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COMPRESSIBLE FLOW ABOUT SYMMETRICAL JOUKOWSKI PROFILES

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SUMMARY

The method of Poggi is employed for the determination of the effects of compressibility upon the flow past an obstacle. A general expression for the velocity increment due to compressibility is obtained. This general result holds whatever the shape of the obstacle; but, in order to obtain the complete solution, it is necessary to know a certain Fourier expansion of the square of the velocity of flow past the obstacle. An application is made to the case of a symmetrical Joukowski profile with a sharp trailing edge, fixed in a stream of velocity \( v_0 \) at an arbitrary angle of attack and with the circulation determined by the Kutta condition. The results are obtained in a closed form and are exact insofar as the second approximation to the compressible flow is concerned, the first approximation being the result for the corresponding incompressible flow. Formulas for the lift and moment analogous to the Blasius formulas in incompressible flow are developed and are applied to thin symmetrical Joukowski profiles for small angles of attack.

Since actual experimental data for Joukowski profiles are lacking, the theoretical results are applied to a thin and a thick profile at zero angle of attack, and the velocity and pressure distributions are calculated and compared with those for the corresponding incompressible cases. The critical values for the ratio of the stream velocity \( v_0 \) to the velocity of sound in the stream \( c_0 \), corresponding to the attainment of the local velocity of sound \( c \) by the fluid on the surface of the airfoils, are also obtained.

INTRODUCTION

When a compressible fluid streams past a fixed body with a velocity small enough so that nowhere in the fluid is the local velocity of sound exceeded, the flow may be represented by a velocity potential. The effect of compressibility is to distort the streamline picture associated with the corresponding incompressible flow. This distortion has been calculated by Janzen (reference 1) and Rayleigh (reference 2) for circular cylinders and spheres and recently for elliptical cylinders by Hooker (reference 3). The methods used by these authors, however, are not feasible for the determination of the flow about obstacles other than the simple ones mentioned. On the other hand, a method introduced by Poggi (reference 4) may be used in determining the flow about shapes resembling airfoil profiles.

The method due to Poggi is as follows: When the fluid is compressible, the equation of continuity may be written as

\[
\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = -\frac{1}{\rho} \frac{D\rho}{Dt}
\]

where the symbol \( D/Dt \) denotes, as usual, the operator \( \frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} \); \( v_x, v_y \) the fluid velocity components; and \( \rho \), the variable density of the fluid.

This divergence will introduce extra terms in the expressions for the velocity components, the divergence at an element \( dx \, dy \) being equivalent to a simple source of strength \( \frac{1}{2\pi} \frac{D\rho}{Dx} \) of strength \( \frac{1}{2\pi} \frac{D\rho}{Dy} \). Poggi thus replaces the compressible flow by an incompressible flow due to a suitable distribution of sources throughout the region of flow.

If the motion of the fluid is steady, then the equation of continuity and Euler's differential equations of motion become:

\[
\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = -\frac{v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_y}{\partial y}}{\rho} \quad \text{and} \quad \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = -\frac{1}{\rho} \frac{\partial \rho}{\partial y}
\]

Assuming the pressure \( p \) to be a function of the density \( \rho \) only and introducing the local velocity of sound

\[
c \left( = \sqrt{\frac{\partial \rho}{\partial p}} \right)
\]

equations (2) yield the following:

\[
\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = \frac{1}{c^2} \left( v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_x \frac{\partial v_y}{\partial y} + v_y \frac{\partial v_x}{\partial x} \right)
\]

or if

\[
\mathbf{v}^* = v_x + v_y
\]

then

\[
\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = \frac{1}{2c^2} \left( v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_x \frac{\partial v_y}{\partial y} + v_y \frac{\partial v_x}{\partial x} \right)
\]

If it is further assumed that the motion of the fluid is irrotational, then

\[
\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0
\]
and a velocity potential $\phi$ may be introduced, where

$$
\nu_x = -\frac{\partial \phi}{\partial x}, \quad \nu_y = -\frac{\partial \phi}{\partial y}
$$

The strength of the source at a point $(x, y)$, given by the expression on the right-hand side of equation (3), then becomes

$$
\frac{1}{4\pi e^2} \left( \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{\partial \phi}{\partial y} \right)
$$

Suppose now that $(\xi, \eta)$ and $(x, y)$ are the rectangular coordinates of points in the $\xi$ and $\eta$ planes, respectively, and furthermore that these two planes are conformally related, that is

$$
\xi = f(x)
$$

where $\xi = \xi + i\eta$, $x = x + iy$. Let the $\xi$ plane be the plane of the profile and the $\eta$ plane, the plane of the circle into which the profile is mapped by the foregoing conformal transformation. It is well known that, at a pair of corresponding points at which $\xi$ and $\eta$ possess no singularities, a source at one such point corresponds to a source of equal strength at the other. It follows then that at corresponding elements

$$
\frac{1}{4\pi e^2} \left( \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{\partial \phi}{\partial y} \right) d\xi d\eta
$$

where, in the expression on the right-hand side, $\phi$ is the velocity potential in the $\eta$ plane while $v$ is the magnitude of the velocity in the $\xi$ plane.

In polar coordinates $(r, \theta)$ the strength of a source at an element $d\theta d\phi$ of the $\eta$ plane is

$$
\frac{1}{4\pi e^2} \left( \frac{\partial \phi}{\partial r} \frac{\partial \phi}{\partial r} + \frac{\partial \phi}{\partial \theta} \frac{\partial \phi}{\partial \theta} \right) r^2 d\theta d\phi
$$

or, introducing a new variable $\lambda = \frac{R}{r}$ (where $R$ is the radius of the circle into which the profile is mapped), and

$$
\nu_r = -\frac{\partial \phi}{\partial r}, \quad \nu_\theta = -\frac{1}{r} \frac{\partial \phi}{\partial \theta}
$$

this expression becomes

$$
\frac{1}{4\pi e^2} \left( \nu_r \frac{\partial \phi}{\partial \lambda} + \nu_\theta \frac{\partial \phi}{\partial \theta} \right) \lambda d\lambda d\theta
$$

With the source distribution known in the plane of the circle and given by equation (5), the induced tangential velocity at the circular boundary may be calculated.

