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POTENTIAL FLOW ABOUT ELONGATED BODIES OF REVOLUTION

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SUMMARY

This report presents a method of solving the problem of axial and transverse potential flows around arbitrary elongated bodies of revolution. The solutions of Laplace's equation for the velocity potentials of the axial and transverse flows, the system of coordinates being an elliptic one in a meridian plane, are known to be of the following form:

\[ \phi = \sum_{n=1}^{\infty} A_n Q_n(\lambda) P_n(\mu) \] (axial flow)

\[ \phi = \sum_{n=1}^{\infty} A_n Q_n(\lambda) P_n(\mu) \cos \theta \] (transverse flow)

If a power-series development of \( \lambda \) in \( \mu \) is assumed as the equation of the meridian profile in elliptic coordinates, the boundary conditions of the two types of flow yield linear equations for the determination of the coefficients \( A_n \) and \( A_m \). It is further shown that a knowledge of these coefficients leads directly to the sink-source and doublet distributions corresponding to the axial and transverse flows, respectively.

The theory is applied to a body of revolution obtained from a symmetrical Joukowsky profile, a shape resembling an airship hull. The pressure distribution and the transverse-force distribution are calculated and serve as examples of the procedure to be followed in the case of an actual airship. A section on the determination of inertia coefficients is also included in which the validity of some earlier work is questioned.

INTRODUCTION

There are two methods of handling the problem of potential flow about a body of revolution. One, the indirect method first used by Taylor (reference 1) and by G. Fuhrmann (reference 2) who computed the pressure distribution by the method of sources and sinks suggested by Rankine. Fuhrmann assumed certain sink-source distributions and calculated the pressure distribution for the streamline body resulting from the assumed sink-source system. He also constructed models of the calculated shapes and measured the pressure distributions over them when placed in a wind tunnel.

The other method, developed by von Kármán (reference 3), considered the direct problem; i.e., the calculation of the pressure distribution over a given streamline shape. He approximated the requisite sink-source distribution by a computed continuous system of sinks and sources arranged in stepwise constant intensity. The various strengths were determined from the condition that the airship hull is a streamline surface in the parallel flow and the flow induced by the sinks and sources. By satisfying this condition at an arbitrary number of points equal to the number of unknown sink and source segments, von Kármán obtained a system of linear equations for the determination of the unknown strengths of the sink-source distribution. He also treated the case of transverse flow (references 3 and 4) by the distribution of doublets along the axis of symmetry of the body of revolution and calculated the strengths of the various doublet segments in a manner similar to that used for the sink-source intensities.

The present paper is an attempt to treat the direct problem according to the methods of the potential theory. Thus, Laplace's equation for the velocity potential is set up in a system of elliptic-cylindrical coordinates and solved in conjunction with the appropriate boundary conditions for axial and transverse flows. It is then assumed that a power-series development of \( \lambda \) in \( \mu \) represents the meridian profile of the elongated body of revolution. The boundary conditions for the two types of flow may then be expressed in the form of power series in \( \mu \) valid for the entire range of \( \mu \). This method leads to two sets of linear equations, each set infinite in number of equations and each equation containing an infinite number of unknown coefficients which serve to determine the velocity potentials for the axial and transverse flows. Incidental to the major task of determining these coefficients, the sink-source and doublet distributions corresponding to the axial and transverse flows are also determined. Thus the results of this method are essentially the same as those obtained by the method of von Kármán but are obtained in a more rigorous and
direct manner. In von Kármán's method, approximations are made prior to the analysis; whereas, in the method presented in this paper, approximations are made after the analysis has been carried through in a rigorous manner.

FUNDAMENTAL EQUATIONS

The fluid motion is assumed to be steady and irrotational. There then exists a velocity potential $\phi$, which is, in general, a function of the rectangular Cartesian coordinates $(x, y, z)$. In cases of rotational symmetry, however, it is appropriate to introduce the cylindrical coordinates $(z, \rho, \theta)$ where $z$ denotes the distance along the axis of symmetry, $\rho(=\sqrt{x^2+y^2})$ the perpendicular distance from this axis, and $\theta$ the angle between the $(z, \rho)$ and $(x, y)$ planes. (See fig. 1.)

Furthermore, since only elongated surfaces of revolution are to be considered it is natural to introduce a prolate-elliptic coordinate system in the $(z, \rho)$ plane. The equations of transformation from the coordinates $(z, \rho)$ to the prolate-elliptic coordinates $(t, \eta)$ are:

$$
\begin{align*}
    z &= 2a \cosh t \cos \eta, \\
    \rho &= 2a \sinh t \sin \eta
\end{align*}
$$

where $0 \leq t \leq \infty$ and $0 \leq \eta < 2\pi$

Thus $t=\text{constant}$ and $\eta=\text{constant}$ represent confocal ellipses and hyperbolae, respectively, the distance between the foci being $4a$.

For any point in space $P(x, y, z)$ then

$$
\begin{align*}
    x &= 2a(\lambda^2-1)^{1/2}(1-\mu^2)^{1/2} \cos \theta, \\
    y &= 2a(\lambda^2-1)^{1/2}(1-\mu^2)^{1/2} \sin \theta, \\
    z &= 2a \lambda \mu
\end{align*}
$$

where $\lambda = \cosh t$ and $\mu = \cos \eta$.

If, furthermore, the fluid is incompressible the velocity potential $\phi$ satisfies Laplace's equation $\Delta \phi = 0$ and since the $(\lambda, \mu, \theta)$ system of coordinates is an orthogonal one, takes the form:

$$
\frac{\partial}{\partial \lambda} \left[ (\lambda^2-1) \frac{\partial \phi}{\partial \lambda} \right] + \frac{\partial}{\partial \mu} \left[ (1-\mu^2) \frac{\partial \phi}{\partial \mu} \right] + \left( \frac{1}{\lambda^2-1} + \frac{1}{1-\mu^2} \right) \frac{\partial^2 \phi}{\partial \theta^2} = 0
$$

FLOW PARALLEL TO THE AXIS OF SYMMETRY

In this case the flow is the same for all meridian planes $(z, \rho)$ and therefore the velocity potential $\phi$ is a function only of $\lambda$ and $\mu$. Equation (3) then reduces to

$$
\frac{\partial}{\partial \lambda} \left[ (\lambda^2-1) \frac{\partial \phi}{\partial \lambda} \right] + \frac{\partial}{\partial \mu} \left[ (1-\mu^2) \frac{\partial \phi}{\partial \mu} \right] = 0
$$

(4)

If this equation is to be satisfied by a product

$$
\phi = L(\lambda) M(\mu)
$$

it follows that

$$
\frac{1}{L(\lambda)} \frac{d}{d\lambda} \left[ (\lambda^2-1) \frac{dL(\lambda)}{d\lambda} \right] = -\frac{1}{M(\mu)} \frac{d}{d\mu} \left[ (1-\mu^2) \frac{dM(\mu)}{d\mu} \right]
$$

which separates into two ordinary differential equations

$$
\frac{d}{d\lambda} \left( (1-\lambda^2) \frac{dL(\lambda)}{d\lambda} \right) + cL = 0
$$

$$
\frac{d}{d\mu} \left( (1-\mu^2) \frac{dM(\mu)}{d\mu} \right) + cM = 0
$$

(5)

where $c$ is an arbitrary constant.

Furthermore, if $c=-n(n+1)$, each of these equations is of the Legendre type and therefore the general solution of equation (4) is

$$
\phi = \sum_{n=0}^{\infty} A_n P_n(\lambda) P_1(\mu)
$$

(6)

This expression for $\phi$ has a singularity at infinity since $P_n(\lambda)$ is a polynomial of the $n$th degree in $\lambda$ and is therefore infinite for $\lambda=\infty$. Since the region outside a surface is to be considered and since it must include the region at infinity, another solution for $L(\lambda)$ is required. This solution, linearly independent of $P_n(\lambda)$, is the zonal harmonic of the second kind and is denoted by $Q_n(\lambda)$ where

$$
Q_n(\lambda) = \frac{d}{d\lambda} \int_{\lambda}^{\infty} \frac{d\lambda}{\left[ P_n(\lambda)^2 \right]} (\lambda^2-1)
$$

(7)

It vanishes for $\lambda=\infty$ but has a singular point for $\lambda=\pm1$ where it is infinite like $\log(\lambda \pm 1)$.

Thus, for example, since $P_0(\lambda)=1$, $P_1(\lambda)=\lambda$, it is found that

$$
Q_0(\lambda) = \int_{\lambda}^{\infty} \frac{d\lambda}{\lambda^2-1} = \frac{1}{2} \log \frac{\lambda+1}{\lambda-1} = \frac{1}{\lambda} \frac{1}{3\lambda^4} + \frac{1}{5\lambda^6} + \ldots
$$

where $|\lambda|>1$ and

$$
Q_1(\lambda) = \lambda \int_{\lambda}^{\infty} \frac{d\lambda}{\lambda^2(\lambda^2-1)} = \frac{1}{2} \log \frac{\lambda+1}{\lambda-1} - \frac{1}{\lambda} \frac{1}{3\lambda^4} + \frac{1}{5\lambda^6} + \ldots
$$

It may also be shown that

$$
Q_n(\lambda) = \frac{1}{2} P_n(\lambda) \log \frac{\lambda+1}{\lambda-1} - K_n(\lambda)
$$

where $K_n(\lambda)$ is a polynomial of the $(n-1)$th degree.
Another useful expression for \( Q_\alpha(\lambda) \) is that due to F. Neumann (reference 5); namely

\[
Q_\alpha(\lambda) = \frac{1}{2} \int_{-1}^{1} \frac{P_\alpha(\lambda)}{\lambda - \lambda_1} d\lambda_1
\]  

(8)

Expanding \( \frac{1}{\lambda - \lambda_1} \) in decreasing powers of \( \lambda \),

\[
Q_\alpha(\lambda) = \frac{1}{2} \sum_{i=0}^{n} \frac{1}{\lambda^{i+1}} \int_{-1}^{1} \lambda^i P_\alpha(\lambda_1) d\lambda_1
\]

Expressing \( \lambda^i \) in terms of zonal harmonics (reference 5)

\[
\lambda^i = \sum_{k=0}^{\frac{i}{2}} \frac{(2i-4k+1)i!}{(2i-2k+1)[2k] !} P_{i-2k}(\lambda_1)
\]

where the upper limit \( \frac{i}{2} \) used depends on whether \( i \) is even or odd and where \([2n]=2\cdot4\cdot6\ldots2n; [2n-1]=1\cdot3\cdot5\ldots(2n-1)\); \([0]=[1]=-1\). Also \((2n-1)!=[2n-1][2n-2]\).

