NATIONAL ADVISORY COMMITTEE
FOR AERONAUTICS

REPORT 934

RECOMMENDATIONS FOR NUMERICAL SOLUTION
OF REINFORCED-PANEL
AND FUSELAGE-RING PROBLEMS

By N. J. HOFF and PAUL A. LIBBY

1949
# Aeronautical Symbols

## 1. Fundamental and Derived Units

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Metric</th>
<th>English</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length</td>
<td>( l )</td>
<td>meter ( m )</td>
</tr>
<tr>
<td>Time</td>
<td>( t )</td>
<td>second ( s )</td>
</tr>
<tr>
<td>Force</td>
<td>( F )</td>
<td>weight of 1 kilogram ( \text{kg} )</td>
</tr>
<tr>
<td>Power</td>
<td>( P )</td>
<td>horsepower (metric) ( \text{hp} )</td>
</tr>
<tr>
<td>Speed</td>
<td>( V )</td>
<td>kilometers per hour ( \text{km/hr} )</td>
</tr>
</tbody>
</table>

## 2. General Symbols

- \( W \): Weight \( = mg \)
- \( g \): Standard acceleration of gravity \( = 9.80665 \text{ m/s}^2 \)
  - or \( 32.174 \text{ ft/sec}^2 \)
- \( m \): Mass \( = \frac{W}{g} \)
- \( I \): Moment of inertia \( = mL^2 \) (Indicate axis of radius of gyration \( k \) by proper subscript.)
- \( \mu \): Coefficient of viscosity

## 3. Aerodynamic Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S )</td>
<td>Area</td>
</tr>
<tr>
<td>( S_e )</td>
<td>Area of wing</td>
</tr>
<tr>
<td>( G )</td>
<td>Gap</td>
</tr>
<tr>
<td>( b )</td>
<td>Span</td>
</tr>
<tr>
<td>( c )</td>
<td>Chord</td>
</tr>
<tr>
<td>( A )</td>
<td>Aspect ratio ( = \frac{b}{S} )</td>
</tr>
<tr>
<td>( V )</td>
<td>True air speed</td>
</tr>
<tr>
<td>( q )</td>
<td>Dynamic pressure ( = \frac{1}{2} \rho V^2 )</td>
</tr>
<tr>
<td>( L )</td>
<td>Lift, absolute coefficient ( C_L = \frac{L}{qS} )</td>
</tr>
<tr>
<td>( D )</td>
<td>Drag, absolute coefficient ( C_D = \frac{D}{qS} )</td>
</tr>
<tr>
<td>( D_p )</td>
<td>Profile drag, absolute coefficient ( C_{D_p} = \frac{D_p}{qS} )</td>
</tr>
<tr>
<td>( D_i )</td>
<td>Induced drag, absolute coefficient ( C_{D_i} = \frac{D_i}{qS} )</td>
</tr>
<tr>
<td>( D_s )</td>
<td>Parasite drag, absolute coefficient ( C_{D_s} = \frac{D_s}{qS} )</td>
</tr>
<tr>
<td>( C )</td>
<td>Cross-wind force, absolute coefficient ( C = \frac{C}{qS} )</td>
</tr>
</tbody>
</table>

- \( \alpha \): Angle of attack
- \( \beta \): Angle of downwash
- \( \alpha_e \): Angle of attack, infinite aspect ratio
- \( \alpha_i \): Angle of attack, induced
- \( \alpha_a \): Angle of attack, absolute (measured from zero-lift position)
- \( \gamma \): Flight-path angle

- \( \rho \): Kinematic viscosity
- \( \rho \): Density (mass per unit volume)
- \( R \): Reynolds number, \( \rho V_l \) where \( l \) is a linear dimension (e.g., for an airfoil of 1.0 ft chord, 100 mph, standard pressure at 15° C, the corresponding Reynolds number is 335,400; or for an airfoil of 1.0 m chord, 100 mps, the corresponding Reynolds number is 6,865,000)
REPORT 934

RECOMMENDATIONS FOR NUMERICAL SOLUTION
OF REINFORCED-PANEL
AND FUSELAGE-RING PROBLEMS

By N. J. HOFF and PAUL A. LIBBY

Polytechnic Institute of Brooklyn
National Advisory Committee for Aeronautics

Headquarters, 1794 F Street NW., Washington 25, D. C.

Created by act of Congress approved March 3, 1915, for the supervision and direction of the scientific study of the problems of flight (U. S. Code, title 50, sec. 151). Its membership was increased from 12 to 15 by act approved March 2, 1929, and to 17 by act approved May 25, 1948. The members are appointed by the President, and serve as such without compensation.

JEROME C. HUNSAKER, Sc. D., Cambridge, Mass., Chairman

ALEXANDER WEITEMORE, Sc. D., Secretary, Smithsonian Institute, Vice Chairman

ROY B. ALISON, Assistant Secretary of Commerce,
DETHEY W. BRONK, PH. D., President, Johns Hopkins University,
KARL T. COMPTON, PH. D., Chairman, Research and Development Board, Department of Defense,
EDWARD U. CONDON, PH. D., Director, National Bureau of Standards,
JAMES H. DOOLITTLE, Sc. D., Vice President, Shell Union Oil Corp.,
R. M. HAZEN, B. S., Director of Engineering, Allison Division, General Motors Corp.,
WILLIAM LITTLEWOOD, M. E., Vice President, Engineering, American Aircrane, Inc.,
THEODORE C. LONNOYER, Rear Admiral, United States Navy, Deputy and Assistant Chief of the Bureau of Aeronautics.

JOHN W. CROWEY, JR., M. S., Associate Director for Research

E. H. CHAMBERLIN, Executive Officer

HENRY J. E. REID, D. Eng., Director, Langley Aeronautical Laboratory, Langley Field, Va.,

SMITH J. DERANCE, B. S., Director, Ames Aeronautical Laboratory, Moffett Field, Calif.,

EDWARD R. SHARP, Sc. D., Director, Lewis Flight Propulsion Laboratory, Cleveland Airport, Cleveland, Ohio

TECHNICAL COMMITTEES

AERODYNAMICS

Power Plants for Aircraft
Aircraft Construction

Operating Problems

Industry Consulting

Coordination of Research Needs of Military and Civil Aviation

Preparation of Research Programs
Allocation of Problems
Prevention of Duplication
Consideration of Inventions

LANGLEY AERONAUTICAL LABORATORY,
Langley Field, Va.

LEWIS FLIGHT PROPULSION LABORATORY,
Cleveland Airport, Cleveland, Ohio

AMES AERONAUTICAL LABORATORY,
Moffett Field, Calif.

Conduct, under unified control, for all agencies, of scientific research on the fundamental problems of flight

Office of Aeronautical Intelligence
Washington, D. C.

Collection, classification, compilation, and dissemination of scientific and technical information on aeronautics
REPORT 934

RECOMMENDATIONS FOR NUMERICAL SOLUTION OF REINFORCED-PANEL AND FUSELAGE-RING PROBLEMS

By N. J. Hoff and Paul A. Libby

SUMMARY

Procedures are recommended for solving the equations of equilibrium of reinforced panels and isolated fuselage rings as represented by the external loads and the operations table established according to Southwell's method. From the solution of these equations the stress distribution can be easily determined. The recommendations are based on the experience of the past 4 years in applying numerical procedures to monocoque stress analysis at the Polytechnic Institute of Brooklyn Aeronautical Laboratories. The method of systematic relaxations, the matrix-calculus method, and several other methods applicable in special cases are discussed.

Definite recommendations are made for obtaining the solution of reinforced-panel problems which are generally designated as shear lag problems. The procedures recommended are demonstrated in the analysis of a number of panels, several of which were discussed in previous PIBAL reports, whereas others are shown for the first time.

In the case of fuselage rings it is not possible to make definite recommendations for the solution of the equilibrium equations for all rings and loadings. However, suggestions based on the latest experience are made and demonstrated on several rings.

INTRODUCTION

The application of the indirect methods of Cross (reference 1) and Southwell (reference 2) to the analysis of monocoque structures has been shown in a series of investigations (references 3 to 8) carried out at the Polytechnic Institute of Brooklyn Aeronautical Laboratories. These indirect methods are likely to lead to solutions of problems in stress analysis that are intractable by direct analytical methods because the structure is tapered, it has large cutouts, its reinforcing elements are distributed irregularly, or the like.

The distorted shape corresponding to equilibrium under the applied loads is determined first in the indirect methods. From it the stresses, forces, and moments required can be calculated without difficulty. This approach is justified by the comparative ease with which the stresses in a complex structure can be determined for an individual displacement of one point and with which the final distorted shape of a complex structure can be represented by a summation of such individual displacements.

The complete structure is considered to be composed of appropriate elements and its degrees of freedom are the displacements of the several reference points on the boundary of each element. Each of these points is displaced in turn and the reactions at the reference points caused by the displacement are listed. If by suitable displacements of all points the reaction forces and moments are made equal and opposite to the external loads at each point, the whole structure is in equilibrium and its distorted shape is determined.

In applying the indirect methods to monocoque structures the terminology of Southwell (reference 2) has been retained. Thus, the elements which compose the complete structure are "units" and the determination of the forces and moments due to a displacement of a boundary point of such units is termed the "unit problem." The magnitudes of these forces and moments are given by "influence coefficients." The complete effect of a displacement is given in an "operations table," and the step-by-step process, which can be employed to determine the equilibrium distorted shape, is called the "method of systematic relaxations." At each step of this process forces and moments referred to as "residuals" remain unbalanced at each point in the structure. A running account of the residuals and of the displacements or "operations" undertaken is kept in the "relaxation table."

The operations table along with the external forces constitutes a system of linear equations, which are equal in number to the degrees of freedom of the structure and which have as variables the displacements. Each equation represents the condition of equilibrium for the force or moment associated with one degree of freedom. When the method of systematic relaxations is applied an approximate solution to this system of equations and accordingly an approximate equilibrium state of the structure are found.

The indirect method of analysis just outlined has been applied at PIBAL to the reinforced-panel and ring components of a monocoque structure as well as to complete circular cylinders with and without cutouts. In references 3 and 4 the stress distribution in the sheet and stringers of a reinforced panel was determined under loads applied parallel to the stringers. Fuselage rings with and without internal bracing elements were investigated in reference 5. The determination of the influence coefficients for the ring unit problem was found to involve considerable computational work and therefore appropriate graphs and tables are given in reference 6 to facilitate their calculation. In references 7 and 8 the elements, namely, the reinforced panel and the ring, are combined into a circular cylinder and the stress distribution in the cylinder was investigated for the case when the loading is a pure bending moment.
In the application of the indirect-stress-analysis methods to the problems mentioned the major obstacle has been to find an approximate solution of the system of equations with a reasonable expenditure of effort. In each problem it has been readily possible to establish satisfactory units and to combine them to represent the complex structure. During the past 4 years considerable experience has been gained at PIBAL in overcoming this obstacle to the wider application of numerical procedures in the analysis of monocoque structures. On the basis of this experience some recommendations can be made as to the most expeditious method of solving reinforced-panel and fuselage-ring problems after the operations table has been established as described in references 3 to 5.

In many problems solution of the set of linear equations by means of matrix algebra was found easier and less time consuming than the determination of the displacements by systematic relaxations. In other cases special methods, such as the growing-unit method, proved to be most expeditious.

It is assumed that the reader is familiar with the terminology of Southwell's relaxation method and with the solution of the unit problem as well as the establishment of the operations table for both the reinforced-panel and fuselage-ring problems. Complete details of these are given in references 3 to 6.

This work, carried out at the Polytechnic Institute of Brooklyn, was sponsored by and conducted with the financial assistance of the National Advisory Committee for Aeronautics. Mr. Arnold O. Ostrand contributed the growing-unit method for reinforced panels. The authors also wish to acknowledge their indebtedness to the following members of the staff of the Polytechnic Institute of Brooklyn: Professors George B. Hoadley and William MacLean of the Department of Electrical Engineering for their work on the electric analogue, Mr. Burton Erickson for carrying out the major portion of the computations, and Dr. Bruno A. Boley for his editorial advice.

**SYMBOLS**

\( A \) cross-sectional area of stringer and effective sheet

\( A \) to \( Q \) points on a ring or a reinforced panel; group operations

\( A^* \) effective shear area of ring section

\( a \) distance between adjacent longitudinal stringers

\( b \) distance between adjacent transverse stringers

\( C \) electrical conductance

\( E \) Young's modulus of elasticity

\( F \) tensile force in stringer; applied external load

\( G \) shear modulus of elasticity

\( H \) horizontal direction

\( I \) moment of inertia of cross section; electrical current

\( I \) to XX group operations

\( L \) length of straight bar or length of arc of curved bar

\( M \) bending moment

\( N \) moment acting on a joint

\( q \) shear flow

\( R \) radial force acting at a joint; electrical resistance

\( T \) tangential force acting at a joint

\( t \) sheet thickness

\( u \) displacement of a joint in tangential direction

\( V \) electrical potential; vertical direction

\( v \) displacement of a joint in radial direction; displacement of a joint in vertical direction

\( n_{che} \) vertical block displacement

\( w \) rotation of a joint

\( x \) magnitude of group operation to be determined

\( x, y \) rectangular coordinates

\( Y \) force in y-axis direction

\( \beta \) angle subtended by ring segment

\( \gamma \) section-length parameter \((AL^2/l)\)

\( \xi \) ratio of effective shear area to tension area \((A^*/A)\)

\( \Sigma \) summation

**REINFORCED PANELS**

**INTRODUCTION**

In this section plane and slightly curved reinforced panels are discussed when the loads are applied in the plane of the flat panels or tangentially to the surface of the slightly curved panels.

In most airplane structures there is a predominant direction in which the major forces act and in which the major reinforcing elements lie. When the panel is symmetric and symmetrically loaded experience has shown that it suffices to consider displacements and force equilibrium in the predominant direction only. Even when the structure or the loads are nonsymmetric, the displacements and forces in the transverse direction are usually of secondary importance but they may be considered in a more refined analysis.

In references 3 and 4 numerical procedures for the determination of the stress distribution in reinforced panels subjected to axial stringer loads are developed and demonstrated on several flat and curved panels with and without cutouts. The results obtained by means of these procedures are in good agreement with those of tests.

Solution of the system of equations represented by the operations table and the external forces can be found by several methods, five of which are described herein. The various conditions of loading and structure which suggest the use of one method rather than another are discussed.

**RELAXATION METHOD**

For most reinforced-panel problems the relaxation method of solution is the most suitable. Simple group and block operations lead to a rapid elimination of the residuals and require little initiative on the part of the computer familiar with the sequence of step-by-step operations. The method, however, is not efficient in the case of panels with many bays in the direction of the stringer loads or panels with sheet covering of large shearing rigidity, since large forces are then introduced into adjacent stringers when one stringer is balanced. These forces in turn must be liquidated in successive operations with the consequence that the procedure becomes time consuming. Also in problems involving many loading conditions it may be expedient to use the electric-analogy method described in the section entitled “Electric Analogue,” since in the relaxation method each new loading requires new step-by-step operations.
In this section panels are discussed which are not excluded from application of the relaxation method by the foregoing considerations. They may be classified according to the boundary conditions of the stringers into four groups. Recommendations for each group follow with a fifth subsection added containing suggestions for panels in which transverse forces and displacements are considered.

