SUMMARY

A method is presented for determining the time-dependent flow over a rectangular wing moving with a supersonic forward speed and undergoing small vertical distortions expressible as polynomials involving spanwise and chordwise distances. The solution for the velocity potential is presented in a form analogous to that for steady supersonic flow having the familiar "reflected area" concept discovered by E. Parshall. Particular attention is paid to indicial-type motions and results are expressed in terms of generalized indicial forces. Numerical results for Mach numbers equal to 1.1 and 1.2 are given for polynomials of the first and fifth degree in the chordwise and spanwise directions, respectively, on a wing having an aspect ratio of 4.

INTRODUCTION

One of the basic problems arising in the analysis of wing flutter boundaries is the calculation of the aerodynamic forces on wings undergoing small but arbitrary spanwise and chordwise distortions. When the wing aspect ratio is large (actually, when the distance between spanwise nodal lines is large), these forces are usually estimated by some strip theory in which the loading on each spanwise section is approximated from that on a two-dimensional wing having the same chordwise distortion. This report is concerned with low-aspect-ratio rectangular wings for which tip effects are important and the full three-dimensional theory must be used.

The exact linearized solution for the forces on thin rectangular wings (limited, however, to the range where effective aspect ratio \( \sqrt{A^2 - 1} \leq 1 \)) traveling at supersonic speeds has been presented by both Gardner (ref. 1) and Miles (ref. 2) in terms of multiple integrals involving arbitrary surface undulations. However, the use of such solutions in evaluating, numerically say, the forces induced by arbitrary surface undulations presents some difficulties. It is the purpose of this report to discuss certain techniques that can simplify the labor involved in these calculations and to present numerical tables for the forces induced by a class of surface deformations, a class general enough to represent the first few mode shapes of rectangular plates.

Mathematically the problem is to find and analyze a solution to the four-dimensional wave equation

\[ \varphi_{xx} + \varphi_{yy} + \varphi_{zz} + \frac{1}{a_0^2} \varphi_{tt} = 0 \]  

(1a)

(where \( a_0 \) is the speed of sound, \( t' \) is the time, and \( x, y, z \) are space coordinates) that satisfies the appropriate boundary conditions. The particular form of the solution to be analyzed differs from those presented by Gardner and Miles but its development is based on the method due to Gardner.

Hadamard (ref. 4) studied a generalized form of equation (1a) in which the number of dimensions was arbitrary. His solutions to these generalized equations are fundamentally different, depending on whether the total number of dimensions is odd or even. In fact, the methods Hadamard developed apply directly only to equations for which the total number of dimensions is odd. Solutions for the even cases (such as eq. (1a)) are determined by a "method of descent"; that is, the solution for the next higher odd-dimensioned equation is found and then reduced by (made independent of) one dimension. It is apparent, however, that such a technique is in itself by no means unique. Thus, Hadamard found the solution to equation (1a) by descending from a solution to the equation

\[ \varphi_{xx} + \varphi_{yy} + \varphi_{zz} + \frac{1}{a_0^2} \varphi_{tt} = 0 \]  

(1b)

but there are many other partial differential equations and groups of partial differential equations governing a five-dimensional \((x, y, z, \xi, t)\) space all of which satisfy equation (1a) in a plane \( \xi = \text{constant} \). Gardner discovered a set of equations containing equation (1a) in a \( \xi = \text{constant} \) plane which are simpler than equation (1a) in that solutions could be found and adapted to the boundary conditions for time-dependent motion by methods well known to aerodynamicists who have studied the flow about wings in steady supersonic flight. This is the essential part of Gardner’s contribution and it represents the technique upon which the development of the solution presented in this report is based. Actually, Gardner first applied a Lorentz transformation to equation (1a) and then used his method outlined above. The application of such a transformation is unnecessary and has the disadvantage that the resulting coordinates have lost their direct physical significance. We will apply Gardner’s method of descent directly to equation (1a) and then proceed to analyze the solutions so obtained.

In order to simplify the analysis as much as possible, we will limit solutions to the plane of the wing, and, further, consider only indicial-type boundary conditions; in other words, unsteady motions in which the wing attains instantaneously, at the time zero, a certain spanwise and chordwise distortion which is thereafter fixed. It is well known that the transient responses to these indicial motions can be
used, in a superposition integral, to obtain responses to many other types of unsteady motion; in particular, responses to the harmonic oscillations of nonrigid wings.

Finally, the principal interpretation of the results will be made in terms of generalized forces, since these can be used directly in either flutter or gust studies, and it will be shown that the amount of labor required to calculate such forces is reduced by using reciprocity relations derived from the general theorems presented in reference 5.

**LIST OF IMPORTANT SYMBOLS**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$A$</td>
<td>aspect ratio</td>
</tr>
<tr>
<td>$a_0$</td>
<td>speed of sound</td>
</tr>
<tr>
<td>$B(x,y,z)$</td>
<td>beta function (See eq. (2a).)</td>
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<tr>
<td>$B_{1-2}(x,y,z)$</td>
<td>complete beta function (See eq. (2b).)</td>
</tr>
<tr>
<td>$C_{1}(x,y,z)$</td>
<td>influence function for effect of side edge (See eq. (A10).)</td>
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<tr>
<td>$C_L$</td>
<td>lift coefficient, $\frac{L}{q_0 S}$</td>
</tr>
<tr>
<td>$C_{\alpha_{L}}$</td>
<td>indicial lift coefficient due to angle-of-attack change, without pitching, $C_{\alpha_{L}} = \frac{\partial C_L}{\partial \alpha}$</td>
</tr>
<tr>
<td>$C_{\alpha_{L}}'$</td>
<td>indicial lift coefficient due to pitching for a wing rotating about its leading edge, $C_{\alpha_{L}}' = \frac{\partial C_L}{\partial \alpha}$</td>
</tr>
<tr>
<td>$C_m$</td>
<td>pitching-moment coefficient, positive when trailing edge tends to sink relative to leading edge, $C_m = \frac{\partial M}{\partial \alpha}$</td>
</tr>
<tr>
<td>$C_{\alpha_m}$</td>
<td>indicial pitching-moment coefficient due to angle-of-attack change (without pitching) measured about the leading edge, $C_{\alpha_m} = \frac{\partial C_m}{\partial \alpha}$</td>
</tr>
<tr>
<td>$C_{\alpha_m}'$</td>
<td>indicial pitching-moment coefficient due to pitching measured about the leading edge for a wing rotating about its leading edge, $C_{\alpha_m}' = \frac{\partial C_m}{\partial \alpha}$</td>
</tr>
<tr>
<td>$c$</td>
<td>wing chord</td>
</tr>
<tr>
<td>$F_{\alpha}(\theta)$</td>
<td>generalized indicial force coefficient (See eq. (36).)</td>
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<tr>
<td>$f_{\alpha}(\theta)$</td>
<td>generalized indicial force coefficient (See eq. (37).)</td>
</tr>
<tr>
<td>$h(x,y,\xi)$</td>
<td>distance of wing camber line from $z=0$ plane</td>
</tr>
<tr>
<td>$M$</td>
<td>Mach number</td>
</tr>
<tr>
<td>$\Delta p$</td>
<td>loading coefficient (pressure on the lower surface minus pressure on the upper surface divided by free-stream dynamic pressure)</td>
</tr>
<tr>
<td>$\binom{n}{m}$</td>
<td>binomial coefficient, $\binom{n}{m} = \frac{n!}{m!(n-m)!}$</td>
</tr>
<tr>
<td>$q$</td>
<td>dimensionless rate of pitching, $\frac{\partial \theta}{\partial \xi}$</td>
</tr>
<tr>
<td>$q_0$</td>
<td>free-stream dynamic pressure, $\frac{1}{2} \rho M^2 s^2$</td>
</tr>
<tr>
<td>$q_r$</td>
<td>generalized coordinate</td>
</tr>
<tr>
<td>$Q_{r}$</td>
<td>generalized force corresponding to the generalized coordinate $q_r$.</td>
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**STATEMENT OF THE PROBLEM**

**THE GOVERNING EQUATION**

Assuming a wing's vertical motion is of such a nature that the velocities induced in the fluid are small relative to the magnitude of the wing's steady forward motion, the normalized form of equation (1a)

$$\varphi_{xy} + \varphi_{yy} + \varphi_{xx} - \varphi_{tt} = 0$$

(1c)
where \( t = ad' \), can be used as the governing partial differential equation of the flow field. This equation applies to the determination of the velocity potential when the body or wing in question moves through the fluid, the axes remaining fixed with respect to the still fluid infinitely distant from the origin. For convenience we place the wing leading edge on the \( y \) axis at \( t = 0 \) and the side edge on the \( x \) axis. The wing flies at a constant forward (in the negative \( z \) direction) speed so that at subsequent times the leading edge lies along the line \( z = -Mt \), where \( M \) is the Mach number, and the side edge moves along the \( z \) axis as shown in figure 1.

**Figure 1.—Wing in fixed coordinates system.**

### The Boundary Conditions

The fluid velocity normal to the surface of a solid moving in a frictionless fluid must be zero. If the equation of the solid's surface is represented by

\[
G(x, y, z, t') = 0
\]

this boundary condition can be expressed mathematically, in terms of the coordinate system used in equation (1c), as

\[
\frac{\partial G}{\partial t'} + \frac{\partial G}{\partial x} \frac{\partial x}{\partial t'} + \frac{\partial G}{\partial y} \frac{\partial y}{\partial t'} + \frac{\partial G}{\partial z} \frac{\partial z}{\partial t'} = 0
\]

Consider a thin surface near the \( z = 0 \) plane. The equation of the camber line of this surface can then be expressed in the form

\[
G(x, y, z, t') = z - h(x, y, t') = 0
\]

and, assuming that thickness and lifting effects can be separated linearly, the boundary condition for the camber line becomes

\[
\frac{\partial h}{\partial t'} + \frac{\partial G}{\partial x} \frac{\partial x}{\partial t'} + \frac{\partial G}{\partial y} \frac{\partial y}{\partial t'} + \frac{\partial G}{\partial z} \frac{\partial z}{\partial t'} = 0
\]

If the derivatives of \( h \) with respect to each of the coordinates are small, the two middle terms can be neglected and the expression for the boundary condition reduces to

\[
\frac{\partial h}{\partial t'} = w_u(x, y, t')
\]

We wish to simulate a rectangular wing deformed indicially by bending in the spanwise and chordwise directions. For this purpose, on the portion of the \( z = 0 \) plane occupied by the wing plan form, the vertical velocity, which determines the wing shape according to the previous equation, is assumed to have the form

\[
w_u = \begin{cases} 
0 & t < 0 \\
\sum \sum a_{l,n} \left( \frac{x+Mt}{c} \right)^l \left( \frac{y}{c} \right)^n & t > 0
\end{cases}
\]

where \( c \) is chord length, \( a_{l,n} \) is a constant and \( l \) and \( n \) are integers \( \geq 0 \).