Substituting this expression for \( \lambda^i \) in the foregoing equation for \( Q_\alpha(\lambda) \) it follows that

\[
Q_\alpha(\lambda) = \frac{1}{2} \sum_{i=0}^{n} \frac{1}{\lambda^{i+1}} \sum_{k=0}^{\frac{i}{2}} \frac{(2i-4k+1)i!}{(2i-2k+1)[2k] !} P_{i-2k}(\lambda_1)
\]

The zonal harmonics \( P_\alpha(\lambda_1) \) are orthogonal functions and satisfy the following relations:

\[
\int_{-1}^{1} P_\mu P_\nu \lambda^i d\lambda = \begin{cases} 0 & \text{if } \mu \neq \nu \\ 2 & \text{if } \mu = \nu \\ \frac{1}{2r+1} & \text{if } \mu = \nu = 1 \end{cases}
\]

Expanding the preceding expression for \( Q_\alpha(\lambda) \) with regard to \( i \) and writing the terms with equal indices of \( k \) in columns and adding these columns, there is obtained, using the orthogonal property of the \( P_\alpha(\lambda_1) \)'s, the following equation:

\[
Q_\alpha(\lambda) = \sum_{k=0}^{n} \frac{(n+2k)!}{(2n+2k+1)[2k] !} \frac{1}{\lambda^{n+2k+1}}
\]

This expression is convergent for \(|\lambda|>1\) and divergent for \(|\lambda| \leq 1\).

Instead of being given by equation (6) the velocity potential is now given by the following expression:

\[
\phi = \sum_{\alpha=0}^{n} A_\alpha Q_\alpha(\lambda) P_\alpha(\mu)
\]

(10)

which gives the general solution of equation (4) for regions outside a surface of revolution and extending to infinity.

In cases of rotational symmetry where the lines of flow are in meridian planes, it is convenient to introduce Stokes' stream function \( \psi \). This function arises from the statement that the fluid is incompressible (equation of continuity) and is related to the velocity potential \( \phi \) according to the following equations:

\[
\frac{\partial \psi}{\partial \rho} = \rho \frac{\partial \phi}{\partial z} \quad \text{and} \quad \frac{\partial \psi}{\partial z} = -\rho \frac{\partial \phi}{\partial \rho}
\]

(11)

The lines \( \psi = \text{constant} \) represent the streamlines. It may be remarked that, unlike the two-dimensional case where both the stream function and the velocity potential satisfy Laplace's equation, Stokes' stream function does not satisfy it.

The introduction of the variables \( \lambda, \mu \) into equations (11) by means of equations (1) leads to the following relations:

\[
\frac{\partial \psi}{\partial \lambda} = 2a(1-\mu^2) \frac{\partial \phi}{\partial \mu} \quad \text{and} \quad \frac{\partial \psi}{\partial \mu} = -2a(\lambda^2-1) \frac{\partial \phi}{\partial \lambda}
\]

(12)

If a substitution is made for \( \phi \) from equation (10) and \( P_\alpha(\mu) \) is replaced by its value obtained from Legendre's differential equation, that is:

\[
P_\alpha(\mu) = -\frac{1}{n(n+1)} \frac{d}{d\mu} \left[ (1-\mu^2) \frac{dP_\alpha}{d\mu} \right]
\]

it is found that

\[
\psi = 2a(1-\mu^2)(\lambda^2-1) \sum_{n=1}^{\infty} A_n \frac{dP_n}{d\mu} \frac{dQ_n}{d\lambda} + \text{constant}
\]

(13)

Furthermore, if the body of revolution is moving with a velocity \( U \) in the direction of the axis of symmetry \( z \), it may be conveniently supposed to be at rest and the fluid to have a translation \(-U\) superposed on its actual motion. This consideration adds a term \( 2a\lambda U\mu \) to the velocity potential and \( 2a^2U(1-\mu^2)(\lambda^2-1) \) to the stream function. Therefore

\[
\psi = 2a^2(1-\mu^2)(\lambda^2-1)U \left[ \sum_{n=1}^{\infty} A_n \frac{dP_n}{d\mu} \frac{dQ_n}{d\lambda} + \text{constant} \right]
\]

(14)

At the surface of the fixed body of revolution the normal velocity of the fluid must be zero and therefore the boundary must coincide with a streamline \( \psi = \text{constant} \), say \( 0 \). Hence the boundary condition at the surface is given by

\[
\sum_{n=1}^{\infty} A_n \frac{dP_n}{d\mu} \frac{dQ_n}{d\lambda} + aU = 0
\]

(15)

In order to find the velocity components \( u_\lambda, u_\mu \) in the directions of the coordinate lines \( \lambda, \mu \) respectively, it is to be noted that since the system of coordinates is an orthogonal one,

\[
u_\lambda = -\frac{\partial \phi}{\partial \lambda} \quad \text{and} \quad u_\mu = -\frac{\partial \phi}{\partial \mu}
\]

where

\[
d\lambda^2 = dx^2 + dy^2 = ds_\lambda^2 + ds_\mu^2
\]
By means of the equations of transformation (1), it is found that
\[ d\phi_a = 2a \left( \frac{\lambda^2 - \mu^2}{\lambda^2 - 1} \right) d\lambda. \]
and
\[ d\phi_a = 2a \left( \frac{\lambda^2 - \mu^2}{1 - \mu^2} \right) d\mu. \]
Therefore:
\[ \phi_a = -\frac{1}{2a} \left( \frac{\lambda^2 - 1}{\lambda^2 - \mu^2} \right) \frac{\partial \phi}{\partial \lambda} \]
and
\[ \phi_a = -\frac{1}{2a} \left( \frac{1 - \mu^2}{\lambda^2 - \mu^2} \right) \frac{\partial \phi}{\partial \mu}. \]
Hence:
\[ \phi^2 = \frac{1}{4a^2(\lambda^2 - \mu^2)} \left( \frac{\partial \phi}{\partial \lambda} \right)^2 + \left( 1 - \mu^2 \right) \frac{\partial \phi}{\partial \mu}^2 \]

**Sink-source distribution**

The distribution of sinks and sources is assumed to lie along the segment of the axis of symmetry \( z, \) \(-2a \leq z \leq 2a,\) and to be of intensity \( I(z) \) per unit length. At any point \((z, \rho)\) in any meridian plane the velocity potential due to this distribution is given (reference 6, p. 60) by
\[ \phi = \frac{1}{4\pi} \int_{-2a}^{2a} \frac{I(z)dz}{(z-z_1)^2 + \rho^2}. \]

For points lying on the \( z \) axis but outside the distribution, the velocity potential is given by the simplified expression
\[ \phi = \frac{1}{4\pi} \int_{-2a}^{2a} \frac{I(z_1)dz_1}{(z-z_1)^2}. \]

Substituting for \( z \) and \( z_1 \) \( 2a \lambda \) and \( 2a \lambda_i, \) respectively, the preceding equation takes the form
\[ \phi = \frac{1}{4\pi} \int_{-1}^{1} \frac{I(2a\lambda_i)dz}{\lambda_i}. \]

Finally, substituting for \( \phi \) from equation (10) and noting that \( P_n(1) = 1 \) for all values of \( n, \)
\[ \sum_{n=1}^{\infty} A_n Q_n(\lambda) = \frac{1}{4\pi} \int_{-1}^{1} \frac{I(2a\lambda_i)dz}{\lambda_i} \]
This is an integral equation for the unknown function \( I(2a\lambda_i). \) It may be solved in the following manner:

From F. Neumann's expression for \( Q_n(\lambda) \) given by equation (8) the following development is suggested for the distribution function:
\[ I(2a\lambda_i) = \sum_{n=0}^{\infty} a_n P_n(\lambda_i) \]
where
\[ -1 \leq \lambda_i \leq 1 \]
It then follows directly from equation (8) that
\[ \sum_{n=1}^{\infty} \left( A_n - \frac{1}{2\pi} a_n \right) Q_n(\lambda) = 0 \]
for all values of \( \lambda. \)
Hence
\[ a_n = 2\pi A_n \]
and
\[ I(2a\lambda_i) = 2\pi \sum_{n=1}^{\infty} A_n P_n(\lambda_i) \]

Thus, given the potential function \( \phi, \) that is the \( A_n^s, \) this expression determines the equivalent sink-source distribution.

**Flow normal to the axis of symmetry**

The differential equation for the velocity potential in the case of transverse flow is given by equation (3). Recalling that this expression is Laplace's equation in the coordinates \( \lambda, \mu, \theta, \) it may be solved by supposing \( \phi \) to be a product \( N(\lambda, \mu) R(\theta). \) Replacing \( \phi \) in equation (3) by this product, the following pair of differential equations is obtained:
\[ \frac{\partial^2 R}{\partial \lambda^2} + k^2 R = 0 \]
\[ \frac{\partial^2 R}{\partial \mu^2} \left( \lambda^2 - 1 \right) \frac{\partial N}{\partial \lambda} + \frac{\partial}{\partial \mu} \left( 1 - \mu^2 \right) \frac{\partial N}{\partial \mu} = 0 \]

The general solution of the first equation is given by
\[ R = A \cos k \theta + B \sin k \theta \]
where \( A \) and \( B \) are arbitrary constants. Putting \( N(\lambda, \mu) = L(\lambda) M(\mu) \) in the second equation leads to the following pair of ordinary differential equations:
\[ \frac{d^2 L}{d\lambda^2} \left( 1 - \lambda^2 \right) + \left( 1 - \mu^2 \right) \frac{d^2 M}{d\mu^2} = 0 \]

where \( c \) is an arbitrary constant.

