0	6.10918, 0
.12	1.69092, 0	2.01455, 0	2.40043, 0	2.86014, 0	3.40790, 0	4.06058, 0	4.83623, 0	5.76483, 0	6.86889, 0	8.18440, 0	9.75184, 0
.14	2.01367, 0	2.49115, 0	3.08200, 0	3.81189, 0	4.71736, 0	5.83624, 0	7.22049, 0	8.93306, 0	1.10618, 1	1.36731, 1	1.69181, 1
.15	2.29388, 0	2.89702, 0	3.65875, 0	4.62078, 0	5.83874, 0	7.37080, 0	9.30806, 0	1.17555, 1	1.48465, 1	1.87502, 1	2.38803, 1
.16	2.85518, 0	3.68764, 0	4.76280, 0	6.15143, 0	7.94492, 0	1.02613, 1	1.32531, 1	1.71172, 1	2.21078, 1	2.85532, 1	3.68784, 1
.165	3.52301, 0	4.73706, 0	6.19364, 0	8.09814, 0	1.06862, 1	1.38440, 1	1.81009, 1	2.36069, 1	3.09441, 1	4.04692, 1	5.28999, 1
.17	3.13201, 0	4.16093, 0	5.50817, 0	7.29103, 0	9.66300, 0	1.28066, 1	1.69729, 1	2.24946, 1	2.98127, 1	3.95116, 1	5.23776, 1
.175	2.22438, 0	2.98379, 0	4.00240, 0	5.38866, 0	7.20117, 0	9.65907, 0	1.29557, 1	1.73771, 1	2.33071, 1	3.12604, 1	4.19268, 1
.18	1.95367, 0	2.64805, 0	3.58888, 0	4.86301, 0	6.59052, 0	8.92985, 0	1.20985, 1	1.63901, 1	2.22019, 1	3.00721, 1	4.07277, 1
.185	1.80549, 0	2.46998, 0	3.37822, 0	4.61938, 0	6.31491, 0	8.63065, 0	1.17928, 1	1.61087, 1	2.19985, 1	3.00337, 1	4.09918, 1
.19	1.70780, 0	2.33524, 0	3.24646, 0	4.47253, 0	6.15830, 0	8.47457, 0	1.16551, 1	1.60146, 1	2.20045, 1	3.02049, 1	4.14328, 1
.195	1.63709, 0	2.27384, 0	3.15467, 0	4.37273, 0	6.05468, 0	8.37431, 0	1.15690, 1	1.59828, 1	2.19986, 1	3.02693, 1	4.15914, 1
.20	1.58294, 0	2.21194, 0	3.08592, 0	4.29791, 0	5.97506, 0	8.28063, 0	1.14793, 1	1.58689, 1	2.18561, 1	3.00408, 1	4.11712, 1
.21	1.50468, 0	2.12355, 0	2.98495, 0	4.18271, 0	5.83304, 0	8.06765, 0	1.11719, 1	1.53270, 1	2.08520, 1	2.81642, 1	3.75387, 1
.22	1.45352, 0	2.06036, 0	2.90731, 0	4.07042, 0	5.64724, 0	7.76109, 0	1.08013, 1	1.39997, 1	1.88915, 1	2.52198, 1	3.35563, 1
.23	1.40978, 0	2.01020, 0	2.83288, 0	3.95646, 0	5.37584, 0	7.18046, 0	9.81063, 0	1.16732, 1	1.34700, 1	1.39261, 1	1.35513, 1
.24	1.37862, 0	1.96707, 0	2.75477, 0	3.76751, 0	4.99251, 0	6.32542, 0	7.47022, 0	7.75729, 0	5.81731, 0	-9.84541, -1	-5.54233, 0
.25	1.35388, 0	1.92738, 0	2.68741, 0	3.55274, 0	4.47279, 0	5.12803, 0	4.84686, 0	2.25240, 0	-5.32010, 0	-2.28444, 1	-5.95406, 1
.26	1.33378, 0	1.88884, 0	2.56694, 0	3.28372, 0	3.79822, 0	3.53484, 0	1.31193, 0	-5.22684, 0	-2.05578, 1	-5.29048, 1	-1.17059, 2
.27	1.31797, 0	1.85100, 0	2.46219, 0	2.95532, 0	2.94386, 0	1.50128, 0	-3.24484, 0	-1.49390, +1	-4.04255, 1	-9.21896, 1	-1.92550, 2
.28	1.30333, 0	1.81384, 0	2.31654, 0	2.55543, 0	1.89634, 0	-1.01922, 0	-8.91958, 0	-2.70640, 1	-6.52494, 1	-1.41217, 2	-2.86510, 2
.29	1.29167, 0	1.76740, 0	2.16325, 0	2.08880, 0	6.47883, -1	-4.06081, 0	-1.57760, 1	-4.17329, 1	-9.52252, 1	-2.00088, 2	-3.97822, 2
.30	1.28157, 0	1.72193, 0	1.98746, 0	1.54135, 0	-8.27091, -1	-7.64462, 0	-2.39065, 1	-5.90469, 1	-1.30406, 2	-2.68352, 2	-5.24766, 2
.32	1.26572, 0	1.62467, 0	1.56737, 0	2.02063, -1	-4.46807, 0	-1.65319, 1	-4.39125, 1	-1.01213, 2	-2.14066, 2	-4.28186, 2	-8.01279, 2
.34	1.25376, 0	1.50328, 0	1.04595, 0	-1.48988, 0	-9.08129, 0	-2.77068, 1	-6.54998, 1	-1.51680, 2	-3.08170, 2	-5.84329, 2	-1.04319, 3
.36	1.24468, 0	1.38550, 0	4.15995, -1	-3.54961, 0	-1.45620, 1	-4.09645, 1	-9.67291, 1	-2.04918, 2	-3.97017, 2	-7.04020, 2	-1.12320, 3
.38	1.23773, 0	1.20538, 0	-3.30892, -1	-5.98851, 0	-2.11583, 1	-5.68241, 1	-1.26235, 2	-2.54098, 2	-4.57013, 2	-7.15660, 2	-8.88330, 2
.40	1.23235, 0	1.02855, 0	-1.20024, 0	-8.80068, 0	-2.84587, 1	-7.15425, 1	-1.53925, 2	-2.88345, 2	-4.57857, 2	-5.45223, 2	-1.75978, 2
.42	1.22815, 0	8.18566, -1	-2.19844, 0	-1.19750, 1	-3.63030, 1	-8.70278, 1	-1.75624, 2	-2.48243, 2	-3.65640, 2	-1.20798, 2	1.12514, 3
.44	1.22484, 0	5.82181, -1	-3.32932, 0	-1.54729, 1	-4.44558, 1	-1.00856, 2	-1.86263, 2	-2.58679, 2	-1.49106, 1	6.02397, 2	2.99872, 3
.46	1.22215, 0	3.14783, -1	-4.59635, 0	-1.92528, 1	-5.25145, 1	-1.11209, 2	-1.80158, 2	-1.66787, 2	2.12700, 2	1.61487, 3	5.22668, 3
.48	1.21989, 0	1.35241, -2	-7.72300, 0	-2.32465, 1	-5.99988, 1	-1.16370, 2	-1.04856, 2	-8.32572, 0	7.20692, 2	2.82444, 3	7.33202, 3
.50	1.21792, 0	-3.24763, -1	-9.27825, 0	-2.73737, 1	-6.63670, 1	-1.13918, 2	-9.54518, 1	-2.21895, 2	1.34628, 3	4.03728, 3	9.29452, 3

TABLE 12.- $\tilde{F}^{(o)}_{\nu,1}(\tau) = h(\tau)\ T^{\nu}(\tau) \longrightarrow h(\tau)\ T^{\nu}(\tau)\ \cos\left(\nu\omega - \mu - \frac{\pi}{4}\right)$