The expression \((x+Mt)^l\) is used so that for \( l > 0 \) the tangent to the wing camber line at the leading edge is tangent to the flight-path angle of the leading edge. Consider, for example, the case \( l = 1, n = 0 \). The downwash

\[
w_u = \frac{a_{1,0}}{c} (x+Mt)
\]

represents an infinite class of surface shapes having the form

\[
h(x, y, t) = \frac{a_{1,0}}{2cU_0} (x+Mt)^2 + f(x, y)
\]

where \( f(x, y) \) is an arbitrary function and \( h \) is, by definition, the distance of the wing's camber line from the \( z = 0 \) plane. Since, within the accuracy of linearized theory, the solution for the flow about the wing depends only upon the value of \( w_u(x, y, t) \), the loading on all the wings represented by the above equation is the same.

Let us inspect the two special cases

(i) \( f(x, y) = -x^2 \)

(ii) \( f(x, y) = 0 \)

For case (i)

\[
h(x, y, t) = \frac{a_{1,0} M^2}{2cU_0}
\]

and the wing is a flat plate pitching at a uniform rate about its leading edge which is following the flight path

\[
\left( \frac{\partial h}{\partial t'} \right)_{u} = \frac{a_{1,0} M^2}{2cU_0}
\]

as shown in figure 2. Hence, at time \( t \) the tangent to the flight path of the leading edge is

\[
\frac{\partial h}{\partial t'} - \frac{a_{1,0} M^2}{2cU_0}
\]

The slope of the leading edge of the plate at the same time is

\[
\frac{\partial h}{\partial x} = \frac{a_{1,0} M^2}{c}
\]

and the two slopes are seen to be equivalent.

\footnote{The scale in both figures 2 and 3 is purposely distorted in order to make the drawings clear. A basic assumption used in setting up the boundary-value problem, by means of which the leading was determined, was that the surface of the wing must remain near the \( z = 0 \) plane.}
The problem is linear, so it will be sufficient to determine a solution for arbitrary \( l \) and \( n \), and then add results for any combination of terms as desired. Thus, the complete boundary conditions to be studied are

\[
w_n(x, y, 0) = \frac{1}{2cU_0} (x + \alpha x)^2
\]

and the wing is a plate which obtained a sudden parabolic camber at \( t=0 \), a shape it maintained thereafter as shown in figure 3.

The potential is required, together with the system of axes; also, traces in the \( z=0 \) plane of the wave system set up by the indicial motion of the wing are indicated. The wave pattern for only two edges is shown; the flight speed is supersonic so the trailing edge has no effect on the velocities induced over the wing surface, and the results are valid (in their entirety) only for \( \beta A \geq 1 \), so the opposite edge either has no effect or one that can be incorporated by simple superposition.

The wave traces divide the wing area into several regions, indicated by the Roman numerals, in each of which the analytical formulation for the potential is different. Region \( I \) consists of that part of the wing where the effect of neither the side edge nor leading edge has yet been felt. In region \( II \), the side-edge influence is acting (the line \( y=t \) is the trace of the starting cylindrical wave from the side edge \( y=0 \)) but not the leading edge. Region \( III \) is the part within the starting cylindrical wave from the leading edge, but outside the influence of the side edge. This region, and region \( V \), are further subdivided for reasons that will appear later. Region \( IV \) is a compound region; potential there can be found by adding the potentials for regions \( II \) and \( III \) and subtracting the potential for region \( I \). Region \( V \) consists of the portion of the wing within the spherical wave originating at the wing corner. The flow over the part of the wing comprising regions \( VI \) and \( VII \) has reached a steady state relative to a point on the wing, and the potential there is just that for the corresponding parts of a rectangular wing with the proper downwash distribution in steady motion. Finally, region VIII is again a composite region, its potential being the sum of potentials for regions \( III \) and \( VII \), less the potential for region \( VI \).

All the regions just listed, with the exception of region \( V \), are actually governed by the three-dimensional wave equation and the potential therein could be obtained by methods applicable to this simpler equation. However, in this report we shall present a unified approach and the problem will be solved by the same method in all regions.
As in steady-state wing theory, there is a formula for time-dependent flows that relates the velocity potential to a distribution of time-dependent sources and doublets over a certain region in the wing plane. This formula is due to Kirchhoff, and some of its aerodynamic uses are discussed in reference 6. Kirchhoff's result is immediately applicable in the study of unsteady lifting-surface problems when the potential can be represented by sources alone, that is, when the upper and lower surfaces of the wing do not interact, as is the case in regions I, III, and VI of figure 4.

Kirchhoff's formula for source distributions can be written

$$\phi(x,y,0,t) = \frac{1}{2\pi} \int_{S_a} \left[ \frac{w_s}{r_s^2} \right] dx dy,$$  \hspace{1cm} (3)

where

$$r_s^2 = (x-x_1)^2 + (y-y_1)^2.$$

The brackets on $$w_s$$ indicate that the retarded value is to be taken

$$[w_s] = w_s(x_1,y_1,t-r_s)$$

and $$S_a$$ indicates that the region of integration is the acoustic plan form corresponding to the event $$(x,y,0,t)$$. These concepts are discussed at length in reference 6.

As has been pointed out, equation (3) holds for each of the regions I, III, and VI, but the area of integration $$S_a$$ differs considerably from one of these regions to another. Consider, for example, the determination of $$\phi$$ for region III, denoted $$\phi_{III}$$.

Part of the boundary of the acoustic plan form $$S_a$$ is found by eliminating $$T$$ between the equation of the leading edge, $$x_1 = -MT$$, and the expression

$$(x-x_1)^2 + (y-y_1)^2 = (t-T)^2$$

which gives the outer boundary, at "time" $$t$$, of all the disturbances that, operating at "time" $$T$$, can produce an effect at the point $$(x,y)$$. This boundary is the ellipse

$$\left( \frac{\beta}{M} x - x_m \right)^2 + (y-y_1)^2 = t_m^2$$  \hspace{1cm} (4a)

where

$$x_m = \frac{Mx + t}{\beta}, \hspace{0.5cm} t_m = \frac{x + Mt}{\beta}.$$

If the point $$(x,y)$$ lies within the cylindrical wave from the leading edge, that is, $$-t < x < t$$, the ellipse of equation (4a) comprises only part of the acoustic plan form, the remainder being bounded by so much of the circle

$$(x-x_1)^2 + (y-y_1)^2 = \rho$$  \hspace{1cm} (4b)

as lies on the wing at time zero. Figure 5 shows the three possible acoustic plan forms for points in region III. The limits for the three types are

(i) $$t \geq x \geq 0$$

(ii) $$0 \geq x \geq -t/M$$

(iii) $$-t/M \geq x \geq -t$$

and these correspond to the subregions IIIa, IIIb, and IIIc identified in figure 4. Using equation (3), we can write the potential in, say, region III as

$$\phi_{III} = \frac{1}{2\pi} \int_{S_a} \left[ \frac{w_s}{r_s^2} \right] dx dy,$$

where

$$X_1(y-y_1) = \frac{M}{\beta} \left[ x_m - \sqrt{\rho} \right] (y-y_1)^2.$$
Gardner's method for solving equation (1c) is to define a five-dimensional space in which a potential function $\psi$ is governed by the equations

$$\psi_{tt} - \psi_{xx} - \psi_{tt} = 0$$

$$\psi_{tt} - \psi_{xx} - \psi_{tt} = 0$$

and show that solutions to equations (6) in this space are general enough to contain general solutions to equation (1c) in a plane $\xi = \text{constant}$. We shall, therefore, proceed by analyzing these equations and eventually let $\xi$ approach a plane in which the boundary conditions of equations (2a) and (2b) are satisfied. For convenience, the latter plane is taken to be the $\xi = 0$ plane.

Since equations (6a) and (6b) are linear, a number of possibilities exist for the choice of the dependent variable $\psi(x, y, z, 0, t)$. Aside from the more obvious choice $\psi(x, y, z, 0, t) = \varphi(x, y, z, t)$, where $\varphi$ is the velocity potential of equation (1c); for example, one could let $\psi(x, y, z, 0, t) = \varphi(x, y, z, t)$ or again, $\psi(x, y, z, 0, t) = \varphi(x, y, z, t)$. These various choices amount only to relatively minor differences in the detailed technique of the subsequent analysis. If, in imposing the boundary conditions of equations (2), one is to use only source-type solutions for both equations (6a) and (6b), the last choice is sufficient. Therefore, set

$$\psi(x, y, z, 0, t) = \varphi(x, y, z, t)$$

Now differentiate equation (6a) with respect to $z$ and set $z = 0$.

Defining

$$W(\xi, x, y, z, 0, t) = \frac{\partial \psi}{\partial \xi} |_{\xi = 0}$$

equation (6a) can be expressed in the form

$$W_{tt} - W_{xx} - W_{tt} = 0$$

and the boundary conditions for equation (9) are given directly by equations (2). Thus on the wing

$$\frac{\partial W}{\partial \xi} |_{\xi = 0} = \frac{\partial \varphi}{\partial \xi} |_{\xi = 0} = W_0(x, y, z, t) = a_{1N} \left( \frac{x + M_1 t}{c} \right)^{\frac{\gamma}{\gamma - 1}}$$

and off the wing

$$\frac{\partial W}{\partial \xi} |_{\xi = 0} = \varphi(x, y, 0, t) = 0$$

Assuming equation (9) to have been solved for the boundary conditions given by equations (10), we return to the second of the set of partial differential equations (6), specifically,

$$\psi_{tt} - \psi_{xx} - \psi_{tt} = 0$$

From equation (8), it is seen that the solution to equation (9) yields the result

$$\varphi(\xi, x, y, 0, t)$$

Further, the boundary conditions for the original problem in $(x, y, z, t)$ space require that $\varphi$ be an odd function with respect to $z$, and continuous across the $z = 0$ plane except over the wing plan form. Thus $\varphi$ must be zero for $z = 0$ except over the wing plan form. The continuation of this condition into $(x, y, z, t)$ space then implies, according to equation (7), that off the wing

$$\frac{\partial \varphi}{\partial \xi} |_{\xi = 0} = 0$$

Hence, both the second partial differential equation and its boundary conditions are identical in form to the first set given by equations (9) and (10), respectively. Applying equation (7) to their dual solution, we obtain the desired result

$$J \varphi(x, y, 0, t)_{|_{\xi = 0}} = \varphi(x, y, 0, t)$$

for the potential on a rectangular wing (with $\beta \geq 1$) in supersonic unsteady motion.

**THE GENERAL EXPRESSION FOR THE POTENTIAL**

The method outlined in the preceding section will now be applied to obtain integral expressions for the potential in any region of the rectangular wing shown in figure 4. Consider first equation (9) for $W(\xi, x, y, t)$. This equation is the same partial differential equation as that which governs supersonic steady flow. Further, the boundary values in the $(\xi, x, y, t)$ space are identical to those representing a thin planar wing in a steady supersonic flow. Since the Mach number in the steady-flow analog is $\sqrt{2}$, the equivalent plan-form of this wing (shown in fig. 6) is a swept-forward wing tip having all supersonic edges (i.e., the component of the free-stream velocity normal to all edges is supersonic).