Both of the latter equations are of the form of the differential equation for the associated Legendre functions provided that \( c = n(n + 1). \) Accordingly,
\[ M(\mu) = P_n^*(\mu) \]
and \( L(\lambda) = P_n^*(\lambda) \)

where, for example,
\[ P_n^*(\mu) = (1 - \mu^2)^{-\frac{1}{2}} \frac{d^n P_n(\mu)}{d\mu^n} \]

The general solution of equation (3) may then be written as
\[ \phi = \sum_{n=0}^{\infty} \sum_{\lambda=0}^{\infty} P_n^*(\mu) P_n^*(\lambda) [A_{n\lambda} \cos k \theta + B_{n\lambda} \sin k \theta] \]

This expression, however, has a singularity at infinity and since only the region outside a given surface of revolution is of interest, the infinite region, or the neighborhood of \( \lambda = \infty, \) must be considered. Therefore \( P_n^*(\lambda) \) is replaced by the associated Legendre function of the second kind \( Q_n^*(\lambda), \) where by definition,
\[ Q_n^*(\lambda) = (\lambda^2 - 1)^{\frac{1}{2}} \frac{d^n Q_n(\lambda)}{d\lambda^n} \]

Then
\[ \phi = \sum_{n=0}^{\infty} \sum_{\lambda=0}^{\infty} P_n^*(\mu) Q_n^*(\lambda) [A_{n\lambda} \cos k \theta + B_{n\lambda} \sin k \theta] \]

If the body of revolution moves with a uniform velocity \( V \) in the direction of the \( z \) axis, it may be
supposed to be at rest and the fluid to have a translation \(-V\) superposed on its actual motion. Then

\[ \phi = \phi_1 + zV \quad (23) \]

Consider the body profile in any one of the meridian planes \(\theta\). At any arbitrary point of it the normal derivative of \(\phi\) is given by

\[ -\frac{\partial \phi}{\partial n} = \frac{\partial \phi}{\partial s_\lambda} ds_\lambda - \frac{\partial \phi}{\partial s_\mu} ds_\mu \quad (24) \]

Since the normal velocity along the meridian curve is zero, it follows from equations (23) and (24) that

\[ \left( \frac{\partial \phi_1}{\partial s_\lambda} + V \frac{\partial z}{\partial s_\lambda} \right) ds_\mu = -\left( \frac{\partial \phi_1}{\partial s_\mu} + V \frac{\partial z}{\partial s_\mu} \right) ds_\lambda = 0 \]

Also, \( z = \rho \cos \theta \), so that

\[ \frac{\partial z}{\partial s_\lambda} = \cos \theta \frac{\partial \rho}{\partial s_\lambda} \quad \text{and} \quad \frac{\partial z}{\partial s_\mu} = \cos \theta \frac{\partial \rho}{\partial s_\mu} \]

Therefore

\[ \left( \frac{\partial \phi_1}{\partial s_\lambda} + V \cos \theta \frac{\partial \rho}{\partial s_\lambda} \right) ds_\mu = -\left( \frac{\partial \phi_1}{\partial s_\mu} + V \cos \theta \frac{\partial \rho}{\partial s_\mu} \right) ds_\lambda \quad (25) \]

In order that the condition of no flow normal to the body of revolution be valid for all values of \(\theta\), there must be chosen from among all the solutions given by equation (22) that one which has \(\cos \theta\) as a factor; namely:

\[ \phi_1 = \sum_{n=1}^\infty A_n P_n'(\mu) Q_n^1(\lambda) \cos \theta \]

or

\[ \phi_1 = 2aV \cos \theta (\lambda^2 - 1)^{\frac{1}{2}} (1 - \mu^2) \sum_{n=1}^\infty C_n \frac{dP_n}{d\mu} \frac{dQ_n}{d\lambda} \quad (26) \]

where \( C_n = \frac{A_n}{2aV} \)

Furthermore

\[ ds_\lambda = 2a \left( \frac{\lambda^2 - \mu^2}{\lambda^2 - 1} \right)^{\frac{1}{2}} d\lambda \quad \text{and} \quad ds_\mu = 2a \left( \frac{\lambda^2 - \mu^2}{1 - \mu^2} \right)^{\frac{1}{2}} d\mu \]

so that equation (25) becomes

\[ \frac{\partial}{\partial \lambda} (\phi_1 + \rho V \cos \theta) = \frac{1 - \mu^2}{\lambda^2 - 1} \frac{d\lambda}{d\mu} \]

Finally, by means of equation (26) and the differential equation for the Legendre polynomials the foregoing boundary condition takes the following form:

\[ \sum_{n=1}^\infty C_n \left[ \frac{dP_n}{d\mu} \frac{dQ_n}{d\lambda} - n(n+1) \frac{d}{d\mu} (P_nQ_n) \right] = d(\lambda) \quad (27) \]

**DISTRIBUTION OF DOUBLETs**

The doublets are assumed to have their axes in the \(z\) direction and to lie along the segment \(-2a \leq z \leq 2a\) of the axis of symmetry \(z\). The velocity potential at any point \((\xi, \rho)\) of some meridian plane \(\theta\) then takes the form (see reference 6):

\[ \phi_1 = \frac{\rho \cos \theta}{4\pi} \int_{-2a}^{2a} J(\xi) ds_\lambda \]

where \( J(\xi) \) is the intensity of the doublets per unit length.

Substituting for \(\phi_1\) from equation (26) it follows that

\[ V \sum_{n=1}^\infty C_n \frac{dP_n}{d\mu} \frac{dQ_n}{d\lambda} = \frac{1}{4\pi} \int_{-2a}^{2a} J(\xi) ds_\lambda \]

For points lying on the \(z\) axis but outside the distribution this equation takes the following simplified form:

\[ V \sum_{n=1}^\infty \frac{n(n+1)}{2} C_n \frac{dQ_n}{d\lambda} = \frac{1}{16\pi a^2} \int_{-a}^{a} J(2a\lambda) d\lambda \quad (28) \]

where \( z_1 \) is replaced by \( 2a\lambda_1 \), \( z \) by \( 2a\lambda \), and \( \frac{dP_n}{d\mu} \) by \( \frac{n(n+1)}{2} \). This is an integral equation for the unknown function \(J(2a\lambda_1)\). In the solution of this integral equation it is necessary that a development of \( \frac{1}{(\lambda - \lambda_1)^3} \) as a series of Legendre polynomials in \(\lambda_1\) be obtained. The form of this development is suggested by Neumann's equation (8). Thus assume that

\[ \frac{1}{(\lambda - \lambda_1)^3} = \sum_{n=1}^\infty b_n P_n(\lambda_1) Q_n(\lambda) \]

Then substituting this expression for \( \frac{1}{(\lambda - \lambda_1)^3} \) in Neumann's equation and making use of the orthogonality relations satisfied by the Legendre polynomials, it is found that

\[ b_n = 2n+1 \]

Therefore

\[ \frac{1}{(\lambda - \lambda_1)^3} = \sum_{n=1}^\infty (2n+1) P_n(\lambda_1) Q_n(\lambda) \]

Differentiating this last expression once with regard to \(\lambda\) and once with regard to \(\lambda_1\), it follows that

\[ \frac{1}{(\lambda - \lambda_1)^3} = -\sum_{n=1}^\infty \frac{2n+1}{2} \frac{dP_n(\lambda_1)}{d\lambda_1} \frac{dQ_n(\lambda)}{d\lambda} \]

Equation (28) then becomes

\[ V \sum_{n=1}^\infty C_n \frac{n(n+1)}{2} \frac{dQ_n}{d\lambda} = -\sum_{n=1}^\infty \frac{2n+1}{2} \frac{dP_n(\lambda_1)}{d\lambda_1} \frac{dQ_n(\lambda)}{d\lambda} \quad (29) \]

It is now obvious that the following assumption must be made:

\[ J(2a\lambda_1) = -8\pi a^2 V (1 - \lambda_1^2) \sum_{n=1}^\infty c_n \frac{dP_n}{d\lambda_1} \]
and substituting this expression in equation (29) it follows that
\[ \sum_{n=1}^{\infty} (C_n - c_n) \frac{dQ_n}{d\lambda} = 0 \]
In order that this equation be valid for arbitrary values of \( \lambda \),
\[ c_n = C_n \]
and therefore
\[ J(2a\lambda) = -8\pi a^2 V(1 - \lambda^2) \sum_{n=1}^{\infty} C_n \frac{dP_n}{d\lambda} \]  
(30)

Thus, given the velocity potential \( \phi \), that is, the \( C_n \)'s, this expression determines the equivalent doublet distribution.

**DETERMINATION OF THE COEFFICIENTS \( A_* \) AND \( C_* \)**

Any symmetrical profile may be represented by a power series in \( \mu (= \cos \eta) \). That is
\[ \lambda = \sum_{r=0}^{\infty} a_r \mu^r \]  
(31)
The rapidity of convergence of this series depends, however, on the choice of origin with respect to the profile. Since \( \lambda = a_0 \) defines an ellipse, the rapidity of convergence of the foregoing series may be looked upon as a measure of the resemblance of the profile to the ellipse \( \lambda = a_0 \). The proper choice of origin may be attained in the following manner. The radius of curvature \( R \) of an ellipse at the end of its major axis is given by
\[ R = \frac{B^2}{A} \]
where \( A \) and \( B \) are its semimajor and semiminor axes, respectively.

Eliminating \( \eta \) from equations (1), the following equation of a system of confocal ellipses results:
\[ \frac{x^2}{2a \cosh \xi} + \frac{y^2}{2a \sinh \xi} = 1. \]  
(The distance between foci is \( 4a \)).