τ	$\tilde{F}^{(o)}_{1/2,1}(\tau)$	$\tilde{F}^{(o)}_{3/2,1}(\tau)$	$\tilde{F}^{(o)}_{5/2,1}(\tau)$	$\tilde{F}^{(o)}_{7/2,1}(\tau)$	$\tilde{F}^{(o)}_{9/2,1}(\tau)$	$\tilde{F}^{(o)}_{11/2,1}(\tau)$	$\tilde{F}^{(o)}_{13/2,1}(\tau)$	$\tilde{F}^{(o)}_{15/2,1}(\tau)$	$\tilde{F}^{(o)}_{17/2,1}(\tau)$	$\tilde{F}^{(o)}_{19/2,1}(\tau)$	$\tilde{F}^{(o)}_{21/2,1}(\tau)$
.02	1.0890, 0	1.0618, 0	1.0364, 0	1.0096, 0	9.8433, -1	9.5977, -1	9.3583, -1	9.1248, -1	8.8971, -1	8.6751, -1	8.4587, -1
.04	1.1968, 0	1.1366, 0	1.0782, 0	1.0236, 0	9.7212 -1	9.2306, -1	8.7648, -1	8.3224, -1	7.9024, -1	7.5036, -1	7.1250, -1
.06	1.3274, 0	1.2257, 0	1.1318, 0	1.0450, 0	9.6493, -1	8.9098, -1	8.2269, -1	7.5964, -1	7.0142, -1	6.4767, -1	5.9803, -1
.08	1.4954, 0	1.3404, 0	1.2016, 0	1.0771, 0	9.5549, -1	8.6493, -1	7.7580, -1	6.9542, -1	6.2338, -1	5.5579, -1	5.0090, -1
.10	1.7213, 0	1.4949, 0	1.2983, 0	1.1275, 0	9.7919, -1	8.5039, -1	7.3854, -1	6.4139, -1	5.5702, -1	4.8375, -1	4.2013, -1
.12	2.0522, 0	1.7224, 0	1.4455, 0	1.2132, 0	1.0182, 0	8.5453, -1	7.1718, -1	6.0190, -1	5.0516, -1	4.2398, -1	3.5562, -1
.14	2.6233, 0	2.1204, 0	1.7139, 0	1.3853, 0	1.1197, 0	9.0507, -1	7.3156, -1	5.9131, -1	4.7795, -1	3.8632, -1	3.1226, -1
.15	3.1468, 0	2.4911, 0	1.9725, 0	1.5617, 0	1.2365, 0	9.7910, -1	7.7825, -1	6.1585, -1	4.8605, -1	3.8488, -1	3.0475, -1
.16	4.2322, 0	3.2768, 0	2.5371, 0	1.9644, 0	1.5209, 0	1.1749, 0	9.1175, -1	7.0593, -1	5.4658, -1	4.2319, -1	3.2736, -1
.165	5.7440, 0	4.3932, 0	3.3800, 0	2.5898, 0	1.9555 0	1.5032, 0	1.1497, 0	8.7932, -1	6.7252, -1	5.1436, -1	3.9340, -1
.17	1.0065, 0	7.5871, 0	5.7252, 0	4.3205, 0	3.2600, 0	2.4600, 0	1.8562, 0	1.4007, 0	1.0569, 0	7.9755, -1	6.0182, -1
.175	6.0065, 0	4.4232, 0	3.3486, 0	2.4996, 0	1.8660 0	1.3912, 0	1.0399, 0	7.7630, -1	5.7964, -1	4.3255, -1	3.2287, -1
.18	4.6172, 0	3.4233, 0	2.5378, 0	1.8810, 0	1.3940, 0	1.0329, 0	7.5816, -1	5.6684, -1	4.1991, -1	3.1065, -1	2.3016, -1
.185	3.7891, 0	2.7991, 0	2.0666, 0	1.5260, 0	1.1247, 0	8.2918, -1	6.1067, -1	4.4990, -1	3.3120, -1	2.4371, -1	1.7925, -1
.19	3.2093, 0	2.3710, 0	1.7492, 0	1.2895, 0	9.4870, -1	6.9703, -1	5.1174, -1	3.7535, -1	2.7503, -1	2.0140, -1	1.4725, -1
.195	2.7704, 0	2.0545, 0	1.5586, 0	1.1202, 0	8.2404, -1	6.0484, -1	4.4298, -1	3.3031, -1	2.3634, -1	1.7218, -1	1.2525, -1
.20	2.4217, 0	1.8094, 0	1.3442, 0	9.9432, -1	7.3220, -1	5.3715, -1	4.0082, -1	2.9454, -1	2.0802, -1	1.5080, -1	1.0905, -1
.21	1.8962, 0	1.4642, 0	1.0994, 0	8.2052, -1	5.8496, -1	4.4602, -1	3.2403, -1	2.3453, -1	1.6889, -1	1.2095, -1	8.6292, -2
.22	1.5163, 0	1.2128, 0	9.3938, -1	7.1075, -1	5.2818, -1	3.6692, -1	2.8018, -1	2.0090, -1	1.4286, -1	1.0085, -1	7.0741, -2
.23	1.2280, 0	1.0419, 0	8.3174, -1	6.3850, -1	4.7587, -1	3.4743, -1	2.4948, -1	1.7727, -1	1.2568, -1	8.6207, -2	5.8114, -2
.24	1.0332, 0	9.4666, -1	7.5710, -1	5.8852, -1	4.4124, -1	3.2115, -1	2.2918, -1	1.5756, -1	1.0828, -1	7.2785, -2	4.9361, -2
.25	8.2084, -1	8.3498, -1	7.0474, -1	5.5284, -1	4.1170, -1	2.9800, -1	2.0937, -1	1.4159, -1	9.5259, -2	6.3131, -2	4.1300, -2
.26	6.7298, -1	7.3780, -1	6.6788, -1	5.3522, -1	3.9055, -1	2.7705, -1	1.9041, -1	1.2866, -1	8.3852, -2	5.3639, -2	3.4375, -2
.27	5.5037, -1	7.0728, -1	6.4208, -1	5.0845, -1	3.7532, -1	2.6084, -1	1.7413, -1	1.1417, -1	7.3274, -2	4.6939 -2	2.8321, -2
.28	4.4824, -1	6.7019, -1	6.2417, -1	4.9404, -1	3.5868, -1	2.4624, -1	1.6234, -1	1.0364, -1	6.4390, -2	3.9062, -2	2.3149, -2
.29	3.6158, -1	6.4314, -1	6.1199, -1	4.8259, -1	3.4558, -1	2.3056, -1	1.4977, -1	9.2824, -2	5.5976, -2	3.2742, -2	1.8587, -2
.30	2.8353, -1	6.2479, -1	6.0468, -1	4.7354, -1	3.3375, -1	2.1994, -1	1.3793, -1	8.3091, -2	4.8275, -2	2.7080, -2	1.4171, -2
.32	1.7133, -1	6.0890, -1	5.9844, -1	4.5910, -1	3.1176, -1	1.9881, -1	1.1589, -1	6.5189, -2	3.4925, -2	1.7776, -2	8.5209, -3
.34	8.4097, -2	8.0474, -1	5.0014, -1	4.4730, -1	2.9088, -1	1.7294, -1	9.5719, -2	4.9586, -2	2.3953, -2	1.0601, -2	4.1877, -3
.36	1.8153, -2	6.1568, -1	5.0628, -1	4.3634, -1	2.7023, -1	1.5098, -1	7.7256, -2	3.8119, -2	1.5094, -2	5.2777, -3	1.1831, -3
.38	-3.2075, -2	8.3622, -1	6.1547, -1	4.2180, -1	2.4961, -1	1.2994, -1	6.0050, -2	2.4706, -2	8.1829, -3	1.5089, -3	-0.3728, -4
.40	-7.0246, -2	6.6297, -1	5.2523, -1	4.1362, -1	2.2843, -1	1.0961, -1	4.5369, -2	1.5128, -2	2.9145, -3	-9.3337, -4	-1.8039, -3
.42	-9.9071, -2	6.9397, -1	5.3666, -1	4.0154, -1	2.0715, -1	9.0312, -2	3.1924, -2	7.3605, -3	-8.6088, -4	-2.4543, -3	-1.3661, -3
.44	-1.2043, -1	7.3411, -1	6.4808, -1	3.8739, -1	1.8567, -1	7.1974, -2	2.0151, -2	1.2420, -3	-3.3780, -3	-3.1336, -3	-1.9107, -3
.46	-1.3563, -1	7.7697, -1	6.5965, -1	3.7242, -1	1.6381, -1	5.4743, -2	1.0025, -2	-3.6631, -3	-4.8441, -3	-3.2263, -3	-1.6021, -3
.48	-1.4566, -1	8.2454, -1	7.3456, -1	3.5610, -1	1.4191, -1	3.8697, -2	-4.1007, -3	-6.6184, -3	-5.4560, -3	-2.9123, -3	-1.1672, -3
.50	-1.5119, -1	8.7676, -1	7.3424, -1	3.3825, -1	1.2008, -1	2.3874, -2	-6.4212, -3	-8.6539, -3	-5.0848, -3	-2.3428, -3	-7.0109, -4

TABLE 13.- $\tilde{F}^{(o)}_{-\nu,1}(\tau) = h(\tau)\ T^{-\nu}(\tau) \longrightarrow \frac{1}{2}h(\tau)\ T^{-\nu}(\tau)\ \cos\left(\nu\omega + \mu + \frac{\pi}{4}\right)$