Since all edges of the equivalent wing plan-form are supersonic, the solution for $W$ can be written immediately

$$W(x, y, z, t) = \frac{\partial \varphi}{\partial \xi} |_{\xi = 0}$$

and

$$\varphi(\xi, x, y, 0, t)$$

**FIGURE 6.**—Equivalent plan form in $\xi, x, y, t$ space.
in terms of "sources" only, their strength being given by equation (10a). Thus, by analogy with the well-known results of supersonic wing theory, we have

\[
W(\xi, x, t) = -\frac{1}{p} \int \frac{\nu(x, t)}{\sqrt{(\xi - \xi_0)^2 - \xi^2 - (x - x_0)^2}} \, dx dt \tag{11}
\]

where \( \tau \) is the area on the wing cut out by the forecone from the point \((\xi, x, t)\). The analytic form of \( W \) will differ considerably in each of the three regions above the equivalent wing shown in figure 7.

The value of \( W \) given by equation (11) now becomes a boundary condition for the solution of equation (6b). Thus, over the portion of the \( z=0 \) plane for which \( y \geq 0, \xi \geq 0 \), the variation of \( \frac{\partial \psi}{\partial z} \mid_{x=0} \) is now known and for \( y < 0, \xi \geq 0 \) the condition \( \frac{\partial \psi}{\partial \xi} \mid_{x=0} = 0 \) applies. (These conditions are still not sufficient to determine a unique solution unless the further restriction is imposed that the loading falls to zero as the edge \( y=0 \) is approached, i.e., as \( y \rightarrow 0^+ \).) Again we observe that these boundary conditions and the partial differential equation (6b) are identical to those studied in connection with a stationary planar wing in a supersonic stream. As shown in figure 7, solutions from the \( t, x, \xi \) space above the \( \xi=0 \) plane are referred to as \( W_A, W_B, \) and \( W_C \) occurring along constant \( z \) lines in the \( x, \xi \) plane and the corresponding regions in figure 4 for which each applies. Each of these five problems is directly analogous to the boundary-value problem encountered in steady-state lifting-surface theory, of a planar, rectangular lifting surface in a steady supersonic stream. The "leading edges" of these analogous rectangular plan forms lie along the lines \( \xi = \xi_1, \xi = \sqrt{\beta - x^2} \) or \( \xi = \xi_m \), depending on the value of \( x \).
and the "side edge" lies along the line \( y = 0 \). Hence, by means of this steady-flow analog, we can immediately write the solution to equation (6b) in the form

\[
\varphi(x, y, 0, \xi, t) = \frac{1}{\pi} \int \frac{W(x, y, \xi, t) \, dx \, dy}{\sqrt{\xi^2 - (x-x_0)^2}}
\]

(12)

where only the area of integration \( \sigma \) must be discussed.

Two possibilities exist for the shape of \( \sigma \). First, if the point \( \xi, y \) lies to the right of the dashed lines in figure 8, which in the analogous steady-flow problem represent the traces of the Mach cones from the leading-edge tips, \( \sigma \) is the triangular area shown (for region III\(_t\)) in figure 9, part (i). If however, \( \xi, y \) lies between this line and the side edge, \( y = 0 \), \( \sigma \) is the trapezoidal area shown (for region V\(_t\)) in figure 9, part (ii). The latter is a well-known result used in steady supersonic lifting-surface theory and first developed by Evvard (ref. 8). The division of the five kinds of problems illustrated in figure 8 into the final twelve, represented by the regions in figure 4, is brought about by the various combinations of \( W_A, W_B, \) and \( W_C \) that can occur in the area \( \sigma \) as the point \( \xi, y \) assumes all necessary values on the wing.

When \( \varphi \) has been determined, the potential in the physical plane is found by equation (7), or, combining equations (11) and (12),

\[
\varphi(x, y, 0, \xi, t) = \frac{1}{\pi} \lim_{\xi \to 0} \frac{\partial}{\partial \xi} \int \frac{\partial \mu}{\partial \xi} \, dx \, dy,
\]

(13)

\[
\int \frac{w(x + \Delta t, y) \, dx \, dy}{\sqrt{(x-x_0)^2 - (y-y_0)^2}}
\]

A detailed analysis of equation (13) for a point \( x, y, t \) in region \( V_t \) of figure 4 is given in Appendix A, and a study of this analysis enables one to write the results for all regions without difficulty.

**INTERPRETATION OF THE RESULTS**

The results of the rather involved analysis given in Appendix A can be interpreted in terms of the known solutions for simpler boundary conditions. These latter solutions have already been reviewed in a previous section in which it was shown that the potential on a lifting surface with all supersonic edges can be written in the form

\[
\varphi(x, y, 0, t) = -\frac{1}{2\pi} \int \frac{w(x, y) \, dx \, dy}{r_0}
\]

From Appendix A it is found that the potential at a point on a rectangular lifting surface can always be expressed as the sum of two parts

\[
\varphi(x, y, 0, t) = \varphi^{\alpha}(x, y, 0, t) - \varphi^{\alpha}(x, y, 0, t)
\]

where

\[
\varphi^{\alpha}(x, y, 0, t) = -\frac{1}{2\pi} \int \frac{w(x, y) \, dx \, dy}{r_0}
\]

and

\[
\varphi^{\alpha}(x, y, 0, t) = -\frac{1}{2\pi} \int \frac{C(x, y) \, dx \, dy}{r_0}
\]

The value of \( C(x, y) \) is given by equation (A10) in Appendix A and the areas of integration, \( S_0 \) and \( S_r \), are illustrated for the various regions I through VIII in figure 16.

Let us first inspect equations (15) in light of their possible analogy with the familiar solution for the steady-state, rectangular lifting surface. If a rectangular wing having arbitrary twist and camber is placed in a steady supersonic flow, the solution for the potential on its surface can also be expressed as the sum of two parts

\[
\varphi(x, y, 0, t) = \varphi^\alpha(x, y, 0, t) - \varphi'^\alpha(x, y, 0, t)
\]

where, if

\[
r^2 = (x-x_0)^2 - (y-y_0)^2
\]

\[
\varphi^\alpha(x, y, 0, t) = -\frac{1}{\pi} \int \frac{w(x, y) \, dx \, dy}{r_0}
\]

and

\[
\varphi'^\alpha(x, y, 0, t) = -\frac{1}{\pi} \int \frac{w(x, y) \, dx \, dy}{r_0}
\]

These equations can be construed in the following simple way: Equation (17a) represents the potential induced at \( x, y, 0 \) by a distribution of sources over the wing plan form, each source having a strength proportional to the local streamwise slope of the upper surface. The area \( S_0 \), as shown in figure 10, is the portion of the wing within the Mach forecone from \( x, y, 0 \). Equation (17b) has a similar interpretation; it also represents a distribution of sources over the wing, each having a strength proportional to the local slope of the upper surface. But the area of integration \( S_r \) is now that portion of the wing within the Mach forecone from the point \( x, -y, 0 \);
that is, within the cone which forms a mirror image of the physical Mach forecone in the vertical plane containing the wing’s side edge. The potential \( \varphi^{(x,y,0)}(x,y,0) \) represents the difference between the potentials for a wing with a vertically symmetrical thickness distribution and a surface with no thickness having the same shape as the upper surface of the nonlifting wing.

Let us return now to equations (15). Just as in the steady-state case, \( \varphi^{(x,y,0,t)}(x,y,0,t) \) represents the potential induced at \( x,y,0 \) by a distribution of sources (see eq. (3)) over the wing plan form, each proportional to the local slope of the wing, but now, since the wing is in motion, with the added condition that they be local slopes at the appropriate time. The area \( S_v \), shown in figure 11, is just the acoustic plan form defined earlier in the discussion of equations (3) and (4). Physically, \( S_v \) represents those points on the wing from which disturbances can, at the time \( t \), influence the flow at \( x,y,0 \). It is the generalization, in the stationary coordinate system, of the wing area bounded by the Mach forecone.

The relation between \( \varphi^{(0)}(x,y,0,t) \) and \( \varphi^{(0)}(x,y,0,t) \) is similar to that between their steady-state analogs. Thus, again, \( \varphi^{(0)}(x,y,0,t) \) represents the difference between the potentials for an uncambered nonlifting wing and a lifting surface having the same shape as the top of the nonlifting wing. A more striking similarity lies in the relation between \( S_v \) and \( S_v \).

We have already seen that \( S_v \) is the acoustic plan form, and, as it turns out, \( S_v \) is the reflection of the acoustic plan form in the vertical plane containing the side edge (see fig. 12)—a
the reflected plan form is not the same as it is for the slopes in the basic acoustic plan form; the influence in the former case now being given by the integral \( C(x_1, y_1) \) defined in equation (A10).

One can show, by simply referring the results given in equations (15) to a coordinate system fixed on the wing, that equations (15a) and (15b) are identical, respectively, to equations (17a) and (17b) when they apply to regions VII and VI in figure 4; regions in which, for indicial-type motions, the flow is steady relative to the wing. Hence, equations (15a) and (15b) extend Eyrard's "reflected area" concept to all parts of a rectangular wing in supersonic unsteady motion.\(^4\)

**THE GENERALIZED FORCES**

**REVIEW OF LAGRANGE'S EQUATIONS OF MOTION**

In order to define more clearly the subsequent concepts and notation, we will briefly review Lagrange's equations of motion as applied to distorting wings and will examine a simple application to a rectangular wing.

Lagrange's equations are usually written

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_r} \right) - \frac{\partial U}{\partial q_r} = Q_r; \quad r = 1, 2, \ldots
\]

where
- \( T \) = kinetic energy of wing
- \( U \) = potential energy of wing
- \( Q_r \) = a generalized (external) force
- \( q_r \) = a generalized coordinate

In the present application, \( q_r \) is the amplitude at a given time of a polynomial measuring \( h \), the vertical displacement of the wing's camber line from the \( z=0 \) plane. Thus, relative to a \( x_3, y_3 \) coordinate system that is fixed on the wing, see figure 13

\[
h(x_3, y_3, t') = \sum q_r(t') P_r(x_3, y_3)
\]

The wing's kinetic energy can be written

\[
T = \int_0^1 \frac{1}{2} h^2 m(x_3, y_3) dx_3 dy_3
\]

where \( m \) is the wing mass per unit plan form area. Using equation (19), we find

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_r} \right) = \int_0^1 \frac{1}{2} \dot{q}_r \int P_r(x_3, y_3) m(x_3, y_3) dx_3 dy_3
\]

The potential energy is usually difficult to evaluate analytically. However, it can often be determined experimentally (as will be seen) by measuring the frequencies of the free vibration modes. For the present assume that the wing is a homogeneous plate of constant thickness. The potential energy for such a wing can be expressed as (ref. 9)

\[
U = \int \left\{ (\nabla^2 h)^2 - 2(1 - \mu) \left( \frac{\partial h}{\partial y_3} \frac{\partial h}{\partial y_3} \right) \right\} dx_3 dy_3
\]

where \( \mu \) is Poisson's ratio, \( \nabla^2 = \frac{\partial^2}{\partial x_3^2} + \frac{\partial^2}{\partial y_3^2} \), and

\[
D = \frac{2(\text{Young's modulus}) \text{(plate thickness)}^3}{3(1 - \mu^2)}
\]

Now, if the generalized coordinates have been normalized so that each measures the amplitude of a free vibration mode, all terms in equations (21) and (23) involving the integral of the product of \( P_r \) and \( P_{r'} \) are zero. Assuming, henceforth, such normalization, we can write

\[
\dot{q}_r \int_0^1 P_r(x_3, y_3) m(x_3, y_3) dx_3 dy_3 + D \dot{q}_r \int \left\{ (\nabla^2 h)^2 - 2(1 - \mu) \left( \frac{\partial h}{\partial y_3} \frac{\partial h}{\partial y_3} \right) \right\} dx_3 dy_3
\]

Finally, dividing through by the coefficient of \( \dot{q}_r \), and expressing a generalized force as the integral over the wing plan form of the product of the \( r \)th mode shape and the loadings \( \Sigma(\Delta p) \), induced on the wing by each of the mode shapes considered, we find

\[
\dot{q}_r + \omega_r^2 \int_0^1 P_r(x_3, y_3) m(x_3, y_3) dx_3 dy_3 = Q_r; \quad r = 1, 2, \ldots
\]

where \( \omega_r \) is the frequency of the \( r \)th free vibration mode.\(^5\)

\( ^4 \) It is of further interest to notice that equation (16b) can be reduced to a double integral involving \( w(\xi, \eta) \) by using, for example, the transformations \( t = x_1 + My_1 \) and \( r = t - \eta \) and integrating with respect to \( \eta \).