In terms of elliptic coordinates then
\[ R = 2a \frac{\sinh \xi}{\cosh \xi} \]
Furthermore, for an elongated ellipse the semimajor axis \( 2a \cosh \xi \) is large compared to the semiminor axis \( 2a \sinh \xi \). This limitation means that \( \xi \) is small. Neglecting powers of \( \xi \) higher than the second it follows that (see reference 7)
\[ R = 2a \xi^2 \text{ (approximately)} \]
The ends of the ellipse are at
\[ \pm 2a \cosh \xi = \pm 2a \left( 1 + \frac{\xi^2}{2} + \cdots \right) = \pm \left( 2a + \frac{R}{2} \right) \]  
(approximately)
and therefore the focus of an elongated ellipse very nearly bisects the line joining the end of the semimajor axis and the center of curvature. Thus the proper choice of origin is the point bisecting the line of length \( 4a \) extending from the point midway between the leading edge and the center of curvature of that edge to a point midway between the trailing edge and the center of curvature of that edge. Having thus chosen a reference frame \((z, \rho)\) in which to present the profile, the next step is to obtain the series equation (31). This equation may be obtained with the help of the following expressions. From equation (1) it can be found that
\[ \lambda = \frac{1}{2} \left[ \sqrt{\left( \frac{2a + 1}{2a} \right)^2 + \left( \frac{\rho}{2} \right)^2} + \sqrt{\left( \frac{2a - 1}{2a} \right)^2 + \left( \frac{\rho}{2} \right)^2} \right] \]
(32)
where \(-1 \leq \mu \leq 1\).

A series of corresponding values of \( \lambda \) and \( \mu \) are thus obtained. In order to express \( \lambda \) as a polynomial in \( \mu \) of, say, degree \( n \), it is most convenient to employ the method of least squares for determining the \( (n+1) \) constants \( a_r \) (reference 8).

**FLOW PARALLEL TO THE AXIS OF SYMMETRY**

The boundary condition for this type of flow is given by equation (15). In that expression functions of the \( \alpha \) type \( \phi \) appear and these are to be expressed as power series in \( \mu \).

Suppose the meridian profile to be given by the following analytic expression:
\[ \lambda = a_0 + \mu \sum_{n=0}^{\infty} a_{1, n} \mu^n \]  
(33)
Then on the profile, \( \frac{dQ_n}{d\lambda} \) may be looked upon as a function of \( \mu \) and can be developed in a Taylor series in the neighborhood of \( \mu = 0 \) or \( \lambda = a_0 \). That is,
\[ \frac{dQ_n}{d\lambda} = \sum_{p=1}^{\infty} \frac{\lambda - a_0}{p!} \frac{d^{p+1}Q_n}{d\lambda^{p+1}} \]
where \( \frac{d^{p+1}Q_n}{d\lambda^{p+1}} \) can be expressed as power series in \( \mu \).

Substituting for \( \lambda - a_0 \) from equation (33), it follows that
\[ \frac{dQ_n}{d\lambda} = \sum_{p=1}^{\infty} \frac{\mu^p}{p!} \left( \sum_{n=0}^{\infty} a_{1, n} \mu^n \right)^p \frac{d^{p+1}Q_n}{d\lambda^{p+1}} \]
In the following the expansion of \( S' \) in powers of \( \mu \) is to be determined (reference 9, p. 122), \( p \) being any positive integer and where
\[ S = \sum_{s=0}^{\infty} a_{1, n} \mu^n \]
Thus

\[ S^2 = \sum_{q=0}^{\infty} \sum_{r=0}^{\infty} a_{1,r} \mu^q \]

or

\[ S^2 = \sum_{r=0}^{\infty} \sum_{p=0}^{\infty} a_{1,p-r} \mu^p \]

Expanding \( S^2 \) with respect to \( r \) and writing the terms with equal indices of \( p \) in columns and adding these columns,

\[ S^2 = \sum_{p=0}^{\infty} \mu^p \sum_{r=0}^{p} a_{1,r} \mu^{p-r} = \sum_{p=0}^{\infty} a_{2,p} \mu^p \]

where

\[ a_{2,p} = \sum_{r=0}^{p} a_{1,r} a_{1,r} \]

In a similar manner,

\[ S^2 = \sum_{p=0}^{\infty} a_{3,p} \mu^p \]

where

\[ a_{3,p} = \sum_{r=0}^{p} a_{1,r} a_{2,r} \]

and, in general, it is permissible to write

\[ S^2 = \sum_{p=0}^{\infty} a_{i,p} \mu^p \]

where

\[ a_{i,p} = \sum_{r=0}^{p} a_{i-1,r} a_{1,r} \text{ and } a_{0,r} = 0 \text{ if } r \neq 0 \]

Hence

\[ \frac{dQ_{E}}{d\lambda} = \sum_{p=0}^{\infty} \sum_{q=0}^{p} \mu^q \frac{d^{q+1}Q_{E}}{d\alpha_{q+1}} a_{p,q} \mu^p \]

Expanding according to \( p \) and writing the terms with equal indices of \( q \) in columns and adding these columns,

\[ \frac{dQ_{E}}{d\lambda} = \sum_{p=0}^{\infty} \sum_{q=0}^{p} \mu^q \frac{d^{q+1}Q_{E}}{d\alpha_{q+1}} a_{p,q} \mu^p \]

Then,

\[ \frac{dP_{E}}{d\mu} = \sum_{j=0}^{\infty} (-1)^j \frac{[2n-2j-1]}{(n-2j-1)!2j^2} \mu^{n-2j-1} \]

(37)

Hence

\[ P_{n}(\mu) = \sum_{j=0}^{\infty} (-1)^j \frac{[2n-2j-1]}{(n-2j-1)!2j^2} \mu^{n-2j-1} \]

(36)

The boundary condition also contains terms of the type \( \frac{dP_{E}}{d\mu} \) where \( P_{n}(\mu) \) is the Legendre polynomial in \( \mu \) of degree \( n \) and is given by

\[ P_{n}(\mu) = \sum_{j=0}^{\infty} (-1)^j \frac{[2n-2j-1]}{(n-2j-1)!2j^2} \mu^{n-2j-1} \]

Substituting for \( \frac{dQ_{E}}{d\lambda} \) and \( \frac{dP_{E}}{d\mu} \) the expressions given by equations (34) and (37) into the boundary condition equation (15),

\[ \sum_{n=0}^{\infty} \frac{A_{n}}{n(n+1)} \sum_{j=0}^{n-2j-1} (-1)^j \frac{[2n-2j-1]}{(n-2j-1)!2j^2} \mu^{n-2j-1} \sum_{p=0}^{\infty} \mu^p \sum_{q=0}^{p} \frac{d^{q+1}Q_{E}}{d\alpha_{q+1}} a_{p,q} \mu^p + aU = 0 \]

Expanding according to powers of \( n \) and writing the terms with equal indices of \( j \) in columns and then adding these columns,

\[ \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \frac{A_{n}}{n(n+1)} (-1)^j \frac{[2n-2j-1]}{(n-2j-1)!2j^2} \mu^{n-2j-1} \sum_{p=0}^{\infty} \mu^p \sum_{q=0}^{p} \frac{d^{q+1}Q_{E}}{d\alpha_{q+1}} a_{p,q} \mu^p + aU = 0 \]
Putting \( n-2j-1+q=m \) this becomes
\[
\sum_{j=0}^{m} \sum_{q=0}^{m} \sum_{m=0}^{\infty} \mu^{m} (-1)^{j} \frac{A_{k+j+1+m}}{(2j+1+m-q)(2j+2+m-q)} \frac{[2j+1+2m-2q]}{(m-q)!} \frac{1}{[2j]} \sum_{p=0}^{q} \frac{a_{p+2+j+1+m}}{p!} \frac{d^{p+1}Q_{k+j+1+m-\tau}}{d\theta_{m}^{p+1}} + aU = 0
\]

Expanding with respect to \( q \) and writing the terms with equal indices of \( m \) in columns and adding these columns,
\[
\sum_{m=0}^{\infty} \sum_{q=0}^{m} \sum_{j=0}^{m} (-1)^{j} \frac{A_{k+j+1+m}}{(2j+1+m-q)(2j+2+m-q)} \frac{[2j+1+2m-2q]}{(m-q)!} \frac{1}{[2j]} \sum_{p=0}^{q} \frac{a_{p+2+j+1+m}}{p!} \frac{d^{p+1}Q_{k+j+1+m-\tau}}{d\theta_{m}^{p+1}} + aU = 0
\]

If this expression is to be valid everywhere on the boundary surface, it must hold for the entire range of \( \mu \). It follows that the coefficients of the various powers of \( \mu \) are identical zero. Finally, the introduction of \( k \) and \( n \) by means of the substitutions \( q=m-k \) and \( p=m-n \), respectively, leads to the following expression of the boundary condition:
\[
\sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{a_{n-m-k}}{k!(m-n)!} \sum_{j=0}^{m} (-1)^{j} \frac{[2j+1+2k]}{(2j+1+k)(2j+2+k)[2j]} \frac{d^{n-k+1}Q_{k+j+1+k}}{d\theta_{n-k+1}} = -aU \delta_{m}^{n}
\]

Equation (39) represents a set of linear equations infinite in number and containing an infinite number of unknowns \( A_{m} \). It provides a formal and rigorous solution of the problem of potential flow about a body of revolution, parallel to its axis of symmetry.

In the foregoing equations the only unknowns are the \( A_{k} \)’s. The \( a_{p} \)’s are related to the coefficients of the power series of \( \lambda \) in \( \mu \) (giving the meridian profile equation (33)) and are evaluated by means of equations (35). Finally, the \( Q_{n} \)’s and their derivatives are well-defined Legendre functions.