τ	$\tilde{F}^{(o)}_{-1/2,1}(\tau)$		$\tilde{F}^{(o)}_{-3/2,1}(\tau)$		$\tilde{F}^{(o)}_{-5/2,1}(\tau)$		$\tilde{F}^{(o)}_{-7/2,1}(\tau)$		$\tilde{F}^{(o)}_{-9/2,1}(\tau)$		$\tilde{F}^{(o)}_{-11/2,1}(\tau)$		$\tilde{F}^{(o)}_{-13/2,1}(\tau)$		$\tilde{F}^{(o)}_{-15/2,1}(\tau)$		$\tilde{F}^{(o)}_{-17/2,1}(\tau)$		$\tilde{F}^{(o)}_{-19/2,1}(\tau)$		$\tilde{F}^{(o)}_{-21/2,1}(\tau)$	
.02	1.1189,	0	1.1455,	0	1.1748,	0	1.2048,	0	1.2357,	0	1.2673,	0	1.2997,	0	1.3330,	0	1.3671,	0	1.4021,	0	1.4379,	0
.04	1.2594,	0	1.3263,	0	1.3968,	0	1.4711,	0	1.5492,	0	1.6315,	0	1.7183,	0	1.8096,	0	1.9068,	0	2.0071,	0	2.1158,	0
.06	1.4376,	0	1.5689,	0	1.6862,	0	1.8261,	0	1.9777,	0	2.1418,	0	2.3196,	0	2.5121,	0	2.7207,	0	2.9465,	0	3.1910,	0
.08	1.6662,	0	1.8810,	0	2.0760,	0	2.3160,	0	2.5837,	0	2.8823,	0	3.2155,	0	3.5871,	0	4.0017,	0	4.4642,	0	4.9802,	0
.10	1.9820,	0	2.2822,	0	2.5279,	0	3.0259,	0	3.4942,	0	4.0119,	0	4.6195,	0	5.3192,	0	6.1248,	0	7.0625,	0	8.1206,	0
.12	2.4453,	0	2.9135,	0	3.4718,	0	4.1364,	0	4.9286,	0	5.8726,	0	6.9972,	0	8.3373,	0	9.9340,	0	1.1837,	1	1.4103,	1
.14	3.2455,	0	4.0155,	0	4.9676,	0	6.1458,	0	7.6036,	0	9.4070,	0	1.1638,	1	1.4398,	1	1.7813,	1	2.2039,	1	2.7266,	1
.16	3.9730,	0	5.0176,	0	6.3369,	0	8.0031,	0	1.0107,	1	1.2766,	1	1.6121,	1	2.0380,	1	2.5714,	1	3.2475,	1	4.1014,	1
.18	5.4681,	0	7.0597,	0	9.1181,	0	1.1778,	1	1.5210,	1	1.9645,	1	2.5372,	1	3.2770,	1	4.2324,	1	5.4663,	1	7.0801,	1
.185	7.5195,	0	9.8195,	0	1.2839,	1	1.6767,	1	2.1949,	1	2.8696,	1	3.7522,	1	4.9080,	1	6.4146,	1	8.3669,	1	1.0966,	2
.17	6.6624,	0	8.8295,	0	1.1701,	1	1.5506,	1	2.0548,	1	2.7252,	1	3.6089,	1	4.7825,	1	6.3382,	1	8.3994,	1	1.1131,	2
.175	4.0228,	0	5.3875,	0	7.2156,	0	9.6669,	0	1.2942,	1	1.7337,	1	2.3210,	1	3.1080,	1	4.1618,	1	5.5730,	1	7.4650,	1
.18	3.1143,	0	4.1990,	0	5.6581,	0	7.6259,	0	1.0276,	1	1.3844,	1	1.8648,	1	2.5112,	1	3.3812,	1	4.5536,	1	6.1258,	1
.185	2.5651,	0	3.4655,	0	4.6826,	0	6.3229,	0	8.5318,	0	1.1504,	1	1.5500,	1	2.0868,	1	2.8065,	1	3.7714,	1	5.0681,	1
.19	2.1699,	0	2.9279,	0	3.9419,	0	5.2981,	0	7.1102,	0	9.5257,	0	1.2718,	1	1.6928,	1	2.2471,	1	2.9735,	1	3.9229,	1
.195	1.8617,	0	2.5757,	0	3.5245,	0	4.4129,	0	5.8082,	0	7.6445,	0	1.0451,	1	1.2860,	1	1.6436,	1	2.0737,	1	2.5737,	1
.20	1.6103,	0	2.1257,	0	2.8998,	0	3.8006,	0	4.8483,	0	5.7800,	0	7.1050,	0	8.4735,	0	9.6697,	0	1.0277,	1	9.5511,	0
.21	1.2133,	0	1.6140,	0	1.8242,	0	2.0874,	0	2.1774,	0	1.8554,	0	8.2515,	-1	-1.6642,	0	-6.6914,	0	-1.5089,	1	-3.2854,	1
.22	9.0427,	-1	9.9909,	-1	9.8253,	-1	6.1244,	-1	-3.6941,	-1	-2.4701,	0	-6.5994,	0	-1.4259,	1	-2.7938,	1	-5.1715,	1	-9.2215,	1
.23	6.5060,	-1	5.4114,	-1	1.1911,	-1	-9.2106,	-1	-3.1171,	0	-7.3626,	0	-1.5359,	1	-2.9663,	1	-5.4746,	1	-9.7910,	1	-1.6603,	2
.24	4.3456,	-1	1.2106,	-1	-5.7453,	-1	-2.4924,	0	-6.0921,	0	-1.2942,	1	-2.5514,	1	-4.6919,	1	-8.7346,	1	-1.5815,	2	-2.7029,	2
.25	2.9049,	-1	-2.7419,	-1	-1.5128,	0	-3.9343,	0	-9.3088,	0	-1.9105,	1	-3.7087,	1	-6.9218,	1	-1.2566,	2	-2.2314,	2	-3.8915,	2
.26	7.7457,	-2	-8.5220,	-1	-2.3325,	0	-5.8288,	0	-1.2786,	1	-2.5870,	1	-4.9951,	1	-8.8200,	1	-1.6914,	2	-3.0083,	2	-5.2420,	2
.27	-7.5387,	-2	-1.0180,	0	-2.1516,	0	-7.6306,	0	-1.6450,	1	-3.3188,	1	-6.4007,	1	-1.1937,	2	-2.1675,	2	-3.8463,	2	-6.6906,	2
.28	-2.1586,	-1	-1.3739,	0	-4.0026,	0	-9.2861,	0	-2.0306,	1	-4.0977,	1	-7.9008,	1	-1.4724,	2	-2.6674,	2	-4.7133,	2	-8.1388,	2
.29	-3.4674,	-1	-1.7262,	0	-4.3560,	0	-1.1408,	1	-2.4401,	1	-4.9151,	1	-9.4521,	1	-1.7587,	2	-3.1891,	2	-5.5499,	2	-9.4528,	2
.30	-4.6578,	-1	-2.0737,	0	-5.7237,	0	-1.3398,	1	-2.9610,	1	-5.7640,	1	-1.1516,	2	-2.0423,	2	-3.6441,	2	-6.2849,	2	-1.0489,	3
.32	-6.9816,	-1	-2.7621,	0	-7.4987,	0	-1.7479,	1	-3.7254,	1	-7.4439,	1	-1.4104,	2	-2.5448,	2	-4.3684,	2	-7.0907,	2	-1.0728,	3
.34	-9.1032,	-1	-3.4502,	0	-9.3159,	0	-2.1667,	1	-4.5850,	1	-9.0166,	1	-1.6591,	2	-2.8492,	2	-4.5038,	2	-4.7079,	2	-7.1038,	2
.36	-1.1144,	0	-4.1433,	0	-1.1187,	1	-2.5838,	1	-5.3870,	1	-1.0287,	2	-1.7953,	2	-2.8037,	2	-3.6759,	2	-3.1445,	2	2.0631,	2
.38	-1.3099,	0	-4.8496,	0	-1.3041,	1	-2.9857,	1	-6.0735,	1	-1.0927,	2	-1.7572,	2	-2.2484,	2	-1.5332,	2	3.0880,	2	1.7608,	3
.40	-1.5042,	0	-5.5636,	0	-1.4896,	1	-3.3528,	1	-6.5651,	1	-1.1062,	2	-1.4800,	2	-1.0453,	2	2.1517,	2	1.2490,	3	3.8506,	3
.42	-1.7007,	0	-6.2976,	0	-1.6731,	1	-3.6724,	1	-6.7977,	1	-1.0062,	2	-9.0994,	1	2.0764,	2	7.3761,	2	2.4363,	3	6.1266,	3
.44	-1.9017,	0	-7.0612,	0	-1.8511,	1	-3.9228,	1	-6.6891,	1	-7.9219,	1	-8.6908,	-1	3.8704,	2	1.3792,	3	3.6887,	3	7.9344,	3
.46	-2.1098,	0	-7.8268,	0	-2.0201,	1	-4.0817,	1	-6.1600,	1	-4.3841,	1	1.2361,	2	6.5598,	2	2.0621,	3	4.7035,	3	8.3832,	3
.48	-2.3272,	0	-8.6267,	0	-2.1317,	1	-4.1284,	1	-5.1338,	1	6.4875,	0	3.5609,	2	1.0487,	3	2.6812,	3	5.0741,	3	6.3901,	3
.50	-2.5575,	0	-9.5268,	0	-2.2206,	1	-4.0322,	1	-3.5417,	1	7.2131,	1	4.6080,	2	1.4009,	3	3.0106,	3	4.3463,	3	1.1653,	3