Thus consider a unit source located at a point $Q$ of the $\eta$ plane. In the presence of a circular boundary of radius $R$, the velocity induced at any point $P$ external to or on the boundary is given by

$$
\frac{dw}{dz} = \frac{1}{z-Q} + \frac{1}{z-S} - \frac{1}{z}
$$

where $S$ is the point inverse to $Q$ in the circle. (See fig. 1.) Since the normal velocity at the boundary is zero, the velocity there is wholly tangential and is given by

$$
\frac{dw}{dz} \mid_{\text{on boundary}} = -\frac{2\lambda \sin (\theta - \theta)}{R[1 - 2\lambda \cos (\theta - \theta) + \lambda^2]}
$$

where $S = R e^{i\theta}$; $Q = r e^{i\theta}$; and $z_S = R \frac{R}{r} e^{i\theta}$

Hence, the total velocity induced at any point of the circular boundary by the system of sources given by equation (5) is

$$
\Delta v = \frac{1}{2\pi e^2} \int_0^{2\pi} \int_0^R \frac{\partial \phi}{\partial \lambda} \frac{\partial \phi}{\partial \theta} \left( 1 - 2\lambda \cos (\theta - \theta) + \lambda^2 \right) d\lambda d\theta
$$

The justification for replacing $c$ by $c$ in equation (7) may be shown in the following way. From the Euler equations of motion (2) and the condition for irrotational motion, it follows that

$$
\frac{1}{2} d\phi + \frac{1}{\rho} dp = 0
$$

Then when adiabatic conditions prevail so that the relation between $p$ and $\rho$ is

$$
\rho = \rho_0 \left( \frac{\rho}{\rho_0} \right)^{\gamma - 1}
$$

it follows by integration that

$$
c^2 = c^2 \left[ 1 + \frac{v}{v\gamma - 1} \left( 1 - \frac{\rho^2}{\rho_0^2} \right) \right]
$$

where the zero subscripts denote the corresponding magnitudes in the undisturbed stream. From the foregoing equation it is seen that $c$ has a maximum value at the stagnation point where $v = 0$, that is

$$
c_{max} = c_0 \left( 1 + \frac{v}{2} \frac{1}{v\gamma - 1} \right)
$$
Furthermore, as the streamline corresponding to the boundary of the obstacle is traversed, a point is reached where, for a definite value of the ratio $v_0/c_0$, the velocity of the fluid equals that of the local velocity of sound. This critical velocity is obtained from equation (8) by putting $v=c$ and solving for $c$. Thus

$$v_{cri} = \frac{2}{\gamma + 1} c^2 \left( 1 + \frac{\gamma - 1}{2} \right)$$

For example, let $v_0/c_0 = 0.75$. Then with $\gamma = 1.408$ (for air),

$$c_{max} = 1.065 c_0$$

and

$$c_{max} = v_{cri} = 0.982 c_0$$

Away from the obstacle, $v$ approaches $c_0$ and $c$ approaches $c_0$. Thus it is seen that the variation of the local velocity of sound from $c_0$ is, in general, small enough to permit replacing $v$ by $c_0$, at least to a first approximation.

Equation (7) is a functional equation for the fluid velocity $v$ and may be solved by a method of successive substitutions. The procedure, due to Poggi, is to substitute for $v_0$, $c_0$, and $c^2$ values pertaining to the corresponding incompressible flow and thus obtain a first approximation to the sink-source distribution in the plane of the circle. The method thus considers the incompressible flow to be the first approximation to the compressible flow. The second approximation is then obtained by superposing on the incompressible flow the effect of the sink-source distribution as given by equation (7); that is

$$\xi_{comp} = \xi_{incomp} + \Delta \xi$$

**GENERAL DEVELOPMENTS**

Before equation (7) is applied to any particular case, it is expedient to consider it first in a general way. Thus, suppose that $c^2$ can be developed in a Fourier series so that

$$c^2 = \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)$$

where the $a_n$, $b_n$ are functions of $\lambda$ and also contain the parameters of the shape.

Then

$$\frac{\partial c^2}{\partial \lambda} = \sum_{n=1}^{\infty} n[a_n \cos n\theta + b_n \sin n\theta]$$

$$\frac{\partial c^2}{\partial \theta} = \sum_{n=1}^{\infty} n[b_n \cos n\theta - a_n \sin n\theta]$$

where the primes denote differentiation with regard to $\lambda$. Expressions for $v_r$ and $v_\theta$ are obtained from the complex potential of the flow past a circular cylinder of radius $R$, with the circulation determined by the Kutta condition. Let the stream of velocity $v_0$ make an angle $\alpha$ with the negative direction of the $x$ axis. Then the potential is

$$\psi = \psi_0 \left( z e^{i\alpha} + \frac{R}{z} \right) + \psi_R (e^{i\alpha} - e^{-i\alpha}) \log \frac{z}{R}$$

and

$$\frac{d\psi}{dz} = v_0 (z + R) (ze^{i\alpha} - Re^{-i\alpha})$$

Then

$$v_r = -v_0 (1 - \lambda^2) \cos (\theta + \alpha) = v_0 (c_0 \cos \theta + c_3 \sin \theta)$$

and

$$v_\theta = v_0 (1 - \lambda^2) \sin (\theta + \alpha) + 2v_0 \lambda \sin \alpha$$

where

$$c_0 = (1 - \lambda^2) \cos \alpha$$

$$c_3 = (1 - \lambda^2) \sin \alpha$$

$$d_3 = 2\lambda \sin \alpha$$

$$d_4 = (1 + \lambda^2) \sin \alpha$$

$$d_5 = (1 + \lambda^2) \cos \alpha$$

Therefore

$$\frac{\partial \psi}{\partial \lambda} = \frac{1}{2} v_0 a'_0 (c_1 \cos \theta + c_3 \sin \theta)$$

$$+ \frac{1}{2} v_0 \sum_{n=1}^{\infty} \left[ a_n c_0 \cos (n-1)\theta + \sin (n+1)\theta \
+ b_n c_0 \sin (n+1)\theta - \sin (n-1)\theta \
+ a_n c_3 \cos (n-1)\theta - \sin (n+1)\theta \
+ b_n c_3 \sin (n+1)\theta + \sin (n-1)\theta \right]$$

and

$$\frac{\partial \psi}{\partial \theta} = \frac{v_0}{2\lambda^{n+1}} \sum_{n=1}^{\infty} \left[ a_n d_0 \sin (n+1)\theta + \sin (n-1)\theta \
- b_n d_0 \cos (n-1)\theta + \cos (n+1)\theta \
+ a_n d_3 \cos (n-1)\theta - \cos (n+1)\theta \
- b_n d_3 \sin (n+1)\theta - \sin (n-1)\theta \right]$$

The following definite integrals will be found useful in evaluating equation (7):

$$\int_0^{2\pi} \frac{\sin (\theta - \delta) \psi}{1 - 2\lambda \cos (\theta - \delta) + \lambda^2} \sin n\theta \, d\theta = 0 \quad \text{if} \quad n = 0$$

$$\int_0^{2\pi} \frac{\sin (\theta - \delta) \psi}{1 - 2\lambda \cos (\theta - \delta) + \lambda^2} \sin n\theta \, d\theta = 0 \quad \text{if} \quad n \geq 1$$

Then substituting the foregoing expressions for $v_r$, $v_\theta$, $\frac{\partial \psi}{\partial \lambda}$ and $\frac{\partial \psi}{\partial \theta}$ into equation (7) and integrating with regard to $\theta$, it follows, after replacing the derivatives $a_n'$, $b_n'$ by $a_n$, $b_n$ by means of partial integrations, that
\[
\Delta w = \frac{\pi \delta}{4 \alpha_0} \left[ -(a_0)_{\lambda=0} \sin (\beta + \alpha) - (a_0)_{\lambda=0} \sin (\beta - \alpha) \
+ (b_0)_{\lambda=0} \cos (\beta - \alpha) - \sum_{n=1}^{\infty} 2n \cos (n\delta + \alpha) \int_0^{1} \lambda^n b_{n+1} d\lambda \
+ \sum_{n=1}^{\infty} 2n \sin (n\delta - \alpha) \int_0^{1} \lambda^n a_{n-1} d\lambda \
- \sum_{n=1}^{\infty} 2n \sin (n\delta - \alpha) \int_0^{1} \lambda^n a_{n-1} d\lambda \
+ \sin \alpha \sum_{n=1}^{\infty} 4n \sin n\delta \int_0^{1} \lambda^{n-1} b_n d\lambda \
+ \sin \alpha \sum_{n=1}^{\infty} 4n \cos n\delta \int_0^{1} \lambda^{n-1} a_n d\lambda \right] 
\]