For example, if the meridian profile is an ellipse \( \lambda=1 \), then equation (39) becomes
\[
\sum_{j=0}^{m} (-1)^{j} \frac{A_{j+k+1+m}}{(2j+1+m)(2j+2+m)} \frac{[2j+1+2m+1]}{m!} \frac{1}{[2j]} \frac{d^{k+1}Q_{j+k+1+m}}{d\theta_{j+k+1+m}} = -aU \delta_{m}^{k+1}
\]

For \( m=1, 2, 3, \ldots \) this is an infinite set of linear homogeneous equations for the unknowns \( A_{3}, A_{2}, \ldots \) and, since the determinant of the coefficients is different from zero, the only solution is that \( A_{3}, \ldots \ldots \ldots \ldots \ldots \) are zero. From the first equation, i.e., \( m=0 \), it is then found that
\[
A_{0} = \frac{2aU}{2} \frac{d^{0}Q_{0}}{d\theta_{0}} = \frac{2aU}{2} \frac{d^{0}Q_{0}}{d\theta_{0}}
\]

Hence
\[
\phi = \frac{2aU}{2} \log \frac{a_{0}+1}{a_{0}} (\frac{1}{2})^{\lambda} \log \frac{\lambda+1}{\lambda-1}
\]

If \( A \) and \( B \) are the semimajor and semiminor axes of the meridian ellipse and \( e \) its eccentricity, then
\[
2a = Ae, \ a_{0} = \frac{1}{e} 2a(a_{0}^2-1)^{1/2} = B
\]

so that
\[
\phi = \frac{2aU}{2} \frac{d^{0}Q_{0}}{d\theta_{0}} (\frac{1}{2})^{\lambda} \log \frac{\lambda+1}{\lambda-1}
\]

This result agrees with the well-known expression for the velocity potential of a prolate ellipsoid of revolution (reference 10, p. 132).

\section*{Flow Normal to the Axis of Symmetry}

The case of flow normal to the axis of symmetry will now be treated in a manner similar to the case of parallel flow. The boundary condition is given by equation (27):
\[
\sum_{n=0}^{\infty} \sum_{\lambda} \left[ \frac{d(\lambda\mu)}{d\mu} \frac{dP \lambda}{d\mu} \frac{dQ \lambda}{d\lambda} - n(n+1) \frac{d}{d\mu} (P \lambda Q \lambda) \right] = \frac{d(\lambda\mu)}{d\mu}
\]

From equation (33),
\[
\frac{d(\lambda\mu)}{d\mu} = a_{0} + \sum_{n=0}^{\infty} (n+2)a_{n+1}^{n+1}/(n+1)
\]
Referring to equation (38) it follows that
\[
\sum_{n=1}^{\infty} C_n \frac{d\left(\frac{\partial P_2}{\partial \mu} - \frac{\partial Q_n}{\partial \lambda}\right)}{d\mu} = a_0 \sum_{m=0}^{\infty} \sum_{q=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{\frac{[2j+1+2m-2q]}{[2j]}} C_{2j+1+m-m} \frac{q}{p} \int \frac{dP_2}{d\alpha_0} Q_{2j+1+m-m} = 0
\]
\[
+ \sum_{n=0}^{\infty} (n+2)a_{1, n+1} \frac{dP_2}{d\mu} = a_0 \sum_{m=0}^{\infty} \sum_{q=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{\frac{[2j+1+2m-2q]}{[2j]}} C_{2j+1+m-m} \frac{q}{p} \int \frac{dP_2}{d\alpha_0} Q_{2j+1+m-m} = 0
\]

or
\[
\sum_{n=1}^{\infty} C_n \frac{d\left(\frac{\partial P_2}{\partial \mu} - \frac{\partial Q_n}{\partial \lambda}\right)}{d\mu} = a_0 \sum_{m=0}^{\infty} \sum_{q=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{\frac{[2j+1+2m-2q]}{[2j]}} C_{2j+1+m-m} \frac{q}{p} \int \frac{dP_2}{d\alpha_0} Q_{2j+1+m-m} = 0
\]

Analogous to equation (34)
\[
Q_n = \sum_{m=0}^{\infty} \sum_{q=0}^{\infty} a_{n, m-q} \frac{dP_2}{d\alpha_0}
\]

Hence, in a manner similar to the derivation of equation (38)
\[
\sum_{n=1}^{\infty} n(n+1) C_n \frac{d\left(\frac{\partial P_2}{\partial \mu} - \frac{\partial Q_n}{\partial \lambda}\right)}{d\mu} = 0
\]

Substituting equations (40), (41), and (42) in the boundary condition and equating the coefficients of the various powers of \(\mu\) equal to zero, the following set of equations is obtained:
\[
\sum_{n=1}^{\infty} (-1)^{\frac{[2n+1]}{[2n]}} \left[ C_{2a+1} \frac{dQ_{2a+1}}{d\alpha_0} - (2n+1)(2n+2)Q_{2a+1} \right] - 2nai \frac{dQ_{2a}}{d\alpha_0} = a_0 \text{ for } h=0
\]

and (after rearranging as for equation (39))
\[
\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \sum_{q=0}^{\infty} \sum_{j=0}^{\infty} \frac{(h-k+1)}{[k]} C_{2k+1+m} \frac{d^{h-n+1}Q_{2k+1+m}}{d\alpha_0^{h-n+1}} = 0
\]

where \(h=1, 2, 3, \ldots \infty\).

This set of equations represents a formal and rigorous solution of the problem of potential flow about a body of revolution with flow normal to the axis of symmetry. The only unknown quantities are the infinite number of \(C_n\)'s. The other quantities appearing in the equations are determined as in equations (39).

If the meridian profile is an ellipse \(\lambda=a_0\), the \(a_{1,s}\)'s are all zero and, from the second expression of equation (43),
\[
\sum_{n=0}^{\infty} (-1)^{\frac{[2n+2h+1]}{[2n+2h]}} \left[ a_{0} \frac{dQ_{2a+1}+1}{d\alpha_0} \right] C_{2a+1+1} = 0
\]

where \(h=1, 2, \ldots \infty\).
This expression represents an infinite set of linear homogenous equations in the unknowns $C_2, C_3, \ldots$ and, since the determinant of the coefficients is different from zero, it immediately follows that the only solution is $C_2=C_3=C_4=\ldots=0$. From the first expression of equation (43), it follows that

$$C_1=\frac{a_0}{a_0 dQ_1 da_0 - 2Q_1(a_0)}$$

This result could have been easily obtained from the general expression for the velocity potential given by equation (26). Thus, assume that

$$\frac{\partial \phi}{\partial n}=2aV \cos \theta \left(\frac{1-\mu^2}{\lambda^2-1}\right)^{\frac{1}{2}} \left[ C_1(a_0^2-1) \frac{dQ_1}{da_0} + C_1 a_0 \frac{dQ_3}{da_0} + a_0 \right] d\lambda = 0$$

From Legendre's equation

$$(a_0^2-1) \frac{dQ_1}{da_0^2} = -2a_0 \frac{dQ_1}{da_0} + 2Q_1(a_0)$$

Hence (see reference 10, p. 133)

$$C_1=\frac{a_0}{a_0 \frac{dQ_1}{da_0}} = \frac{a_0}{2 \log \frac{a_0+1}{a_0-1} + \frac{a_0}{a_0-1}}$$

In the appendix an application of the boundary condition (equations (39) and (43)) for axial and transverse flows, respectively, is made to a body of revolution obtained from a symmetrical Joukowksy profile.

**INERTIA COEFFICIENTS OF BODIES OF REVOLUTION**

It is of some interest to obtain the coefficients of inertia for axial and transverse flows and also to compare them with those of an ellipsoid of revolution of equal fineness ratio (references 11 and 12).

When a body moves in a fluid at rest at infinity the total kinetic energy of the fluid is given by

$$2T=-\sigma \int \int \phi \frac{\partial \phi}{\partial n} dS$$

where $\phi$ is the velocity potential of the fluid motion, $\frac{\partial \phi}{\partial n}$ the normal derivative of $\phi$ where the positive direction of the normal to the surface of the body is into the fluid and the integration is performed over the surface of the body; $\sigma$ denotes the density of the fluid.

If $M$ is the mass of fluid displaced by the body, then the coefficient of inertia $k_2$ is the quantity multiplying $MU^2$ in the expression for $2T$.

If the body is a prolate spheroid $\lambda=a_0$ the foregoing expression for $2T$ becomes:

$$2T = \frac{4}{3} \pi a^2 \sigma U^2 (a_0^2-1) \frac{1}{2} a_0 \log \frac{a_0+1}{a_0-1} - \frac{1}{4} \frac{a_0}{a_0^2-1} \frac{1}{2} \log \frac{a_0+1}{a_0-1}$$

But $2a=\Delta e$, $a_0=\frac{1}{e}$ and $2a(a_0^2-1)=B$ where $A$, $B$ are the semimajor and semiminor axes, respectively, and $e$ the eccentricity of the elliptical meridian section. Therefore

$$2T = \frac{1}{2} \frac{1}{a^2} \frac{1}{1-e^2} - \frac{1}{1-e} \frac{1}{4} \pi AB^2 e U^2 = k_2 MU^2$$
The coefficient of inertia for a prolate ellipsoid in uniform axial motion is then given by

\[ k_a = \frac{1}{2\varepsilon} \log \frac{1+\varepsilon}{1-\varepsilon} - \frac{1}{e^2} \frac{1}{2\varepsilon} \log \frac{1+\varepsilon}{1-\varepsilon} \]

(47)

(See reference 10, p. 144)

Equation (46) is now evaluated for the case of a body of revolution obtained from the Joukowsky profile \( \epsilon_1 = 0.15, \epsilon_2 = 0.10 \). (See appendix.) The volume of this body is found to be

\[ Q = \frac{4}{3} \pi (2a)^3 \times 0.05342 \]

so that the expression for the coefficient of inertia for axial flow is then

\[ 2T = \frac{4}{3} \pi (2a)^3 \times 0.03136 \]

or \( k_a = 0.0881 \). (See table I.)

Compare this value of \( k_a \) with that of a prolate spheroid whose fineness ratio is the same as for the above-mentioned body of revolution. The fineness ratio \( f \) is defined as the ratio of the length to the maximum diameter of the body. The maximum diameter is obtained from equation (53) by means of the condition \( \frac{d\rho}{d\tau} = 0 \) and the length of the body is given by \( l = 2a(\lambda_{\rho-1} + \lambda_{\rho-1}) \). By means of these expressions it is found that \( f = 4.208 \). The fineness ratio for an ellipse is given by

\[ f = \frac{A}{B} = \frac{1}{\sqrt{1-e^2}} \]

or

\[ e = \sqrt{1 - \frac{1}{f^2}} = 0.971 \]

where \( e \) is the eccentricity of the ellipse.