(a) **Panels with boundary conditions at both ends of stringers specified in terms of force.**—The following two procedures are recommended for liquidating the residuals on a panel of this group:

**First procedure:**
1. Consider each stringer isolated by cutting the sheet and the transverse reinforcing elements. Select the stringer for which the algebraic sum of the external forces is the largest. Displace the entire stringer as a rigid body (block displacement) until this sum vanishes.
2. Balance one end joint of the stringer by displacing the adjacent joint on the same stringer.
3. After step 2 is completed the end joint is balanced but the joint that was moved is unbalanced. Displace the third joint on the same stringer until the second joint is balanced.
4. Continue the procedure until the second end joint is moved. In this last step both the end joint and the adjacent one will be approximately balanced at the same time since the algebraic sum of all the forces acting upon the stringer was zero after completion of step 1 and this equilibrium has been disturbed only slightly by the shear forces transmitted by the sheet during the individual operations.
5. Stringer 1 is now approximately balanced. Carry out the same procedure with the other stringers of the panel successively.
6. When all the stringers are approximately balanced, return to the first stringer and balance it again by undertaking steps 1 to 4. Repeat the procedure with the other stringers until all the residual forces can be considered negligible for engineering purposes.

**Second procedure:**
1. Consider each stringer isolated by cutting the sheet and the transverse reinforcing elements. Select the stringer for which the algebraic sum of the external forces is the largest. Displace the entire stringer as a rigid body (block displacement) until this sum vanishes.
2. Displace one end point of this stringer so as to balance the residual thereon.
3. Displace by equal amounts the adjacent joint on the same stringer and the end joint which was balanced in step 2 so as to balance this second joint. The equilibrium of the end joint will be disturbed only by a small amount due to shear in the sheet.
4. Displace by equal amounts the third joint on the same stringer and the two joints that were placed in approximate balance by the operation described in step 3 so as to balance this third joint.
5. Continue this procedure until the joint next to the midpoint of the stringer is balanced by equal displacements of all the joints situated between it and the end joint first displaced.
6. Repeat the process described in steps 2 to 5, starting from the other end joint of the stringer and continuing to the midpoint from this direction. After this step is completed this stringer will be in approximate balance, the only residuals being those introduced by shear in the sheet.
7. Consider next the stringer on either side of the approximately balanced stringer. Undertake a block displacement so as to equilibrate externally the stringer under its residual forces.
8. Start at one end joint of this stringer and apply steps 2 to 6. This second stringer will be placed in approximate balance thereby, while the balance of the first stringer will be disturbed only through the shear in the sheet.
9. Either return to the first balanced stringer or proceed to the next stringer on the other side. Each newly considered stringer is first externally equilibrated under the external and residual forces by a block displacement. Then from each free end the residuals are balanced by group displacements involving equal displacements of all the joints situated between the one in question and the free end. Continue to balance individual stringers until all are balanced.

The relaxation tables for the panel shown in figure 1, for which table 1 is the operations table, are used to demonstrate the first and second procedures and are given as tables 2 and 3, respectively. It will be noticed that this operations table considers the displacements of only the joints on the left half of the panel. The panel is symmetrical and is symmetrically loaded. Therefore, the displacements in the balancing process are undertaken symmetrically and only those of the left side joints need be considered, those of the right being correspondingly equal. Since this panel has only three bays along each axially loaded stringer, the internal balancing process is undertaken from one end of the stringer only.

![Figure 1](image-url)
(b) Panels with boundary conditions at one end of stringers specified in terms of force and at other in terms of displacement.—This type of problem occurs, for instance, when one end of the panel is attached to a rigid body which is either held fixed in its position or is displaced a given amount. The recommended procedure for panels of this group is the same as the second procedure for panels in case (a) with two exceptions: (1) No block displacements are needed (or possible) to equilibrate the stringers externally and (2) the internal balancing process can be started only from the one free end of each stringer.

The method is demonstrated on the panel shown in figure 2. It is identical with the panel used for case (a) with the exception of the fixed lower ends of the vertical stringers. The operations table is identical with that of the previous panel except that no block and no \( r_{11} \) and \( r_{21} \) displacements are admissible. The relaxation table is given as table 4.

(c) Panels with boundary conditions at both ends of stringer specified in terms of displacement.—Experience on panels of this type indicates that, although no systematic process of balancing the residuals can be recommended, the direct relaxation process is rapidly convergent. By starting from the midpoint joints on a stringer and by balancing successive joints toward the two fixed ends, the equilibrium position can be approximated rapidly. A further suggestion regarding this type of panel is contained in the later section "Niles Tables."

(d) Panels with irregularly specified boundary conditions.—For such panels a combination of the methods discussed under cases (a), (b), and (c) is recommended. By judicious use of block and group operations similar to those of cases (a) and (b) rapid convergence of the relaxation procedure will be obtained.

(e) Panels in which transverse displacements and forces are considered.—There are two general procedures for treating panels in which the transverse displacements and forces, usually considered negligible, are treated. These are described in the following paragraphs:

First procedure:

The procedure discussed under cases (a) and (b) can be applied to panels with cutouts. The stringers are approximately balanced in the direction of the major axial forces by these procedures and then the residuals normal to this direction are considered. The same step-by-step operations can be applied in balancing transverse stiffeners under these transverse axial forces. The process of first balancing the stringers in one direction, then balancing the stiffeners in the normal direction, and then returning to the originally balanced stringers will be quite rapidly convergent for panels with sheet of low shearing rigidity.

Second procedure:

For panels with cutouts requiring consideration of the transverse forces another procedure, which is demonstrated in reference 4, can be used. The panel is first considered to have continuous sheet and stringers, as if the cutout did not exist, and the displacements for equilibrium of this panel under the external loads are determined by the usual methods. These displacements are then applied as a first approximation to the distorted shape of the actual panel with cutout. Displacements leading to a closer approximation are then undertaken. This procedure is found to be reasonably successful for the cases investigated in reference 4.

**Matrix-Calculus Method**

The operations table together with the external forces can be considered as a system of linear equilibrium equations with the magnitudes of the displacements as the unknowns. Therefore, the methods of matrix calculus can be applied to find the solution of this system by direct mathematical means. The method described in reference 9 is recommended since a check on the calculations is maintained at each step in the process of solution.

Matrix methods of solution have several advantages. After the operations table is established by trained engineering personnel, the solution can be obtained by computing personnel familiar with the matrix-calculus method. Under some conditions this economic advantage may be important. For reinforced panels with sheet of high shearing rigidity the relaxation procedures are slowly convergent even when the recommendations given in the preceding section are observed. The matrix-calculus method is not affected by this physical characteristic of the structure.

When the number of equations is greater than 30 or 40, the work of computation becomes inconveniently large. Therefore, for panels having a sheet covering of small shearing rigidity relaxation methods are recommended. When the sheet covering is very rigid in shear the matrix method is likely to be more advantageous because the routine operations of the matrix method can always be carried out if enough time is allowed.

The equations of equilibrium for the panel shown in figure 1 are given by table 1 and are presented as follows to illustrate how the operations table and the external forces can be considered as a system of equilibrium equations:

![Figure 2](https://example.com/figure2.png)
RECOMMENDATIONS FOR NUMERICAL SOLUTION OF REINFORCED-PANEL AND FUSELAGE-RING PROBLEMS

\[-55.2v_B + 2.00v_V + 51.2v_R = 0\]
\[2.00v_B - 101.6v_V + 4.00v_V + 46.8v_R + 2.00v_R = 0\]
\[51.2v_B + 4.00v_V - 110.4v_V + 2.00v_V + 51.2v_R = 0\]
\[46.8v_B + 2.00v_V - 101.6v_V + 4.00v_V = 0\]
\[+ 46.8v_V + 2.00v_V = 0\]
\[2.00v_B + 51.2v_V + 4.00v_V - 110.4v_V = 0\]
\[+ 2.00v_V + 51.2v_V = 0\]
\[46.8v_B + 2.00v_V - 50.8v_V = 0\]
\[+ 2.00v_V + 60 \times 10^4 = 0\]
\[2.00v_B + 51.2v_V + 2.00v_V = 0\]
\[-55.2v_V + 60 \times 10^4 = 0\]

\[(1)\]

In considering the operations table and the external forces as a system of equilibrium equations, care must be taken to restrain enough joints so that the position of the panel as a rigid body is fixed. In the present case, \(v_A\) and \(v_B\) are assumed to be zero, and since only displacements in the \(y\) direction are considered in this problem, this restraint is sufficient.

GROWING-UNIT METHOD

For reinforced panels with sheet of high shearing rigidity or with a large number of bays in the direction of the axial forces, the relaxation procedure is not rapidly convergent. In such problems either the matrix-calculus or the growing-unit method is recommended. The latter can be applied only to panels the boundary conditions of which are specified in terms of forces at least at one end of the stringers.

The growing-unit method applied to reinforced panels is as follows: The joint at the free end of an arbitrarily selected unbalanced stringer, called hereinafter the principal joint and the principal stringer, respectively, is displaced so as to liquidate the residual on this joint. At the same time the joints lying on adjacent parallel stringers and the same transverse stiffener are displaced so that the residuals that would be otherwise introduced by shear from the balancing of the principal joint as well as any external forces applied to these joints are likewise liquidated. In the second operation the next joint on the principal stringer is relaxed while the previously balanced joints on the first transverse stiffener and the joints on the second transverse stiffener are kept in balance by suitable displacements. After this second operation no residuals remain at the joints of the first two transverse stiffeners. After a sufficient number of repetitions of the procedure all residuals will be confined to reaction points or will be liquidated; the panel will then be in equilibrium.

This procedure is demonstrated on the panel shown in figure 3. The physical properties of the panel are the same as those of the previously discussed panels except for the additional bay in the direction of the axial forces. Actually the convergence of the relaxation method for this panel would be quite rapid, but for convenience the growing-unit method, applicable when this convergence is slow, is demonstrated thereon. Table 5 is the operations table for this panel and contains not only the individual operations but also the group operations of the growing-unit method.

Table 6 is the relaxation table in which these group operations are used.

The group operations given in table 5 require some explanation. In order to avoid introducing a \(Y_B\)-residual when joint \(A\) is relaxed by application of operation (1), a \(v_B\)-displacement is applied, the magnitude of which can be calculated from the equation

\[-55.2v_B + 2.00 = 0\]

Thus operation (9) is \(v_B = (2/55.2) = 0.0362\) and (10) is a group operation equal to the sum of operations (1) and (9), which liquidates the residual \(Y_A\) without introducing a \(Y_B\)-unbalance.

After operation (10) is used, unbalances exist at joints \(E\) and \(F\), that is, on the second transverse stiffener. In order to balance these without disturbing the recently established balance at \(A\) and \(B\), two group operations are developed: One permitting the balancing of \(E\) and one permitting the balancing of \(F\). The magnitudes of \(v_A\) and \(v_B\) required to maintain the balance of \(A\) and \(B\) when a displacement of \(v_B = 1\) is undertaken are given by the following equations:

\[-50.8v_A + 2.00v_B + 46.8 = 0\]
\[2.00v_A - 55.2v_B + 2.00 = 0\]

These are satisfied by \(v_A = 0.921\), operation (11), and \(v_B = 0.0695\), operation (12). Operation (13) is therefore established as the sum of operations (3), (11), and (12). The magnitudes of \(v_A\) and \(v_B\) required to maintain the balance of \(A\) and \(B\) when a displacement of \(v_B = 1\) is undertaken are given by the following equations:

\[-50.8v_A + 2.00v_B + 2.00 = 0\]
\[2.00v_A - 55.2v_B + 51.2 = 0\]
These are satisfied by \( v_A = 0.0758 \), operation (14), and \( v_B = 0.023 \), operation (15). Operation (16) is the sum of operations (4), (14), and (15). Since group operations (13) and (16) both introduce \( Y_F \) and \( Y_P \)-forces, the magnitudes \( x_{13} \) and \( x_{16} \) of these groups required to liquidate the -111-pound and 9-pound residuals at \( E \) and \( F \), respectively, are given by the following equations:

\[
\begin{align*}
-58.3x_{13} + 9.4x_{16} - 111 &= 0 \\
9.4x_{13} - 62.6x_{16} - 9 &= 0
\end{align*}
\]

Thus \( x_{13} = -1.975 \) and \( x_{16} = -0.444 \). Joints \( E \) and \( F \) are balanced without disturbing the balance of \( A \) and \( B \) by the use of these multiples of operations (13) and (16).

In eliminating the residuals at joints \( J \) and \( K \), multiples of operations (13) and (16) are applied since these operations permit displacements of \( E \) and \( F \) to be undertaken while the balance at \( A \) and \( B \) is left undisturbed. When joint \( J \) is displaced a unit amount, multiples of operations (13) and (16), defined by the following equations, are used so that the balance at \( A \), \( B \), \( E \), and \( F \) is maintained:

\[
\begin{align*}
-58.3x_{13} + 9.4x_{16} + 46.8 &= 0 \\
9.4x_{13} - 62.6x_{16} + 2.00 &= 0
\end{align*}
\]

The solution to these equations is \( x_{13} = 0.828 \), operation (17), and \( x_{16} = 0.158 \), operation (18). Operation (19) is the sum of operations (5), (17), and (18).

In a similar manner all the individual and group displacements described in table 4 are found. It may be mentioned that in the present example no shearing stresses were set up in the middle bays because of the symmetry of structure and loading. The original operations table was already established in a manner which complied with these requirements of symmetry. When such is not the case or when there is a greater number of stringers in the panel, displacements of principal stringer joints will in general, cause residuals to appear at more joints so that three or more, rather than two, simultaneous equations have to be solved at each step.

### Niles Tables

In reference 10, Niles demonstrates for the solution of reinforced-panel problems a method which essentially parallels the previously described relaxation method. The Niles method is a procedure for balancing a stringer by the use of tables which give the displacements of each joint on the stringer required to liquidate a residual on a given joint of the stringer. The tables are worked out for various end conditions and sheet shearing rigidities.

Since reference 10 contains tables only for sheet of relatively low shearing rigidity, the Niles method is limited in this respect in the same way as the relaxation method. However, the tables can be employed on stringers with the boundary conditions at both ends specified in terms of displacement; for such problems no step-by-step routine relaxation method has been recommended. Also by use of the tables exact balance of a stringer is gained after a single displacement of each joint, whereas in the relaxation method, because of the shear, small unbalances remain after each joint is moved.

On the other hand, the relaxation method can be applied to stringers with irregularly spaced joints for which no tables were set up by Niles.

Since in reference 10 several examples of the procedure are given, no application of the Niles method is shown herein.

### Electric Analogue

Another convenient method of solving the problem of force distribution in a reinforced panel is that in which the voltages are measured in an electric network which is so constructed as to make it a complete analogue of the reinforced panel. When suitable electric equipment is available, an analogous network can be hooked up and tested with very little work. A particularly attractive property of the stress-analysis procedure by means of electric measurement is the ease with which the effect upon the stress distribution of changes in loading and in dimensions of the various structural elements of the reinforced panel can be investigated. This permits the development of an efficient design with little analytic work.

The analogy between the forces transmitted through the different structural elements of the reinforced panel and the currents flowing through the various branches of the direct-current network can be explained with the aid of figures 4 and 5. The problem investigated is the so-called "one-dimensional shear lag." It is assumed that the transverse stiffeners are infinitely rigid so that the vertical, or longitudinal, displacements \( v \) alone need to be determined. The portion of the sheet covering considered effective in tension or compression is added to the cross-sectional area of each stringer and the panels of sheet are assumed to carry shear stresses only. A consequence of these assumptions is that the shearing stress must be constant in each panel.

![Figure 4.—Forces transmitted through structural elements of reinforced panel.](image-url)
RECOMMENDATIONS FOR NUMERICAL SOLUTION OF REINFORCED-PANELS AND FUSELAGE-RING PROBLEMS

The analogous direct-current network contains as many binding posts as the number of joints in the reinforced panel. Adjacent binding posts are connected by conductors having prescribed resistances $R$. Predetermined electric currents $I$, which correspond to the forces $F$ applied to joints $A$ and $B$ of the reinforced panel, are introduced into the network at points $A$ and $B$.