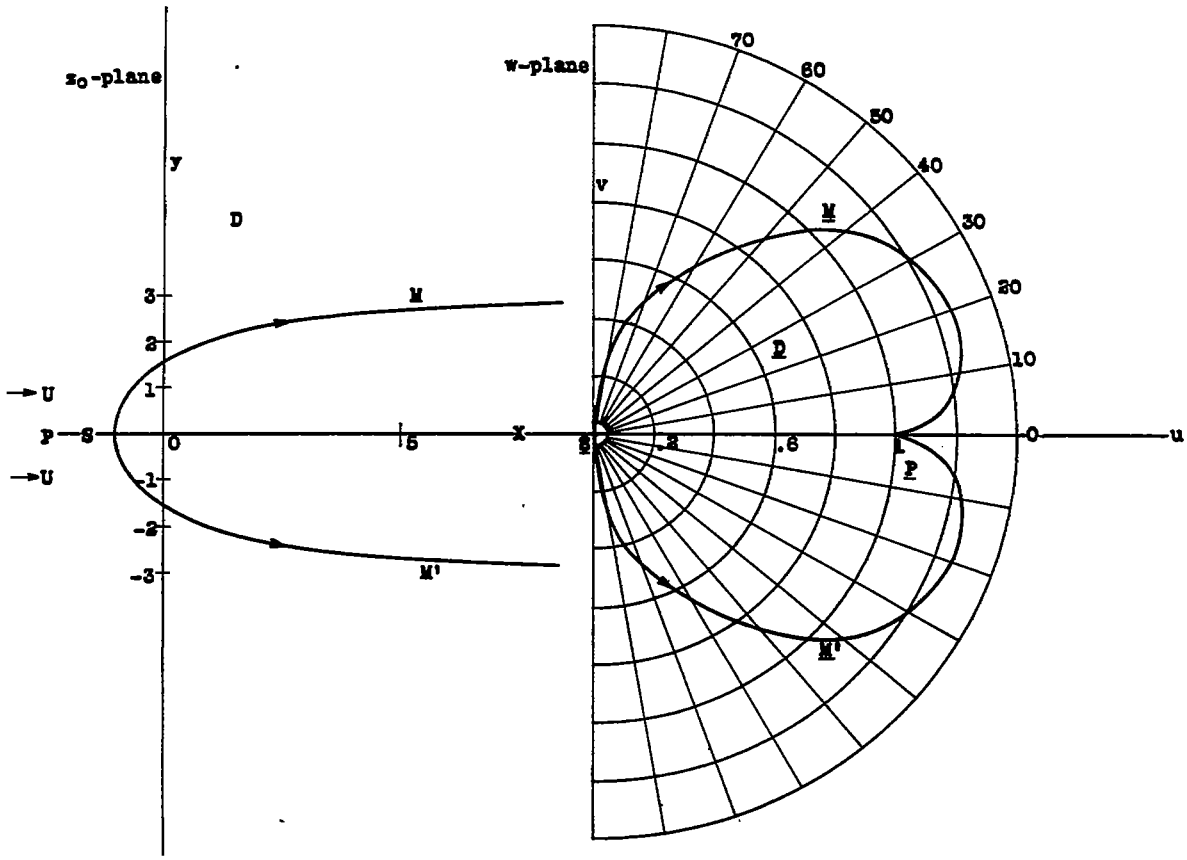

Figure 1.- The mapping of the whole plane D.

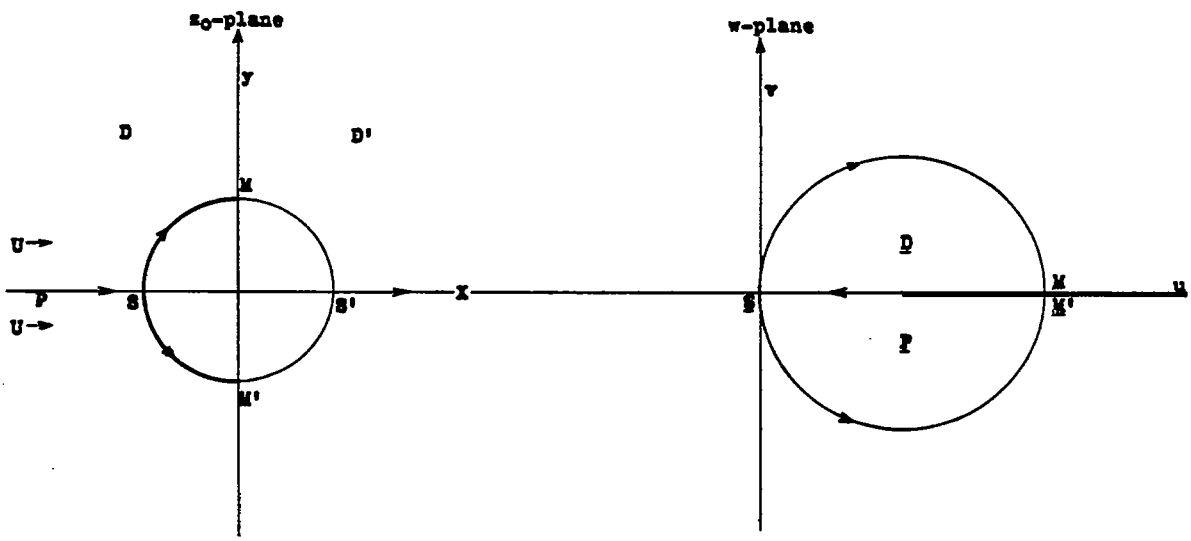

Figure 2.- The mapping of the half plane D.

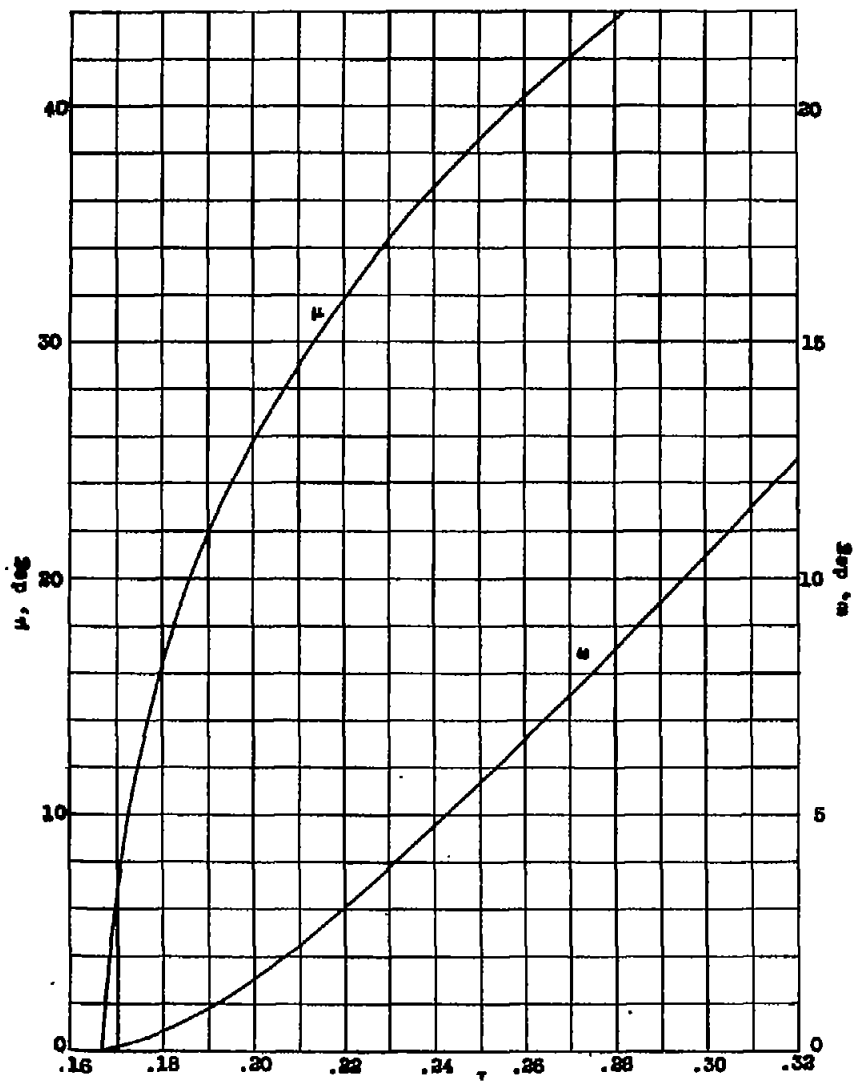

Figure 3.- $\omega(\tau)$; $\mu(\tau)$; $\gamma = 1.405$.