(13)

\[ z' = \alpha + z \]

If \( w \) denotes the complex potential of the incompressible flow in the \( \zeta \) plane, then the complex velocity is given by

\[ \frac{dw}{d\zeta} = \frac{dw}{dz} \frac{dz}{dz'} \frac{dz'}{d\zeta} \]

where

\[ \frac{dz}{dz'} = 1 \quad \text{and} \quad \frac{dz'}{d\zeta} = \frac{z'}{z' + a} (z' - a) \]

According to equation (12)

\[ \frac{dw}{dz} = v_0 \frac{(z + R)(z e^{i\alpha} - R e^{-i\alpha})}{z^2} \]

Hence

\[ \frac{dw}{d\zeta} = v_0 \frac{(z + R)(z e^{i\alpha} - R e^{-i\alpha})z''}{z^2(z' + a)(z' - a)} \]

But

\[ z' + a = z + a(1 + \epsilon) = z + R \]

\[ z' - a = z - a(1 - \epsilon) \]

Therefore

\[ \frac{dw}{d\zeta} = v_0 \frac{(z + a)^2(z e^{i\alpha} - R e^{-i\alpha})}{z^2(z - a(1 - \epsilon))} \]

Figure 2.—Transformation of a symmetrical Joukowski profile into a circular contour.
Introducing 
\[ \lambda = \frac{B}{r}, \quad R = \frac{a}{h}, \text{ and } \frac{a(1-\epsilon)}{R} = k \]
it follows that 
\[ \nu = \nu_0 (1+2h\lambda \cos \theta + k\lambda^2) \left[ 1-2\lambda \cos (\theta+2\alpha + \lambda^2) \right] \]

It is now required to obtain the Fourier series for \( \nu/\nu_0 \). Thus, making use of the following developments:
\[ \frac{\sin \theta}{1-2k\lambda \cos \theta + k^2\lambda^2} = \sum_{n=1}^{\infty} (k\lambda)^{n-1} \sin n\theta \]
\[ \frac{\cos \theta - k\lambda}{1-2k\lambda \cos \theta + k^2\lambda^2} = \sum_{n=1}^{\infty} (k\lambda)^{n-1} \cos n\theta \]
\[ \frac{1}{1-2k\lambda \cos \theta + k^2\lambda^2} = \frac{1}{1-k\lambda^2} \left[ 1+2\sum_{n=1}^{\infty} (k\lambda)^n \cos n\theta \right] \]
it follows that 
\[ \nu = \nu_0 (1+2h\lambda \cos \theta + k\lambda^2)^2 \left[ 1+2\lambda^2 \cos 2\alpha \right] \left[ 1-2h\lambda \cos \theta + k\lambda^2 \right] \]
\[ + 2\lambda \sin 2\alpha \sum_{n=1}^{\infty} (k\lambda)^n \sin n\theta \]
\[ + 2\sum_{n=1}^{\infty} (1+h\lambda^2) \cos 2\alpha \sum_{n=1}^{\infty} (k\lambda)^n \cos n\theta \]
Reducing this expression to the form of a Fourier series, it turns out that, for \( n>2 \)
\[ a_n = 2k^{n-3} (h+k)^2 (1+h\lambda^2)^2 \lambda^2 \left[ \cos 2\alpha + \frac{k(1+\lambda^2) \cos 2\alpha}{1-k\lambda^2} \right] \]
\[ b_n = 2k^{n-3} (h+k)^2 (1+h\lambda^2)^2 \lambda^2 \sin 2\alpha \]
For later use it will be convenient to introduce the following notation for \( n \geq 0 \):
\[ a_n = 2k^{n-3} (h+k)^2 (1+h\lambda^2)^2 \lambda^2 \cos 2\alpha \]
\[ a_n = 2k^{n-3} (h+k)^2 (1+h\lambda^2)^2 \lambda^2 \left[ \frac{k(1+\lambda^2) \cos 2\alpha}{1-k\lambda^2} \right] \]
\[ \bar{b}_n = 2k^{n-3} (h+k)^2 (1+h\lambda^2)^2 \lambda^2 \sin 2\alpha \]
\[ \bar{a}_n = a_n + \bar{a}_n \]
Also
\[ \frac{1}{2} a_0 = AD + (B + Ck\lambda) Fk \lambda \]
\[ a_1 = 2BD + (A + Bk\lambda + Ck^2\lambda) Fk \lambda \]
\[ a_2 = 2CD + (B + Ak\lambda + Bk^2\lambda^2 + Ck^2\lambda^2) Fk \lambda \]
\[ b_1 = E(A + Bk\lambda + Ck^2\lambda^2) \]
\[ b_2 = E(B + Ak\lambda + Bk^2\lambda^2 + Ck^2\lambda^2) \]
where
\[ A = 1+4h^2\lambda^2 + h^4 \lambda^4 \]
\[ B = 2h\lambda (1+h^2\lambda^2) \]
\[ C = h^2\lambda^2 \]
where
\[ a_0 - \tilde{a}_n = \frac{2}{k} \left( (h+k)^3 \cos 2\alpha - k \right) + h^3 + [k(h+k)]^3 \]
\[ -k(1+h)(h+k)^3 + 2hk(h+k)^2 \cos 2\alpha \]
\[ + k^3 \cos 2\alpha \lambda^2 - h^3(k+h)^3 \]
\[ + k^1 + h^2(k+h)^3 \cos 2\alpha \lambda^1 \}
\[ a_1 - \tilde{a}_n = \frac{2 \lambda \cos 2\alpha}{k^2} \{ [h(h+2k) + 2hk\lambda^1] \]
\[ \frac{2h^3 \lambda^2 + (1-2k \cos 2\alpha)}{k^2} \]
\[ \frac{1}{1-k^2 \lambda^2} \}

\[ a_2 - \tilde{a}_n = \frac{2h^2}{k} \lambda \sin 2\alpha \]
\[ b_1 - \tilde{b}_n = \frac{2 \lambda \sin 2\alpha}{k^2} \{ [h(h+2k) + 2hk\lambda^1] \]
\[ \frac{2h^3 \lambda^2 + 2\lambda \cos 2\alpha}{k^2} \]
\[ k^2 \lambda \sin 2\alpha \]
\[ b_0 = \frac{2}{k} (h+k)^3(1+hh\lambda^1)^2 \sin 2\alpha \]