\( ^5 \) We will write \( \Delta p(\gamma) = \gamma (\Delta p(\xi, \eta)) \), where \( \gamma \) is the free-stream dynamic pressure. This is possible without a confusion of notation since the generalized coordinates are expressed as \( \gamma, \eta, \xi, \ldots \) and exclude the term \( \gamma \).
If the free-mode frequencies are experimentally determined, equations—such as equation (23)—giving the wing’s potential energy, never have to be evaluated. Further, in such cases, equation (25) applies to quite general wing structures with varying density. Usually in the application of equation (25), one uses the actual frequency \( \omega \) of the free mode but, in evaluating the aerodynamic forces, uses an analytical expression that only approximates the \( r \)th mode shape. Let us examine the generalized force term in equation (25), taking, for simplicity, only one term of the sum:

\[
Q_r = \frac{g_{08} c}{\frac{\partial}{\partial t}} \int_0^t P_r(z_0, y_0) \left( \frac{z_0}{c} \right)^{r} \left( \frac{y_0}{c} \right)^{s} \, dx_0 dy_0 (26)
\]

According to what has gone before, the mode shape polynomial \( P_r(z_0, y_0) \) has the form

\[
P_r(z_0, y_0) = \left( \frac{z_0}{c} \right)^{r} \left( \frac{y_0}{c} \right)^{s} (27)
\]

while \( \frac{\partial g}{\partial z_0} \) is the loading coefficient corresponding to an indicial deflection (see previous section on boundary conditions)

\[
h = \frac{c}{(z_0 + 1)} g_1(z_0) \left( \frac{y_0}{c} \right)^{s} \left[ \left( \frac{z_0}{c} \right)^{r} + f \left( \frac{z_0}{c} \right)^{r} \left( \frac{z_0}{c} \right)^{r} \right] (28)
\]

which gives a vertical velocity distribution

\[
w_v = u_0 g_1(z_0) \left( \frac{z_0}{c} \right)^{r} \left( \frac{y_0}{c} \right)^{s} (29)
\]

Now a generalized indicial force coefficient can be defined as follows:

\[
\bar{f}_r^{(r)}(t') = \frac{1}{\frac{\partial \bar{f}_r^{(r)}}{\partial t'}} \int_0^t \left[ \frac{\partial \bar{f}_r^{(r)}}{\partial t'} \right] \left( \frac{z_0}{c} \right)^{r} \left( \frac{y_0}{c} \right)^{s} \, dx_0 dy_0 (30)
\]

(The calculation of these quantities \( \bar{f}_r^{(r)}(t') \) will be elaborated in the next section.) Since the generalized force \( Q_r \) is intended to apply to any motion, not necessarily indicial, it is necessary to apply Duhamel’s integral to the indicial force coefficient \( \bar{f}_r^{(r)}(t') \); thus,

\[
Q_r = g_{08} c \frac{d}{dt} \int_0^\infty g_1(t' - r) \left[ \frac{\bar{f}_r^{(r)}}{\bar{g}_1(1)} \right] \, dr' (31)
\]

As an example, consider now a simple one degree of freedom vibrating plate. The plate is fixed to a wall and restrained along its leading edge. The mode shape is assumed to have the form

\[
h = c g_1(t') \left( \frac{z_0}{c} \right)^{r} \left( \frac{y_0}{c} \right)^{s} (32)
\]

so for a plate with uniform density and thickness

\[
m \int_0^s dy_0 \int_0^s dx_0 P_r(x_0, y_0) = \frac{m s c}{25} \left( \frac{c}{z_0} \right)^{r} (33)
\]

Equation (25) now becomes

\[
\bar{g}_1 + \omega^2 \bar{g}_1 = \frac{25}{m s c} \left( \frac{c}{z_0} \right)^{r} Q_1 (34)
\]

For this case, we have the generalized indicial force coefficients \( f_r^{(r)}(t') \) and \( \bar{f}_r^{(r)}(t') \);

\[
Q_r = g_{08} c \frac{d}{dt} \int_0^\infty \left[ 2 g_1(t' - r) \left[ \frac{\bar{f}_r^{(r)}}{\bar{g}_1(1)} \right] \right] \, dr' (35)
\]

Therefore, equation (33) can be written

\[
\bar{g}_1 + \omega^2 \bar{g}_1 = \frac{25 s c}{m} \left( \frac{c}{z_0} \right)^{r} \left[ 2 g_1(t' - r) \left[ \frac{\bar{f}_r^{(r)}}{\bar{g}_1(1)} \right] \right] \, dr' (36)
\]

THE GENERALIZED INDIICAL FORCE COEFFICIENT

It is clear from the previous section that a study of the dynamic behavior of rectangular wings moving at supersonic speeds can be carried out if one can obtain values of the generalized force coefficient, \( f_r^{(r)}(t') \), as defined by equation (30). We will now show how these values can be obtained from the solution to the aerodynamic boundary-value problem represented by equation (14).

It was convenient in developing equation (14) to use a coordinate system—\( x_3, y_3, z_3, t \)—which was fixed in space so that the left edge of the wing moved along the \( z \) axis as shown in figure 1. On the other hand, in studying the dynamic problem it was more convenient to use an \( x_4, y_4, z_4, t \) system which is fixed on the wing. In order to convert the results in one coordinate set to the other, let us first transfer results in the \( x_3, y_3, z_3, t \) set to the \( x_4, y_4, z_4, t \) set (shown in figure 14) and then, finally, transfer to \( x_3, y_3, z_3, t \) coordinates.

For this case, we have the generalized indicial force coefficients \( f_r^{(r)}(t') \) and \( \bar{f}_r^{(r)}(t') \);
The indicial force coefficient \( F_{ij}^*(t') \) is defined as follows:

\[
F_{ij}^*(t') = \frac{1}{\infty} \int_{\infty-M}^{\infty-M_1} dx \int_{-\infty}^{\infty} dy \left( \frac{x+M_1}{c} \right)^{i} \left( \frac{y-s}{c} \right)^{j} \left( \frac{\Delta p}{\rho_0} \right)^{in}
\]  \hspace{1cm} (36)

In order to transfer the WRS from the set shown in figure 1 to the more convenient set of figure 14, so that mode shapes are symmetric or asymmetric about the wing's spanwise center line and the force coefficients denoted \( f_{ij}^* \) can be determined, we proceed as follows. First, the loading coefficient for a wing in the \((x,y)\) system with downwash given by

\[
\frac{w_{e0}}{U_0} = \left( \frac{x+M_1}{c} \right)^{i} \left( \frac{y-s}{c} \right)^{j}
\]

is obtained. This loading coefficient can be written as a sum:

\[
\left( \frac{\Delta p}{\rho_0} \right)^{in} = (-1)^{n} \sum_{\mu=0}^{n} (-1)^{n} \left( \frac{n}{\mu} \right) \left( \frac{A}{2} \right)^{n-\mu} \left( \frac{\Delta p}{\rho_0} \right)^{in}
\]

Now the quantity \( f_{ij}^* \) is defined in the \(x,y,z,t\) system as

\[
f_{ij}^* = \frac{1}{2\pi} \int_{\infty-M}^{\infty-M_1} d\alpha \int_{-\infty}^{\infty} dy \left( \frac{x+M_1}{c} \right)^{i} \left( \frac{y-s}{c} \right)^{j} \left( \frac{\Delta p}{\rho_0} \right)^{in}
\]

This last integral can be written as

\[
f_{ij}^* = \frac{1}{2\pi} \left[ 1 + \left( -1 \right)^{i+n} \right] \int_{\infty-M}^{\infty-M_1} d\alpha \int_{-\infty}^{\infty} dy \left( \frac{x+M_1}{c} \right)^{i} \left( \frac{y-s}{c} \right)^{j} \left( \frac{\Delta p}{\rho_0} \right)^{in}
\]

where all forces are responses to a unit indicial disturbance. Note that if equation (37) is applied in the case of a wing cantilevered on a wall, both \( n \) and \( g \) must be even in order to satisfy the boundary conditions of reflection in the wall.

By superimposing boundary conditions and their resulting solutions, one can further show that the value of \( f_{ij}^* \) given by equation (37) is valid for all reduced aspect ratios \( \beta A \) greater than 1 in spite of the fact that the value of \( F_{ij}^* \) given by equation (36), as it stands, applies only to wings for which \( \beta A \) is greater than 2.

Given \( f_{ij}^*(t') \), one can determine the generalized force associated with the generalized coordinate \( q_t \) by means of the superposition integral as illustrated by equation (34).
Next with \( j = 1 \), and using \( C_{m'} \) to designate the pitching moment measured about the leading edge of the wing,

\[
C_{m'} = -\left( -\frac{1}{a_0(U_0)} \right) f_{m'}^{20} = -\frac{2}{M} \left\{ \left( \frac{1}{2} t_o^2 \right) \right\} \frac{t_o}{3A} \left[ 3 - (M^2 + 1) t_o^2 \right] ; \quad 0 \leq t_o \leq \frac{1}{M+1}
\]

\[
= -\frac{2}{M} \left\{ \frac{1}{\pi} \left[ \left( \frac{1}{2} t_o^2 \right) \cos^{-1} \frac{M t_o - 1}{t_o} \right] + \frac{M}{\beta} \cos^{-1} \left( M - \beta t_o \right) + \frac{1 + M t_o}{2} \sqrt{t_o^2 - (1 - M t_o^2)} \right\}; \quad \frac{1}{M+1} \leq t_o \leq \frac{1}{M-1}
\]

\[
= -\frac{2}{\beta} \left\{ \frac{1}{3} \left[ \frac{t_o}{3A} \right] ^2 \right\}; \quad t_o \geq \frac{1}{M-1}
\]

These expressions agree with those given by Miles in reference 2.