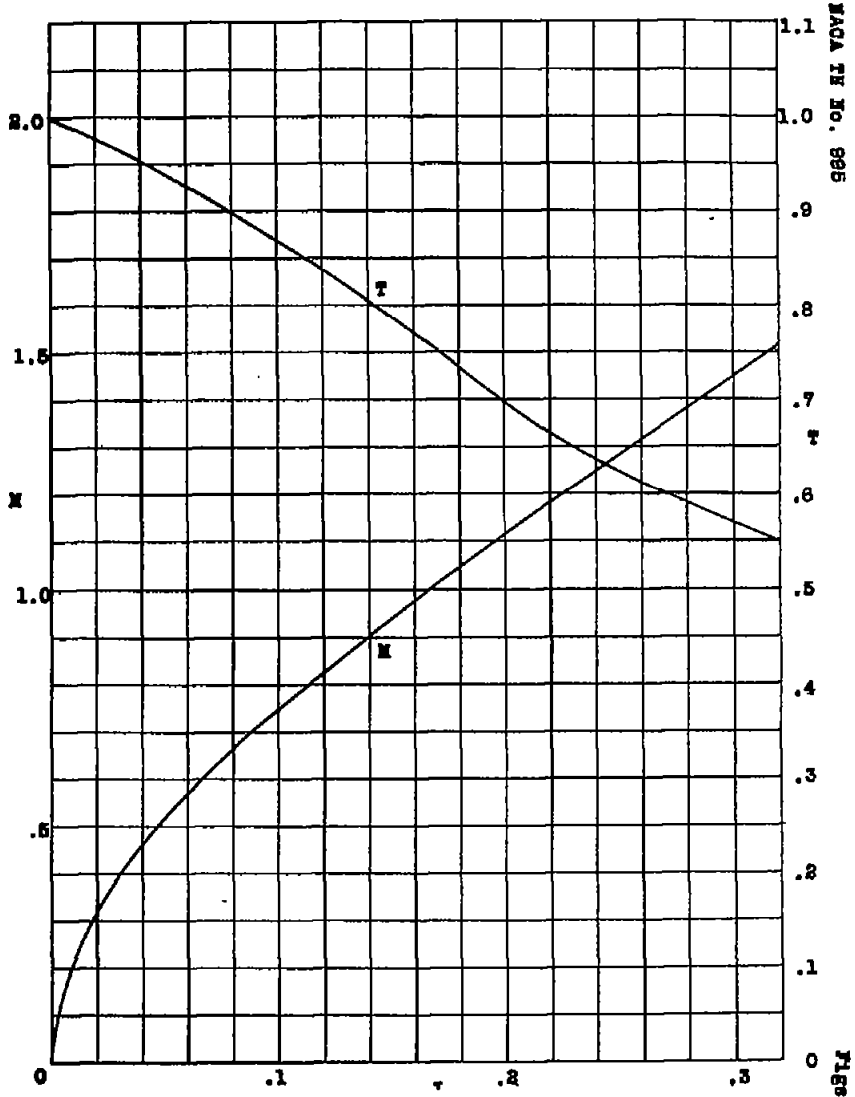

Figure 4.- $T(\tau)$; $M(\tau)$; $\gamma = 1.405$.

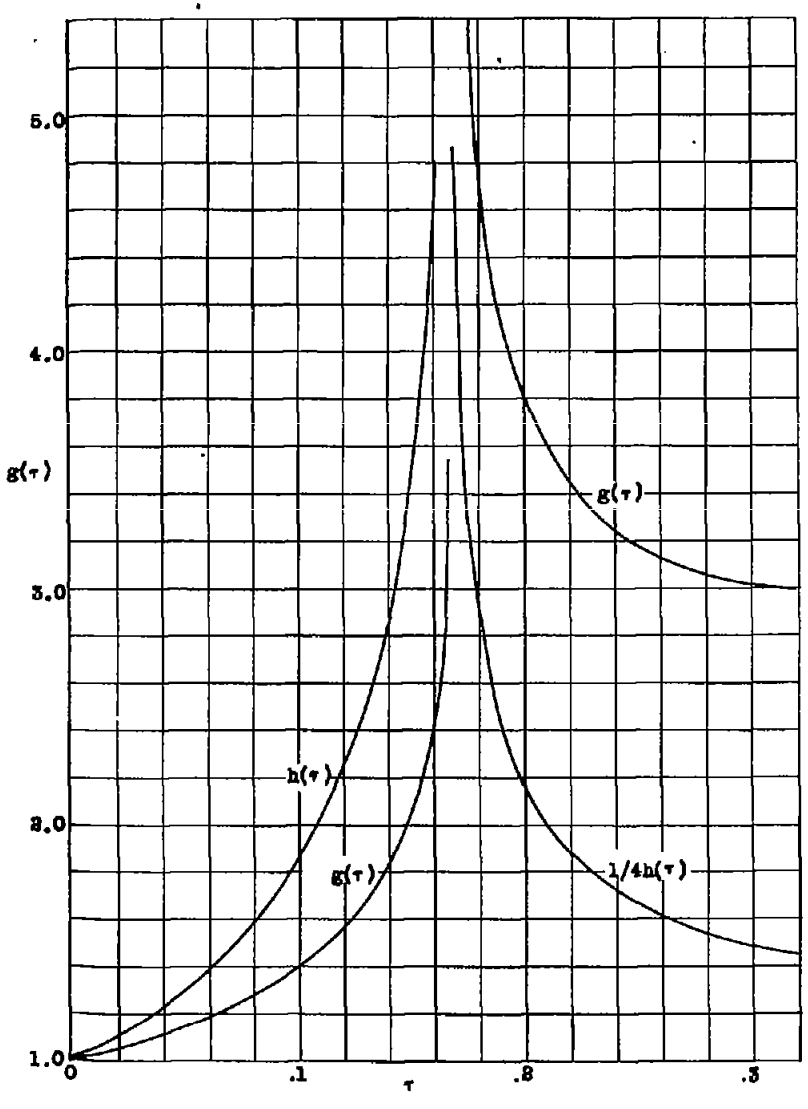

Figure 5.- g(τ); h(τ); γ = 1.405.

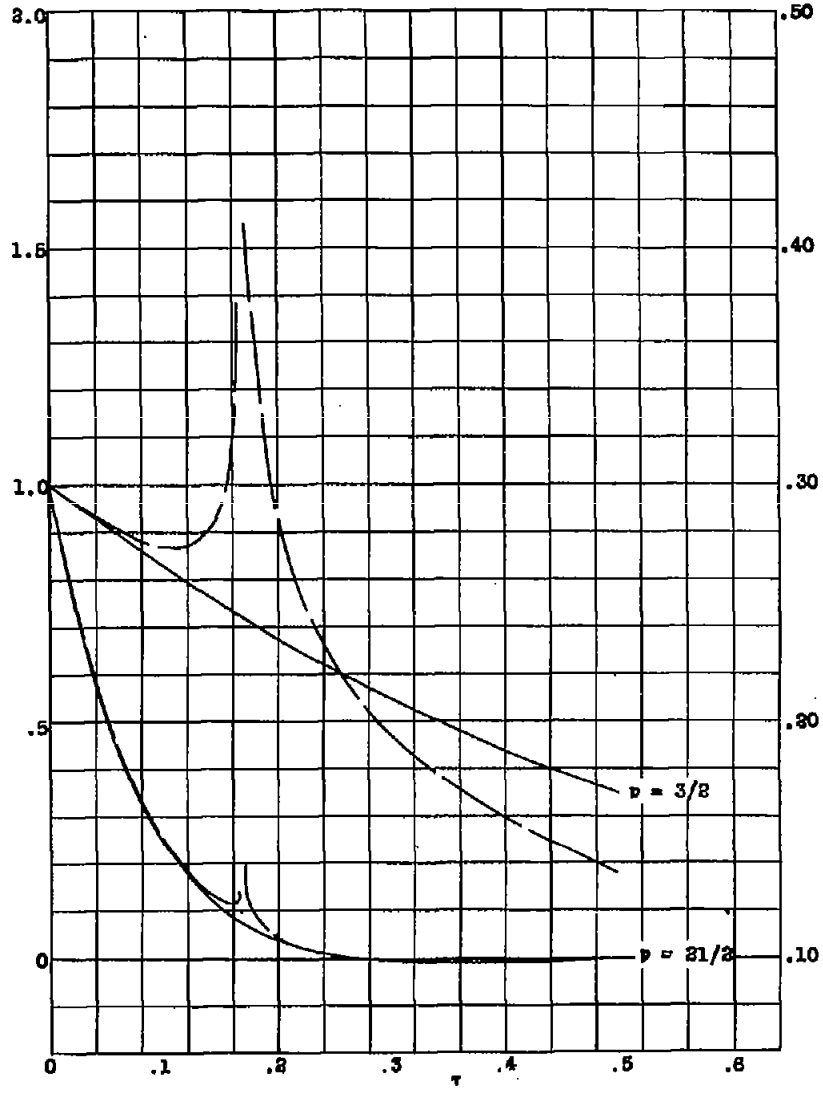

Figure 6.- $E_p(\tau)$; γ = 1.405. The dash line denotes the approximate values of $E_p(\tau)$.