Substituting this value of \( e \) into equation (47), the following value of \( k_a \) is obtained,

\[ k_a = 0.0757 \]

A theorem enunciated by Munk (reference 13) states that when the disturbance caused by a body moving in an infinite fluid is replaced by fictitious sinks and sources, the total mass is the sum of the products obtained by multiplying the intensity of each source or sink by the potential of the parallel flow. This theorem will now be shown to be only a first approximation and to hold exactly only for ellipsoids of revolution. Thus from equation (19),

\[ I(z_1) = 2\pi \sum_{n=1}^{\infty} A_n P_n(\lambda_1) \]

where \( z_1 = 2a\lambda_1 \)

The strength per length \( d\tau \) is then

\[ 4\pi a \sum_{n=1}^{\infty} A_n P_n(\lambda_1) \cdot d\lambda_1 \]

The velocity potential at \( (z_1, 0) \) of the parallel flow is given by

\[ \phi = 2a U \lambda_1 \]

Hence, according to Munk's theorem

\[ 2T_{total} = 8\pi a^2 U \sum_{n=1}^{\infty} A_n \int_{-1}^{1} \lambda_1 P_n(\lambda_1) \cdot d\lambda_1 \]

or

\[ 2T_{total} = \frac{4}{3} \pi (2a)^3 \sigma U^2 \frac{A_1}{2a U} \]

Therefore

\[ 2T_{esd} = \left( \frac{4}{3} \pi (2a)^3 \frac{A_1}{2a U} - 1 \right) M U^2 \]

where \( Q \) is the volume of the body and \( M \) is the mass of the displaced fluid.

The coefficient of inertia for axial flow is therefore

\[ k_a = \frac{4}{3} \frac{\pi (2a)^3 A_1}{Q (2a U)} - 1 \]

This expression for \( k_a \) is valid for a prolate ellipsoid but is not valid for a more general shape.

It is obvious from this expression that Munk's theorem applies exactly only to ellipsoids of revolution since only the coefficient \( A_1 \) appears.\(^1\) In order to provide a numerical comparison between Munk's theorem and the exact method, the foregoing equation is evaluated for the body of revolution whose meridian curve is the Joukowsky profile \( \epsilon_1 = 0.15, \epsilon_2 = 0.10 \). It yields a value \( k_a = 0.0717 \) as compared with the more exact value \( k_a = 0.0867 \) already obtained by means of the fundamental equation (46).

FLOW NORMAL TO THE AXIS OF SYMMETRY

For flow normal to the axis of symmetry the velocity potential depends not only on the elliptic coordinates \( \lambda, \mu \), but also on the cylindrical coordinate \( \theta \). Hence, the equation for the element of surface \( dS \) is

\[ dS = r \rho ds \]

and equation (44) becomes

\[ 2T = -\sigma \int \int \rho \frac{\partial \phi}{\partial \rho} ds \ dn \]

If the body moves in the direction of the transverse axis \( oz \) with a constant velocity \( V \) the boundary condition is

\[ \frac{\partial \phi}{\partial \rho} ds = -V \cos \theta \frac{\partial \rho}{\partial \rho} ds \]

\(^1\) The above criticism of Munk's theorem has been found to be incorrect. (It is to be noted that the volume of the body contains all of the coefficients \( A_n \) implicitly.) This theorem may be readily verified by applying Green's second theorem to the space internal to the shape and using the appropriate distribution of sinks and sources. Then if \( \phi = U \tau \) and \( \phi = \phi \) (where \( \rho \phi = \gamma = 0 \), Green's second theorem together with equation (44) immediately yields the following expression on Munk's theorem:

\[ 2T = \int \int \frac{U \sigma}{r} (\text{volume of body}) - \int \int \frac{U}{r} d\tau = \int \int \frac{U}{r} d\gamma = \int \int \frac{U}{r} d\tau \]

\[ dS = \rho ds \]
Also, in general

\[ \phi = 2aV \cos \theta (\lambda^3 - 1)^{\frac{1}{3}} (1 - \mu^2)^{\frac{1}{3}} \sum_{n=1}^{\infty} C_n \frac{dP_n}{d\mu} \frac{dQ_n}{d\lambda} \frac{d\lambda}{d\mu} \]

Hence

\[ 2T = (2a)^2 \pi \sigma V^2 \int_{-1}^{1} (1 - \mu^2)(\lambda^3 - 1)(\mu \frac{d\lambda}{d\mu} + \lambda) \sum_{n=1}^{\infty} C_n \frac{dP_n}{d\mu} \frac{dQ_n}{d\lambda} \frac{d\lambda}{d\mu} \]

From equation (45), it follows that

\[ \frac{\partial \rho}{\partial \lambda} ds = -8a^2(1 - \mu^2) (\lambda^3 - 1) (\mu \frac{d\lambda}{d\mu} + \lambda) \frac{d\lambda}{d\mu} \]

For a prolate spheroid, \( \lambda = a_0 \) and

\[ \phi = 2aV \cos \theta (\lambda^3 - 1)^{\frac{1}{3}} (1 - \mu^2)^{\frac{1}{3}} \frac{C_1 dQ_1}{d\lambda} \frac{d\lambda}{d\mu} \]

where

\[ C_1 = \frac{a_0}{a_0^2 d\alpha_0 - 2Q_1(a_0)} \]

Therefore

\[ 2T = \frac{a_0}{a_0^2 - 1} \frac{a_0^2 + 1}{2 \log a_0 - 1} \frac{a_0^2}{a_0^2 + 1} \frac{1 - \epsilon^2}{\epsilon^2 + 1 - \epsilon^2} \frac{1 + \epsilon}{\log 1 + \epsilon} \]

or

\[ k_T = \frac{1 - \epsilon^2}{\epsilon^2 + 1 - \epsilon^2} \frac{1 + \epsilon}{\log 1 + \epsilon} \]

(See reference 10, p. 145.)

For the body of revolution whose meridian curve is the Joukowsky profile \( \epsilon_1 = 0.15, \epsilon_2 = 0.10 \) (see appendix) it is seen from equation (48) that

\[ 2T = \frac{\pi V^2 (2a)^3}{3} \times 0.059587 \]

\[ Q = 0.8366 \times MV^2 \]

Therefore \( k_T = 0.8366 \). (See table II.)

According to equation (49) for the prolate ellipsoid of equal fineness ratio \( \epsilon = 0.208 \) and \( k_T = 0.8689 \).

According to Munk’s theorem the inertia coefficient \( k_T \) of transverse flow may be obtained from the doublet distribution along the axis of symmetry. Again, as in the case of axial flow, this theorem is a first approximation and holds exactly only for ellipsoids of revolution since an expression for \( k_T \) is obtained that contains only the coefficient \( C_1 \). Thus from equation (50) it follows that

\[ J(s) = \frac{8 \pi a^2 V(1 - \lambda^2)}{3} \sum_{n=1}^{\infty} C_n \frac{dP_n}{d\lambda} \frac{dQ_n}{d\lambda} \frac{d\lambda}{d\mu} \]

Then according to Munk

\[ 2T_{total} = \sigma V \int_{-1}^{1} J(2a \lambda) 2a \lambda \]

or

\[ 2T_{total} = -16 \pi a^2 V^3 \sum_{n=1}^{\infty} C_n \int_{-1}^{1} (1 - \lambda^2)^2 \frac{dP_n}{d\lambda} \frac{dQ_n}{d\lambda} \frac{d\lambda}{d\mu} \]

since

\[ \frac{dP_1}{d\lambda} = 1 \]

Hence

\[ 2T_{total} = -\frac{8}{3} \pi (2a)^3 V^2 C_1 \]

and

\[ 2T_{fluid} = -\frac{8}{3} \pi (2a)^3 C_1 + \frac{Q}{Q} \]

or

\[ k_T = -\frac{8}{3} \pi (2a)^3 C_1 + \frac{Q}{Q} \]

In order to give a numerical comparison between Munk’s theorem and the exact method, the foregoing equation is evaluated for the body of revolution whose meridian curve is the symmetrical Joukowsky profile \( \epsilon_1 = 0.15, \epsilon_2 = 0.10 \). It yields a value \( k_T = 0.8210 \) as compared with the more exact value \( k_T = 0.8366 \) obtained from the fundamental equation (48).

Langley Memorial Aeronautical Laboratory, National Advisory Committee for Aeronautics, Langley Field, Va., November 12, 1934.
APPENDIX

APPLICATION OF THE ANALYSIS TO SURFACES OF REVOLUTION OBTAINED FROM SYMMETRICAL JOUKOWSKY PROFILES

By means of the mapping function

$$z = z' + \frac{a^2}{\bar{z}}$$

(50)

the circle \(k_1\) of radius \(a\) in the \(z'\) plane is transformed into the line segment \((-2a, 0; 2a, 0)\) in the \(z\) plane and the circle \(k_2\) of radius \((1+\varepsilon_1+\varepsilon_2)a\) with center at \((\varepsilon_1 a, 0)\) is transformed into a symmetrical Joukowsky profile \(J\) in the \(z\) plane. (See fig. 2.)

![Figure 2. Mapping of a circle into a symmetrical Joukowsky profile.](image)

If in the \(z'\) plane \(PQ = ae^{\theta}, PO = ae^{\eta}, \angle POQ = \eta,\) and angle \(PQz' = \phi\) then, according to the law of cosines,

$$e^{\theta - \eta} = 1 + 2a \cos \phi + \delta$$

(51)

where

$$\delta = \frac{\varepsilon_1}{1 + \varepsilon_1 + \varepsilon_2}$$

Again, by the law of sines

$$\tan \eta = \frac{\sin \phi}{\delta + \cos \phi}$$

(52)

Putting \(z' = ae^{\theta + \eta}\) into equation (48),

$$z = 2a \cosh (\xi + i\eta)$$

or \(z = 2a \cosh \xi \cos \eta, \rho = 2a \sin \xi \sin \eta\)

The latter two equations are, in fact, the equations of transformation from the rectangular coordinates \((z, \rho)\) to the elliptic coordinates \((\xi, \eta)\). Since \((z, \rho)\) refer to points of the Joukowsky profile \(J\), using equations (51) and (52), the following parametric equations of the system of symmetrical Joukowsky profiles may be obtained

$$\lambda = \frac{\varepsilon_1(1+2\delta + \delta^2)}{2\delta + \delta^2 + \frac{\delta}{2\varepsilon_1(1+2\delta + \delta^2)}}$$

$$\mu = \frac{\delta + \nu}{(1+2\delta + \delta^2)^2}$$

(53)

where \(\lambda = \cosh \xi, \mu = \cos \eta, \) and \(\nu = \cos \phi\) (the independent parameter).