The above results can be used to demonstrate the usefulness of equation (38a). Taking \( j = n = g = 0 \), \( l = 1 \) in that equation gives

\[
F_{\alpha}^{20} = F_{\alpha}^{00} - F_{\alpha}^{10}
\]

or, for the present case,

\[
f_{\alpha}^{20} = f_{\alpha}^{00} - f_{\alpha}^{10}
\]

which represents the equality

\[
C_{\alpha}' = C_{\alpha} + C_{m'}
\]

\[
\frac{f_{\alpha}^{20}}{a_0(U_0)} = \frac{F_{\alpha}^{20}}{a_0(U_0)} = \frac{4}{M} \left\{ \frac{1}{3} \left( 1 - M t_o^3 \right) - \frac{t_o}{12A} \left[ 4 - M(M^2 + 3)t_o^2 \right] \right\}; \quad 0 \leq t_o \leq \frac{1}{M+1}
\]

\[
= \frac{4}{M} \left\{ \frac{1}{\pi} \left[ \frac{1}{3} \left[ \frac{1 - M t_o^3}{t_o} \cos^{-1} \frac{M t_o - 1}{t_o} \right] + \frac{M}{\beta} \cos^{-1} \left( M - \beta t_o \right) + \frac{1}{24A} \left[ -3t_o^2 + 3(1 - M t_o^3) t_o \right] \right\}; \quad \frac{1}{M+1} \leq t_o \leq \frac{1}{M-1}
\]

\[
= \frac{4}{\beta} \left\{ \frac{1}{3} \left[ \frac{1}{4\beta A} \right] ^2 \right\}; \quad t_o \geq \frac{1}{M-1}
\]

Combining, we find

\[
C_{m'} = -\frac{2}{M} \left\{ \left( \frac{2 + 6 M t_o}{12A} \right) \left[ 8 - 6 M t_o + M(M^2 + 3)t_o^2 \right] \right\}; \quad 0 \leq t_o \leq \frac{1}{M+1}
\]

\[
= \frac{2}{M} \left\{ \frac{1}{\pi} \left[ \frac{2 + 6 M t_o}{3} \cos^{-1} \frac{M t_o - 1}{t_o} + \frac{2 M}{3 \beta} \cos^{-1} \left( M - \beta t_o \right) + \frac{8 - 6 M t_o}{9} \sqrt{t_o^2 - (1 - M t_o^2)} \right] - \frac{1}{24A} \left[ 3 + 8t_o - 6(M - 1)t_o^3 + (M - 1)^2 t_o^4 \right] \right\}; \quad \frac{1}{M+1} \leq t_o \leq \frac{1}{M-1}
\]

\[
= \frac{2}{\beta} \left\{ \frac{2}{3} \left[ \frac{1}{4\beta A} \right] ^2 \right\}; \quad t_o \geq \frac{1}{M-1}
\]

that is, the lift coefficient for a pitching wing equals the sum of the lift and pitching-moment coefficients of a sinking wing (primes indicate the wing is pitching about and moments are measured about the wing leading edge). Hence,

\[
C_{\alpha} = \frac{2}{M} \left\{ \left( 1 + \frac{1}{2} t_o^2 \right) - \frac{1}{A} \left[ 4 t_o - M t_o^2 + \frac{M^2 + 1}{3} t_o^3 \right] \right\}; \quad 0 \leq t_o \leq \frac{1}{M+1}
\]

\[
= \frac{2}{M} \left\{ \frac{1}{\pi} \left[ \left( \frac{1}{2} t_o^2 \right) \cos^{-1} \frac{M t_o - 1}{t_o} + \frac{M}{\beta} \cos^{-1} (M - \beta t_o) + \frac{3 - M t_o}{2} \sqrt{t_o^2 - (1 - M t_o^2)} \right] - \frac{1}{6A} \left[ \frac{1}{M+1} + 3t_o - 3(M - 1)t_o^2 + (M - 1)^2 t_o^3 \right] \right\}; \quad \frac{1}{M+1} \leq t_o \leq \frac{1}{M-1}
\]

\[
= \frac{2}{\beta} \left\{ \frac{1}{3} \left[ \frac{1}{3} \right] \right\}; \quad t_o \geq \frac{1}{M-1}
\]

A further application of equation (38a) provides the pitching-moment coefficient for a pitching flat rectangular wing. Thus, with \( l = j = 1 \), \( n = g = 0 \), equation (38a) gives

\[
F_{\alpha}^{20} = \frac{1}{2} (F_{\alpha}^{00} - F_{\alpha}^{10})
\]

which becomes

\[
f_{\alpha}^{20} = \frac{1}{2} (f_{\alpha}^{00} - f_{\alpha}^{10})
\]

and so

\[
C_{\alpha} = \frac{1}{2} \left( f_{\alpha}^{00} - f_{\alpha}^{10} \right)
\]

From equation (B21) in Appendix B it is found that

\[
f_{\alpha}^{00} = \frac{1}{2} \left( f_{\alpha}^{00} - f_{\alpha}^{10} \right)
\]

and

\[
f_{\alpha}^{10} = \frac{1}{2} \left( f_{\alpha}^{00} - f_{\alpha}^{10} \right)
\]

Combining, we find

\[
C_{m'} = -\frac{2}{M} \left\{ \left( \frac{2 + 6 M t_o}{12A} \right) \left[ 8 - 6 M t_o + M(M^2 + 3)t_o^2 \right] \right\}; \quad 0 \leq t_o \leq \frac{1}{M+1}
\]

\[
= \frac{2}{M} \left\{ \frac{1}{\pi} \left[ \frac{2 + 6 M t_o}{3} \cos^{-1} \frac{M t_o - 1}{t_o} + \frac{2 M}{3 \beta} \cos^{-1} \left( M - \beta t_o \right) + \frac{8 - 6 M t_o}{9} \sqrt{t_o^2 - (1 - M t_o^2)} \right] - \frac{1}{24A} \left[ 3 + 8t_o - 6(M - 1)t_o^3 + (M - 1)^2 t_o^4 \right] \right\}; \quad \frac{1}{M+1} \leq t_o \leq \frac{1}{M-1}
\]

\[
= \frac{2}{\beta} \left\{ \frac{2}{3} \left[ \frac{1}{4\beta A} \right] \right\}; \quad t_o \geq \frac{1}{M-1}
\]
Another relation among the generalized indicial forces $f_{ij}$ can be derived by means of the reciprocity relations given in reference 5. The details of the derivation are given in Appendix C and there results

$$\sum_{\mu=0}^{l}(-1)^{\mu} \left( \frac{j}{\mu} \right) f_{j}^{ij} = \sum_{\nu=0}^{j}(-1)^{\nu} \left( \frac{i}{\nu} \right) f_{i}^{ji}$$

(39)

Equation (39) can be used in two ways; one, as a means for checking the internal consistency of a set of calculated generalized indicial forces, and the other, as a means for expressing a given force in terms of a set of others.

Consider, as an example of the former use, the case for which $l=j=0$. Then

$$f_{i0}^{j0} = f_{0i}^{0j}$$

From equation (37) we can express this relation in terms of the calculated quantities $F_{i}^{j0}$ thus

$$\sum_{r=0}^{n}(-1)^{r} \left( \frac{i}{r} \right) \sum_{\mu=0}^{l}(-1)^{\mu} \left( \frac{j}{\mu} \right) \left( \frac{A}{2} \right)^{r+s-n-\mu} F_{i}^{j0} = \sum_{r=0}^{n}(-1)^{r} \left( \frac{j}{r} \right) \sum_{\nu=0}^{l}(-1)^{\nu} \left( \frac{i}{\nu} \right) \left( \frac{A}{2} \right)^{r+s-n-\nu} F_{i}^{j0}$$

If now $n=1, g=3$ the following relation results

$$\left( F_{i}^{j0} - F_{00}^{j} \right) + \frac{A}{2} \left[ \left( F_{i}^{j0} - F_{00}^{j} \right) + 3 \left( F_{i}^{j0} - F_{00}^{j} \right) \right] + \sum_{r=0}^{p}(-1)^{r} \left( \frac{p}{r} \right) \sum_{\mu=0}^{l}(-1)^{\mu} \left( \frac{\mu}{r} \right) \left( \frac{A}{2} \right)^{r+s-n-\mu} F_{i}^{j0}$$

which provides a useful check on the computed quantities.

Next let us solve equation (39) for a given force. Perform the sum operation

$$\sum_{j=0}^{l}(-1)^{j} \left( \frac{j}{j} \right)$$

on both sides of equation (39), and reverse the order of summation on the left side. There results

$$\sum_{\mu=0}^{l}(-1)^{\mu} \left( \frac{j}{j} \right) f_{j}^{ij} = \sum_{\nu=0}^{j}(-1)^{\nu} \left( \frac{i}{\nu} \right) f_{i}^{ji}$$

(40)

The inner sum on the left can be evaluated. Thus one has

$$x^{p} = \left[ 1 - (1-x)^{p} \right] = \sum_{\mu=0}^{p}(-1)^{\mu} \left( \frac{p}{\mu} \right) (1-x)^{\mu}$$

$$= \sum_{\mu=0}^{p}(-1)^{\mu} \left( \frac{p}{\mu} \right) \sum_{r=0}^{\mu}(-1)^{r} \left( \frac{\mu}{r} \right) x^{r}$$

$$= \sum_{r=0}^{p}(-1)^{r} x^{r} \sum_{\mu=r}^{p}(-1)^{\mu} \left( \frac{p}{\mu} \right) \left( \frac{\mu}{r} \right)$$

Equating coefficients of $x$,

$$\sum_{\mu=r}^{p}(-1)^{\mu} \left( \frac{p}{\mu} \right) \left( \frac{\mu}{r} \right) = \begin{cases} 0; & r < p \\ (-1)^{r}; & r = p \end{cases}$$

and equation (40) becomes

$$f_{i}^{j} = \sum_{j=0}^{l}(-1)^{j} \left( \frac{j}{j} \right) \sum_{\mu=0}^{l}(-1)^{\mu} \left( \frac{\mu}{\mu} \right) f_{j}^{i}$$

CONCLUDING REMARKS

A method is presented for evaluating the generalized forces on a rectangular wing flying at supersonic speeds and having an aspect ratio such that $\beta A \geq 1$. The generalized coordinates used to define the wing's behavior are the amplitudes of downwash distributions expressed in terms of polynomials in $x$ and $y$, the chordwise and spanwise directions, respectively.

Numerical results are presented in Table I for generalized indicial forces on a wing having an aspect ratio of 4 and flying at a Mach number equal to 1.1 and 1.2; the polynomial coverage being $0 \leq l \leq 1$ and $0 \leq n \leq 5$, where $w \sim x^{l} y^{n}$.