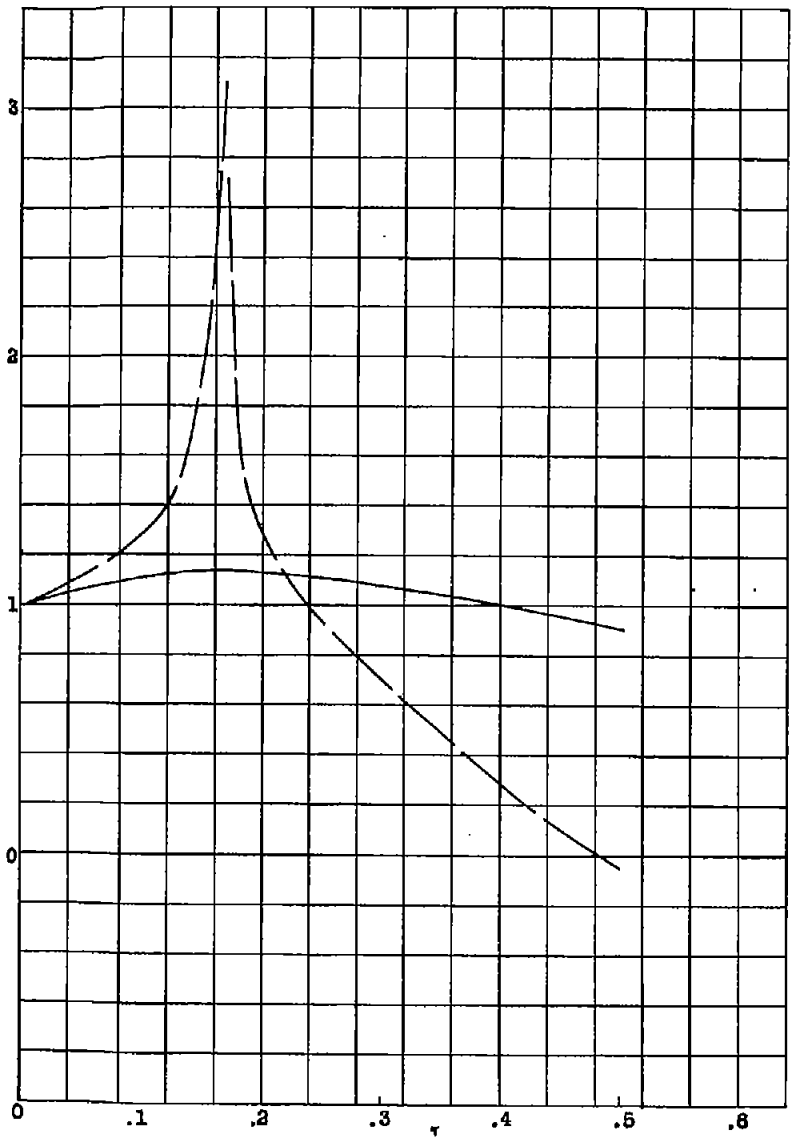

Figure 7.- $\underline{F} - 3/2(\tau)$; $\gamma = 1.405$.

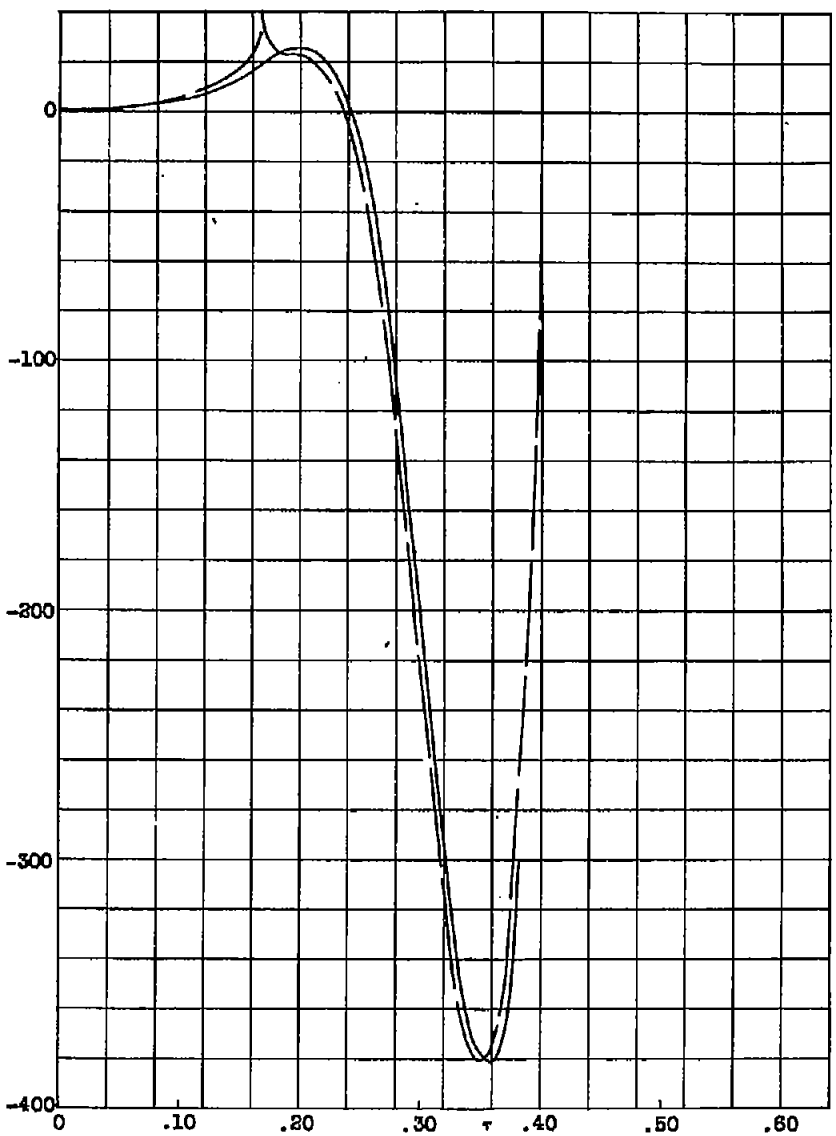

Figure 8.- $\underline{F} - 21/2(\tau)$; $\gamma = 1.405$.

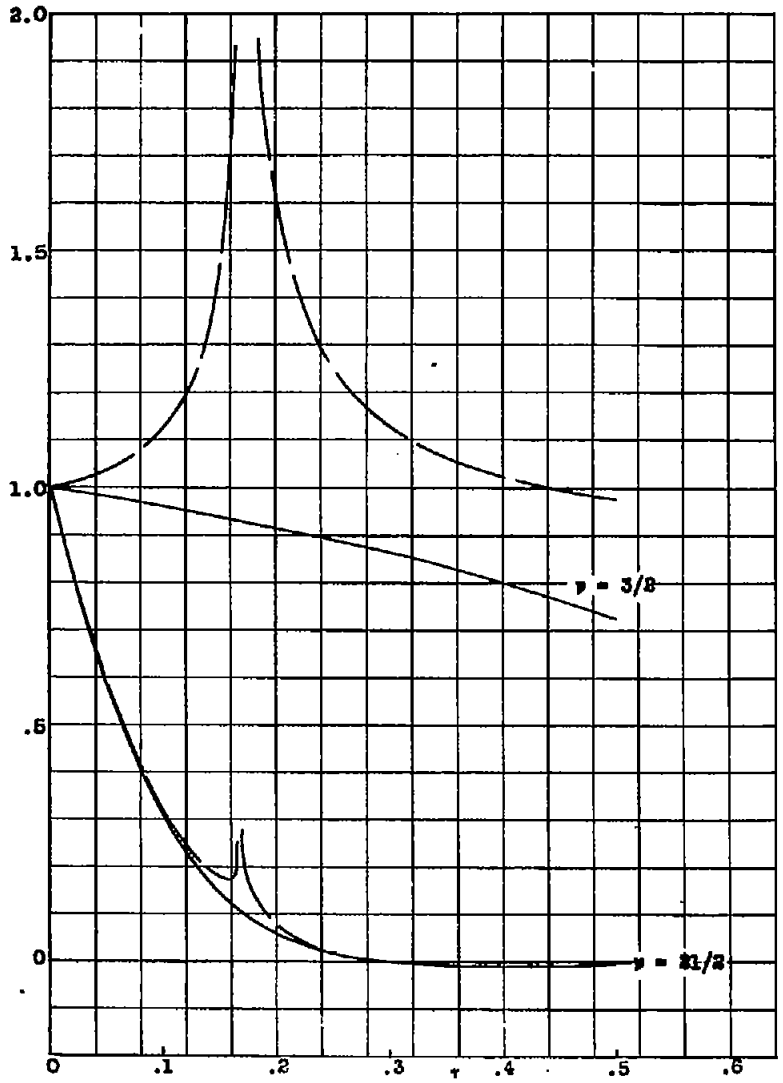

Figure 9.- $\bar{F}_\nu(\tau)$; $\gamma = 1.405$.