It is now recalled that in the relaxation method the joints of the panel are first assumed to be rigidly fixed to a rigid wall behind the panel. The external loads are first applied to these rigid pegs, referred to as the "constraints." The panel is obviously in equilibrium under these conditions but this artificial equilibrium is entirely different from that prevailing in the actual panel, which is not attached to any rigid wall. The actual state of equilibrium is approached by the step-by-step procedure of the relaxation method, in each step of which one single constraint is removed and the corresponding joint is displaced until it reaches its equilibrium position in the system in which all the other joints are still rigidly fixed.

For instance when joint 1 of the reinforced panel is moved through a distance $v$ in the positive direction, this displacement imposes forces upon all the adjacent joints numbered from 2 to 9. Three typical forces are given by the equations:

$$F_{81} = v \left( \frac{EA}{b} - \frac{Gbt}{2a} \right)$$  \hspace{1cm} (7)

$$F_{91} = v \frac{Gbt}{4a}$$  \hspace{1cm} (8)

$$F_{61} = v \frac{Gbt}{2a}$$  \hspace{1cm} (9)

where

- $F_{81}$, $F_{91}$, $F_{61}$ forces acting upon joints 8, 9, and 6, respectively, because of displacement of joint 1
- $E$ modulus of elasticity of stringer
- $G$ shear modulus of sheet
- $t$ thickness of sheet
- $v$ displacement of joint 1

In the case of the analogous network it can be assumed that the potential of each binding post is zero at the outset. If there is no potential difference, no current flows between the posts. It can be imagined that the currents introduced at points $A$ and $B$ are taken out of the system by means of some imaginary conductors. However, the actual distribution of currents in the network prevails without the aid of the imaginary conductors. This actual state can be approached also by means of a step-by-step, approximation-type calculation. For instance it can be assumed first that the potential of binding post 1 is elevated to the value $V$. After this change there is a potential difference between binding posts 1 and 8 and consequently a current will flow from post 1 to post 8. The magnitude of this current can be calculated from the equation

$$I_{81} = \frac{V}{R_{81}} = C_{81} V$$  \hspace{1cm} (10)

where $R_{81}$ is the resistance and $C_{81} = 1/R_{81}$ is the conductance of the conductor between posts 1 and 8. Similarly the current flowing from post 1 to post 9 is

$$I_{91} = C_{91} V$$  \hspace{1cm} (11)

The current flowing from post 1 to post 6 is

$$I_{61} = C_{61} V$$  \hspace{1cm} (12)

Comparison of equations (7) to (9) with equations (10) to (12) reveals an analogy between the effects of a displacement $v$ of joint 1 and the raising of the voltage of binding post 1 by an amount $V$. The current caused by the change in potential corresponds to the force caused by the displacement, provided that the conductance of each conductor is made equal to the influence coefficient in the corresponding force equation. Hence

$$C_{81} = \frac{EA}{b} - \frac{Gbt}{2a}$$  \hspace{1cm} (13)

$$C_{91} = \frac{Gbt}{4a}$$  \hspace{1cm} (14)

$$C_{61} = \frac{Gbt}{2a}$$  \hspace{1cm} (15)

In the relaxation procedure the equilibrium state is approached by displacing individually the joints and summing the effects of each displacement. In exactly the same way the actual distribution of the currents in the network can be determined by changing individually the voltages of each binding post and summing the effects of these changes. In the reinforced panel equilibrium is obtained when at each joint the sum of the external forces and of all the internal
forces caused by the displacements is zero. The forces are considered positive if they are directed as the positive displacements. In the form of an equation,
\[ \Sigma F = 0 \] (16)

An analogous equation in the direct-current network is furnished by Kirchhoff's first law, according to which the sum of the currents flowing into any binding post must be zero. Currents in the direction of any binding post are considered as positive. In the form of an equation,
\[ \Sigma I = 0 \] (17)

Comparison of the last two equations reveals that the conditions of equilibrium for the reinforced panel and Kirchhoff's first law in the case of the direct-current network complete the analogy of the two systems considered. It is possible therefore to construct an electric network with the same configuration of binding posts as that of the joints of the reinforced panel. The conductances of the conductors connecting the binding posts must be so chosen as to make them proportional to the corresponding influence coefficients in the operations table of the reinforced panel. If then currents are introduced at the binding posts which correspond to the joints at which external loads are applied, the distribution of the currents in the network will be the same as the distribution of the forces between the various structural elements of the reinforced panel.

In the first applications of the relaxation process to reinforced panels each joint was displaced until equilibrium was established. It was noted in the section dealing with the solution of the problem by matrix methods that this procedure permitted rigid body displacements of the structure. Rigid body displacements can be eliminated if one or more joints are considered as rigidly fixed. In the case of the reinforced panel of figure 4 the degree of freedom of motion of each joint is one, because the problem is considered as a one-dimensional shear lag problem. Consequently it suffices to fix one single joint so that it is prevented from displacing vertically. However, if joint C, for instance, is fixed, the symmetry of the structure and loading requires the simultaneous fixation of joint D.

In the analogous network binding posts C and D are given predetermined values of the potentials by connecting them to the ground. It is customary to attribute the value zero to the potential of the ground. Consequently \( V_C \) and \( V_D \) are zero just as in the reinforced panel \( \epsilon_C \) and \( \epsilon_D \) are zero.

It will be noticed that in figure 4 the direction of \( F \) at joints A and B is upward, whereas the direction of \( I \) at binding posts A and B in figure 5 is downward. This corresponds to the difference in the sign convention in the two systems. In the panel upward forces were considered positive and in the network currents flowing toward the binding posts were given the positive sign. The directions of the forces and the currents at points C and D are the same. This again corresponds to the correct signs required by the sign convention since the downward forces at these points are negative just as the currents which flow away from the binding posts are negative. Hence the reinforced panel is under the action of external tensile forces, whereas through the network currents are flowing in the downward direction.

In the case under discussion it is easy enough to introduce the two equal currents at posts A and B and to regulate their magnitude by means of an adjustable rheostat. However, when there are a number of impressed currents of different magnitude stipulated, their adjustment may become a lengthy trial-and-error procedure. In such cases it is advantageous to employ a number of commercially available electronic devices, known as constant-current generators, which have the property of maintaining a constant current independently of the properties of the network.

When the construction of the network is completed and the required external currents are introduced, the deflection of any joint of the reinforced panel can be obtained by measuring the potential of the corresponding post in the network with respect to the ground. This quantity multiplied by the conversion factor is the relative displacement of the corresponding joint of the reinforced panel with respect to the fixed points C and B. In most cases, however, the displacement quantities are of interest only indirectly and the main quantities sought are the forces in the stringers and the shear stresses in the sheet. These quantities can be obtained in a simple manner by multiplying potential differences by the appropriate conductances and by the conversion factor.

For instance when the force in stringer segment 1-8 is sought, the voltage drop between posts 1 and 8 must be measured and multiplied by the conductance \( C_{18} \) and the conversion factor. This is a consequence of equations (7) and (10). Similarly when the shear stress in panel 1689 is required, the voltage drops in conductors 1-6 and 8-9 have to be measured. From figure 4 the average displacement of stringer segment 6-9 is \( (v_8 + v_9)/2 \) and the average displacement of stringer segment 1-8 is \( (v_1 + v_8)/2 \). The difference of these two average displacements multiplied by \( G_{18} \) is the shear force transmitted from the panel to stringer segment 6-9. Consequently the sum of the displacement differences \( r_8 - r_1 \) and \( r_9 - r_8 \) multiplied by the influence coefficient 1-6 is the shear force sought. In other words the sum of the voltage drops from post 1 to post 6 and from post 8 to post 9 multiplied by the conductance \( C_{18} \) and the conversion factor is the shear force in question. This shear force divided by the length \( b \) gives the average shear flow in panel 1689 and this shear flow divided by the thickness of the sheet is the average shear stress.

With the cooperation of the Department of Electrical Engineering a network was constructed at the Polytechnic Institute of Brooklyn which was the analogue of the reinforced panel investigated earlier at PIBAL both experimentally and by relaxation methods. The results of these investigations are described in reference 3. The constant currents were introduced by means of constant-current generators. In the electrical system the unit of the potential was chosen as 1 volt and that of the current as 100 milliamperes. Then the unit of the conductance had to be a millimho and that of the resistance, a kilohm. In the mechanical system the unit displacement was 10\(^{-4}\) inch and the unit force, 1 pound. Consequently in this problem the voltage differences had to be multiplied by the conversion factor 10\(^{-4}\) inch per volt in order to obtain displacements. The factor converting currents into forces was 10 pounds per
ampere. The results of the measurements were in excellent agreement with the results quoted in reference 3.

Similar experiments were carried out by Newton and Engle at the Curtiss-Wright Corporation, Airplane Division, in St. Louis and are described in two reports listed as references 11 and 12. Newton's approach to the problem is fundamentally the same as the argument given herein. However, his electric network is slightly simpler since it does not contain the conductors arranged diagonally in the system shown in figure 5. The network of figure 5 was chosen in this report in preference to Newton's simpler network since by this presentation the identity of the conductances of the network and the influence coefficients used in the other part of this report could be established.

It should be mentioned that in many cases it is possible to construct a dual type of network in which the currents correspond to the displacements of the joints of the reinforced panel and the potential differences correspond to the forces in the stringers and in the sheet covering of the panels. In this type of network the external loads can be introduced more easily as impressed potential differences. However, the network described herein is more advantageous since it can always be constructed directly from the geometry of the reinforced panel.

The usefulness of the analogue with the direct-current network breaks down when the influence coefficient in equation (7) becomes negative. In such a case the conductance and consequently the resistance of the corresponding branch of the network should be negative; this is obviously impossible. However, the situation can be usually remedied in the case of one-dimensional shear lag problems. The fundamental assumptions of the problem are not changed if a number of additional horizontal bracing elements are introduced in the panel since all of them are assumed to be infinitely rigid. If, however, the panel length \( b \) is reduced to one-half its original value, then the negative term in the influence coefficient appearing in equation (7) is halved and the positive term is doubled. In most cases this will suffice to change the sign of the influence coefficient. When such is not the case distance \( b \) can be reduced in any other suitable ratio.

Negative influence coefficients can be realized if the analogous network is fed by an alternating current. The quantity corresponding in an alternating-current circuit to the resistance of the direct-current circuit is the impedance. In the impedance the inductance retards the phase of the current and the capacitance advances it so that the two have opposite effects. If one is designated as positive, the other is negative. However, no inductance is entirely free of resistance and for this reason the accuracy of a complicated alternating-current network may not be sufficient for the solution of some of the problems encountered in practice.

The use of the electric analogue for solution of shear lag problems is recommended when several similar panels with many loading conditions are to be analyzed. For such a case the construction of the analogous network, the variation of the loading by varying the impressed currents, and the determination of the potentials at the binding posts would be simpler than any analytic method of solution.

### FUSELAGE RINGS

**INTRODUCTION**

In reference 5 numerical procedures for the determination of the bending-moment distribution in fuselage rings are developed and demonstrated on several simple and internally braced fuselage rings. The number of redundant internal bracing elements increases little the work involved in establishing the operations table for the ring and affects not at all the amount of numerical work in the solution of the operations table. This nonsensitivity to the number of redundancies constitutes the advantage of this method in the analysis of fuselage rings.

The methods suggested for the solution of the system of equations represented by the operations table and the external forces are three: Relaxation, matrix-calculus, and growing-unit. The latter two may be considered as direct mathematical methods and as in reinforced-panel problems require only computing personnel. For the analysis of isolated fuselage rings of complex shape the use of these direct methods is recommended since an accurate solution is assured in a reasonable length of time, whereas the relaxation method may not lead to sufficiently accurate results even after considerable effort has been expended. However, for simply shaped rings and for problems of stress distribution in sheet, stringer, and ring combinations, application of the relaxation method to fuselage rings is advantageous.

For this reason the relaxation method for fuselage-ring problems is presented and new, more rapidly convergent procedures are developed.

It has not been found possible to make concrete recommendations for relaxation procedures which are rapidly convergent for all types of ring and loading. However, satisfactory procedures for several distinct types of ring and loading are demonstrated and explained in some detail. It is felt that consideration of these examples will suggest to the analyst means of solving more rapidly other ring and cylinder problems which are not efficiently attacked by direct mathematical means. The procedures, which involve essentially appropriate combined operations, are demonstrated on two rings solved in reference 5 by the usual relaxation methods and on a new internally braced ring. Application of the growing-unit and matrix-calculus methods to the latter problem is made to demonstrate these methods and to verify the results of the relaxation procedure.

### TORSION OF A CIRCULAR RING

In reference 5 the bending-moment distribution for a simple circular ring with antisymmetric loading consisting of concentrated forces and distributed constant shear flow is determined by application of numerical methods. The dimensions and loading for this ring are shown in figure 6, and the operations table is given as table 7. Relaxation methods are applied to the solution of this ring problem in reference 5. By a process of increasing all the residuals in such a proportion that one key operation would liquidate them all to within the desired degree of accuracy, the residuals were reduced to within 2 percent of the maximum applied load in 12 operations.
In the present report combined operations which increase
the rate of convergence are demonstrated. Tangential and
angular displacements of A and C balance these points in
four operations and place all remaining residuals at B.
Since no tangential forces exist at A and C, the force residual
at B must be vertical and the moment residual, equal to the
couple of the vertical forces. Suppose the residual moment
at B is liquidated by a rotation of that joint while the balance
of A and C is preserved by suitable displacements of A and C.

In order to balance the residuals at B without disturbing the
balance at A and C obtained by use of operations A to D,
combined operations involving tangential and angular
displacements of A and C and a unit rotation of B are de
veloped. If joint A is to remain in balance when a rotation
of B is undertaken, joint A must be rotated and displaced
in such a manner that the tangential force and the moment
introduced at A by this rotation of B are equilibrated. Since
the angular displacement introduces tangential forces at
A and the tangential displacement introduces moments, two
simultaneous equations must be solved for the unknown
tangential and angular displacements. The equations for
A are:

\[
-281.95w_A + 49.079u_A - 29.966 \times 10^{-2} = 0
-49.079w_A + 52.296u_A + 64.675 \times 10^{-2} = 0
\]

The solution to these equations is \( w_A = -0.38434 \times 10^{-3} \) radian and \( u_A = 1.5974 \times 10^{-2} \) inch. A unit rotation of B and tangential and angular displacements of C are combined in equations (19) so that the tangential force and moment introduced at C by the combined operations are zero.

\[
-157.899w_C - 1.563u_C + 56.5117 \times 10^{-2} = 0
-1.563w_C - 0.322u_C + 6.632 \times 10^{-2} = 0
\]

The solution to these equations is \( w_C = -0.16180 \times 10^{-3} \) radian and \( w_C = 19.811 \times 10^{-3} \) inch.
The relaxation table using these five combined operations, A to E, is given as table 8. The balancing process was carried out on a slide rule and after five operations all the residuals were reduced to negligible quantities. From the magnitudes of these group operations the total individual displacements of A, B, and C can be found and the unknown radial forces at A and C calculated.

The procedure just described involves essentially the development of group operations so that full advantage of the symmetry properties of the ring may be realized. This method is applicable to other rings. The internally braced circular ring subjected to antisymmetric loads and analyzed in reference 5 can be treated in the same way as this simple ring. If these rings had been symmetrically loaded, the force residuals at B, after A and C had been balanced by simple radial displacements, would have a horizontal resultant. By combining radial and tangential displacements of A and C such that the resultant force introduced at B is horizontal and such that A and C remain in balance, the horizontal resultant at B could be liquidated by application of such a combined operation. The moment residual at B is not necessarily eliminated when the force residual at D is balanced. Joint B must be rotated while A and C are displaced radially so that the moment at B is liquidated and joints A and C are kept in balance. If the process of liquidating first the residual force and then the moment at B, preserving in each operation the balance at A and C, is not rapidly convergent, two equations for the equilibrium of B can be established and solved for the required amounts of the combined operations.