AMES AERONAUTICAL LABORATORY

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

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APPENDIX A

EXPRESSIONS FOR THE POTENTIAL

In order to write the expressions for the potential in all regions shown in figure 4, it is sufficient to derive in detail only that for region \( V \). Having carried out this analysis, one can determine the expressions for potential in other regions without difficulty.

Consider, therefore, equation (13) and let \( \sigma \) and \( \tau \) apply to region \( V \). First, it is necessary to determine the potentials \( W_A \) and \( W_B \) in the \( t, x, \xi \) space. From equation (11), in conjunction with figure 7, it is found that

\[
W_A = -\frac{1}{\pi} \int_{\xi - \sqrt{\tau - \xi^2}}^{\xi + \sqrt{\tau - \xi^2}} dx_1 \int_{t - \sqrt{\xi(\tau - \xi)}}^{t + \sqrt{\xi(\tau - \xi)}} \frac{w_0(x_1 + M_1, y_1) dx_1}{(t - t_1)^2 - \xi^2 - (x - x_1)^2}
\]

(A1)

With the values of \( W \) given in equations (A1) and (A2) it is possible now to solve equation (6b) for \( \psi \), figure 8 giving the required data in the \( \xi, y \) plane. Thus, if \( R^2 = (t - t_1)^2 - (y - y_1)^2 \)

\[
W_B = -\frac{1}{\pi} \int_0^\infty dx_1 \int_{\xi - \sqrt{\tau - \xi^2}}^{\xi + \sqrt{\tau - \xi^2}} \frac{w_0(x_1 + M_1, y_1) dx_1}{(t - t_1)^2 - \xi^2 - (x - x_1)^2}
\]

(A2)

Now apply the operation of equation (7) and the potential \( \varphi_r \) is given by

\[
\psi(t, \xi, y, z) = -\frac{1}{\pi} \int_{t-r}^{t} dy_1 \int_{\xi - \sqrt{r - \xi^2}}^{\xi + \sqrt{r - \xi^2}} dx_1 \int_{y_1 - \sqrt{r - y_1^2}}^{y_1 + \sqrt{r - y_1^2}} dz_1 \frac{w_0(x_1 + M_1, y_1) dx_1}{(t - t_1)^2 - \xi^2 - (x - x_1)^2}
\]

(A3)

where \( R^2 = \xi^2 - (y - y_1)^2 \) and the bars on the integrals signify that the finite part of the integral is to be taken in the sense defined \(^1\) in reference 10 and that the order of integration cannot, in general, be reversed.\(^3\)

For convenience set

\[
\varphi_r = -\frac{1}{\pi} \int_{t-r}^{t} dy_1 \int_{\xi - \sqrt{r - \xi^2}}^{\xi + \sqrt{r - \xi^2}} dx_1 \int_{y_1 - \sqrt{r - y_1^2}}^{y_1 + \sqrt{r - y_1^2}} dz_1 \frac{w_0(x_1 + M_1, y_1) dx_1}{(t - t_1)^2 - \xi^2 - (x - x_1)^2}
\]

(A4)

where \( R^2 = \xi^2 - (y - y_1)^2 \) and the bars on the integrals signify that the finite part of the integral is to be taken in the sense defined \(^1\) in reference 10 and that the order of integration cannot, in general, be reversed.\(^3\)

Consider the first of these integral sets. Using equation

\(^1\) For the subsequent analysis to hold, the definition of the finite part given in reference 10 is essential. This definition differs from that given by Hadamard when it applies to multiple integrals.

\(^3\) Since the order of integration plays an important role in the following developments, integration first with respect to \( x \) and then with respect to \( y \) would be denoted \( \int dx \int dy \phi(x, y) \) while integration first with respect to \( y \) and then with respect to \( x \) will be denoted \( \int dy \int dx \phi(x, y) \). When the notation \( \int dx \int dy \phi(x, y) \) is used, the order of integration is immaterial.

\[I_1 = \int_{t-r}^{t} dy_1 \int_{\xi - \sqrt{r - \xi^2}}^{\xi + \sqrt{r - \xi^2}} dz_1 \frac{w_0(x_1 + M_1, y_1) dx_1}{(t - t_1)^2 - \xi^2 - (x - x_1)^2}
\]

(A1), we can write

\[
I_1 = \int_{t-r}^{t} dy_1 \int_{\xi - \sqrt{r - \xi^2}}^{\xi + \sqrt{r - \xi^2}} dz_1 \frac{w_0(x_1 + M_1, y_1) dx_1}{(t - t_1)^2 - \xi^2 - (x - x_1)^2}
\]

(A4)

In order to simplify this expression, the order of these integrals will be rearranged so the integration with respect to \( \xi \) can be carried out first. The technique of changing the order of repeated integrals with strong singularities set forth in reference 10 will be used here. Consider the change of order in the \( \xi, z_1 \) plane. Pretend for the moment, that the \( t \), integration has been carried out. Then the highest order singularity (since \( w_0 \) is bounded) in the \( \xi, z_1 \) plane has the order 3/2 which is weak in the sense that no residual occurs
when the sequence of integration is reversed. The top of
figure 15 shows the area of integration, so immediately

\[ I_1 = \int_{y-t}^{y} dy_1 \int_{x-\sqrt{(y-y_1)^2}}^{x+\sqrt{(y-y_1)^2}} dx_1 \int_{-\infty}^{\infty} \frac{\xi_1 d\xi_1}{[t_1^2 - (y-y_1)^2]^{3/2}} \]

\[ \int_{0}^{\infty} w(x_1 + M t_1, y_1) dt_1 \]

To change order in the \( \xi_1, t_1 \) plane, consult the bottom of
figure 15. In this case an inherent singularity exists at the
confluence of the singularity lines of the integrand; namely,
where \( \xi_1 = y - y_1 \) and \( t_1 = t - \sqrt{(x-x_1)^2 + \xi_1^2} \). The change of
order can therefore not be performed directly, but account
must be taken of the existence of a residual term (see ref. 10).
This residual is defined as the difference between the two
integrals taken in different orders over a vanishingly small
region surrounding the inherent singularity (the dotted
region in bottom of figure 15). The residual \( R_t \) is then,

\[ R_t = \lim_{\epsilon \to 0} \left\{ \int_{y-t}^{y} dy_1 \int_{x-\sqrt{(y-y_1)^2}}^{x+\sqrt{(y-y_1)^2}} dx_1 \int_{-\infty}^{\infty} \frac{\xi_1 d\xi_1}{[t_1^2 - (y-y_1)^2]^{3/2}} \right. \]

\[ \left. \int_{t-r_0}^{t} w(x_1 + M t_1, y_1) dt_1 \int_{x-\sqrt{(y-y_1)^2}}^{x+\sqrt{(y-y_1)^2}} dx_1 \right\} \]

\[ \xi_1 = \sqrt{(x-x_1)^2 + (y-y_1)^2} \]

\[ \theta = \frac{\pi [w_0]}{2 r_0} \]

where \( \theta = \frac{(x-x_1)^2 + (y-y_1)^2}{2} \). The second integral vanishes
(see ref. 10), and, passing to the limit \( \epsilon \to 0 \) in the first integral
there results

\[ R_t = \frac{-\pi}{2} \frac{w_0(x_1 + M t_1 - M t_0, y_1)}{r_0} = \frac{\pi [w_0]}{2 r_0} \]

where the square brackets again mean that the retarded
value is to be taken. Thus, the integral \( I_1 \) can be reduced to

\[ I_1 = \frac{-\pi}{2} \int_{y-t}^{y} dy_1 \int_{x-\sqrt{(y-y_1)^2}}^{x+\sqrt{(y-y_1)^2}} dx_1 \frac{[w_0]}{r_0} \]

In the same way, the integral \( I_2 \) can be reduced, and

\[ I_1 + I_2 = \frac{-\pi}{2} \int_{y-t}^{y} dy_1 \int_{x-\sqrt{(y-y_1)^2}}^{x+\sqrt{(y-y_1)^2}} dx_1 \frac{[w_0]}{r_0} \]

which is recognized as Kirchhoff's formula, equation (3),
with an acoustic plan form bounded by the circle

\[ (x-x_1)^2 + (y-y_1)^2 = \frac{t^2}{4} \]

The reduction of the integrals \( I_3, I_4, I_5, \) and \( I_6 \) is quite
similar, leading to the sum

\[ \sum_{i=1}^{6} I_s = \frac{-1}{2 \pi} \int_{0}^{\infty} dy_1 \int_{x-\sqrt{(y-y_1)^2}}^{x+\sqrt{(y-y_1)^2}} dx_1 \frac{[w_0]}{r_0} \int_{0}^{\infty} dy_1 \int_{-\sqrt{(y-y_1)^2}}^{\sqrt{(y-y_1)^2}} dx_1 \frac{[w_0]}{r_0} \]

Examination of the limits on these integrals shows their
total area of integration is that shown in figure 11. But
this area corresponds exactly to the acoustic plan form \( S_a \)
for a point in region \( V_a \)l. Hence, denoting the combination
of terms in equation (A7) by \( \phi_a^{(a)} \) we can write simply

\[ \phi_a^{(a)} = \frac{-1}{2 \pi} \int_{(a)_{r_0}} \frac{[w_0]}{r_0} dx_1 dy_1 \]

It now remains to calculate the integrals \( I_7 \) through \( I_{10} \).
Designating their total effect on the potential, \( \phi_a^{(a)} \) one can
readily show (since no inherent singularities arise in these
cases) that
GENERALIZED INDICIAL FORCES ON DEFORMING WINGS

\[ \varphi_{w} = \frac{1}{\pi} \int_{0}^{r_{1}} dy_{1} \int_{-\infty}^{+\infty} \sqrt{1 - \left( \varphi_{w} \right)^{2}} \, dx_{1} \]

where \( r_{1} = (z - z_{0})^{2} + (y - y_{0})^{2} \). Now let

\[ C(x_{1}, y_{1}) = \begin{cases} \int_{-t_{1}/M}^{t_{1}/M} \frac{4g(y_{1}) w_{w}(x_{1} + M t_{1}, y_{1}) dt_{1}}{[(t_{1} - t_{1})^{2} - r_{1}^{2}]} \, dx_{1} & z_{1} < 0 \\ \int_{0}^{t_{1}/M} \frac{4g(y_{1}) w_{w}(x_{1} + M t_{1}, y_{1}) dt_{1}}{[(t_{1} - t_{1})^{2} - r_{1}^{2}]} \, dx_{1} & z_{1} > 0 \end{cases} \tag{A10} \]

In terms of this expression, equation (A9) can be written simply

\[ \varphi_{w} = \frac{1}{\pi} \int_{S_{2}} C(x_{1}, y_{1}) \, dx_{1} \, dy_{1} \tag{A11} \]

where the area \((S_{2})_{r_{0}}\) is illustrated in figure 12.