It is a great simplification to replace \( \tilde{a}_n \) by \( a_a + a_a \) in the last two integrals of equation (18) before performing the integrations. Then
\[ \sum_{n=1}^{\infty} 2n \cos (n\delta + \alpha) \int_0^1 \left( \lambda^2 \tilde{b}_{n+1} + \lambda^{n-1} \tilde{b}_n \right) d\lambda \]
\[ + \sum_{n=1}^{\infty} 2n \sin (n\delta + \alpha) \int_0^1 \left( \lambda^2 \tilde{a}_{n+1} + \lambda^{n-1} \tilde{a}_n \right) d\lambda \]
\[ = (1+k)(h+k)^3 \sum_{n=1}^{\infty} 4nk^{n-4} \sin (n\delta - \alpha) \]
\[ \int_0^1 (1+hk\lambda^3)^{2\lambda^{n-1}} d\lambda \]

and
\[ \sum_{n=1}^{\infty} 2n \cos (n\delta - \alpha) \int_0^1 \left( \lambda^2 \tilde{b}_{n+1} + \lambda^{n-1} \tilde{b}_n \right) d\lambda \]
\[ - \sum_{n=1}^{\infty} 2n \sin (n\delta - \alpha) \int_0^1 \left( \lambda^2 \tilde{a}_{n+1} + \lambda^{n-1} \tilde{a}_n \right) d\lambda \]
\[ = -(1+k)(h+k)^3 \sum_{n=1}^{\infty} 4nk^{n-4} \sin (n\delta - \alpha) \]
\[ \int_0^1 (1+hk\lambda^3)^{2\lambda^{n-1}} d\lambda \]

Also
\[ \sum_{n=1}^{\infty} 2n \sin (n\delta + \alpha) \int_0^1 \left( \lambda^2 \tilde{a}_{n-1} + \lambda^{n-1} \tilde{a}_n \right) d\lambda \]
\[ = -(1+k)(h+k)^3 \sum_{n=1}^{\infty} 4nk^{n-4} \sin (n\delta + \alpha) \]
\[ \int_0^1 (1+hk\lambda^3)^{2\lambda^{n-1}} d\lambda \]
\[ + (1-2k \cos 2\alpha + k^2)(1+k)(h+k)^3 \]
\[ \sum_{n=1}^{\infty} 4nk^{n-6} \sin (n\delta + \alpha) \int_0^1 \left( \lambda \tilde{a}_{n+1} + \lambda^{n-1} \tilde{a}_n \right) d\lambda \]

and
\[ \sum_{n=1}^{\infty} 2n \sin (n\delta - \alpha) \int_0^1 \left( \lambda^2 \tilde{a}_{n-1} + \lambda^{n-1} \tilde{a}_n \right) d\lambda \]
\[ = -(1+k)(h+k)^3 \sum_{n=1}^{\infty} 4nk^{n-4} \sin (n\delta - \alpha) \]
\[ \int_0^1 (1+hk\lambda^3)^{2\lambda^{n-1}} d\lambda \]
\[ + (1-2k \cos 2\alpha + k^2)(1+k)(h+k)^3 \]
\[ \sum_{n=1}^{\infty} 4nk^{n-6} \sin (n\delta - \alpha) \int_0^1 \left( \lambda \tilde{a}_{n+1} + \lambda^{n-1} \tilde{a}_n \right) d\lambda \]

Consider now the integrals
\[ J_2 = \sum_{n=1}^{\infty} 4nk^n \sin (n\delta + \beta) \int_0^1 (1+hk\lambda^3)^{2\lambda^{n-1}} d\lambda \]
and
\[ J_1 = \sum_{n=1}^{\infty} 4nk^n \cos (n\delta - \beta) \int_0^1 (1+hk\lambda^3)^{2\lambda^{n-1}} d\lambda \]

Then
\[ J_1 + iJ_2 = \sum_{n=1}^{\infty} \frac{1}{2} \left[ (1+hk\lambda^3)^{2\lambda^{n-1}} \right] d\lambda \]

Let \( s = k \lambda^2 \). Then
\[ J_1 + iJ_2 = \frac{e^{is}}{2} \int_0^1 e^{is(1+hk)} d\delta = \frac{e^{is(1+hk)}}{2} \int_0^1 \left[ k \sin (1+hk) \lambda^2 + \frac{1}{2} \lambda^2 \sin 2\delta \right] d\lambda \]

Hence
\[ J_2 = \frac{k(1+hk)^3 \sin (\beta + \delta) - k \sin \beta}{2} \]
\[ \frac{1}{1-2k \cos \delta + k^2} \]
\[ - k \left[ \cos (\beta - \delta) \tan^{-1} \frac{k \sin \delta}{1-k \cos \delta} \right] \]
\[ \frac{1}{2} \sin (\beta - \delta) \cos (\beta - \delta) \tan^{-1} \frac{k \sin \delta}{1-k \cos \delta} \]
\[ + \frac{1}{2} \sin (\beta - \delta) \cos (\beta - \delta) \tan^{-1} \frac{k \sin \delta}{1-k \cos \delta} \]

Replacing \( \beta \) by \( \alpha_1 - \alpha_2 \), or \( -3\alpha \), the corresponding integrals in equations (20) and (21) are obtained.

Consider finally the integrals
\[ I_1 = \sum_{n=1}^{\infty} 4nk^n \cos n\delta \int_0^1 (1+hk\lambda^3)^{2\lambda^{n-1}} d\lambda \]
and
\[ I_2 = \sum_{n=1}^{\infty} 4nk^n \sin n\delta \int_0^1 (1+hk\lambda^3)^{2\lambda^{n-1}} d\lambda \]

Then
\[ I_1 + iI_2 = \int_0^1 (1+hk\lambda^3)^{2\lambda^{n-1}} \sum_{n=1}^{\infty} n(k\lambda)^n e^{i\delta} d\lambda \]
Again let \( s = kh^2 \). Then

\[
I_1 + iI_2 = \frac{e^{i\theta}}{2} \int_0^1 \frac{(1 + kh)^2}{(1 - ke)(1 - se^\theta)} ds = \\
-\frac{h(h + e^\theta)}{2k(e^\theta - e^{-\theta})^2} \log(1 - k^2) \\
+ \frac{h(h + e^\theta)}{2e^\theta(k - e^{-\theta})^2} \log(1 - ke^\theta) \\
+ \frac{(h + k)(h + e^\theta)}{2(k - e^{-\theta})^2} \log(1 - ke^\theta) \\
- \frac{k(h + e^\theta)}{2(1 - ke^\theta)(k - e^\theta)^2}.
\]