From these equations \(\lambda\) can be expressed as a power series in \(\mu\) by means of a Maclaurin’s expansion in the neighborhood of \(\mu = 0\) (i.e. \(\nu = -\delta\)). Thus,

$$\lambda = a_0 \left(1 + \alpha \gamma + \frac{1}{8} \gamma^4 + \frac{1}{16} \gamma^6 + \ldots \right)$$

(54)

where

$$a_0 = \frac{\varepsilon_1^2(1-\delta^2)+1}{2\varepsilon_1(1-\delta^2)^2}, \alpha = \frac{\varepsilon_1^2(1-\delta^2)-1}{\varepsilon_1^2(1-\delta^2)+\delta} \text{ and } \gamma = \frac{\delta}{\sqrt{1-\delta^2}}$$

FLOW PARALLEL TO THE AXIS OF SYMMETRY

Equation (39) is a set of linear equations for the infinite number of unknown coefficients \(A_n\) and provides a solution of the problem of axial potential flow about a body of revolution. In practice it is necessary to evaluate only the first few coefficients \(A_n\). From equation (39), neglecting the \(A_n\)'s after \(A_6\), the following equations are obtained:

1 This power series suggests the form

$$\lambda = a_0(\alpha + \sqrt{1+\gamma^2})$$

In fact, if the expression for \(\mu\) from equation (53) is herein substituted the equation for \(\lambda\) is obtained.
The coefficients of the unknown \( A_i \)'s can be calculated simply by a knowledge of power-series development of \( x \) (i.e., the quantities \( a_0, a_1, a_2, \ldots \) are obtained from equation (33)). The zonal harmonics \( Q_0(\theta, \lambda) \) are given by means of the recursion formula:

\[
(\lambda+1)Q_{\lambda+1}(\theta, \lambda) - (2\lambda+1)Q_\lambda(\theta, \lambda) \ni (\lambda+1)Q_{\lambda+1}(\theta, \lambda) - (2\lambda+1)Q_\lambda(\theta, \lambda) = 0
\]

\[
8(2a_{i+1} \frac{d^2 Q_i}{d \theta^2} + a_i \frac{d Q_i}{d \theta}) A_i + 16a_i \frac{d Q_i}{d \theta} A_i + 2 \left( 10 \frac{d Q_i}{d \theta} - 2a_{i+1} \frac{d^2 Q_i}{d \theta^2} - a_i \frac{d Q_i}{d \theta} \right) A_i = -12a_i \frac{d Q_i}{d \theta} A_i = 0
\]

\[
8 \left( 6a_{i+1} \frac{d^2 Q_i}{d \theta^2} + 6a_i \frac{d Q_i}{d \theta} \right) A_i + 24 \left( 2a_{i+1} \frac{d^2 Q_i}{d \theta^2} + a_i \frac{d Q_i}{d \theta} \right) A_i + 6 \left( a_{i+1} - 14a_i \frac{d^2 Q_i}{d \theta^2} + 6a_i \frac{d Q_i}{d \theta} \right) A_i = 0
\]

\[
8 \left( 24a_{i+1} \frac{d^2 Q_i}{d \theta^2} + 12(2a_i a_{i+1} + a_i) \frac{d^2 Q_i}{d \theta^2} + 12a_i \frac{d Q_i}{d \theta} \right) A_i + 32 \left( 6a_{i+1} \frac{d^2 Q_i}{d \theta^2} + 6a_i \frac{d Q_i}{d \theta} \right) A_i = 0
\]

\[
+ 2 \left( 24(5a_{i+1} - a_i) \frac{d^2 Q_i}{d \theta^2} + 12(5a_i^2 - 2a_i a_{i+1} - a_i) \frac{d^2 Q_i}{d \theta^2} - 12a_{i+1} a_i \frac{d^2 Q_i}{d \theta^2} - a_i \frac{d Q_i}{d \theta} \right) A_i = 0
\]

\[
+ 24 \left( 2(7a_{i+1} - 3a_i) \frac{d^2 Q_i}{d \theta^2} + 6a_i a_{i+1} \frac{d^2 Q_i}{d \theta^2} - a_i \frac{d Q_i}{d \theta} \right) A_i = 0
\]

The coefficients of the unknown \( A_i \)'s can be calculated simply by a knowledge of power-series development of \( x \) (i.e., the quantities \( a_0, a_1, a_2, \ldots \) are obtained from equation (33)). The zonal harmonics \( Q_\lambda(\theta, \lambda) \) are given by means of the recursion formula:

\[
(n+1)Q_{n+1}(\theta, \lambda) - (2n+1)Q_n(\theta, \lambda) + nQ_{n-1}(\theta, \lambda) = 0
\]

\[
\frac{dQ_{n+1}}{d \theta} - \frac{dQ_n}{d \theta} = (1 + \frac{1}{2}) \frac{dQ_{n-1}}{d \theta} + Q_0(\theta, \lambda) + 3Q_1(\theta, \lambda) + 5Q_2(\theta, \lambda) + \ldots + (2n+1)Q_n(\theta, \lambda)
\]

where it is necessary to determine \( \frac{dQ_n}{d \theta} \) independently.

In order to calculate the higher derivatives the preceding recursion formula may be repeatedly differentiated with regard to \( \theta \). The higher derivatives of \( Q_0 \) and \( Q_1 \) are obtained independently by means of the \( (r-1) \)th derivative of Legendre's differential equation:

\[
(a^2 - 1) \frac{d^{r+1}Q_r}{d \theta^{r+1}} + 2ra_r \frac{d^r Q_r}{d \theta^r} - (n+r)(n-r+1) \frac{d^{r-1}Q_r}{d \theta^{r-1}} = 0
\]

If the constants \( a_0, a_1, a_2, \ldots \) and the various derivatives of \( Q_\lambda \) are known, the coefficients of the unknowns \( A_i \) in equation (54) are easily calculated. The resulting system of linear equations can then be solved for the \( A_i \)'s which in turn determine the potential function \( \phi \) given by equation (10). A knowledge of the potential function yields directly the velocity \( u \) given by equation (16). Finally, according to Bernoulli's equation, the pressure \( p \) on the surface is given by

\[
p + \frac{1}{2} \rho u^2 = \rho_0 + \frac{1}{2} \rho U^2
\]

or

\[
p = \frac{\rho - \rho_0}{\rho_0} = \left( \frac{u}{U} \right)^2
\]

The numerical work is straightforward but somewhat tedious owing largely to a lack of tables of the zonal harmonics of the second kind.

As an illustration of the procedure here outlined, consider the body of revolution whose meridian curve is
the Joukowsky profile (reference 14) defined by $\xi=0.15$ and $\eta=0.10$. (See fig. 2.) From equation (53) there can be written then for the power-series development of $\lambda$:

$$\lambda = 0.02340 + 0.02630\mu + 0.007476\mu^2 - 0.0000273\mu^4 + \ldots ,$$  \hspace{1cm} (56)

a very rapidly convergent series.

Here

$$a_0 = 0.02340, \quad a_{1,0} = 0.0263, \quad a_{1,1} = 0.007476, \quad a_{1,2} = 0.0000273, \ldots$$

The zonal harmonics of the second kind and their derivatives are given by table III.

Substituting these numerical values into equations (55) the following set of five linear equations is obtained for $A_1$, $A_2$, $A_3$, $A_4$, and $A_5$:

\[
\begin{align*}
19.385 A_1 - 3.780 A_2 + 1.400 A_3 &= 2aU, \\
10.372 A_1 - 7.635 A_2 - 2.342 A_3 + 4.333 A_4 + A_5 &= 0, \\
-1.152 A_1 + 1.294 A_2 - 0.810 A_3 - 0.850 A_4 + A_5 &= 0, \\
-0.500 A_1 + 0.685 A_2 - 0.763 A_3 + 0.300 A_4 + A_5 &= 0, \\
-27.957 A_1 + 39.785 A_2 - 57.432 A_3 - 63.549 A_4 + A_5 &= 0.
\end{align*}
\]

The solution is given by

$$A_1 = 0.0573 \times 2aU, \quad A_2 = 0.0726 \times 2aU, \quad A_3 = 0.0296 \times 2aU, \quad A_4 = 0.0065 \times 2aU, \quad A_5 = 0.0015 \times 2aU$$

The sink-source distribution obtained from equation (19) is:

$$I(2a\lambda) = 4\pi aU (-0.0339 + 0.0157 A_1 + 0.0066 A_2 + 0.0605 A_3 + 0.0283 A_4 + 0.0119 A_5)$$  \hspace{1cm} (58)

Figure 3 shows a graph of this function with $I(2a\lambda)/4\pi aU$ as ordinate and $\lambda_1$ as abscissa.