In order to give expressions for the potential in every region of the wing shown in figure 4, one can show that it is only necessary to vary the areas over which the double integration in equations (A8) and (A11) are carried out. This is evident in connection with the source portion \( \varphi^{(0)} \), for in every case

\[ \varphi^{(0)} = -\frac{1}{2\pi} \int_{S_{2}} \left[ \frac{w_{w}}{r_{0}} \right] \, dx_{1} \, dy_{1} \tag{A12} \]

and only the acoustic plan form \( S_{2} \) changes with the region. In the case of \( \varphi^{(0)} \), the part of the potential due to the existence of the side edge of the wing, equation (A11) can be generalized and written

\[ \varphi^{(0)} = \frac{1}{\pi} \int_{S_{2}} C(x_{1}, y_{1}) \, dx_{1} \, dy_{1} \tag{A13} \]

where the integrands are defined in each case by equation (A10) and only the “reflected” acoustic plan form \( S_{2} \) changes with the region. The region \( S_{2} \) is always bounded by portions of the “reflected” circle

\[ (x - x_{0})^{2} + (y - y_{0})^{2} = r^{2} \]

and the “reflected” ellipse

\[ \frac{\beta}{M} (x - x_{0})^{2} + (y - y_{0})^{2} = r_{0}^{2} \]

Figure 16 shows sketches of both \( S_{2} \) and \( S_{2} \) for all regions in figure 4. The absence of a sketch indicates that the corresponding integral does not exist for that region.

APPENDIX B

THE GENERALIZED INDICIAL FORCES

THE LOADING COEFFICIENT

In order to determine total forces acting on the wing, it is first necessary to obtain expressions for the loading coefficient \( \Delta p / g_{0} \). According to the linear theory

\[ \frac{\Delta p}{g_{0}} = \frac{4}{U_{0} M} \, \frac{\partial \varphi}{\partial t} \tag{B1} \]

so it is necessary to differentiate each of the expressions for potential. As an example, consider, as in Appendix A, just region \( V_{5} \) of figure 4. The loading coefficient will be divided into two parts \( \Delta p^{(0)} / g_{0} \) and \( \Delta p^{(0)} / g_{0} \) to correspond to the potentials \( \varphi^{(0)} \) and \( \varphi^{(0)} \). Thus, using equation (A11)

\[ \left. \frac{\partial \Delta p}{\partial t} \right|_{r_{0}} = \frac{4}{U_{0} M} \left\{ \int_{0}^{t_{1}} dy_{1} \int_{-\infty}^{+\infty} \sqrt{1 - \left( \varphi_{w} \right)^{2}} \, dx_{1} - \int_{0}^{t_{1}} dy_{1} \int_{-\infty}^{+\infty} \sqrt{1 - \left( \varphi_{w} \right)^{2}} \, dx_{1} + \right\} \tag{B2} \]

since the derivative passes the \( z_{1} y_{1} \) integration without effect. Referring to equation (A10) for the function \( C(x_{1}, y_{1}) \) we next find its derivative with respect to \( t \). Write \( r = t - t_{1} \); then for \( z_{1} < 0 \)

\[ C(x_{1}, y_{1}) = \int_{t_{1}/M}^{t_{1}/M} \frac{4g(y_{1}) w_{w}(x_{1} + M t_{1}, y_{1}) dt_{1}}{[(t_{1} - t_{1})^{2} - r_{1}^{2}]} \]

and

\[ \frac{\partial C}{\partial t} = \frac{\sqrt{4g(y_{1}) w_{w}(0, y_{1})}}{[(t + x_{1})^{2} - r_{0}^{2}]} \left( t + x_{1} \right) \, dt \]

\[ + \int_{t_{1}/M}^{t_{1}/M} \frac{4g(y_{1}) \, dt}{[(t_{1} - t_{1})^{2} - r_{1}^{2}]} \int_{t_{1}/M}^{t_{1}/M} \frac{4g(y_{1}) \, dt}{[(t_{1} - t_{1})^{2} - r_{1}^{2}]} \ int_{t_{1}/M}^{t_{1}/M} \frac{4g(y_{1}) \, dt}{[(t_{1} - t_{1})^{2} - r_{1}^{2}]} \]

\[ \frac{\partial C}{\partial t} = \frac{\sqrt{4g(y_{1}) w_{w}(x_{1}, y_{1})}}{[(t - t_{1})^{2} - r_{1}^{2}]} \, dt \]

Notice that if \( w_{w} \) does not depend on \( x_{1} + M t_{1} \) the integral term in equation (B3) vanishes, while if it does, then the integrated term is zero. Next, for \( z_{1} > 0 \),

\[ C(x_{1}, y_{1}) = \int_{t_{0}}^{t_{1}/M} \frac{\sqrt{4g(y_{1}) w_{w}(x_{1} + M t_{1}, y_{1})}}{[(t_{1} - t_{1})^{2} - r_{1}^{2}]} \, dt \]

and

\[ \frac{\partial C}{\partial t} = \frac{\sqrt{4g(y_{1}) w_{w}(x_{1}, y_{1})}}{[(t - t_{0})^{2} - r_{0}^{2}]} \, dt \]

In this case, both terms exist unless \( w_{w} \) is not a function of \( x_{1} + M t_{1} \), in which case the integral vanishes.

Substitution of equations (B3) and (B4) into equation (B2) will now yield an expression for the loading coefficient corresponding to the influence of the side edge;
The explicit form of \( w_{uu} \) given by equation (2), has been inserted and it is assumed that \( l \geq 1 \).

The portion of the loading coefficient corresponding to \( \varphi_{\kappa}^{(l)} \) can be found readily and is

\[
\left( \frac{\Delta p}{g_0} \right) |_{\kappa}^{(l)} = \frac{2a_{km}}{\pi U_0 c^{l+1}} \left\{ \int_{0}^{\tau_0} y_1 \left( x - \sqrt{(t - r_0)^2 - (y - y_0)^2} \right) \frac{1}{\sqrt{t^2 - (y - y_0)^2}} \, dy_1 + ight. \\
\left. Ml \int_{0}^{\tau_0} y_1 \, dy_1 \int_{x - \sqrt{(t - r_0)^2 - (y - y_0)^2}}^{x + \sqrt{(t - r_0)^2 - (y - y_0)^2}} \frac{[x_1 + M(t - r_0)]^{-1}}{r_0} \, dx_1 - ight. \\
\left. Ml \int_{0}^{\tau_0} y_1 \, dy_1 \int_{x - \sqrt{(t - r_0)^2 - (y - y_0)^2}}^{x + \sqrt{(t - r_0)^2 - (y - y_0)^2}} \frac{[x_1 + M(t - r_0)]^{-1}}{r_0} \, dx_1 - ight. \\
\left. Ml \int_{0}^{\tau_0} y_1 \, dy_1 \int_{x_1}^{x_1 + \sqrt{(t - r_0)^2 - (y - y_0)^2}} \frac{[x_1 + M(t - r_0)]^{-1}}{r_0} \, dx_1 - ight. \\
\left. \int_{0}^{\tau_0} y_1 \left( x - \sqrt{(t^2 - (y - y_0)^2)} \right) \frac{1}{\sqrt{t^2 - (y - y_0)^2}} \, dy_1 \right\} 
\]
It is clear that, even for small values of the indices \( l \) and \( n \), the required integrations for the determination of total forces on the wing pose formidable problems. There is, however, a property of the loading coefficient corresponding to vertical velocity distributions of the type chosen here (eq. (2)) that will materially shorten the requisite labor. This may be expressed as follows, adopting the convention that \( \Delta p^2 / q_0 \) corresponds to a downwash distribution proportional to \((x+\mathcal{M})y^n\):

\[
\frac{\partial}{\partial x} \frac{\Delta p^2}{q_0} = \frac{l}{c} \frac{\Delta p^{1-1,s}}{q_0}, \quad l > 0 \tag{B7}
\]

or

\[
\frac{\Delta p^2}{q_0} = \frac{l}{c} \int _{-\mathcal{M}} ^{x_1} \frac{\Delta p^{1-1,s}}{q_0} (x,y,t) \, dx, \quad l > 0 \tag{B8}
\]

**DETAILS OF EVALUATING THE GENERALIZED INDICIAL FORCES**

In calculating the generalized indicial forces by means of equation (36), it has been shown that only the value zero need be taken for the index \( l \). Thus we must find

\[
P_{ij} = \frac{2}{bc}\frac{\Delta p^2}{q_0} \int _{x_1} ^{x_2} (x+\mathcal{M})^l dx \int _{y_1} ^{y_2} \frac{\Delta p^2}{q_0} dy \tag{B9}
\]

The values of the loading coefficient \( \Delta p^2 / q_0 \) are found by differentiating the expressions for potential given in the first part of this appendix.

It is convenient, in evaluating equation (B9), to consider the integration with respect to \( y \) first. Setting

\[
L = \int _{0} ^{y} \frac{\Delta p^2}{q_0} \, dy \tag{B10}
\]

it is found that \( L \) seems to have different representations according to the interval in which \( z \) lies. These expressions can, however, all be expressed by the same formula. The portions of \( L \) corresponding to the parts \( \varphi_0 \) and \( \varphi_{2a} \) of the potential are similarly simplified, and we have

\[
L^{(1)} = \frac{2 \partial \varphi_0}{\pi \mathcal{M} \mathcal{C}^{n+s} z} \left\{ (-1)^n \frac{n!}{(n+g+1)!} [K_0(n+g) + K_M(n+g)] - 2 \sum _{\mu = 0} ^{[g/2]} \frac{(g)_{n+g+1-s}^2}{n+g+1-2\mu} [K_0(2\mu-1) + K_M(2\mu-1)] \right\} \tag{B11}
\]

\[
L^{(2)} = \frac{2 \partial \varphi_0}{\pi \mathcal{M} \mathcal{C}^{n+s} z} \left\{ [K_0(n+g) + K_M(n+g)] \right\} \tag{B12}
\]

where

\[
K_0(n+g) = t^{n+s+1} R. P. \int _{0} ^{\cos (\pi/z)} \sin ^{n+s+1} \theta d\theta
\]

\[
K_M(n+g) = \frac{M}{\mathcal{C}^{n+s+1} R. P.} \int _{0} ^{\cos (\pi/z)} \sin ^{n+s+1} \theta d\theta
\]

\[
J(n,g) = \frac{2}{\pi} \int _{0} ^{1} \frac{d\eta}{\sqrt{1-\eta^2}} \int _{\eta} ^{1} \frac{\eta(\eta+y)^g(\eta+y)^n}{\sqrt{1-\eta^2}} \sqrt{\frac{1}{\eta^2} - \eta^2} \, d\eta
\]

and \([n/2]\) means the greatest integer contained in \( n/2 \). The function \( J(n,g) \) may be expressed as summations, and it has the property

\[
J(n,g) = J(g,n) \tag{B13}
\]

The sum formula is, with \( g+p=n \)

\[
J(g,n) = (-1)^s \sum _{i=0} ^{[g/2]} \left( \begin{array} \frac{p}{2i} \end{array} ^s \right) \frac{1}{\pi} \sum _{i=0} ^{[p/2i]} \left( \begin{array} \frac{p}{2i} \end{array} ^s \right) (-1)^{s-i} \sum _{i=0} ^{[g/2i]} \left( \begin{array} \frac{p}{2i} \end{array} ^s \right) \left( \begin{array} \frac{2ij+3}{2i} \end{array} ^s \right) \left( \begin{array} \frac{2g+1}{2i} \end{array} ^s \right) \left( \begin{array} \frac{2g+1}{2i} \end{array} ^s \right) \tag{B14}
\]