Therefore

\[
I_1 = \frac{(h + k)^2}{2k} \frac{2k - (1 + k^2) \cos \delta}{(1 - 2k \cos \delta + k^2)} \log(1 - k^2) \\
- \frac{h}{4} \frac{(hk - 1) \cos \delta + k - h \cos 2\delta}{1 - 2k \cos \delta + k^2} \tan^{-1} \frac{k \sin \delta}{1 - k \cos \delta} \\
+ \frac{h}{2} \frac{(hk - 1) \sin \delta - h \sin 2\delta}{1 - 2k \cos \delta + k^2} \tan^{-1} \frac{k \sin \delta}{1 - k \cos \delta} \\
+ \frac{h + k}{4} \frac{[hk^2 - 2k + (1 - 2hk + k^2) \cos \delta + h \cos 2\delta]}{(1 - 2k \cos \delta + k^2)^2} \log(1 - 2k \cos \delta + k^2) \\
- \frac{h + k}{2} \frac{(1 - 2hk - k^2) \sin \delta + h \sin 2\delta}{(1 - 2k \cos \delta + k^2)^2} \tan^{-1} \frac{k \sin \delta}{1 - k \cos \delta} \\
+ \frac{h + k}{2} \frac{(1 - h^2) \sin \delta + k \sin 2\delta}{(1 - 2k \cos \delta + k^2)^2} \tan^{-1} \frac{k \sin \delta}{1 - k \cos \delta} \\
- \frac{h + k}{4} \frac{(1 - 2hk - k^2) \sin \delta + h \sin 2\delta}{(1 - 2k \cos \delta + k^2)^2} \log(1 - 2k \cos \delta + k^2) \\
+ \frac{k}{2} \frac{(1 - h^2) \sin \delta}{1 - 2k \cos \delta + k^2}.
\]

From equations (18), (19), and (20) and the integrals \( J_3, I_1, \) and \( I_2 \), it follows that

\[
\Delta \varphi = \frac{\mu}{r_0} \left[ -\frac{1}{3} \left( 3h^2 + 3hk + k^2 \right) + \frac{2h}{3} \left( h + k \right) \log \left( 1 - \frac{2h^2}{k^2} \right) \right] \\
\sin(\beta + \alpha) + \left[ \frac{2}{3} \left( 3h^2 + 3hk + k^2 \right) - \frac{2h}{3} \left( h + k \right) \right] \sin(\beta + \alpha) \\
+ \left[ \frac{10}{3} \frac{h^2}{k^2} \sin(2\beta + \alpha) + \frac{4}{3} \frac{h^2}{k^2} \sin(3\beta + 3\alpha) \right] \\
+ \frac{4h}{k} \sin(\beta - \alpha) + \frac{8}{3} \frac{(h + k)^2}{k^2} J_3(\alpha) \\
- \frac{4}{k} \frac{(h + k)^2}{k^2} J_3(\beta - \alpha) \\
\right] \frac{2}{k} \left( 1 - 2k \cos(2\alpha + \beta) \right) I_1 \sin \alpha \\
+ \frac{4}{k} \frac{(1 - h^2)^2}{k^2} \left( 1 - 2k \cos(2\alpha + \beta) \right) I_2 \sin \alpha
\]

where \( J_3(\beta - \alpha) \), for example, means that in the expression for \( J_3(\beta) \), \(-\alpha\) has been substituted for \( \beta \); and \( \mu = \left( \frac{\varphi_0}{c_0} \right)^2 \). There is no difficulty in evaluating \( \Delta \varphi / \varphi_0 \) for any value of \( h \) except \( h = 0 \). For \( h = 0, k = 1 \), however, the Joukowski profile degenerates into a line segment and

\[
\Delta \varphi = \frac{\mu}{r_0} \sin \alpha \left[ \frac{\cos(\beta + 2\alpha)}{1 - \cos \delta} + 2 \sin \alpha \frac{\sin(\beta + \alpha) + \sin \alpha}{1 - \cos \delta} \\
\right] \\
- \frac{2}{3} \frac{\sin^2 \alpha}{1 - \cos \delta} \log 2(1 - \cos \delta) - \frac{4}{3} \frac{\sin^2 \alpha}{1 - \cos \delta} \frac{1}{\log n^n}
\]

The last term in this expression contains the divergent series \( \sum_{n=1}^{\infty} \frac{1}{n} \), which approaches infinity like \( \lim_{n \to \infty} \log n. \) This infinite term shows that streamline flow cannot be maintained about a straight-line profile except for the trivial case of zero angle of attack.

**Correction for the Circulation**

It is noted that the expression for the complex velocity about the circular profile given by equation (12) was obtained with the circulation fixed by the Kutta condition. When \( \Delta \varphi \), representing the effect of compressibility, is added to the incompressible velocity obtained from equation (12) to yield the compressible velocity, the Kutta condition no longer holds. In order to restore the Kutta condition, an additional circulation \( \Delta \Gamma \) is added to the incompressible one. Thus, the
velocity at the boundary of the circular profile is given by
\[ \frac{v_e}{v_0} = 2 \sin (\delta + \alpha) + \frac{\Gamma_0 + \Delta \Gamma}{2 \pi R v_0} + \frac{\Delta \nu}{v_0} \]  
(24)
where \( \Gamma_0 + \Delta \Gamma = \Gamma \) and \( \Gamma_0 = 4 \pi R v_0 \sin \alpha \), the circulation in the incompressible flow. The Kutta condition, i.e.,
\[ \left( \frac{v_e}{v_0} \right)_{x=m} = 0, \]
thus serves to evaluate \( \Delta \Gamma \). For \( \delta = \pi \) the expressions for \( J_2(\phi) \), \( I_1 \), and \( I_3 \) simplify considerably. Thus
\[ (J_2(\phi))_{x=m} = \frac{k(1 + h k)^2}{2(1 + k)} + h \log (1 + k) - k \]
\[ + h \frac{k}{2} \log (1 + k) + \frac{1}{gk} \log (1 - k) - k \]
(25)

The relation between the velocities in the plane of the circle and the plane of the profile is given by
\[ \frac{d\omega}{d\xi} = \frac{d\omega'}{d\xi'}, \]
where \( \omega' = \omega + a \delta \) and \( \xi' = \xi + \frac{q^2}{2} \).

Then
\[ \left| \frac{dz}{d\xi} \right| = 1 \quad \text{and} \quad \left| \frac{dz'}{d\xi'} \right| = \frac{1 + 2k \cos \delta + h^2 \lambda^2}{\sqrt{1 + 2k \cos \delta + h^2 \lambda^2}} \]
It follows that on the profile where \( \lambda = 1 \)
\[ \frac{v_p}{v_0} = \frac{1 + 2k \cos \delta + h^2}{\sqrt{1 - 2k \cos \delta + h^2 \lambda^2}} \]
where \( v_p \) is the velocity on the profile corresponding to \( v_0 \) on the circle.

When the profile is assumed to be thin so that only the first power of \( h \) is retained and the angle of attack is small so that \( \cos \alpha \approx 1 \) and \( \sin \alpha \approx \kappa \alpha \), then the Kutta condition leads to the following expression for \( \Delta \Gamma/2\pi R v_0 \)
\[ \frac{\Delta \Gamma}{2 \pi R v_0} = \mu (1 + h \kappa) \]
(26)
It then follows that
\[ \frac{\Gamma}{\Gamma_0} = 1 + \frac{1 + h \kappa}{2} \mu \]
(27)
This value for the ratio \( \Gamma/\Gamma_0 \) corroborates Glauert's result (reference 6)
\[ \frac{\Gamma}{\Gamma_0} = \frac{1}{\sqrt{1 - \mu}} = 1 + \frac{1}{2} \mu + \ldots \]
when the profile is very thin, i.e., when \( h \) is negligible in comparison with unity.