Thus the foregoing procedures for both the symmetrical and antisymmetrical loading can be applied to any ring singly symmetrical with only one joint between the center line of symmetry joints. It may, therefore, be advantageous in some ring problems to combine several bars, as in the method of the growing unit, such that only one joint between the boundary joints has independent degrees of freedom. This will permit use of the foregoing procedure.

Sufficient accuracy for most engineering purposes can be obtained in the computations of this procedure by the use of a slide rule throughout. Although the combined operations shown herein were obtained by the use of a computing machine carrying five significant figures, the procedure was first demonstrated with the use of a slide rule for all calculations. The results of the two sets of calculations are in good agreement, thus indicating the sufficiency of slide-rule accuracy.

**EGG-SHAPED RING**

Figure 7 shows the dimensions of, and loading on, a ring which is analyzed in reference 5. The operations table for this ring is given as table 9. In this ring there are two points B and C between the center line of symmetry points A and D. By making the degrees of freedom of either point B or C dependent on the other and on the adjacent center line of the symmetry point, one point with independent degrees of freedom is established between A and D and the method discussed previously can be used.

![Diagram of Egg-Shaped Ring](image)
However, in order to demonstrate the simplicity and effectiveness of group operations, another approach is used. The center lines of symmetry points A and D are balanced by simple radial displacements of A and B. The midpoint of bar BC is assumed restrained tangentially so that only equal and opposite tangential displacements of B and C are undertaken. Because of the large extensional stiffness of bar BC as compared with the bending rigidity of the circular segments and because the ring is almost symmetrical about a horizontal axis, such displacements of B and C liquidate approximately equal and opposite tangential residual forces at B and C, such as those which will be obtained at these points when the residuals associated with the other degrees of freedom are small.

If the balance at A and D is preserved by appropriate combinations of the radial displacements of A and D with the required displacements of B and C and if the tangential residuals at B and C are not considered until the foregoing operation will liquidate them both, main attention is focused on the radial force and moment residuals at B and C. In order to balance these, no specific method of convergence is used but the state of the residuals after each step is considered before the next operation is selected. In this problem of egg-shaped rings and many other rings and in the complete cylinder problems this approach, utilizing physical properties of the system and eliminating or reducing extraneous forces and moments at each step in the relaxation process, may be the most satisfactory method of solution.

Table 10 is the relaxation table for the ring in question. The first two operations involve only radial displacements which balance the 500-pound forces at A and D. The largest residual then is the radial force of 451 pounds at C. If point C is displaced radially so as to balance this residual, a large moment and a large radial force are introduced at B. In order to reduce these extraneous forces and moments and to keep joints A and D balanced, radial displacements of A, B, and D and a rotation of B are combined as shown by the following operations:

\[
\begin{align*}
-3.34833v_A + 8.92216w_A - 2.69614v_C &= 0 \\
8.92216v_A - 327.866w_A + 11.4697v_C + 8.10267 \times 10^{-4} &= 0 \\
-2.69614v_A + 11.4697w_A + 4.00991v_B + 0.66158 \times 10^{-4} &= 0 \\
-12.2400v_A - 1.11900 \times 10^{-4} &= 0
\end{align*}
\]

The solution of this system of equations is: \( v_A = -0.26384 \times 10^{-4} \) inch, \( w_A = 0.03279 \times 10^{-4} \) radian, \( v_B = 0.43618 \times 10^{-4} \) inch, and \( w_B = -0.9024 \times 10^{-4} \) inch.

The forces and moments introduced by each of the individual operations and by the combination are given in the following table:

<table>
<thead>
<tr>
<th>Operation</th>
<th>( v_A )</th>
<th>( N_A )</th>
<th>( D_A )</th>
<th>( T_A )</th>
<th>( N_C )</th>
<th>( R_C )</th>
<th>( T_C )</th>
<th>( R_D )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v_A = 10^{-4} ) in</td>
<td>-0.60522</td>
<td>8.10267</td>
<td>0.66158</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( v_A = 0.60522 \times 10^{-4} ) in</td>
<td>2.69614</td>
<td>-3.53420</td>
<td>0.26570</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( v_A = -0.9024 \times 10^{-4} ) in</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( v_A = -0.26384 \times 10^{-4} ) in</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( v_A = 0.43618 \times 10^{-4} ) in</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( v_A = -0.9024 \times 10^{-4} ) in</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The use of combined operation \( F \) is desirable in balancing the radial residual force at C, since it also reduces the moment residual at C and adjusts the tangential residuals at B and C in the desired manner.

The residual considered after use of operation \( F \) is \( R_B = 402 \) pounds. In order to balance it by a displacement \( v_B \) while the balance at A is preserved, a \( v_A \)-displacement must be undertaken as well. If \( v_B = 10^{-4} \) inch, then \( v_A = -3.34833 \times 10^{-4} = -0.80522 \times 10^{-4} \) inch. The forces and moments introduced by these individual operations as well as by the combinations are given in the following table:

<table>
<thead>
<tr>
<th>Operation</th>
<th>( v_A )</th>
<th>( N_A )</th>
<th>( D_A )</th>
<th>( T_A )</th>
<th>( N_C )</th>
<th>( R_C )</th>
<th>( T_C )</th>
<th>( R_D )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v_B = 10^{-4} ) in</td>
<td>-2.69614</td>
<td>11.4697</td>
<td>-4.00991</td>
<td>3.4302</td>
<td>-8.10267</td>
<td>0.66158</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( v_B = -0.9024 \times 10^{-4} ) in</td>
<td>2.69614</td>
<td>-7.1843</td>
<td>2.1710</td>
<td>-3.1949</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \Sigma ) + Operation ( G )</td>
<td>0</td>
<td>4.2854</td>
<td>-1.83991</td>
<td>0.2403</td>
<td>-8.10267</td>
<td>0.66158</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Consider the effect of eliminating the \( R_B \)-residual by use of operation \( G \). The moment residual \( N_B \) would also be reduced by roughly 1000 inch-pounds, the \( T_A \)-residual would be brought in closer agreement with the \( T_C \)-residual, an \( R_C \)-residual of about 30 percent of the previous \( R_C \)-residual of 451 pounds would be introduced, and a large \( N_C \)-residual would be introduced. The last two effects are undesirable. However, by use of operation \( F \) again, the \( R_C \)-residual can be
balanced without introducing a new $R_p$-residual. The large $N_C$-residual is not so easily balanced unless a new combination involving joints A, B, and D is evolved.

Suppose, therefore, that a rotation of C and a radial displacement of D are combined so that a moment at C can be eliminated and so that the balance of D is preserved by use of the combination. The individual operations and the combinations are given in the following table:

<table>
<thead>
<tr>
<th>Operation</th>
<th>Forces and moments</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Sigma = \text{Operation M} = 1$</td>
<td>$R_A$</td>
</tr>
<tr>
<td>$\omega_c = 10^{-4}$ radian</td>
<td>0</td>
</tr>
<tr>
<td>$\omega_c = -5.9294 \times 10^{-4}$ in.</td>
<td>0</td>
</tr>
</tbody>
</table>

If operations G and H are combined so that the moment at C introduced by the combination is zero, the resulting forces and moments are given in the following table:

<table>
<thead>
<tr>
<th>Operation</th>
<th>Forces and moments</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Sigma = \text{Operation M} = 1$</td>
<td>$R_A$</td>
</tr>
<tr>
<td>$1 \times (G)$</td>
<td>0</td>
</tr>
<tr>
<td>$-0.03385 \times (H)$</td>
<td>0</td>
</tr>
<tr>
<td>$\Sigma = \text{Operation M} = 1$</td>
<td>0</td>
</tr>
</tbody>
</table>

Use of operation I results in liquidation of the $R_B$-residual, in reduction in the $N_B$-residual, in adjustment of the $T_B$- and $T_C$-residuals toward the desired equality, and in introduction of an $R_C$-residual of 138 pounds. The latter can be balanced by the use of operation F, which will preserve the balance of A and D and will not affect the $N_B$- and $R_P$-residuals.

After this fifth operation the $T_B$- and $T_C$-residuals are approximately equal and opposite as desired. Therefore, a group operation, involving equal and opposite tangential displacements of B and C and sufficient radial displacements of A and D so that the latter remain balanced, is developed in the following table:

<table>
<thead>
<tr>
<th>Operation</th>
<th>Forces and moments</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Sigma = \text{Operation M} = 1$</td>
<td>$R_A$</td>
</tr>
<tr>
<td>$u_B = 10^{-4}$ in.</td>
<td>0</td>
</tr>
<tr>
<td>$\omega_c = 10^{-4}$ radian</td>
<td>0</td>
</tr>
<tr>
<td>$\omega_c = 1.1850 \times 10^{-4}$ in.</td>
<td>0</td>
</tr>
<tr>
<td>$\omega_c = 0.83669 \times 10^{-4}$ in.</td>
<td>0</td>
</tr>
</tbody>
</table>

Use of operation J liquidates the $T_B$- and $T_C$-residuals and affects little the balance in the other degrees of freedom. The remaining residuals are considered negligibly small, the moment of 309 inch-pounds being approximately 3 percent of the maximum moment in the ring. As in the previous problem the individual displacements can be determined from the magnitudes of the group operations and thus the unknown moments and tangential forces at A and D calculated.

Although the calculations of the group operations shown herein have been carried out on a computing machine with five significant figures maintained wherever possible, sufficient accuracy for engineering purposes can be obtained by the use of a slide rule. In developing this procedure a slide rule was used for all computations and the results agreed satisfactorily with those shown herein.

**OVAL-SHAPED RING WITH INTERNAL BRACING**

The ring shown in figure 8 is used as a third example of the new relaxation procedures. As a check on the results of this procedure the system of equations given by the operations table and external forces is also solved by the exact mathematical methods of matrix calculus and of the growing-unit method. In order that the charts and tables of reference 6 could be used in determining the influence coefficients, the following physical characteristics of the elements of the ring are assumed:

Segments AB and EF:

$$\gamma = \frac{4L^2}{I} = 500$$

$$\zeta = \frac{A\xi}{A} = 0.10$$
\[ \beta = 45^\circ \]

\[ EI = 10^6 \text{ lb-in.}^2 \]

\[ L = 18.85 \text{ in.} \]

Segments BC, CD, and DE:

\[ \gamma = \frac{AL^2}{I} = 500 \]

\[ \xi = \frac{A^e}{A} = 0.10 \]

\[ \beta = 30^\circ \]

\[ EI = 10^6 \text{ lb-in.}^2 \]

\[ L = 18.85 \text{ in.} \]

Segment EG:

\[ \gamma = \frac{AL^2}{I} = 400 \]

\[ \xi = \frac{1}{A^e} = 0.10 \]

\[ \beta = 0^\circ \]

\[ EI = 10^6 \text{ lb-in.}^2 \]

\[ L = 16.97 \text{ in.} \]

Because of the symmetry about a line through AGF only one-half of the ring need be considered. Joints A, G, and F are then restrained from rotating or displacing tangentially and cannot be subjected to radial forces. The assumed positive directions of the displacements and of the forces and moments at each joint are shown in figure 8. From the foregoing assumptions, the influence coefficients and the operations table given in table 11 are determined.

The horizontal external forces of 1000 pounds at C and D are resolved into their tangential and radial components. Thus the external forces are:

\[ \begin{align*}
R_C &= 965.93 \text{ lb} \\
T_C &= 258.82 \text{ lb} \\
R_D &= -965.93 \text{ lb} \\
T_D &= -258.82 \text{ lb}
\end{align*} \]

The matrix-calculus solution of the system of equations given by these external forces and by the operations table is first obtained so that the equilibrium of the ring as given by this solution will provide a check on the whole setup. Joint G is considered fixed so that a unique solution to this system of equations is obtained; thus there are 14 degrees of freedom to be considered. The 14 unknowns are found by the method of reference 9 to be:

\[ \begin{align*}
r_A &= -605.73 \times 10^{-3} \text{ in.} \\
w_B &= 40.825 \times 10^{-3} \text{ radian} \\
r_B &= 35.144 \times 10^{-3} \text{ in.} \\
u_B &= -300.06 \times 10^{-3} \text{ in.} \\
w_C &= -11.445 \times 10^{-3} \text{ radian} \\
R_C &= 664.55 \times 10^{-3} \text{ in.} \\
u_C &= -72.282 \times 10^{-3} \text{ in.} \\
o_G &= -22.975 \times 10^{-3} \text{ radian} \\
R_G &= -94.783 \times 10^{-3} \text{ in.} \\
u_G &= 6.2337 \times 10^{-3} \text{ radian} \\
r_E &= -42.621 \times 10^{-3} \text{ in.} \\
\end{align*} \]

These displacements give the following values of the unknown moments and tangential reactions at A, F, and G on the bars rather than on the joints:

\[ \begin{align*}
N_A &= -3118.8 \text{ in.-lb} \\
T_A &= 492.51 \text{ lb} \\
N_F &= -105.49 \text{ in.-lb} \\
T_F &= -182.57 \text{ lb} \\
N_G &= -371.06 \text{ in.-lb} \\
R_F &= 0.43 \text{ lb} \\
T_F &= 584.98 \text{ lb}
\end{align*} \]
Figure 9 is the bending-moment diagram for the ring with these reactions applied.

By examining the equilibrium of one-half the ring under these reactions and the external forces, the accuracy of the operations table is established. Since \( R_A \) and \( R_F \) are zero, the summation of forces in the vertical direction is simply:

\[
\sum F_v = R_G = 0.43 \text{ lb} \tag{24}
\]

The summation of forces in the horizontal direction is:

\[
\sum F_H = T_A - T_F - T_G = 0.10 \text{ lb} \tag{25}
\]

The summation of moments about point \( G \) is:

\[
\sum M_G = N_A + T_A \left[ 24 + \frac{12}{0.70711} + 24(0.70711) \right] +
\]

\[
N_G + N_F + T_F(24)(1 - 0.70711) - 1000(2 \times 36 \times 0.25882)
- 3118.91 + 402.49(57.941) - 371.06 + 18635.04
= 17.45 \text{ in.-lb} \tag{26}
\]

The equilibrium conditions for the half ring are approximately satisfied, the maximum percent error being a moment of less than 0.1 percent of the applied couple of 18,635 inch-pounds. It is considered that the accuracy of the operations table is established by this equilibrium check.

Approximately 20 man-hours by an unskilled computing-machine operator were required to solve this system of 14 equations. It is estimated that a skilled operator familiar with the Crout method would require about 10 man-hours.

In applying the Crout method to this problem the coefficients of the linear equations are assumed to be mathematically exact and, therefore, as many figures as could be carried on the 100-bank computing machine are used throughout the computation. In this way an accurate solution is obtained and the additional computing work is not great. Afterward the values of the unknowns can be rounded off to the physically correct number of significant figures.

Use of the growing-unit method of solution on this ring is demonstrated as follows. This method is described in detail on pages 39 to 46 of reference 5. It is demonstrated on this new ring as an application of the procedure to a ring with many intermediate joints between the center line of symmetry points. In applying the growing-unit method to this ring the units are combined into bars of increasing length until displacements of all points are known such that the only unbalanced forces remaining act in the radial direction at \( A \) and \( F \) when unit radial displacements are undertaken at \( A \) and \( F \). Then these forces at \( A \) and \( F \) can be eliminated by appropriate radial displacements of \( A \) and \( F \) and the final distorted shape determined.