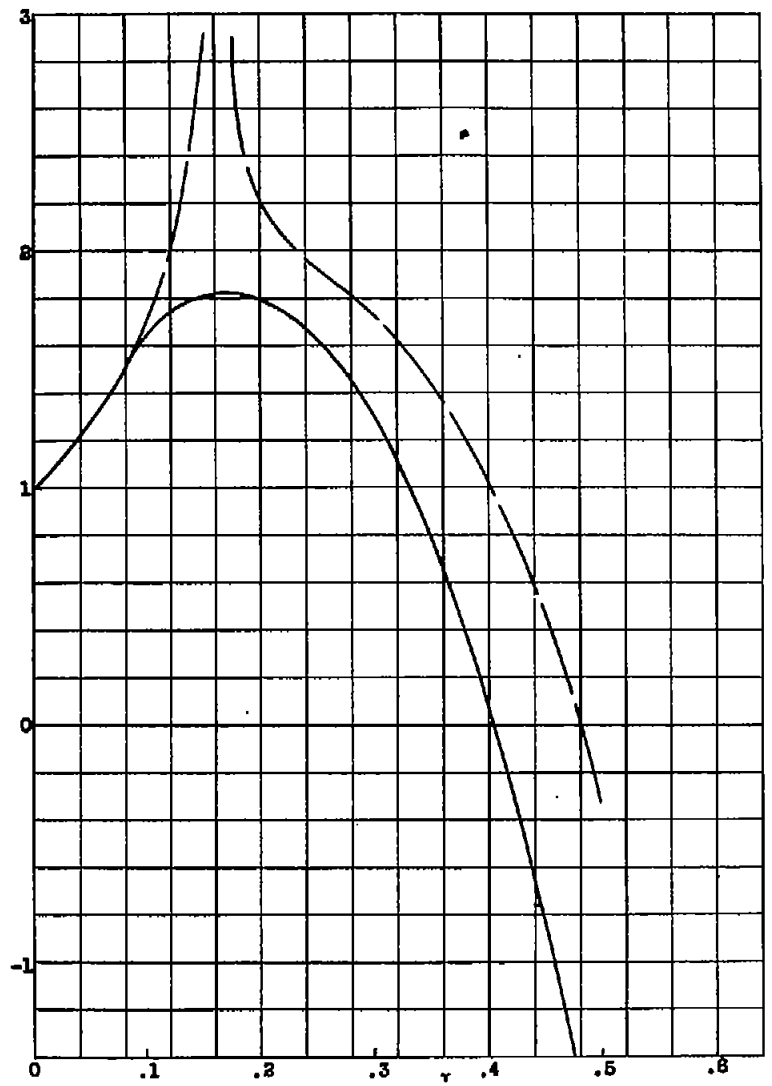

Figure 10.- $\bar{F} - 3/2(\tau)$; $\gamma = 1.405$.

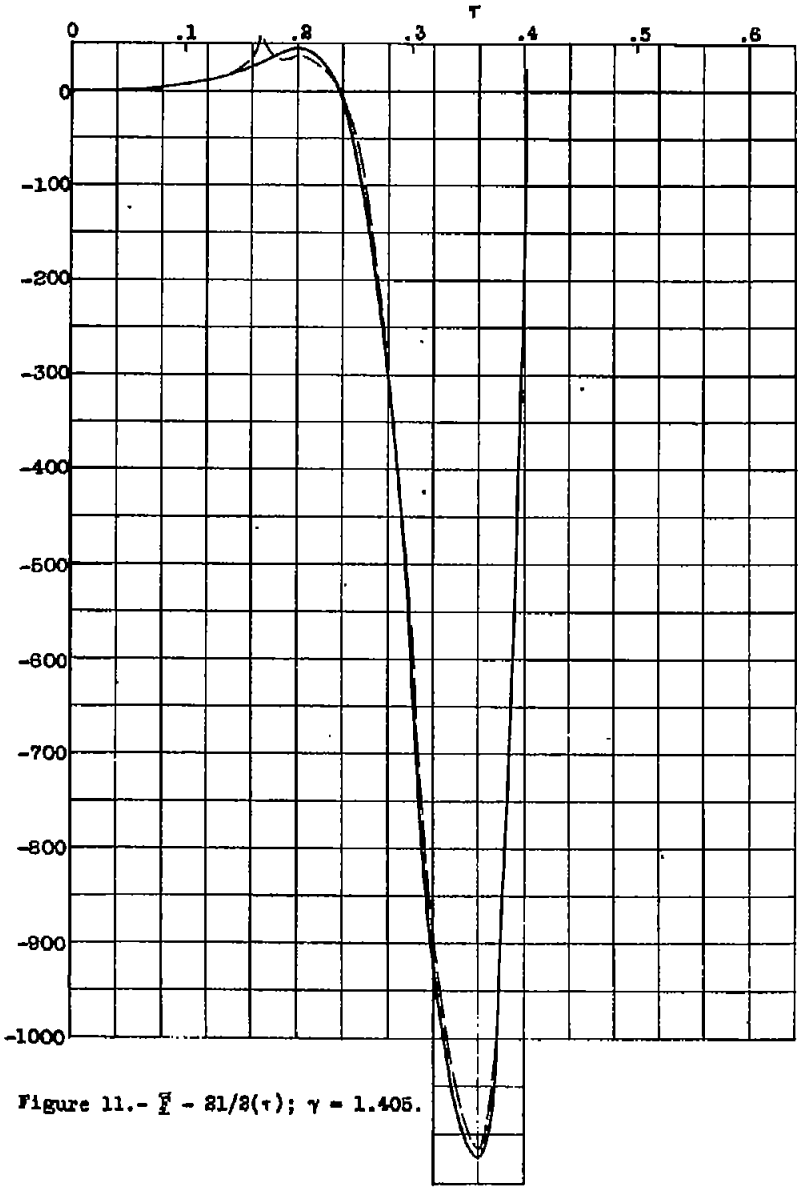

Figure 11.- $\bar{F}$ - 21/2(τ); γ = 1.405.

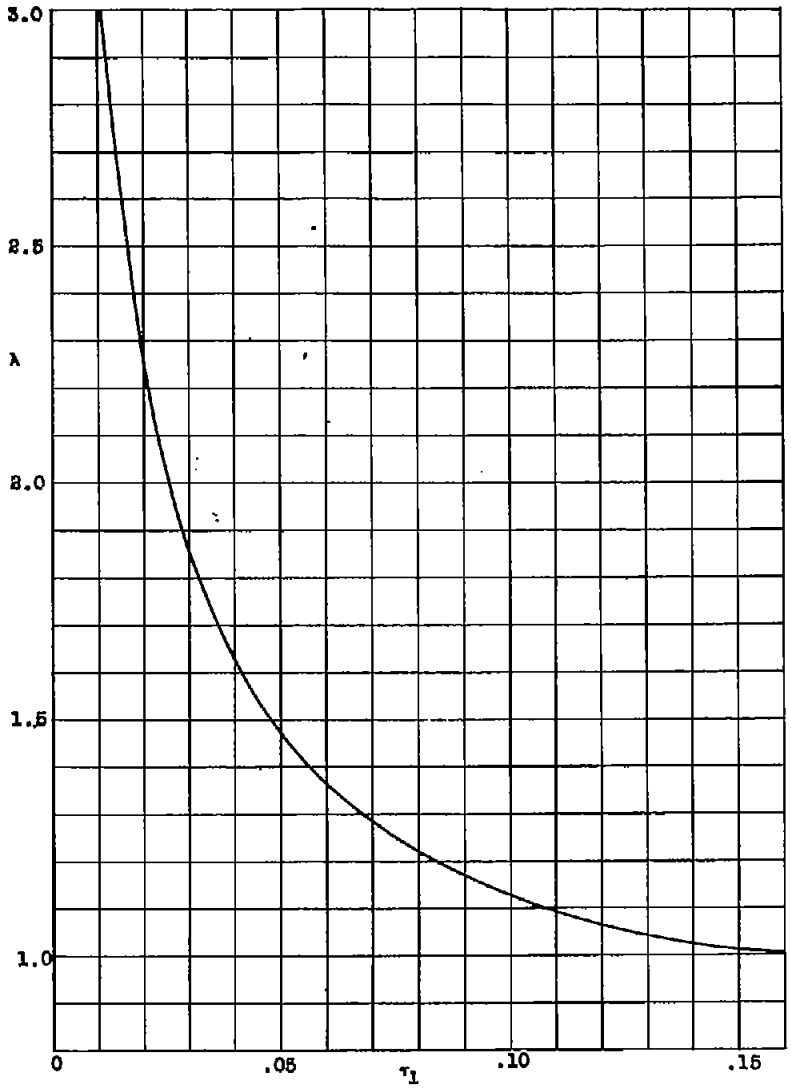

Figure 12.- λ(τ_1); γ = 1.405.