Since the rigorous expressions are available, it may be interesting to compare the approximate result given by equation (26) with the exact result obtained from equation (22). Thus, for a very thin profile defined by \( \mu = 0.01 \left( h = \frac{1}{101}, k = \frac{99}{101} \right) \) and for the more conventional profiles defined by \( \mu = 0.05 \left( h = \frac{1}{21}, k = \frac{19}{21} \right) \) and \( \epsilon = 0.10 \left( h = \frac{1}{11}, k = \frac{9}{11} \right) \), at angles of attack \( \alpha = 10^\circ \) and \( \alpha = 5^\circ \), the following table presents the results:

<table>
<thead>
<tr>
<th>( \alpha ) (deg.)</th>
<th>( \epsilon = 0.01 )</th>
<th>( \epsilon = 0.05 )</th>
<th>( \epsilon = 0.10 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta \Gamma / (2 \pi R v_0) ) (approximate)</td>
<td>( \Delta \Gamma / (2 \pi R v_0) ) (exact)</td>
<td>( \Delta \Gamma / (2 \pi R v_0) ) (approximate)</td>
<td>( \Delta \Gamma / (2 \pi R v_0) ) (exact)</td>
</tr>
<tr>
<td>( 10 )</td>
<td>( 0.1705% )</td>
<td>( 0.2070% )</td>
<td>( 0.2236% )</td>
</tr>
<tr>
<td>( 5 )</td>
<td>( 0.0214% )</td>
<td>( 0.0915% )</td>
<td>( 0.0905% )</td>
</tr>
</tbody>
</table>

It is to be noticed that for \( \alpha = 10^\circ \), the exact evaluation of \( \Delta \Gamma/2\pi R v_0 \) yields a greater value for \( \epsilon = 0.01 \) than for \( \epsilon = 0.05 \), a fact not given by the approximate equation (26). This reversal appears, in general, for larger values of \( \epsilon \) as the angle of attack increases; e.g., for \( \epsilon = 0.05 \) at \( \alpha = 20^\circ \). This feature of the exact expression for the additional lift has no practical significance, however, insofar as the lift is concerned, since the appropriate combination of \( \epsilon \) and \( \alpha \) showing this reversal is outside the practical range.

In the calculation of the local velocities and pressures on the surface of the airfoil, the rigorous expressions for \( \Delta v/v_0 \) are to be used. The rigorous derivation, however, of the total integrated lift and moment on the airfoil involves great mathematical difficulties. A simplified form for \( \Delta v/v_0 \) may, however, be obtained for a thin Joukowski profile at small angles of attack. Its use in integrating for the lift yields, as will be shown later, the expected result that
\[ \text{Lift} = \rho v_0 \Gamma \]
where \( \Gamma = \Gamma_0 \left( 1 + \frac{1 + h}{2} \mu \right) \)

This result justifies the use of the simplified form of \( \Delta v/v_0 \) in calculating the lift, but its use in integrating for the moment, although reasonable, is somewhat uncertain.

If, then, only the first power of \( h \) is retained and the angle of attack taken small enough so that \( \cos \alpha \approx 1 \) and \( \sin \alpha \approx \kappa \alpha \), it follows from equation (22) that
\[ \Delta v/v_0 = \mu \left( \cos \delta + h(4 + 8 \cos \delta + 3 \cos 2 \delta) \alpha + h(\sin \delta + \sin 2 \delta) \right) \]

The expression for \( v_0/v_0 \), replacing \( \Delta \Gamma/2\pi R v_0 \) by the value given by equation (26), then becomes:
\[ \frac{v_0}{v_0} = 2 \sin \delta + 2 \alpha (1 + \cos \delta) \]
\[ + \mu \left( \cos \delta + h(4 + 8 \cos \delta + 3 \cos 2 \delta) \alpha + h(\sin \delta + \sin 2 \delta) \right) \]
(28)
CALCULATION OF THE PRESSURE AND LIFT ON THE AIRFOIL

According to equation (8)

\[ c_0 = c_0 \left[ 1 + \gamma \frac{1 - v_0^2}{2} \left( 1 - \frac{v_0^2}{c_0^2} \right) \right] \]

Then from the adiabatic equation of state

\[ p = p_0 \left( \frac{p}{p_0} \right)^{\gamma} \]

and the definition of the local velocity of sound \( c \)

\[ c = \sqrt{\frac{dp}{d\rho}} \]

it follows that

\[ \frac{c^2}{c_0^2} = \left( \frac{p}{p_0} \right)^{\gamma - 1} = \left( \frac{p}{p_0} \right)^{\gamma - 1} \]

Therefore

\[ \frac{p}{p_0} = \left[ 1 + \gamma \frac{1 - v_0^2}{2} \left( 1 - \frac{v_0^2}{c_0^2} \right) \right]^{\gamma - 1} \] (29)

Expanding the right-hand side of the foregoing equation according to powers of \( \frac{v_0^2}{c_0^2} \) it follows that

\[ p = p_0 + \frac{1}{2} \rho c_0^2 \left( 1 - \frac{v_0^2}{c_0^2} \right)^2 + \cdots \] (30)

The pressure distribution may be calculated by means of equation (29) together with the values for \( v_0 \) obtained from equations (22), (24), and (25). Equation (30) will be used in obtaining the total lift and moment on the airfoil.

Since the profile is a streamline, the normal velocity \( \partial \phi / \partial n = 0 \) and, accordingly, if \( \eta \) denotes the length along it, then Bernoulli's equation may be written

\[ p = \text{constant} - \frac{1}{2} \rho \left( 1 + \frac{1}{2} \mu \right) \left( \frac{\partial \phi}{\partial \xi} \right)^2 + \frac{1}{8} \rho \mu \left( \frac{\partial \phi}{\partial \xi} \right)^4 + \cdots \]

Let \( \eta \) denote the inward-drawn normal to the contour. Then from figure 2, it is seen that \( p \cos (\xi, \eta) \) and \( p \cos (\eta, \eta) \) are the components of the pressure along the \( \xi \) and \( \eta \) axes, respectively. Accordingly, the force on the airfoil is given by

\[ \mathcal{P} = \mathcal{P}_1 - i\mathcal{P}_2 = -i\rho \left( 1 + \frac{1}{2} \mu \right) \oint_C \left( \frac{\partial \phi}{\partial \xi} \right)^2 \cos (\xi, \eta) \]

\[ -i \cos (\eta, \eta) ds + \frac{i \rho \mu}{8 \rho_0^2} \oint_C \left( \frac{\partial \phi}{\partial \xi} \right)^4 \cos (\xi, \eta) \]

\[ -i \cos (\eta, \eta) ds \]

where the profile \( C \) is traversed in the counterclockwise positive sense. On the other hand

\[ ds = d\xi \cos (\eta, \eta), d\xi = ds \cos (\eta, \eta) \]

and therefore

\[ \mathcal{P} = \frac{i \rho \mu}{8 \rho_0^2} \oint_C \left( \frac{\partial \phi}{\partial \xi} \right)^4 (d\xi - id\eta) \]