The first units to be combined are \( AB \) and \( BC \). In order to effect this combination, the displacements of \( B \) required to maintain the balance of \( B \) while point \( A \) is displaced radially \( 10^{-3} \) inch are given by the equations:

\[
\begin{align*}
N_B &= -454.34w_B + 6.7238v_B - 78.411u_B + 5.9020 \times 10^{-3} = 0 \\
R_B &= 6.7238w_B - 12.093v_B + 0.55690u_B - 4.5778 \times 10^{-3} = 0 \\
T_B &= -78.411w_B + 0.55690v_B - 84.510u_B + 14.662 \times 10^{-3} = 0 
\end{align*}
\]

The solution to these equations is: \( w_B = -0.026434 \times 10^{-3} \) radian, \( v_B = -0.38424 \times 10^{-3} \) inch, and \( u_B = 0.19549 \times 10^{-3} \) inch.

If the forces and moments at points \( A \) and \( C \) due to a displacement \( v_A = 10^{-3} \) inch and due to the foregoing displacements \( w_B, v_B, \) and \( u_B \) are summed, the following equations are obtained:

\[
\begin{align*}
R_A &= -2.6618 \text{ lb} \\
N_C &= 10.699 \text{ in.-lb} \\
R_C &= -1.8871 \text{ lb} \\
T_C &= 3.2614 \text{ lb} 
\end{align*}
\]

The displacements \( N_B, R_B, \) and \( T_B \) are zero since that is the condition satisfied by equations (27).

The displacements of \( B \) required to maintain balance at \( B \) during unit rotational, radial, or tangential displacements of \( C \) are determined in a similar manner and are collected in table 12.

The forces and moments given in the last seven rows of this table constitute the influence coefficients for a new unit of the ring, namely, the segment \( ABC \). This unit is not a bar, the center line of which is an arc of a circle, but rather one composed of two arcs of circles. This combining of units, extended until the entire ring is one segment, is the main principle of the growing-unit method.

Each column of table 12 represents a group displacement made up of individual displacements of points \( A, B, \) and \( C \). Let these group displacements be identified by the Roman numeral given at the head of each column. For example, group II is made up of the displacements \( w_C = 10^{-3} \) radian, \( v_C = -0.21631 \times 10^{-3} \) radian, \( v_B = -0.23971 \times 10^{-3} \) inch, \( u_A = 0.74197 \times 10^{-3} \) inch, \( v_A = v_C = u_B = 0. \) The moment at \( C \), for instance, caused by the application of \( x_{II} \) units of the group displacement II is then:

\[
N_C = -389.56x_{II} \tag{29}
\]

With a similar notation for all other forces and group displacements, equations (30) may be set up representing the requirements for equilibrium of joint \( C \) under the external forces acting at that point, balance of \( B \) being maintained.

\[
\begin{align*}
N_C &= -389.56x_{II} - 10.093x_{III} - 53.771x_{IV} = 0 \\
R_C &= -10.093x_{II} - 6.7529x_{III} - 10.615x_{IV} + 965.93 = 0 \\
T_C &= -53.771x_{II} - 10.615x_{III} - 54.199x_{IV} - 258.82 = 0
\end{align*}
\]

(30)
The solution to this system is \( x_{II} = 1.0476, x_{III} = 217.61, \) and \( x_{IV} = -48.436, \) and the following forces and moments are introduced at A and D:

\[
\begin{align*}
R_A &= -557.43 \text{ lb} \\
N_D &= -2,666.6 \text{ in.-lb} \\
R_D &= 184.60 \text{ lb} \\
T_D &= 626.46 \text{ lb}
\end{align*}
\] (31)

The forces and moments at D are added to the external forces applied to the ring at D and are balanced after the unit problem for the segment at ABCD is established. The \( R_A \) force is not balanced in the complete ring is one segment and until the \( R_A \) and \( R_D \) residuals can be balanced together.

The next unit to be considered is the combination of the ABC segment with bar CD into the segment ABCD. The problem is to find the forces and moments at A, D, and E due to (1) a unit radial displacement of A with joint D fixed and (2) unit radial, tangential, and rotational displacements of D with A and E fixed. Joints B and C are free to displace so as to maintain the balance at B and C in each of these four cases. By determining the magnitudes of \( x_{II}, x_{III}, \) and \( x_{IV} \) required to balance C in each of these four cases, the required displacements of both B and C are implicitly determined and the unit problem for segment ABCD is solved.

The magnitudes of the \( x_{IV}, x_{III}, \) and \( x_{IV} \) operations required to balance joint C when A is displaced radially \( 10^{-3} \) inch and B permitted to displace so as to remain in balance are given by the following equations:

\[
\begin{align*}
N_C &= -389.56 x_{II} - 10.093 x_{III} - 53.771 x_{IV} + 10.699 = 0 \\
R_C &= -10.093 x_{II} - 6.7529 x_{III} - 10.615 x_{IV} - 1.5871 = 0 \\
T_C &= -53.771 x_{II} - 10.615 x_{III} - 54.199 x_{IV} + 3.2615 = 0
\end{align*}
\] (32)

The forces and moments at C to be balanced are given in group I in table 12. The solution to these equations is \( x_{II} = 0.021497, x_{III} = -0.53843, \) and \( x_{IV} = 0.14430. \) Use of these multiples of operations II, III, and IV and of a unit amount of group I results in the following forces and moments at A and D:

\[
\begin{align*}
R_A &= -0.94505 \text{ lb} \\
N_D &= 6.7928 \text{ in.-lb} \\
R_D &= -0.74924 \text{ lb} \\
T_D &= 0.77749 \text{ lb}
\end{align*}
\] (33)

The forces and moments given by groups V, VI, VII, and VIII in table 12 are the influence coefficients for segment ABCD. For example, the forces and moments introduced at A, D, and E due to a unit radial displacement of D with A and E fixed and with B and C in balance are given by VII. With these sets of coefficients it is possible to balance joint D while the balance of B and C is preserved. The forces and moments to be balanced at D are (1) the external forces on the ring at D and (2) the forces and moments which are introduced at D by the balancing of C and which are given by equations (31). The residuals to be balanced at D are

\[
\begin{align*}
N_D &= -2666.6 \text{ in.-lb} \\
R_D &= -460.93 + 184.60 = -781.27 \text{ lb} \\
T_D &= -258.82 + 626.46 = 367.64 \text{ lb}
\end{align*}
\] (34)

The equations which condition the balancing of joint D, from consideration of groups VI, VII, and VIII, are seen to be:

\[
\begin{align*}
N_D &= -346.88 x_{VI} - 16.697 x_{VII} - 43.745 x_{VIII} - 2666.6 = 0 \\
R_D &= -16.697 x_{VI} - 5.6500 x_{VII} + 12.458 x_{VIII} - 781.27 = 0 \\
T_D &= -43.745 x_{VI} - 12.458 x_{VII} - 50.817 x_{VIII} + 367.64 = 0
\end{align*}
\] (35)

The solution to these equations is \( x_{VI} = -3.3100, x_{VII} = -328.09, \) and \( x_{VIII} = 90.518, \) which give the following forces and moments:

\[
\begin{align*}
R_A &= 293.71 \text{ lb} \\
N_E &= 4889.4 \text{ in.-lb} \\
R_E &= -522.69 \text{ lb} \\
T_E &= -149.69 \text{ lb}
\end{align*}
\] (36)

As in the balancing of C, a tangential force and moment are introduced at A by this balancing of D, but because of symmetry the equilibrium of A is not disturbed by these. The \( R_A \) forces will be balanced later and the residuals at E will be balanced when the influence coefficients for segment ABCDE have been determined.

In order to find the influence coefficients for bar ABCDE, the forces and moments at A and E due to a radial displacement of A with E fixed and at A, E, and F due to unit radial, tangential, and rotational displacements of E with A and F fixed must be determined. By determining the magnitudes of groups VI, VII, and VIII required to balance D in each of these four cases, the required displacements of B, C, and D and the required forces and moments are determined.

The magnitudes of the groups VI, VII, and VIII required to balance D when joint A is moved radially \( 10^{-3} \) inch are given by the following equations:

\[
\begin{align*}
N_D &= -346.88 x_{VI} - 16.697 x_{VII} - 43.745 x_{VIII} + 6.7928 = 0 \\
R_D &= -16.697 x_{VI} - 5.6500 x_{VII} - 12.458 x_{VIII} - 0.74924 = 0 \\
T_D &= -43.745 x_{VI} - 12.458 x_{VII} + 50.817 x_{VIII} + 0.77749 = 0
\end{align*}
\] (37)

The forces and moments at D to be balanced by groups VI, VII, and VIII are given by V in table 12 and are the constant terms in equation (31). The solution of these equations is \( x_{VI} = 0.028042, x_{VII} = -0.42660, \) and \( x_{VIII} = 0.09574. \) The summation of forces and moments due to a unit magnitude
RECOMMENDATIONS FOR NUMERICAL SOLUTIONS OF REINFORCED-PANEL AND FUSELAGE-RING PROBLEMS

The forces and moments introduced at joint E are:

\[
\begin{align*}
R_A &= -0.36050 \text{ lb} \\
N_E &= 4.1055 \text{ in.-lb} \\
R_E &= -0.31703 \text{ lb} \\
T_E &= 0.19226 \text{ lb}
\end{align*}
\]  

(38)

In a similar manner the complete set of influence coefficients for segment ABCDE is determined and is given in Table 14. For example, the forces and moments in group XI are the forces and moments introduced at A, E, F, and G by a unit radial displacement of E with A, F, and G fixed and with points B, C, and D free to displace so as to remain in equilibrium. With these influence coefficients joint E can be balanced while the balance of B, C, and D is preserved.

The forces and moments to be balanced at joint E are those introduced by the balancing of joint D with groups VI, VII, and VIII and are given by equation (36).

The equations in \(x_X\), \(x_{X1}\), and \(x_{XII}\) balancing joint E under these loads are:

\[
\begin{align*}
N_E &= -533.92x_X - 38.099x_{X1} - 29.579x_{XII} + 4889.4 = 0 \\
R_E &= -38.099x_X - 49.295x_{X1} - 53.432x_{XII} - 522.69 = 0 \\
T_E &= -29.579x_X - 53.432x_{X1} - 76.322x_{XII} - 149.69 = 0
\end{align*}
\]  

The solution to these equations is \(x_X = 11.184\), \(x_{X1} = -51.518\), and \(x_{XII} = 29.771\) and the forces introduced at A, F, and G by this balancing of E are:

\[
\begin{align*}
R_A &= 67.974 \text{ lb} \\
R_F &= -266.68 \text{ lb} \\
R_G &= 70.668 \text{ lb}
\end{align*}
\]  

(40)

The tangential forces and the moments introduced at A, F, and G are not considered in this balancing of the half ring, since these are equilibrated by the forces and moments from the other half of the ring.

The final combination of units will be the combination of bar EF with the unit ABCDE. When this union is effected, the influence coefficients for the half ring as a unit will have been determined and the radial forces at A and F can be balanced simultaneously. The radial forces at joints A and F due to a unit radial displacement of A with F fixed and to a unit radial displacement of F with A fixed must be determined. In both cases joints B, C, D, and E are displaced so as to remain balanced.

The equations giving the magnitudes of groups X, XI, and XII required to balance joints B, C, D, and E when joint A is displaced radially as in group IX are:

\[
\begin{align*}
N_E &= -533.92x_X - 38.099x_{X1} - 29.579x_{XII} + 4.1055 = 0 \\
R_E &= -38.099x_X - 49.295x_{X1} - 53.432x_{XII} - 0.31703 = 0 \\
T_E &= -29.579x_X - 53.432x_{X1} - 76.322x_{XII} + 0.19226 = 0
\end{align*}
\]  

The solution to these equations is \(x_X = 0.0094398\), \(x_{X1} = -0.051804\), and \(x_{XII} = -0.035128\) and the forces introduced by a unit magnitude of IX and by these multiples of groups X, XI, and XII are:

\[
\begin{align*}
R_A &= -0.29857 \text{ lb} \\
R_F &= -0.33361 \text{ lb} \\
R_G &= 0.035176 \text{ lb}
\end{align*}
\]  

(42)

The equations giving the magnitudes of groups X, XI, and XII required to balance joints B, C, D, and E when joint F is displaced radially 10^{-3} inch are:

\[
\begin{align*}
N_E &= -533.92x_X - 38.099x_{X1} - 29.579x_{XII} - 5.9029 = 0 \\
R_E &= -38.000x_X - 49.295x_{X1} - 53.432x_{XII} - 4.5778 = 0 \\
T_E &= -29.579x_X - 53.432x_{X1} - 76.322x_{XII} - 14.662 = 0
\end{align*}
\]  

The solution to these equations is \(x_X = -0.017182\), \(x_{X1} = 0.50354\), and \(x_{XII} = -0.53797\) and the forces introduced at A, F, and G by this balancing of A, F, and G introduced by the foregoing multiples of groups X, XI, and XII are:

\[
\begin{align*}
R_A &= -0.33361 \text{ lb} \\
R_F &= -1.4470 \text{ lb} \\
R_G &= 1.1156 \text{ lb}
\end{align*}
\]  

(44)

The forces given by equations (42) and (44) represent the influence coefficients for the entire half ring and are labeled groups XIII and XIV, respectively. These forces permit calculation of the multiples of groups XIII and XIV required to balance the radial forces at A and F. These forces are the total forces remaining from the balancing of G, C, and D; \(R_A\) is given by the sum of the \(R_A\)-forces of equations (31), (36), and (40) and is:

\[
\begin{align*}
R_A &= -557.43 + 293.71 + 67.974 = -195.75 \text{ lb}
\end{align*}
\]

The \(R_F\)-force is the force introduced by the balancing of E alone and is given by equation (40). It is:

\[
\begin{align*}
R_F &= -266.68 \text{ lb}
\end{align*}
\]

The equations giving the magnitudes of groups XIII and XIV required to balance joints A and F under those loads are:

\[
\begin{align*}
R_A &= -0.29857x_{XIII} - 0.33361x_{XIV} - 195.75 = 0 \\
R_F &= -0.33361x_{XIII} - 1.4470x_{XIV} - 266.68 = 0
\end{align*}
\]  

(45)

The solution to these equations is \(x_{XIII} = -605.73\) and \(x_{XIV} = -44.646\). The radial force at G introduced by this balancing is -71.114 pounds, but the \(R_F\)-force given by equation (40) in the balancing of E is 70.668 pounds. The difference between the two, -0.446 pound, is considered negligibly small compared to the applied loads of 1000 pounds.

With the balancing of joints A and F and the substantiation of the balance at G, the entire half ring is balanced. The total deflections in each degree of freedom can now be calculated and used to determine the unknown bending.
moments and tangential forces at A, F, and G. In order
to calculate these deflections the balancing equations (30),
(35), (39), and (45) give the magnitudes of the group opera-
tions involved while the equations determining the group
influence coefficients give the individual operations involved
in each group.

Table 15 gives the magnitude of all group displacements
from I to XIV implied in a unit application of any one group.
For example, row X in this table indicates that a unit
magnitude of group X (that is, \(x_X=1\)) is equivalent to the
sum of the effects of

\[X_{VII}=-2.1885, \quad X_{VIII}=-4.6760, \quad X_{X}=-0.16186, \quad X_{XII}=-0.19736, \quad X_{XIV}=-0.16186 \times 10^{-3} \text{ radian}, \quad X_{XV}=-4.6760 \times 10^{-3} \text{ inch}, \quad X_{XVI}=2.1885 \times 10^{-3} \text{ inch}. \]

During the solution of the problem the magnitude of group X which was
explicitly used was 11.184, as given in the last column of
table 15.