In order to obtain the pressure distribution, the following expressions can be evaluated at a sufficient number of points of the boundary:

$$\left(\frac{u}{U}\right)^2 = \frac{1}{\lambda^2 - \mu^2} \left[ (\lambda^2 - 1) \left( \frac{1}{2aU} \sum_{n=1}^{5} A_n \frac{dQ_n}{d\lambda} P_n(\lambda) + \mu \right)^2 + (1 - \mu^2) \left( \frac{1}{2aU} \sum_{n=1}^{5} A_n Q_n(\lambda) \frac{dP_n}{d\mu} + \lambda \right)^2 \right]$$  \hspace{1cm} (59)

Note here that the velocity potential

$$\phi = 2aU [0.0573 \ P_1(\mu)Q_1(\mu) + 0.0726 \ P_3(\mu)Q_3(\mu) + \ldots + 0.0015 \ P_5(\mu)Q_5(\mu)]$$

is exact for the body of revolution obtained by superposing a uniform velocity $U$ on the flow from the sink-source distribution given by equation (58). This body is a very good approximation to the actual body obtained by revolving the Joukowsky profile about the axis of symmetry, so that in calculating the pressure distribution it is permissible to use the $(\lambda, \mu)$ values as given by equation (55).

Table IV shows the sequence of operations to be followed in obtaining the pressure distribution and figure 4 presents graphically the pressure distribution.

---

1. In its exact form the system contains an infinite number of equations with an infinite number of unknowns $A_1, A_2, \ldots$. For practical purposes, however, the following method of solution is suggested. Suppose the system of equations to have been solved to an arbitrary degree of approximation, say three. Then to this solution there corresponds a definite sink-source (or doublet, as the case may be) distribution from which can be obtained the corresponding profile and hence a $(\lambda, \mu)$ curve. This $(\lambda, \mu)$ curve can then be compared to the $(\lambda, \mu)$ curve of the actual profile. In order to improve the approximation, the true $(\lambda, \mu)$ curve can be shifted in such a manner that a repetition of the process of solution, to the same degree of approximation, yields a new system of $(\lambda, \mu)$ values closer to the actual set of $(\lambda, \mu)$ values than the first approximation. In this manner the process can be carried on until the desired degree of accuracy is obtained.
From equations (43), for the first five coefficients $A_1, A_2, A_3, A_4, A_5$ the following set of linear equations can be obtained:

\[
\begin{align*}
A_1 &= a_0 \frac{dQ_0}{da_0} - 2Q_1 \\
A_2 &= 3a_{1,0} \frac{dQ_0}{da_0} \\
A_3 &= -3 \left( \frac{1}{2} a_0 \frac{dQ_0}{da_0} - 6Q_3 \right) \\
A_4 &= -15 a_{1,0} \frac{dQ_0}{da_0} \\
A_5 &= 15 \left( \frac{1}{2} a_0 \frac{dQ_0}{da_0} - 15Q_5 \right)
\end{align*}
\]

From equations (43), for the first five coefficients $A_1, A_2, A_3, A_4, A_5$, the following set of linear equations can be obtained:

\[
\begin{align*}
A_1 &= a_0 \frac{dQ_0}{da_0} - 2Q_1 \\
A_2 &= 3a_{1,0} \frac{dQ_0}{da_0} \\
A_3 &= -3 \left( \frac{1}{2} a_0 \frac{dQ_0}{da_0} - 6Q_3 \right) \\
A_4 &= -15 a_{1,0} \frac{dQ_0}{da_0} \\
A_5 &= 15 \left( \frac{1}{2} a_0 \frac{dQ_0}{da_0} - 15Q_5 \right)
\end{align*}
\]
Again, substituting the numerical values given by equation (53) and table III into the foregoing expressions, there result the following equations:

\[
A_5^*=\frac{3}{4!} \left[ a_{1,2}^4 \frac{d^4 Q_4}{da_4^4} + a_{1,3}^3 \left( \frac{5 a_{1,2} a_{1,3} + a_{1,3}^3}{3!} + a_{2,3}(a_{1,2}^2 + a_{1,3}^2) \right) \frac{d^3 Q_2}{da_3^3} + a_{1,4}(a_{1,2}^2 + a_{1,3}^2) \frac{d^2 Q_3}{da_3^2} 
+ (5a_{1,2} a_{1,3} + 5a_{2,3} a_{1,3} + a_{2,4} a_{1,4} - 10a_{1,4} a_{2,4}) \right] \frac{dQ_3}{da_3} 
+ (5a_{1,2} - 11a_{1,4}) \frac{dQ_4}{da_4} \right]
\]

\[
A_3^* = -\frac{1}{2} \left[ a_{2,4} a_{1,3} \frac{d^4 Q_4}{da_4^3} + a_{1,3}^2 \left( \frac{3a_{2,4} - 13a_{1,3}^2}{2!} \right) \frac{d^3 Q_2}{da_3^2} + 3 \left[ a_{2,4} = -0 \right] \right] \frac{dQ_3}{da_3} 
+ 15(17a_{1,3} - 11a_{1,4}) \frac{dQ_4}{da_4} 
\]

\[
A_4^* = \frac{5}{64} \left[ a_{1,2} a_{1,3} \frac{d^4 Q_4}{da_4^2} + a_{1,3}^2 \left( \frac{6a_{1,2} a_{1,3} - 71a_{1,3}^2}{3!} \right) \frac{d^3 Q_2}{da_3^1} + \frac{3}{16} \left[ a_{1,2} = -0 \right] \right] \frac{dQ_3}{da_3} 
+ 15(13a_{1,3} - 15a_{1,4}) \frac{dQ_4}{da_4} 
\]

\[
A_6^* = 5 \left[ a_{1,2} a_{1,3} \frac{d^4 Q_4}{da_4^1} + a_{1,3}^2 \left( \frac{6a_{1,2} a_{1,3} - 71a_{1,3}^2}{3!} \right) \frac{d^3 Q_2}{da_3^0} + \frac{3}{16} \left[ a_{1,2} = -0 \right] \right] \frac{dQ_3}{da_3} 
+ 15(13a_{1,3} - 15a_{1,4}) \frac{dQ_4}{da_4} 
\]

Again, substituting the numerical values given by equation (53) and table III into the foregoing expressions, there result the following equations:

\[
-22.402 C_1 - 1.363 C_3 + 34.018 C_3 + 2.578 C_4 = 39.602 C_6 = 1.02340
25.036 C_1 - 67.392 C_2 - 45.658 C_3 + 163.240 C_4 + 66.762 C_6 = 0.05260
-20.665 C_1 + 15a_{1,3} C_1 + 3a_{1,3} C_1 - 135a_{1,3} C_4 + 7a_{1,3} C_6 = 0.02243
-15.235 C_1 + 64.050 C_3 - 178.260 C_4 + 202.310 C_4 + 391.330 C_6 = 0
10.437 C_1 - 46.495 C_2 + 150.380 C_2 - 361.150 C_4 + 197.330 C_6 = 0.00014
\]

The solution is given by

\[
C_1 = -0.0486 \quad C_3 = -0.0178 \quad C_4 = -0.0027 \quad C_6 = -0.00005
\]

The doublet distribution function \( J(2a\lambda_1) \) then becomes:

\[
J(2a\lambda_1) = 8\pi\alpha^2 V (1 - \lambda_1^2) (0.0447 + 0.0512\lambda_1 + 0.0187\lambda_1^2 + 0.0049\lambda_1^3 + 0.0022\lambda_1^4)
\]

The graph of this function with \( J/8\pi\alpha^2 V \) as ordinate and \( \lambda_1 \) as abscissa is shown in figure 3.

**Determination of the Transverse-Force Distribution.**

When the axial flow is combined with the transverse flow some information regarding the distribution of forces over the surface of the body can be obtained by introducing the notion of the transverse-force coefficient. For the pressure difference at the surface, according to Bernoulli's equation:

\[
p - p_0 = \frac{\sigma}{2} (U^2 + V^2 - q^2)
\]

where \( p_0 \) is the pressure at an infinite distance from the body and \( q \) is the velocity of the fluid at the surface, supposing the body to be at rest and the fluid to strike it at an angle \( \alpha \) where \( \tan \alpha = \frac{U}{V} \).

Now \( q \) has three components—in the directions \( \delta q_1, \delta q_2, \) and \( \delta q_3 \). They may be denoted by \( q_1, q_2, \) and \( q_3 \) respectively. Also denote by \( u_1, u_2, u_3 \) the velocity components of the axial flow and by \( v_1, v_2, v_3 \) the velocity components of the transverse flow taken in the plane \( \theta = 0 \). Then,

\[
q_1 = u_1 + v_1 \cos \theta \\
q_2 = u_2 + v_2 \cos \theta \\
q_3 = u_3 + v_3 \sin \theta
\]

Introducing these values into Bernoulli's equation,
It is only the term
\[ \sigma(u_x, v_x + u_y, v_y) \cos \theta = \sigma(u_x) \cos \theta \]
that need be considered, for the other terms have equal values for \( \theta \) and \( \pi - \theta \) and accordingly vanish when integrated over the entire cross section. The resulting transverse force, relative to an annular element of width unity, is therefore
\[ \frac{dQ}{dz} = \sigma \int_0^{2\pi} (u_x) \cos \theta \ d\theta = \sigma \pi \rho \ (u_x) \]

Klemperer defines the transverse-force coefficient by
\[ \frac{dQ}{dz} = \frac{2}{\pi \rho \sqrt{U^2 + V^2}} \]

or
\[ \beta = \pi \rho \left( \frac{u_x}{U^2 + V^2} \right) \sin 2\alpha \] (63)

By means of the velocity potentials of the axial and transverse flows this last expression takes the following form:
\[ \beta = \pi \rho \left( \frac{u_x}{U^2 + V^2} \right) \sin 2\alpha \] (64)

Tables IV and V give the numerical data for the evaluation of the right-hand side of equation (61) and figure 5 represents graphically these numerical results.

![Figure 5.—Transverse-force distribution.](image)

with \( \frac{\beta}{2\pi \sin 2\alpha} \) as ordinate and \( \frac{z}{2\alpha} (= \lambda \mu) \) as abscissa.

According to theory the positive and negative areas included by the \( \beta \) curve and the \( z \) axis are equal; that is, there is no resultant lift force but only a simple couple.

REFERENCES


2 Mr. Upon has kindly pointed out to the author the very close check of his approximation formula (equation (10), N. A. C. A. Report 400) for the transverse force with the more exact equation (63).
### TABLE I

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TABLE V

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\[ \beta = \frac{n(n+1)}{2} \frac{dQ_{a}}{dP_{a}} \]