Now, by definition,

\[ \frac{d\omega}{dz} = -i + \omega = \frac{\partial \phi}{\partial \xi} - i \frac{\partial \phi}{\partial \eta} \]

and, since the velocity normal to the profile equals zero, it follows that

\[ \frac{\partial \phi}{\partial \xi} = \frac{\partial \phi}{\partial \xi} \sin (\xi, \eta), \frac{\partial \phi}{\partial \eta} = \frac{\partial \phi}{\partial \eta} \sin (\eta, \eta) \]

Therefore

\[ \frac{d\omega}{dz} = \frac{\partial \phi}{\partial \xi} - i \frac{\partial \phi}{\partial \eta} \]

But

\[ ds = d\xi^2 + d\eta^2 = (d\xi - id\eta)(d\xi - id\eta) \]

or

\[ d\xi - id\eta = \frac{ds}{d\xi} \]

Hence

\[ \frac{\partial \phi}{\partial \xi} = \frac{d\omega}{dz} \]

and

\[ \mathcal{P} = i \rho \left( 1 + \frac{1}{2} \mu \right) \oint_C \left( \frac{d\omega}{dz} \right)^2 d\xi - \frac{i \rho \mu}{8 \rho_0^2} \oint_C \left( \frac{d\omega}{dz} \right)^4 d\xi \] (31)

Referring to the plane of the circle of radius \( R \)

\[ \mathcal{P} = i \rho \left( 1 + \frac{1}{2} \mu \right) \oint_{\text{circle}} \left( \frac{d\omega}{dz} \right)^2 dz - \frac{i \rho \mu}{8 \rho_0^2} \oint_{\text{circle}} \left( \frac{d\omega}{dz} \right)^4 dz \] (32)

where \( z = 4^a \) and \( \delta \) is the polar angle of the circle of radius \( R \).

Since by definition

\[ \frac{d\omega}{dz} = R(-v_x + iv_y) \]

it follows from equation (28) that

\[ \frac{d\omega}{dz} = i Re^{-\mu} a_0 (a_z + a_0 + a_{\pm} z + \frac{a_{\pm}}{z} + \frac{a_{\mp}}{z}) \] (33)

where

\[ a_0 = \left( 1 + \frac{h \mu}{2} \right) + i \alpha \left( 1 + \frac{h \mu}{2} + 4 h \mu \right) \]

\[ a_1 = \frac{h \mu}{2} + \frac{3}{2} h \mu \alpha \]

\[ a_{\pm} = i \alpha (2 + \mu + 5 h \mu) \]

\[ a_{\mp} = -\left( 1 + \frac{h \mu}{2} \right) + i \alpha \left( 1 + \frac{h \mu}{2} + 4 h \mu \right) \]

\[ a_{\pm} = -\frac{h \mu}{2} + \frac{3}{2} i h \mu \alpha \]
Also from the Joukowski transformation
\[ \tau = \zeta' + \frac{1}{\zeta} \]
and the relation
\[ \zeta' = R(z + h) \]
it follows that
\[ \frac{dz}{d\tau} = \frac{1}{R \left( 1 - \frac{1 - (1 - h)^2}{(z + h)^2} \right)} \]
\[ = \frac{1}{R} \left( 1 + \frac{1 - 2h}{\zeta^2} - \frac{2h}{\zeta^2} + \frac{1 - 4h}{\zeta^2} - \frac{4h}{\zeta^2} + \ldots \right) \]
\[ \frac{dz}{d\tau} = \frac{1}{R \left( 1 - \frac{1 - (1 - h)^2}{(1 + h\zeta)^2} \right)} \]
\[ \frac{dz}{dz} = -\zeta^2 \]

Then making use of the well-known relations
\[ \oint z^m dz = 0 \text{ if } m \neq -1 \]
and
\[ \oint_{\partial C} \frac{dz}{z} = 2\pi i \text{ if } m = -1 \]
it turns out, neglecting as usual terms containing powers of \( \mu, h, \) and \( \alpha \) higher than the first, that
\[ \bar{P} = P_t - iP_r = -i\rho \omega \Gamma \left[ 1 + \left( 1 + \frac{1}{2}h \right) \mu \right] + i\rho \omega \Gamma \left( 1 + \frac{1 + h}{2} \mu \right) \]
or
\[ P_r = \rho \omega \Gamma \left( 1 + \frac{1 + h}{2} \mu \right) \]

This last expression shows that, when the angle of attack is assumed small enough so that only the first power of \( \alpha \) is retained, the component \( P_r \) of the lift vanishes in comparison with the component \( P_t \).

Thus
\[ \text{Lift} = P_t = \rho \omega \Gamma \]

(34)

**Calculation of the Moment**

The moment arm \( OQ = m \sin (\sigma - \varphi) \) and the force per unit length along the airfoil is \( pds \) (fig. 2). Hence the total moment about the origin 0 is given by
\[ M = \oint_C p m \sin (\sigma - \varphi) ds = \oint_C p (m \sin \sigma \cos \varphi - m \cos \sigma \sin \varphi) ds \]

But
\[ d\xi = ds \cos \sigma \text{ and } d\eta = -ds \sin \sigma \]

Hence
\[ M = \oint_C p (m \cos \varphi d\xi + m \sin \varphi d\eta) \]
\[ = \oint_C p (\xi d\eta + \eta d\xi) = \frac{1}{2} \oint_C p dm^2 \]

Now
\[ \xi d\eta + \eta d\xi = R.P. \frac{d\xi}{d\tau} \]
and since \( d\xi = \frac{dz}{d\tau} \)
\[ \xi d\eta + \eta d\xi = R.P. \frac{dz}{d\tau} \]

Substituting for the pressure \( p \) the expression
\[ \text{Const.} \frac{1}{2} \rho \left( 1 + \frac{1}{2} \mu \right) \left( \frac{d\varphi}{d\sigma} \right)^2 + \frac{1}{8} \rho \omega \left( \frac{d\varphi}{d\sigma} \right)^4 + \ldots \]
it follows that
\[ M = \frac{1}{2} \rho \left( 1 + \frac{1}{2} \mu \right) R.P. \oint_C \left( \frac{d\varphi}{d\sigma} \right)^2 d\tau \]
\[ + \frac{1}{8} \rho \omega \left( \frac{d\varphi}{d\sigma} \right)^4 R.P. \oint_C \frac{d\varphi}{d\sigma} d\tau \]

(35)

Referring to the plane of the circle of radius \( R \)
\[ M = \frac{1}{2} \rho \left( 1 + \frac{1}{2} \mu \right) R.P. \oint_{\partial C} \left( \frac{d\varphi}{d\sigma} \right)^2 d\tau \]
\[ + \frac{1}{8} \rho \omega \left( \frac{d\varphi}{d\sigma} \right)^4 R.P. \oint_{\partial C} \frac{d\varphi}{d\sigma} d\tau \]

(36)

Performing the integrations in a manner analogous to that for the lift, it turns out that
\[ M = 4\pi \rho R^2 v_0^3 a \left[ 1 + h + \left( 1 + \frac{1}{2} h \right) \mu \right] - \pi \rho R^2 v_0^3 \mu \alpha (2 + 6h) \]
or
\[ M = 4\pi \rho R^2 v_0^3 a \left[ 1 + h + \frac{1 + 6h}{2} \mu \right] \]

(37)

This expression represents the moment about the origin of coordinates, and the moment about the center of the circle of radius \( R \) (into which the profile is mapped) can be obtained at once as
\[ M_e = M - La = 4\pi \rho R^2 v_0^3 a^2 \alpha + 4\pi \rho R^2 v_0^3 a^2 \alpha \left( 1 + \frac{1 + 7h}{2} \mu \right) \]
or
\[ M_e = M_0 \left( 1 + \frac{1 + 7h}{2} \mu \right) \]

(38)

where \( M_0 = 4\pi \rho R^2 v_0^3 a^2 \alpha \) is the moment about the center of the circle of radius \( R \) for the corresponding incompressible flow.