From table 15 the total magnitudes of each group opcra-
tion may be found. For example, the total magnitude of
group VI is:

\[x_{VI}=(1)(-3.3100)+(0.028042)(0)+(-0.16186)(11.184)+\]

\[(0.037474)(-51.515)+(0.0022427)(29.771)+\]

\[(0.024494)(-605.73)+(0.022857)(-44.646)\]

\[= -22.975 \quad (46)\]

The total displacement \(w_D\) is:

\[w_D=(x_{VI}) \times 10^{-3} = -22.975 \times 10^{-3} \text{ radian} \quad (46a)\]

Similarly the displacements of all points except point B
may be calculated from table 15 and are given in the last
row of that table.

Point B was displaced during the application of groups I,
II, III, and IV, and therefore the magnitude of its dis-
placement must be calculated as indicated in the following
equation:

\[w_B = (-0.026434 \times 10^{-3})(-605.73)+\]

\[(-0.21631 \times 10^{-3})(-11.444)+\]

\[(0.017847 \times 10^{-3})(-72.286)\]

\[= 40.825 \times 10^{-3} \text{ inch} \quad (47)\]

where the first number in each product is the magnitude of
\(w_n\) involved in each unit application of groups I, II, III,
and IV, respectively.

The total displacements used are assembled in equations
(47a).

These total displacements constitute the unknowns of the
system of equations given by the operations table and the
external forces; comparison between this growing-unit and
the matrix-calculus solutions given by equations (47a) and
(22), respectively, indicates good agreement for the dis-
placements. In fact, the forces and moments given by the
two methods differ by less than 1 percent and therefore are
given only for the matrix method (equation (23)).

Several general remarks are made about the growing-unit
method:

(a) In determining the influence coefficients and in balanc-
ing the external forces and moments, sets of equations with
the same left-hand sides but with different constant terms
are used several times. This simplifies solution of the equa-
tions and reduces the computational work considerably.

(b) In order to obtain sufficient accuracy of solution for
rings with many joints, calculating machines must be used;
five significant figures were carried throughout the calcu-
lations. However, on the simpler rings such as the circular
ring and the egg-shaped ring discussed previously, slide-rule
accuracy for determining the displacements in a combined
operation is probably sufficient for engineering purposes.

(c) A check on the influence coefficients for composite
bars is obtained by applying Maxwell's theorem of reciprocal
deflections. This is a valuable device for assuring accuracy
at each stage.

In applying the new relaxation procedures to this ring, it
would have been possible to use the general method de-
scribed for the egg-shaped ring, that is, to consider the resi-
duals after each operation and develop a satisfactory com-
bined operation to reduce as many residuals as possible.
However, the number of degrees of freedom involved in this
ring is large and, therefore, the number of residuals to be
RECOMMENDATIONS FOR NUMERICAL SOLUTION OF REINFORCED-PANEL AND FUSELAGE-RING PROBLEMS

The loading on the ring provides a clue to overcoming this difficulty. No external loads are applied at A, B, F, and G; moreover, A, F, and G are points along the center line of symmetry. Therefore, if in balancing D and E the balance at the other joints is preserved by suitable displacements, attention is fixed on the two joints D and E and the procedure described for the egg-shaped ring can be used effectively. It will be recognized that this procedure is essentially a combination of growing-unit and relaxation methods of solution.

In executing the proposed method the bar $\overline{ABCD}$, free only to displace radially at A and fixed at D, is considered first. The equations giving the displacements of A and B required to maintain balance of these points while joint C is rotated through $10^{-3}$ radian are:

$$
R_A = -7.1310\rho_A + 5.9020w_B - 4.5778\rho_B + 
\frac{14.662w_D}{10^{-3}} = 0
$$

$$
N_B = 5.9020\rho_A - 454.34w_B + 6.7238\rho_B - 
\frac{78.411w_D}{10^{-3}} = 0
$$

$$
R_B = -4.5778\rho_A + 6.7238w_B - 12.093\rho_B + 
\frac{0.55690w_D}{10^{-3}} = 0
$$

$$
T_B = -14.662\rho_A - 78.411w_B + 0.55690\rho_B - 
\frac{84.510w_D}{10^{-3}} = 0
$$

The solution to these equations is $\rho_A = 4.0195 \times 10^{-3}$ inch, $w_B = 0.32255 \times 10^{-3}$ radian, $\rho_B = 1.7842 \times 10^{-3}$ inch, and $w_B = 1.5277 \times 10^{-3}$ inch. These displacements combined with the unit rotation of C yield:

$$
N_C = -346.56 \text{ in.-lb}
$$

$$
R_C = -17.678 \text{ lb}
$$

$$
T_C = -40.663 \text{ lb}
$$

$$
N_D = -38.489 \text{ in.-lb}
$$

$$
R_D = 1.8576 \text{ lb}
$$

$$
T_D = 45.876 \text{ lb}
$$

The moment and tangential force introduced at A are not considered until the balancing of the ring is complete.

In a similar manner the forces and moments for unit radial and tangential displacements of C are determined, as shown in table 16. The forces and moments given by groups XV, XVI, and XVII constitute the influence coefficients for the displacements of C with A and D fixed and with joints A and B balanced. Use of these coefficients permits focusing of attention on C and D, the joints at which the external forces are applied when C is being balanced.

The forces and moments introduced at C and D when D is displaced a unit amount in each degree of freedom and when E, F, and G are displaced so as to maintain the balance thereof are calculated and shown in table 17.

Table 18 is an operations table consisting of unit magnitudes of group operations XV to XX. Table 19 is the relaxation table for this ring which uses these group operations. The external forces applied at C and D are given in the first row of table 19.

A discussion of each step in the relaxation process is given as follows:

**Step 1.**—Because of the antisymmetry of the loading and of the quasisymmetry of the ring about a horizontal axis operations $\text{XVI}=1$ and $\text{XXI}=-1$ are applied as a first approximation to the deflected shape. The forces and moments introduced are as given in the following table:

<table>
<thead>
<tr>
<th>Operation</th>
<th>Forces and moments</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{XVI}=1$</td>
<td>$N_C = -17.678$, $R_C = -40.663$, $T_C = 45.876$, $N_D = -38.489$, $R_D = 1.8576$, $T_D = 45.876$</td>
</tr>
<tr>
<td>$\text{XXI}=-1$</td>
<td>$N_C = -1.8576$, $R_C = 45.876$, $T_C = -38.489$, $N_D = 11.535$, $R_D = 13.393$, $T_D = 3.571$</td>
</tr>
</tbody>
</table>

Operation $\text{K}$ is used to balance the $R_C$-residual; the same operation reduces the other force residuals but introduces large $N_C$- and $N_D$-residuals.

**Step 2.**—In order to reduce these moment residuals an antisymmetrical combination of $w_C$ and $w_D$ is made, as shown as group operation L:

<table>
<thead>
<tr>
<th>Operation</th>
<th>Forces and moments</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{XV}=1$</td>
<td>$N_C = -386.29$, $R_C = -13.393$, $T_C = 0$, $N_D = -430.95$, $R_D = 13.393$, $T_D = 3.571$</td>
</tr>
<tr>
<td>$\text{XVII}=1$</td>
<td>$N_C = -386.29$, $R_C = -13.393$, $T_C = 0$, $N_D = -430.95$, $R_D = 13.393$, $T_D = 3.571$</td>
</tr>
</tbody>
</table>

However, use of operation L by itself would reintroduce large $R_C$- and $R_D$-residuals, and therefore operations K and L are combined so that the $R_C$-residual will be smaller and the $R_D$-residual eliminated, as shown in the following table:

<table>
<thead>
<tr>
<th>Operation</th>
<th>Forces and moments</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{XV}=1$</td>
<td>$N_C = -385.94$, $R_C = -13.393$, $T_C = 3.571$</td>
</tr>
<tr>
<td>$\text{XVII}=1$</td>
<td>$N_C = -386.29$, $R_C = -13.393$, $T_C = 0$, $N_D = -430.95$, $R_D = 13.393$, $T_D = 3.571$</td>
</tr>
</tbody>
</table>

The new force residuals introduced by operation M are less than 30 percent of the original residuals and, therefore, the rate of convergence is felt to be adequate.

**Step 3.**—The radial residuals at C and D have the same sign and, therefore, symmetrical displacements $v_C$ and $v_D$ are undertaken. It is seen that such a combination would introduce large tangential residuals at C and D. Therefore, a tangential displacement of C (D could have been chosen instead) such as to eliminate the $T_C$- and $T_D$-forces is undertaken. The forces and moments introduced by the individ-
ual displacements and by the combination are denoted as operation \( N \).

\[
\begin{array}{ccccccc}
\text{Operation} & \text{Forces and moments} & N_c & R_c & T_c & N_d & R_d & T_d \\
-0.93201\times(XVII) & 0.91279 & 6.8773 & 26.709 & -24.406 & 7.3316 & -26.587 \\
\sum \text{Operation } N=1 & 5.811 & -0.7653 & 0 & -14.730 & -1.1400 & -0.100 \\
\end{array}
\]

The use of operation \( N \) reduces substantially all the residuals except \( N_c \).

Step 4.—In order to reduce \( N_c \) and at the same time keep the \( T_c \) and \( T_p \) residuals small, a combination of groups XV and XX is made. Group XX is included since a force increasing the residual \( T_n \) would be introduced by the use of XV alone.

\[
\begin{array}{ccccccc}
\text{Operation} & \text{Forces and moments} & N_c & R_c & T_c & N_d & R_d & T_d \\
(XV)=1 & -340.55 & -17.676 & -40.902 & -38.489 & 1.8750 & 45.876 \\
0.91279\times(XX) & 41.875 & 12.579 & 45.616 & -38.610 & 11.597 & -45.876 \\
\sum \text{Operation } O=1 & -304.67 & -5.099 & 4.954 & -77.105 & 13.455 & 0 \\
\end{array}
\]

Steps 5 and 6.—After operation \( O \) is used, the largest force residual is approximately 6 percent of the applied forces and the moment residuals are small. It was considered desirable to reduce further the force residuals. Therefore, operation \( I \) was used again so as to reduce \( R_d \), the largest force residual, and then \( XVII \) was used so as to reduce the resulting \( T_c \)-residual. After this sixth step the largest residual of 4 percent of the external force is considered small enough.

A check table using the total displacements is used as a check on the accuracy of the combined operations and on the relaxation table. The total individual displacements are calculated as discussed in the previous two examples and are as follows:

\[
\begin{align*}
v_a &= -596.18 \times 10^{-3} \text{ in.} \\
v_B &= 36.779 \times 10^{-3} \text{ in.} \\
v_G &= 0.22394 \times 10^{-3} \text{ in.} \\
v_D &= -304.69 \times 10^{-3} \text{ in.} \\
v_C &= -14.32 \times 10^{-3} \text{ in.} \\
r_C &= 561.66 \times 10^{-3} \text{ in.} \\
u_C &= -114.61 \times 10^{-3} \text{ in.} \\
w_D &= -18.4 \times 10^{-3} \text{ in.} \\
w_C &= -133.66 \times 10^{-3} \text{ in.} \\
w_D &= 3.7242 \times 10^{-3} \text{ in.} \\
w_E &= 4.815 \times 10^{-3} \text{ in.} \\
w_F &= 30.057 \times 10^{-3} \text{ in.} \\
v_B &= -41.708 \times 10^{-3} \text{ in.} \\
v_G &= 59.979 \times 10^{-3} \text{ in.} \\
v_D &= 114.52 \times 10^{-3} \text{ in.}
\end{align*}
\]

It is pointed out that certain of these displacements differ considerably from those given by the exact solutions of the matrix-calculus and growing-unit methods, mainly because the relaxation solution is approximate and in it joint \( C \) is permitted to displace radially.

The unknown reactions given by the foregoing relaxation procedure are:

\[
\begin{align*}
N_a &= -2851.1 \text{ in.-lb} \\
T_a &= 380.21 \text{ lb} \\
N_p &= -140.16 \text{ in.-lb} \\
T_p &= -224.09 \text{ lb} \\
N_d &= -440.45 \text{ in.-lb} \\
T_d &= 648.18 \text{ lb}
\end{align*}
\]

Consideration of the equilibrium of the half ring gives:

\[
\begin{align*}
\sum F_B &= 380.21 + 224.09 - 648.18 = -43.88 \text{ lb} \\
\sum F_V &= 0 \\
\sum M_G &= -2851.1 + 380.21(57.941) - 440.45 + 140.16 - 224.09(7.0294) - 18,635 = -1331.8 \text{ in.-lb}
\end{align*}
\]

The moment equilibrium unbalance is approximately 7 percent of the applied moment and is considered satisfactory for engineering purposes. If a more accurate representation of the final deflected shape and consequently of the bending-moment diagram is desired, several more operations in the relaxation table could be undertaken and the residuals at \( C \) and \( D \) further reduced.

The bending-moment diagram given by the reactions of equation (51) is shown in figure 9 along with that of the exact solutions. The external unbalanced moment of 1331.8 inch-pounds is applied linearly along the ring as a distributed moment. If this unbalance is not distributed in this manner, it would be concentrated at either joint \( A \) or joint \( F \), depending on the direction in which the bending moments are calculated, and would lead to large errors in the bending moment in the neighborhood of that joint. It is seen from figure 9 that the agreement between the exact and relaxation solutions is good.

It is pointed out that, by slightly modifying the determination of the influence coefficient for joint \( D \) when \( E \) is fixed and \( F \) and \( G \) are free to displace radially, a table similar to table 17 could be established and solved by matrix-calculus methods. The slight modification is to make \( V_c = 0 \) in the equations corresponding to table 17. Such a solution is essentially the growing-unit method, except that the ring is combined from joints \( C \) and \( D \) to \( A \) and \( F \), respectively, rather than from \( A \) to \( F \). The total displacements in each degree of freedom will be the same in each approach.

**CONCLUSIONS**

This report contains recommendations as to the choice of the most expeditious method of solution of the simultaneous linear equations represented by the operations table and the external loads. The operations table is first established in
acccordance with Southwell’s suggestions and, together with the external loads, defines completely the problem of stress distribution in a reinforced panel or of the moment distribution in a fuselage ring. However, the following generalized suggestions can be made:

1. In most reinforced-panel problems the use of the relaxation procedure is advantageous.

2. Solution of the equations defining a reinforced-panel problem by means of the electric analogue is advisable when many closely related problems have to be investigated.

3. Ring problems are best solved by matrix methods.

4. In very complicated ring problems a combination of matrix methods with the growing-unit and relaxation methods may become advisable.