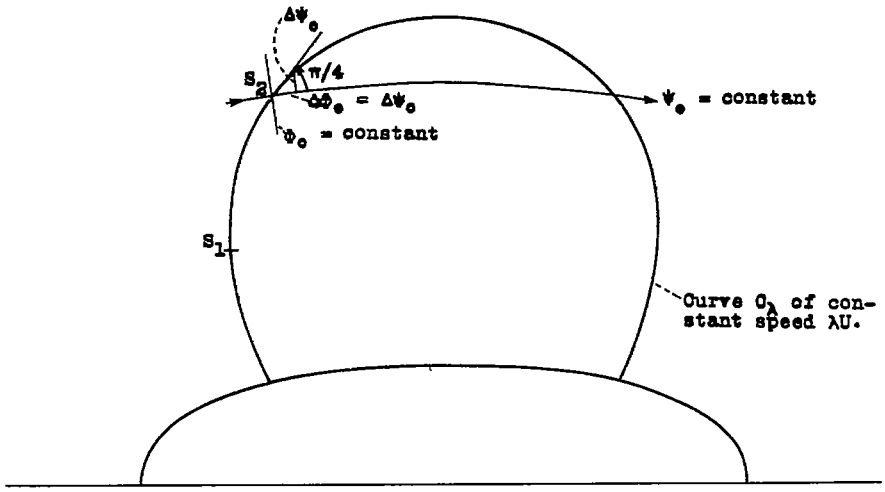

Figure 13.- Geometrical condition of eq. (20.5).

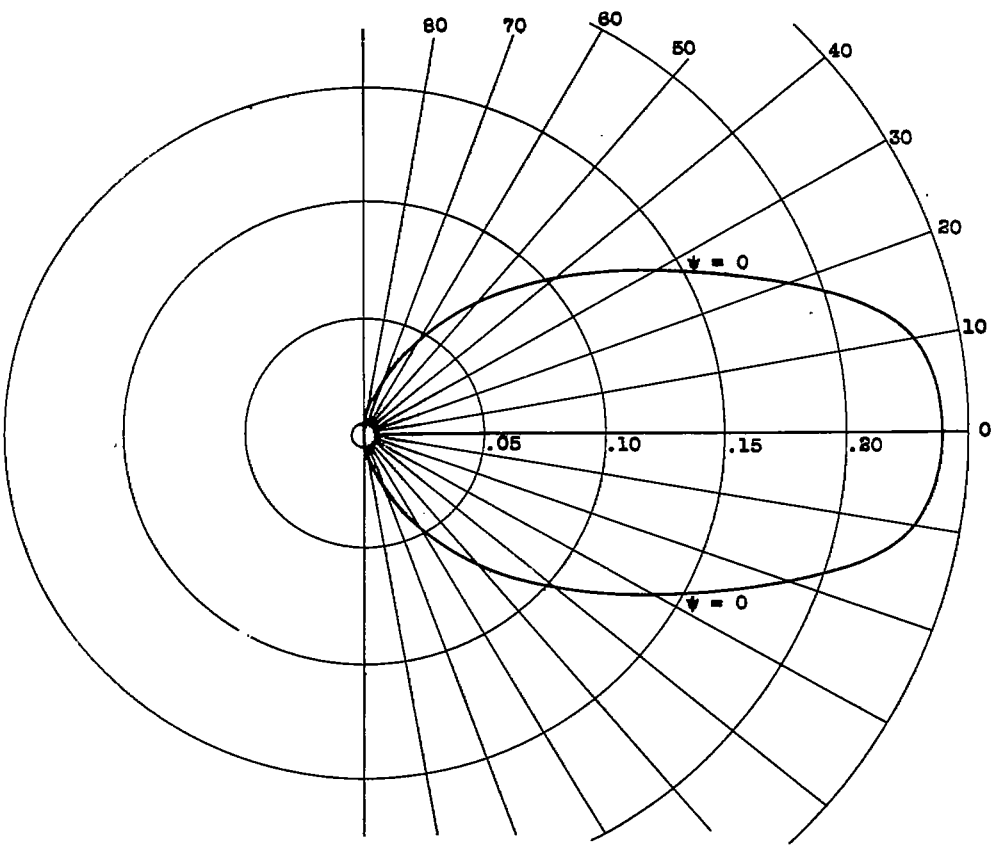

Figure 14.- The compressible flow in τ, θ-plane.
$\epsilon = 1/2$; $M_1 = 0.6$.

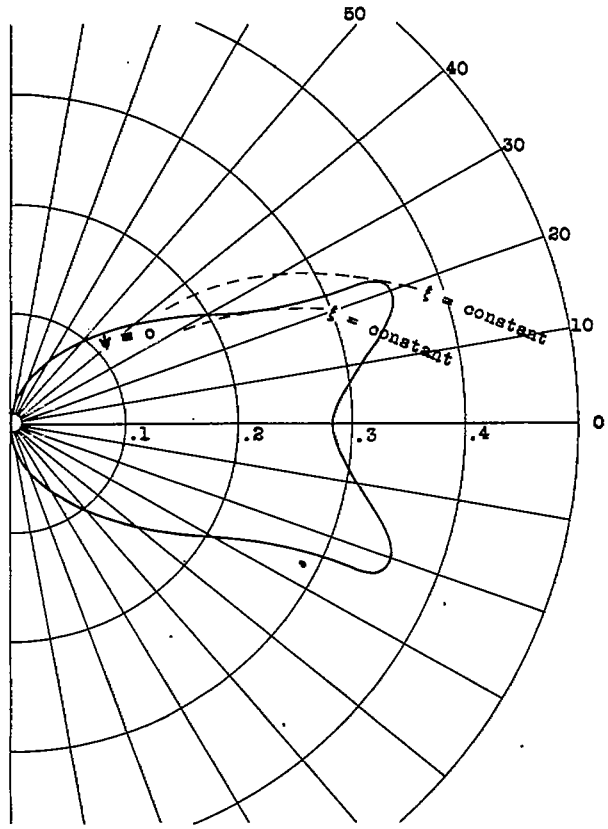

Figure 15.- The compressible flow in τ, θ-plane. $\epsilon = 1/3$; $M = 0.7$.

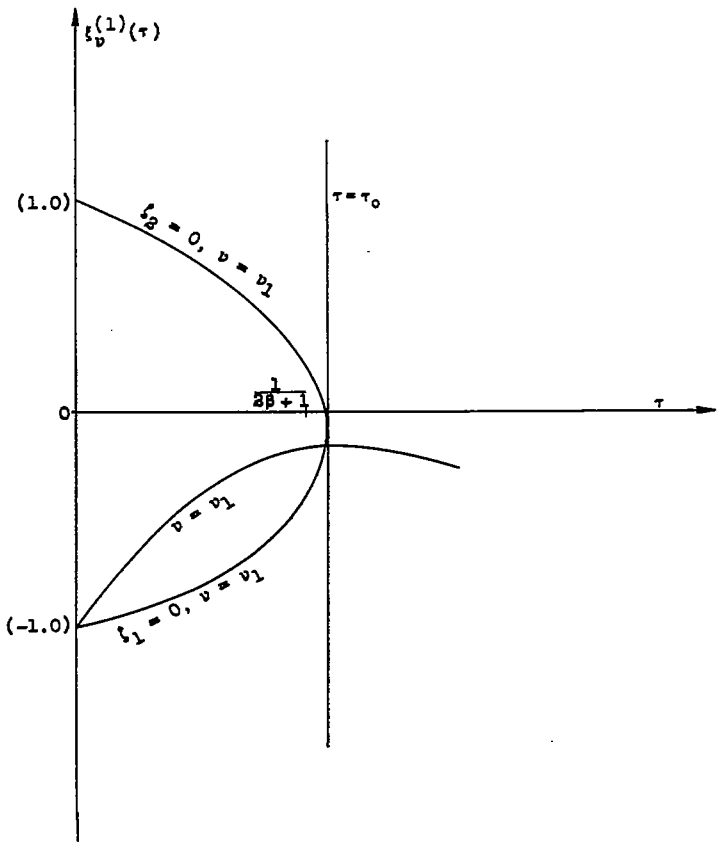

Figure 16.- The behavior of the integral-curve; $\zeta_\nu^{(1)}(\tau)$.