If now \( d \) represents the distance of the center of pressure from the origin of coordinates, then
\[ M = Ld \]

or
\[ d = \frac{4\pi \rho R^2 v_0^3 a \left( 1 + h + \frac{1 + 6h}{2} \mu \right)}{4\pi \rho R^2 v_0^3 a \left( 1 + \frac{1 + 7h}{2} \mu \right)} = a (1 + 3h\mu) \]

(39)
This expression shows that the airfoil has a constant center of pressure at a distance equal to \(1/4 (1 - 3h_p)\) of the chord from the leading edge. For a thin airfoil, say \(e=0.05\) and for a stream velocity \(v_0=0.835 c_0\), the center of pressure, as compared with the corresponding incompressible case, is nearer to the leading edge by about 2.5 percent of the chord.

Langley Memorial Aeronautical Laboratory, National Advisory Committee for Aeronautics, Langley Field, V.A., November 19, 1937.
APPLICATION OF THE THEORETICAL RESULTS

As an example of the application of the theory to any particular case, the flows past a thin and a fairly thick symmetrical Joukowski profile for zero angle of attack will be calculated. Since no experimental results are available for purposes of comparison, it was considered hardly worth while to perform the rather lengthy and tedious calculations associated with angle of attack or circulation.

Equation (28) for \( \Delta p/\rho_0 \) simplifies considerably when the angle of attack \( \alpha \) is taken to be zero. Thus

\[
\frac{\Delta p}{\rho_0} = \frac{\mu}{4} \left[ h(12-21h-4h^2+30h^3+24h^4-8h^5) \sin \delta \right. \\
+ \frac{4h(1-h)^2}{k^4} \sin 2\delta + \frac{2h^2}{k} \sin 3\delta \\
\left. - \frac{32h^3(1-h)^3}{k^5} J_4(0) + \frac{64h^3(1-h)^3}{k^5} I_2 \right]
\]

\( \text{Figure 3.} \text{ Pressure and velocity distribution for the symmetrical Joukowski airfoil section } \varepsilon = 0.05, \mu = 0.70, \alpha = 0^\circ. \)

The profiles chosen are defined by \( \varepsilon = 0.05 \) and \( \varepsilon = 0.15 \) or \( h = 1/21, k = 19/21; \) and \( h = 3/23, k = 17/23, \) respectively. Tables I and II present the calculations and figures 3 and 4 show the velocity and pressure distributions for both incompressible and compressible flow. The values of \( \mu \) chosen were the critical values obtained by plotting \( (\rho_2/\rho_0)_{\text{crit}} \) against \( \rho_2/\rho_0 \) and then noting the intersection of this graph with that of \( (\rho_2/\rho_0)_{\text{crit}} \) against \( \rho_2/\rho_0 \) as given by equation (9). Table III

\[
J_4(0) = \frac{(1+2h)(1-4h^2)(1-h)^3}{2(1-2k \cos \delta + k^2)} \frac{\sin \delta}{1-k \cos \delta} \\
- \frac{h}{2} \left( \sin \delta + h \sin 2\delta \right) \log \left( 1 - 2k \cos \delta + k^2 \right)
\]

\( \text{Figure 4.} \text{ Pressure and velocity distribution for the symmetrical Joukowski airfoil section } \varepsilon = 0.15, \mu = 0.67, \alpha = 0^\circ. \)
The expressions for \( \frac{\Delta P}{\eta_0} \) are given by:

\[
\frac{\Delta P}{\eta_0} = \mu[0.24905 \sin \delta + 0.18053 \sin 2\delta + 0.01151 \sin 3\delta \\
- 0.40583 J_4(0) + 0.10583 I_4]
\]

and for \( \epsilon = 0.15 \)

\[
\frac{\Delta P}{\eta_0} = \mu[0.24905 \sin \delta + 0.18053 \sin 2\delta + 0.01151 \sin 3\delta \\
- 0.40583 J_4(0) + 0.10583 I_4]
\]

It is to be noted in tables I, II, and III that the maximum velocity \( \eta_0' \) for the incompressible flow occurs at about \( \delta = 35^\circ \) and \( \delta = 45^\circ \) for \( \epsilon = 0.05 \) and \( \epsilon = 0.15 \), respectively. It is then assumed that the position of maximum velocity is independent of \( \mu \) and maximum values for \( \eta_0' \) are calculated for various values of \( \mu \). These values of \( \eta_0' \) are given in table III and are used in obtaining the critical values of \( \mu \) as shown in figure 5. The coordinates of the airfoils \( \epsilon = 0.05, \epsilon = 0.15 \) are given in table IV and the corresponding contours in figure 6.
APPENDIX B

NOTATION

\( x, y \) rectangular coordinates in the plane of the circle.
\( \xi, \eta \) rectangular coordinates in the plane of the profile.
\( z = x + iy, \xi = \xi + i\eta \) plane polar coordinates in the \( z \) plane.
\( v_x, v_y \) fluid velocity components along the \( x \) and \( y \) axes, respectively.
\( v_t \) tangential velocity on the circle.
\( v_{pr} \) tangential velocity on the profile corresponding to \( v_t \).
\( v = \sqrt{v_x^2 + v_y^2} \) magnitude of the fluid velocity.
\( c \) local velocity of sound in the fluid.
\( \rho \) density of the fluid.
\( P \) static pressure in the fluid.
\( \rho_0, c_0, \rho_0, P_0 \) corresponding magnitudes in the undisturbed stream.
\( \mu = \left(\frac{v}{c_0}\right)^2 \) correction term to the velocity in incompressible flow due to compressibility.
\( \phi \) velocity potential of the incompressible flow.
\( v_r = -\frac{\partial \phi}{\partial r} \) component of velocity along the radius vector.
\( v_\theta = -\frac{1}{r} \frac{\partial \phi}{\partial \theta} \) component of velocity perpendicular to the radius vector in the sense of \( \theta \) increasing.
\( \lambda = \frac{P}{r} \) force vector on the airfoil.
\( P_x, P_y \) components of \( \mathbf{F} \) along the \( x \) and \( y \) axes, respectively.
\( M_r \) moment about origin of coordinates in the plane of compressible flow.
\( M_{pr} \) moment about center of circle of radius \( R \) in the plane of compressible flow.
\( M_o \) moment corresponding to \( M_r \) in the plane of incompressible flow.

REFERENCES

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$1$ Jacobian $= \frac{2\pi}{\sin(\epsilon)} \sqrt{\frac{\epsilon}{\pi}}$ for $\epsilon < \pi/2$

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