REFERENCES


TABLE 1.—OPERATIONS TABLE FOR REINFORCED PANEL

<table>
<thead>
<tr>
<th>Operation</th>
<th>Force</th>
<th>Y_A</th>
<th>Y_B</th>
<th>Y_C</th>
<th>Y_D</th>
<th>Y_E</th>
<th>Y_F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Force 1</td>
<td>y_A</td>
<td>-50.8</td>
<td>2.00</td>
<td>46.8</td>
<td>2.00</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>Force 2</td>
<td>y_B</td>
<td>46.8</td>
<td>2.00</td>
<td>101.6</td>
<td>4.00</td>
<td>46.8</td>
<td>2.00</td>
</tr>
<tr>
<td>Force 3</td>
<td>y_C</td>
<td>2.00</td>
<td>51.2</td>
<td>4.00</td>
<td>-101.4</td>
<td>2.00</td>
<td>51.2</td>
</tr>
<tr>
<td>Force 4</td>
<td>y_D</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>Force 5</td>
<td>y_E</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>Force 6</td>
<td>y_F</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

TABLE 2.—RELAXATION TABLE FOR REINFORCED PANEL—PROCEDURE 1

<table>
<thead>
<tr>
<th>Operation</th>
<th>Force</th>
<th>Y_A</th>
<th>Y_B</th>
<th>Y_C</th>
<th>Y_D</th>
<th>Y_E</th>
<th>Y_F</th>
</tr>
</thead>
<tbody>
<tr>
<td>External forces</td>
<td>y_A</td>
<td>-100</td>
<td>-10</td>
<td>20</td>
<td>-20</td>
<td>20</td>
<td>-20</td>
</tr>
<tr>
<td>Force 1</td>
<td>y_B</td>
<td>-110</td>
<td>-10</td>
<td>20</td>
<td>-20</td>
<td>20</td>
<td>-20</td>
</tr>
<tr>
<td>Force 2</td>
<td>y_C</td>
<td>-2.36</td>
<td>2.00</td>
<td>2.00</td>
<td>20</td>
<td>-20</td>
<td>20</td>
</tr>
<tr>
<td>Force 3</td>
<td>y_D</td>
<td>0</td>
<td>-5</td>
<td>-5</td>
<td>-5</td>
<td>-5</td>
<td>-5</td>
</tr>
<tr>
<td>Force 4</td>
<td>y_E</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Force 5</td>
<td>y_F</td>
<td>120</td>
<td>5</td>
<td>238</td>
<td>9</td>
<td>110</td>
<td>5</td>
</tr>
</tbody>
</table>

TABLE 3.—RELAXATION TABLE FOR REINFORCED PANEL—PROCEDURE 2

<table>
<thead>
<tr>
<th>Operation</th>
<th>Force</th>
<th>Y_A</th>
<th>Y_B</th>
<th>Y_C</th>
<th>Y_D</th>
<th>Y_E</th>
<th>Y_F</th>
</tr>
</thead>
<tbody>
<tr>
<td>External forces</td>
<td>y_A</td>
<td>-120</td>
<td>-10</td>
<td>20</td>
<td>-20</td>
<td>20</td>
<td>-20</td>
</tr>
<tr>
<td>Force 1</td>
<td>y_B</td>
<td>-110</td>
<td>-10</td>
<td>20</td>
<td>-20</td>
<td>20</td>
<td>-20</td>
</tr>
<tr>
<td>Force 2</td>
<td>y_C</td>
<td>-2.36</td>
<td>2.00</td>
<td>2.00</td>
<td>20</td>
<td>-20</td>
<td>20</td>
</tr>
<tr>
<td>Force 3</td>
<td>y_D</td>
<td>0</td>
<td>-5</td>
<td>-5</td>
<td>-5</td>
<td>-5</td>
<td>-5</td>
</tr>
<tr>
<td>Force 4</td>
<td>y_E</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Force 5</td>
<td>y_F</td>
<td>120</td>
<td>5</td>
<td>238</td>
<td>9</td>
<td>110</td>
<td>5</td>
</tr>
</tbody>
</table>

TABLE 4.—RELAXATION TABLE FOR REINFORCED PANEL—FIXED ENDS

<table>
<thead>
<tr>
<th>Operation</th>
<th>Force</th>
<th>Y_A</th>
<th>Y_B</th>
<th>Y_C</th>
<th>Y_D</th>
<th>Y_E</th>
<th>Y_F</th>
</tr>
</thead>
<tbody>
<tr>
<td>External forces</td>
<td>y_A</td>
<td>-120</td>
<td>-10</td>
<td>20</td>
<td>-20</td>
<td>20</td>
<td>-20</td>
</tr>
<tr>
<td>Force 1</td>
<td>y_B</td>
<td>-110</td>
<td>-10</td>
<td>20</td>
<td>-20</td>
<td>20</td>
<td>-20</td>
</tr>
<tr>
<td>Force 2</td>
<td>y_C</td>
<td>-2.36</td>
<td>2.00</td>
<td>2.00</td>
<td>20</td>
<td>-20</td>
<td>20</td>
</tr>
<tr>
<td>Force 3</td>
<td>y_D</td>
<td>0</td>
<td>-5</td>
<td>-5</td>
<td>-5</td>
<td>-5</td>
<td>-5</td>
</tr>
<tr>
<td>Force 4</td>
<td>y_E</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Force 5</td>
<td>y_F</td>
<td>120</td>
<td>5</td>
<td>238</td>
<td>9</td>
<td>110</td>
<td>5</td>
</tr>
</tbody>
</table>
### Table 5.—Operations Table for Reinforced Panel—Growing-Unit Method

<table>
<thead>
<tr>
<th>Operation</th>
<th>$Y_A$</th>
<th>$Y_B$</th>
<th>$Y_X$</th>
<th>$Y_Y$</th>
<th>$Y_J$</th>
<th>$Y_K$</th>
<th>$Y_N$</th>
<th>$Y_O$</th>
<th>$Y_S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) $F=1$</td>
<td>60.0</td>
<td>2</td>
<td>46.0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2) $F=1$</td>
<td>2</td>
<td>51.2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(3) $F=1$</td>
<td>46.8</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(4) $F=1$</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>110.4</td>
<td>2</td>
<td>3</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(5) $F=1$</td>
<td></td>
<td>51.2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(6) $F=1$</td>
<td></td>
<td>2</td>
<td>2</td>
<td>110.4</td>
<td>2</td>
<td>3</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(7) $F=1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Table 6.—Relaxation Table for Reinforced Panel—Growing-Unit Method

<table>
<thead>
<tr>
<th>Force</th>
<th>$Y_A$</th>
<th>$Y_B$</th>
<th>$Y_X$</th>
<th>$Y_Y$</th>
<th>$Y_J$</th>
<th>$Y_K$</th>
<th>$Y_N$</th>
<th>$Y_O$</th>
<th>$Y_S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>External loads</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-120 \times (10)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-2.37 \times (10)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-1.975 \times (19)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-0.444 \times (19)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-1.657 \times (19)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-0.742 \times (28)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-1.45 \times (25)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-0.925 \times (28)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Table 7.—Operations Table for Circular Ring

<table>
<thead>
<tr>
<th>Force and moments</th>
<th>$N_A$</th>
<th>$T_A$</th>
<th>$N_B$</th>
<th>$T_B$</th>
<th>$N_C$</th>
<th>$T_C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 \times 10^{-4}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2 \times 10^{-4}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$3 \times 10^{-4}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$4 \times 10^{-4}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$5 \times 10^{-4}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$6 \times 10^{-4}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$7 \times 10^{-4}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### Table 8.—Relaxation Table for Circular Ring

<table>
<thead>
<tr>
<th>Operation</th>
<th>External forces</th>
<th>$N_x$ (in.-lb)</th>
<th>$T_x$ (lb)</th>
<th>$N_y$ (in.-lb)</th>
<th>$R_y$ (lb)</th>
<th>$T_y$ (lb)</th>
<th>$N_z$ (in.-lb)</th>
<th>$R_z$ (lb)</th>
<th>$T_z$ (lb)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>-1.84</td>
<td>-8.19</td>
<td>-55.0</td>
<td>39.9</td>
<td>-30.1</td>
<td>-29.9</td>
<td>-0.1</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$-0.9078X(A)$</td>
<td>1.84</td>
<td>0</td>
<td>0.7</td>
<td>-0.1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$-23.9X(B)$</td>
<td>0</td>
<td>-8.75</td>
<td>-54.3</td>
<td>56.4</td>
<td>38.0</td>
<td>-53.1</td>
<td>-23.9</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$-0.3533X(C)$</td>
<td>0</td>
<td>0</td>
<td>-68.3</td>
<td>63.7</td>
<td>38.0</td>
<td>-53.1</td>
<td>-23.9</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$-77.8X(D)$</td>
<td>0</td>
<td>24</td>
<td>-8.6</td>
<td>-2.2</td>
<td>-2.2</td>
<td>-2.2</td>
<td>23.9</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$-2.98X(E)$</td>
<td>0</td>
<td>0</td>
<td>-549</td>
<td>27.5</td>
<td>27.5</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0</td>
<td>6</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Check-table results</td>
<td>0.0171</td>
<td>0.0010</td>
<td>0.5065</td>
<td>-0.0943</td>
<td>-0.0063</td>
<td>-0.0726</td>
<td>-0.0121</td>
<td></td>
</tr>
</tbody>
</table>

### Table 9.—Operations Table for Egg-Shaped Ring

<table>
<thead>
<tr>
<th>Operation</th>
<th>$R_x$ (lb)</th>
<th>$N_x$ (in.-lb)</th>
<th>$R_y$ (lb)</th>
<th>$T_x$ (lb)</th>
<th>$N_y$ (in.-lb)</th>
<th>$R_z$ (lb)</th>
<th>$T_y$ (lb)</th>
<th>$N_z$ (in.-lb)</th>
<th>$R_z$ (lb)</th>
<th>$T_z$ (lb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) $x=10^{-6}$</td>
<td>-3.3412</td>
<td>8.92219</td>
<td>-2.25014</td>
<td>3.46771</td>
<td>-61.242</td>
<td>8.10267</td>
<td>-5.2485</td>
<td>-592</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(2) $x=10^{-8}$</td>
<td>-3.3412</td>
<td>8.92219</td>
<td>-2.25014</td>
<td>3.46771</td>
<td>-61.242</td>
<td>8.10267</td>
<td>-5.2485</td>
<td>-592</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(3) $x=10^{-10}$</td>
<td>-3.3412</td>
<td>8.92219</td>
<td>-2.25014</td>
<td>3.46771</td>
<td>-61.242</td>
<td>8.10267</td>
<td>-5.2485</td>
<td>-592</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(4) $x=10^{-10}$</td>
<td>-3.3412</td>
<td>8.92219</td>
<td>-2.25014</td>
<td>3.46771</td>
<td>-61.242</td>
<td>8.10267</td>
<td>-5.2485</td>
<td>-592</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(5) $x=10^{-10}$</td>
<td>-3.3412</td>
<td>8.92219</td>
<td>-2.25014</td>
<td>3.46771</td>
<td>-61.242</td>
<td>8.10267</td>
<td>-5.2485</td>
<td>-592</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(6) $x=10^{-10}$</td>
<td>-3.3412</td>
<td>8.92219</td>
<td>-2.25014</td>
<td>3.46771</td>
<td>-61.242</td>
<td>8.10267</td>
<td>-5.2485</td>
<td>-592</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(7) $x=10^{-10}$</td>
<td>-3.3412</td>
<td>8.92219</td>
<td>-2.25014</td>
<td>3.46771</td>
<td>-61.242</td>
<td>8.10267</td>
<td>-5.2485</td>
<td>-592</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(8) $x=10^{-10}$</td>
<td>-3.3412</td>
<td>8.92219</td>
<td>-2.25014</td>
<td>3.46771</td>
<td>-61.242</td>
<td>8.10267</td>
<td>-5.2485</td>
<td>-592</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

### Table 10.—Relaxation Table for Egg-Shaped Ring

<table>
<thead>
<tr>
<th>Operation</th>
<th>$R_x$ (lb)</th>
<th>$N_x$ (in.-lb)</th>
<th>$R_y$ (lb)</th>
<th>$T_x$ (lb)</th>
<th>$N_y$ (in.-lb)</th>
<th>$R_z$ (lb)</th>
<th>$T_y$ (lb)</th>
<th>$N_z$ (in.-lb)</th>
<th>$R_z$ (lb)</th>
<th>$T_z$ (lb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>External forces</td>
<td>-500</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-500</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>500</td>
<td>-1330</td>
<td>402</td>
<td>-502</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>500</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>-1330</td>
<td>402</td>
<td>-502</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>500</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$-0.00X(0)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$130X(F)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$20X(1)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$400X(1)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$9.42X(J)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>-0.455</td>
<td>300.771</td>
<td>-2.251</td>
<td>0.038</td>
<td>-290.495</td>
<td>-6.302</td>
<td>0.251</td>
<td>-0.279</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### TABLE 11.—OPERATIONS TABLE FOR OVAL-SHAPED RING

<table>
<thead>
<tr>
<th>Operation</th>
<th>R_x (lb)</th>
<th>N_x (in.-lb)</th>
<th>R_y (lb)</th>
<th>T_y (lb)</th>
<th>N_y (in.-lb)</th>
<th>R_z (lb)</th>
<th>T_z (lb)</th>
<th>N_z (in.-lb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_0 = 10^{-3} ) in.</td>
<td>-7.1310</td>
<td>5.9020</td>
<td>-4.6778</td>
<td>14.602</td>
<td>-38.489</td>
<td>1.8576</td>
<td>45.876</td>
<td>-1.8576</td>
</tr>
<tr>
<td>( \Phi_0 = 10^{-3} ) radian</td>
<td>-454.34</td>
<td>-6.7248</td>
<td>-12.6228</td>
<td>-78.411</td>
<td>-38.469</td>
<td>-2.2277</td>
<td>10.731</td>
<td>-2.2277</td>
</tr>
<tr>
<td>( x_0 = 10^{-3} ) in.</td>
<td>-3.5778</td>
<td>-28.411</td>
<td>10.839</td>
<td>-65.9600</td>
<td>-1.8576</td>
<td>0</td>
<td>0</td>
<td>-1.8576</td>
</tr>
<tr>
<td>( \Phi_0 = 10^{-3} ) radian</td>
<td>-38.489</td>
<td>-1.8576</td>
<td>45.876</td>
<td>-42.247</td>
<td>0</td>
<td>0</td>
<td>-77.623</td>
<td>-38.489</td>
</tr>
<tr>
<td>( x_0 = 10^{-2} ) in.</td>
<td>0</td>
<td>0</td>
<td>1.8576</td>
<td>45.876</td>
<td>13.781</td>
<td>0</td>
<td>49.074</td>
<td>-77.623</td>
</tr>
<tr>
<td>( \Phi_0 = 10^{-2} ) radian</td>
<td>45.876</td>
<td>13.781</td>
<td>-432.37</td>
<td>-77.623</td>
<td>0</td>
<td>0</td>
<td>-100.34</td>
<td>45.876</td>
</tr>
</tbody>
</table>

### TABLE 12.—GROUP OPERATIONS IN GROWING-UNIT METHOD FOR SEGMENT ABC

\[ N_x = 454.34 \text{ in} + 6.7239 \text{ lb} = 78.411 \text{ in} \text{ (R.H.S. in } N_x \text{ equation) = 0} \]
\[ R_x = 6.7239 \text{ in} + 12.956 \text{ lb} + 6.5569 \text{ lb} = 78.411 \text{ in} \text{ (R.H.S. in } R_x \text{ equation) = 0} \]

<table>
<thead>
<tr>
<th>Group</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( x_0 = 10^{-3} ) in.</td>
<td>( \Phi_0 = 10^{-3} ) radian</td>
<td>( x_0 = 10^{-2} ) in.</td>
<td>( \Phi_0 = 10^{-2} ) radian</td>
</tr>
</tbody>
</table>

- **Operation**: \( x_0 = 10^{-3} \) in.
- **R_x (lb)**: 5.9020
- **N_x (in.-lb)**: -38.469
- **R_y (lb)**: 1.8576
- **T_y (lb)**: -2.2277
- **N_y (in.-lb)**: 45.876
- **R_z (lb)**: 13.781
- **T_z (lb)**: 0
- **N_z (in.-lb)**: 49.074

---

**Resultant forces and moments:**

- **R_x (lb)**: -2.6018
- **R_y (lb)**: 10.699
- **R_z (lb)**: -2.3648
- **T_x (lb)**: 0
- **T_y (lb)**: 0
- **T_z (lb)**: 0
- **N_x (in.-lb)**: -1.8871
- **N_y (in.-lb)**: 10.615
- **N_z (in.-lb)**: 45.876

---

**Forms and**

\[ (10)^3 \text{ displacements of joint } B: \]

- **R_x (lb)**: -2.6018
- **R_y (lb)**: 10.699
- **R_z (lb)**: -2.3648
- **T_x (lb)**: 0
- **T_y (lb)**: 0
- **T_z (lb)**: 0
- **N_x (in.-lb)**: -1.8871
- **N_y (in.-lb)**: 10.615
- **N_z (in.-lb)**: 45.876
### TABLE 13.—GROUP OPERATIONS IN GROWING-UNIT METHOD FOR SEGMENT ABCD

<table>
<thead>
<tr>
<th>Group</th>
<th>V</th>
<th>VI</th>
<th>VII</th>
<th>VIII</th>
</tr>
</thead>
<tbody>
<tr>
<td>Displacement</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| Operation | | | | |
| | | | | |

### TABLE 14.—GROUP OPERATIONS IN GROWING-UNIT METHOD FOR SEGMENT ABCDE

<table>
<thead>
<tr>
<th>Group</th>
<th>IX</th>
<th>X</th>
<th>XI</th>
<th>XII</th>
</tr>
</thead>
<tbody>
<tr>
<td>Displacement</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| Operation | | | | |
| | | | | |

### Notes

1. \( NR = -346.88 \text{ radian} - 16.69 \text{ in.} \) \( R = -43.745 \text{ lb} \)
2. \( RD = -10.699 \text{ in.} - 16.69 \text{ in.} \) \( R = -43.745 \text{ lb} \)
3. \( TD = -43.745 \text{ in.} - 16.69 \text{ in.} \) \( R = -43.745 \text{ lb} \)
TABLE 15.—DETERMINATION OF TOTAL DISPLACEMENTS IN GROWING-UNIT METHOD

<table>
<thead>
<tr>
<th>Related tables</th>
<th>w_p, u_p, and u_p from table 19</th>
<th>(II), (III), and (V) from table 12 or 16</th>
<th>Magnitudes of group displacements explicitly used in balancing</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prime displacement</td>
<td>r_x=10^-4 in.</td>
<td>r_e=10^-4 radian</td>
<td>r_c=10^-4 in.</td>
</tr>
<tr>
<td>Group operations</td>
<td>I</td>
<td>II</td>
<td>III</td>
</tr>
<tr>
<td>I</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>II</td>
<td>0.231407</td>
<td>-0.58843</td>
<td>0.14430</td>
</tr>
<tr>
<td>III</td>
<td>0.544927</td>
<td>0.606211</td>
<td>-0.596672</td>
</tr>
<tr>
<td>IV</td>
<td>-0.095469</td>
<td>0.853465</td>
<td>0.583933</td>
</tr>
<tr>
<td>V</td>
<td>-0.017316</td>
<td>1.75022</td>
<td>2.9219</td>
</tr>
<tr>
<td>VI</td>
<td>0.019749</td>
<td>-0.76475</td>
<td>-0.34536</td>
</tr>
<tr>
<td>VII</td>
<td>0.019749</td>
<td>0.78438</td>
<td>0.44799</td>
</tr>
<tr>
<td>VIII</td>
<td>-0.0898407</td>
<td>-0.502111</td>
<td>0.49175</td>
</tr>
<tr>
<td>IX</td>
<td>-0.0017462</td>
<td>0.05456</td>
<td>-0.47063</td>
</tr>
<tr>
<td>Total displacements</td>
<td>-605.73</td>
<td>-11.444</td>
<td>664.55</td>
</tr>
</tbody>
</table>

TABLE 16.—GROUP OPERATIONS IN RELAXATION METHOD FOR SEGMENT ABCD

\[
\begin{align*}
R_A &= 1.1316 w + 5.9020 x - 4.5778 y + 14.662 w = (R.H.S. in R_A) = 0 \\
N_A &= 5.9020 w + 45.349 z - 8.7326 y + 78.411 w = (R.H.S. in N_A) = 0 \\
R_B &= 4.5778 w + 12.093 z = (R.H.S. in R_B) = 0 \\
T_B &= 14.662 w + 78.411 z = (R.H.S. in T_B) = 0 \\
\end{align*}
\]

<table>
<thead>
<tr>
<th>Related tables</th>
<th>w_p, u_p, and u_p from table 19</th>
<th>(VI), (VII), and (VIII) from table 14 or 15</th>
<th>(IX), (XI), and (XII) from table 15</th>
<th>Magnitudes of group displacements explicitly used in balancing</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prime displacement</td>
<td>r_x=10^-4 in.</td>
<td>r_e=10^-4 radian</td>
<td>r_c=10^-4 in.</td>
<td>( r_1 = 1 )</td>
</tr>
<tr>
<td>Group operations</td>
<td>I</td>
<td>II</td>
<td>III</td>
<td>IV</td>
</tr>
<tr>
<td>I</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>II</td>
<td>0.051804</td>
<td>-0.12578</td>
<td>-0.03518</td>
<td>1</td>
</tr>
<tr>
<td>III</td>
<td>0.028042</td>
<td>-0.16186</td>
<td>0.037474</td>
<td>1</td>
</tr>
<tr>
<td>Total displacements</td>
<td>-605.73</td>
<td>6.2331</td>
<td>-42.020</td>
<td>32.511</td>
</tr>
</tbody>
</table>

TABLE 16.—GROUP OPERATIONS IN RELAXATION METHOD FOR SEGMENT ABCD

\[
\begin{align*}
R_A &= 1.1316 w + 5.9020 x - 4.5778 y + 14.662 w = (R.H.S. in R_A) = 0 \\
N_A &= 5.9020 w + 45.349 z - 8.7326 y + 78.411 w = (R.H.S. in N_A) = 0 \\
R_B &= 4.5778 w + 12.093 z = (R.H.S. in R_B) = 0 \\
T_B &= 14.662 w + 78.411 z = (R.H.S. in T_B) = 0 \\
\end{align*}
\]
TABLE 17.—GROUP OPERATIONS IN RELAXATION METHOD FOR SEGMENT CDEF-G

\[ \begin{align*}
N_p &= -649.24 \text{ in.-lb} - 18.056 \text{ in.} - 67.081 \text{ in.} = -5.920 \text{ in.-lb} + 16.025 \text{ in.-lb} \\
R_p &= 18.056 \text{ in.-lb} + 40.020 \text{ in.-lb} = -67.081 \text{ in.-lb} - 14.602 \text{ in.-lb} + 1.892 \text{ in.-lb} + 1.954 \text{ in.-lb} \\
T_p &= -67.081 \text{ in.-lb} - 40.020 \text{ in.-lb} = 136.570 \text{ in.-lb} - 14.602 \text{ in.-lb} + 1.892 \text{ in.-lb} - 1.954 \text{ in.-lb} \\
R_g &= 18.056 \text{ in.-lb} + 4.575 \text{ in.-lb} - 16.025 \text{ in.-lb} = -7.319 \text{ in.-lb} \\
\end{align*} \]

\[ \begin{align*}
N &= 18.056 \text{ in.-lb} + 1.305 \text{ in.-lb} - 1.892 \text{ in.-lb} = -1.388 \text{ in.-lb} \\
R &= 18.056 \text{ in.-lb} + 1.305 \text{ in.-lb} - 1.892 \text{ in.-lb} = -1.388 \text{ in.-lb} \\
\end{align*} \]

<table>
<thead>
<tr>
<th>Group</th>
<th>Displacement</th>
<th>Operation</th>
<th>XVII</th>
<th>XIX</th>
<th>XX</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-10^3) X right-hand side in equation:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$N$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R_0$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ \begin{align*}
N_p &= 1.8576 \\
R_p &= 45.876 \\
T_p &= -1.8576 \\
R_0 &= 0 \\
\end{align*} \]

\[ \begin{align*}
&\text{Forces and moments:} \\
&N_c, \text{ in.-lb} \\
&K_c, \text{ lb} \\
&T_c, \text{ lb} \\
&N_o, \text{ in.-lb} \\
&R_o, \text{ lb} \\
&T_o, \text{ lb} \\
\end{align*} \]

\[ \begin{align*}
&\text{XVII} \quad \text{XIX} \quad \text{XX} \\
&10^{-3} \text{ radian} \quad 10^{-3} \text{ in.} \quad 10^{-3} \text{ in.} \\
&\begin{array}{cccc}
-38.489 & -1.8576 & 45.876 \\
45.876 & 0 & 0 \\
6.49270 & 0.19069 & 49.944 \\
9.74456 & 0.22018 & 49.944 \\
1.1111 & 0.32802 & 45.876 \\
0 & 0 & 0 \\
-2.7533 & -0.5706 & 45.876 \\
\end{array} \\
\end{align*} \]

TABLE 18.—GROUP-OPERATIONS TABLE FOR RELAXATION METHOD

[Forces and moments at joints A, B, E, F, and G are zero for all operations]

<table>
<thead>
<tr>
<th>Forces and moments</th>
<th>Operation</th>
<th>N_c (in.-lb)</th>
<th>K_c (lb)</th>
<th>T_c (lb)</th>
<th>N_o (in.-lb)</th>
<th>R_o (lb)</th>
<th>T_o (lb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-10)^3 X displacements of joints E, F, and G:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$w_p$, in.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T_p$, in.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R_p$, in.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R_0$, in.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ \begin{align*}
&\text{Forces and moments:} \\
&N_c, \text{ in.-lb} \\
&K_c, \text{ lb} \\
&T_c, \text{ lb} \\
&N_o, \text{ in.-lb} \\
&R_o, \text{ lb} \\
&T_o, \text{ lb} \\
\end{align*} \]

\[ \begin{align*}
&\text{XVII} \quad \text{XIX} \quad \text{XX} \\
&10^{-3} \text{ radian} \quad 10^{-3} \text{ in.} \quad 10^{-3} \text{ in.} \\
&\begin{array}{cccc}
-38.489 & -1.8576 & 45.876 \\
45.876 & 0 & 0 \\
6.49270 & 0.19069 & 49.944 \\
9.74456 & 0.22018 & 49.944 \\
1.1111 & 0.32802 & 45.876 \\
0 & 0 & 0 \\
-2.7533 & -0.5706 & 45.876 \\
\end{array} \\
\end{align*} \]
## TABLE 19. RELAXATION TABLE FOR GROUP OPERATIONS

<table>
<thead>
<tr>
<th>Step</th>
<th>Operation</th>
<th>( R_D ) (lb)</th>
<th>( T_D ) (lb)</th>
<th>( N_R ) (in.-lb)</th>
<th>( R_F ) (lb)</th>
<th>( T_F ) (lb)</th>
<th>( N_F ) (in.-lb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>External loads</td>
<td>-260</td>
<td>-0.71</td>
<td>1218</td>
<td>38</td>
<td>252</td>
<td>1218</td>
</tr>
<tr>
<td>2.</td>
<td>-18.4X(M)</td>
<td>0</td>
<td>0</td>
<td>500</td>
<td>-1</td>
<td>1164</td>
<td>-44</td>
</tr>
<tr>
<td>3.</td>
<td>214X(N)</td>
<td>0</td>
<td>164</td>
<td>1946</td>
<td>45</td>
<td>164</td>
<td>3050</td>
</tr>
<tr>
<td>4.</td>
<td>-0.08X(O)</td>
<td>0</td>
<td>21</td>
<td>1246</td>
<td>45</td>
<td>0</td>
<td>314</td>
</tr>
<tr>
<td>5.</td>
<td>-15.7X(K)</td>
<td>0</td>
<td>21</td>
<td>306</td>
<td>35</td>
<td>50</td>
<td>210</td>
</tr>
<tr>
<td>6.</td>
<td>-0.70X(XVII)</td>
<td>0</td>
<td>21</td>
<td>306</td>
<td>35</td>
<td>50</td>
<td>210</td>
</tr>
<tr>
<td>Check table</td>
<td>0.001</td>
<td>0.310</td>
<td>0.025</td>
<td>0.156</td>
<td>327</td>
<td>224.7</td>
<td>0.047</td>
</tr>
</tbody>
</table>

## Forces and moments

<table>
<thead>
<tr>
<th>Step</th>
<th>Operation</th>
<th>( R_D ) (lb)</th>
<th>( T_D ) (lb)</th>
<th>( N_R ) (in.-lb)</th>
<th>( R_F ) (lb)</th>
<th>( T_F ) (lb)</th>
<th>( N_F ) (in.-lb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>External loads</td>
<td>-260</td>
<td>-0.71</td>
<td>1218</td>
<td>38</td>
<td>252</td>
<td>1218</td>
</tr>
<tr>
<td>2.</td>
<td>-18.4X(M)</td>
<td>0</td>
<td>0</td>
<td>500</td>
<td>-1</td>
<td>1164</td>
<td>-44</td>
</tr>
<tr>
<td>3.</td>
<td>214X(N)</td>
<td>0</td>
<td>164</td>
<td>1946</td>
<td>45</td>
<td>164</td>
<td>3050</td>
</tr>
<tr>
<td>4.</td>
<td>-0.08X(O)</td>
<td>0</td>
<td>21</td>
<td>1246</td>
<td>45</td>
<td>0</td>
<td>314</td>
</tr>
<tr>
<td>5.</td>
<td>-15.7X(K)</td>
<td>0</td>
<td>21</td>
<td>306</td>
<td>35</td>
<td>50</td>
<td>210</td>
</tr>
<tr>
<td>6.</td>
<td>-0.70X(XVII)</td>
<td>0</td>
<td>21</td>
<td>306</td>
<td>35</td>
<td>50</td>
<td>210</td>
</tr>
<tr>
<td>Check table</td>
<td>0.001</td>
<td>0.310</td>
<td>0.025</td>
<td>0.156</td>
<td>327</td>
<td>224.7</td>
<td>0.047</td>
</tr>
</tbody>
</table>
Positive directions of axes and angles (forces and moments) are shown by arrows.

<table>
<thead>
<tr>
<th>Axis</th>
<th>Designation</th>
<th>Symbol</th>
<th>Force (parallel to axis) symbol</th>
<th>Moment about axis</th>
<th>Angle</th>
<th>Velocities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Longitudinal</td>
<td>X</td>
<td>Y</td>
<td>Z</td>
<td>Roll</td>
<td>Pitch</td>
<td>Yaw</td>
</tr>
<tr>
<td>Lateral</td>
<td>X</td>
<td>Y</td>
<td>Z</td>
<td>Pitch</td>
<td>Roll</td>
<td>Yaw</td>
</tr>
<tr>
<td>Normal</td>
<td>Y</td>
<td>Z</td>
<td>X</td>
<td>Roll</td>
<td>Pitch</td>
<td>Yaw</td>
</tr>
</tbody>
</table>

Absolute coefficients of moment:

- \( C_\nu = \frac{L}{qS} \) (rolling)
- \( C_\alpha = \frac{M}{qS} \) (pitching)
- \( C_\phi = \frac{N}{qS} \) (yawing)

Angle of set of control surface (relative to neutral position), \( \phi \). (Indicate surface by proper subscript.)

4. PROPELLER SYMBOLS

- \( P \) - Power, absolute coefficient \( C_P = \frac{P}{\rho n^3 D^5} \)
- \( G_P \) - Speed-power coefficient \( = \frac{\sqrt{P V_s}}{\rho n^2} \)
- \( \eta \) - Efficiency
- \( n \) - Revolutions per second, rps
- \( \phi \) - Effective helix angle \( = \tan^{-1} \left( \frac{V}{2\pi n} \right) \)

5. NUMERICAL RELATIONS

| 1 hp = 76.04 kg-m/s = 550 ft-lb/sec |
| 1 metric horsepower = 0.9863 hp |
| 1 mph = 0.4470 mps |
| 1 mps = 2.2369 mph |

| 1 lb = 0.4536 kg |
| 1 kg = 2.2046 lb |
| 1 mi = 1,609.35 m = 5,280 ft |
| 1 m = 3.2808 ft |