It is convenient, in evaluating equation (B9), to consider the integration with respect to \( y \) first. Setting

\[
L = \int _{0} ^{y} \frac{\Delta p^2}{q_0} \, dy \tag{B10}
\]
Values of the function $J(g,n)$

<table>
<thead>
<tr>
<th>$g$</th>
<th>$0$</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
<th>$4$</th>
<th>$5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0$</td>
<td>$x-2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$1$</td>
<td>$1$</td>
<td>$-\frac{1}{4}x^4 + \frac{4}{3}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2$</td>
<td>$\frac{5}{4}x - \frac{8}{3}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{20}{64}x - \frac{16}{15}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$3$</td>
<td>$\frac{11}{6}x^3 + \frac{8}{5}$</td>
<td>$\frac{1}{3}$</td>
<td>$-\frac{53}{256}x^2 + \frac{32}{25}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$4$</td>
<td>$\frac{189}{64}x^2 - \frac{32}{5}$</td>
<td>$\frac{11}{15}$</td>
<td>$\frac{129}{256}x - \frac{128}{105}$</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{5529}{16384}x - \frac{256}{315}$</td>
<td></td>
</tr>
<tr>
<td>$5$</td>
<td>$\frac{71}{15}$</td>
<td>$-\frac{165}{256}x^4 + \frac{64}{21}$</td>
<td>$\frac{37}{84}$</td>
<td>$-\frac{975}{4096}x^4 + \frac{64}{21}$</td>
<td>$\frac{1}{5}$</td>
<td>$\frac{1}{65536}x^4 + \frac{512}{8192}$</td>
</tr>
</tbody>
</table>

where $\binom{p}{2i}$ is the binomial coefficient

$$\binom{p}{2i} = \frac{p!}{(2i)!(p-2i)!}$$

and $B(p, q)$ is the beta function

$$B(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx$$

$$= 2\int_0^{\pi/2} \sin^{2p-1} \theta \cos^{2q-1} \theta \, d\theta$$

$$= \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

The function $J(g, n)$ has been calculated for $g$, $n$ taken $0, 1, 2, 3, 4, 5$. Because of the property (B13), it is only necessary to give a triangular array, which appears in the above table.

Now consider the functions $K_0(v)$ and $K_\mu(v)$, defined after equation (B12). It is convenient, for computational purposes, to express these in terms of the incomplete beta functions, defined as

$$B_1(x, p, q) = \int_0^{\cos^{-1}(\omega)} \sin^{2p-1} \theta \cos^{2q-1} \theta \, d\theta$$

$$= \int_0^{1-\omega^2} \xi^{p-1}(1-\xi)^{q-1}d\xi$$

A tabulation of the incomplete beta functions is available in reference 11. Note that when the symbol $B$ is written without a subscript, the complete integral is meant, that is, in equation (B15b), $x$ equals 0. It is necessary to exercise some care when interpreting $K_0(v)$ and $K_\mu(v)$ as beta functions because of the upper limit. Thus, since

$$K_\phi(v) = t^{t+1}R.P. \int_0^{\cos^{-1}(\omega)} \sin^{t+1} \omega \, d\omega$$

we have the following cases:

(i) $x\geq t$, $R.P. \cos^{-1}\left(-\frac{x}{t}\right)$ $=$ $\pi$

(ii) $0 \leq x \leq t$, $R.P. \cos^{-1}\left(-\frac{x}{t}\right) = \cos^{-1}\left(-\frac{x}{t}\right) + \cos^{-1}\left(\frac{x}{t}\right)$

$$K_\phi(v) = t^{t+1} \left[ \frac{2B}{\frac{r+2}{2}, \frac{1}{2}} - B_{1\rightarrow 0\phi}\left(\frac{r+2}{2}, \frac{1}{2}\right) \right]$$

(iii) $-t \leq x \leq 0$, $R.P. \cos^{-1}\left(-\frac{x}{t}\right) = \cos^{-1}\left(-\frac{x}{t}\right)$

$$K_\phi(v) = t^{t+1} \left[ \frac{2B}{\frac{r+2}{2}, \frac{1}{2}} - B_{1\rightarrow 0\phi}\left(\frac{r+2}{2}, \frac{1}{2}\right) \right]$$

(iv) $-M \leq x \leq -t$, $R.P. \cos^{-1}\left(-\frac{x}{t}\right) = 0$

$$K_\phi(v) = 0$$

A similar line taken with $K_\mu(v)$ leads to

(i) $x \geq t$, $K_\mu(v) = 0$

(ii) $-t \leq x \leq t$, $K_\mu(v) = \frac{M}{\beta} t^{t+1} B_{1\rightarrow 0\mu}(\frac{r+2}{2}, \frac{1}{2})$  

(iii) $-t \leq x \leq -t$, $K_\mu(v) = \frac{M}{\beta} t^{t+1} B_{1\rightarrow 0\mu}(\frac{r+2}{2}, \frac{1}{2})$

The generalized indicial force $F_{p, q}^\phi$ can now be expressed as

$$F_{p, q}^\phi = \frac{8\sigma_\mu}{\pi M \sigma_x^{2+x+\phi+1}} \left\{ \frac{1}{4} \left[ \frac{J(g, n)}{2^{x+\phi} + 2(-1)^n} \frac{n! g!}{(n+g+1)!} \right] \right\}$$

$$\left\{ [J(g, n) + J_h(g, n)] - \sum_{n=0}^{[n]} \left( \frac{2\mu}{g+n+1-2\mu} \right) \frac{J^{x+n-1}}{g+n+1-2\mu} \right\}$$

B16
where

\[ {I}_0^* (\nu) = \int_{-M+1}^{0} \left( x+M \nu \right)^d x \left[ t^{r+1} R.P. \int_{0}^{\cos^{-1} (-\nu \phi)} \sin^{r+1} \theta \, d\theta \right] \]  
(B17)

\[ {I}_1 (\nu) = \int_{-M+1}^{0} \left( x+M \nu \right)^d x \left[ \frac{M}{\beta} t^{r+1} R.P. \int_{0}^{\cos^{-1} (-\nu \phi)} \sin^{r+1} \theta \, d\theta \right] \]  
(B18)

It is convenient to express these forces in terms of dimensionless quantities. Thus setting

\[ x_0 = \frac{x}{c}, \quad t_0 = \frac{t}{c} \]
we have

\[ *{I}_0^* (\nu) = c^{t+2} \int_{-M+1}^{0} \left( x_0+M t_0 \nu \right)^d x_0 \left[ t_0^{r+1} R.P. \int_{0}^{\cos^{-1} (-\nu \phi)} \sin^{r+1} \theta \, d\theta \right] = c^{t+2} {I}_{0'} (\nu) \]  
(B19)

\[ *{I}_1 (\nu) = c^{t+2} \int_{-M+1}^{0} \left( x_0+M t_0 \nu \right)^d x_0 \left[ \frac{M}{\beta} \left( x_0+M t_0 \nu \right)^d \right] R.P. \int_{0}^{\cos^{-1} \left( \frac{M t_0}{\beta} \nu \right)} \sin^{r+1} \theta \, d\theta = c^{t+2} {I}_{1'} (\nu) \]  
(B20)

and

\[ F_{10}^{m} = \frac{4a_0}{\pi M U_0} \left\{ \frac{1}{2h} \left[ J(g,n) + (-1)^n \frac{2}{g(n+1)-1} \right] \left[ {I}_0^* (g+n) + \frac{(\frac{n}{g})^{\frac{1}{2}}} {g^{\frac{n-1}{2}}} \left[ \frac{n}{g} \left( \frac{(\frac{n}{g})^{\frac{1}{2}}} \right) \right] \right] \right\} \]  
(B21)

The integrals \( {I}_0^* (\nu) \) and \( {I}_1 (\nu) \) can be simplified by reversing the order of integration. This can be accomplished in a straightforward manner by merely inspecting the region of integration in the \( x_0, t_0 \) plane. Consider first the integral \( {I}_0^* (\nu) \). Depending upon the relation between the chord length and the time, we see—from figure 17—that reversing the order of integration results in three different possibilities for the upper limit of the \( \theta \) integral. However, if we define \( x_0 \) such that

\[ x_0 = \frac{x}{c}, \quad t_0 = \frac{t}{c} \]

we have

\[ *{I}_0^* (\nu) = c^{t+2} \int_{-M+1}^{0} \left( x_0+M t_0 \nu \right)^d x_0 \left[ t_0^{r+1} R.P. \int_{0}^{\cos^{-1} (-\nu \phi)} \sin^{r+1} \theta \, d\theta \right] = c^{t+2} {I}_{0'} (\nu) \]  
(B19)

\[ *{I}_1 (\nu) = c^{t+2} \int_{-M+1}^{0} \left( x_0+M t_0 \nu \right)^d x_0 \left[ \frac{M}{\beta} \left( x_0+M t_0 \nu \right)^d \right] R.P. \int_{0}^{\cos^{-1} \left( \frac{M t_0}{\beta} \nu \right)} \sin^{r+1} \theta \, d\theta = c^{t+2} {I}_{1'} (\nu) \]  
(B20)

and

\[ F_{10}^{m} = \frac{4a_0}{\pi M U_0} \left\{ \frac{1}{2h} \left[ J(g,n) + (-1)^n \frac{2}{g(n+1)-1} \right] \left[ {I}_0^* (g+n) + \frac{(\frac{n}{g})^{\frac{1}{2}}} {g^{\frac{n-1}{2}}} \left[ \frac{n}{g} \left( \frac{(\frac{n}{g})^{\frac{1}{2}}} \right) \right] \right] \right\} \]  
(B21)

Figure 17.—Areas of integration used in analysis.

(i) \( x_0 = t_0; \quad 0 \leq t_0 \leq \frac{1}{M+1} \)

(ii) \( x_0 = 1-M t_0; \quad \frac{1}{M+1} \leq t_0 \leq \frac{1}{M-1} \)

(iii) \( x_0 = -t_0; \quad \frac{1}{M-1} \leq t_0 \)

then, in every case, \( {I}_0^* (\nu) \) can be written

\[ {I}_0^* (\nu) = \frac{t^{r+1}}{j+1} \sum_{j=0}^{M} \cos^{-1} (-\nu \phi) \int_{0}^{\cos^{-1} \theta \cos \phi} \sin^{r+1} \theta \, d\theta = \frac{t^{r+1}}{j+1} \sum_{j=0}^{M} \left( \frac{j+1}{r} \right) \frac{M^{j+1}}{r} \int_{0}^{\cos^{-1} (-\nu \phi) \sin \phi \cos \phi} \sin^{r+1} \theta \, d\theta \]  
(B22)

and, similarly, it can be shown that

\[ {I}_1 (\nu) = \frac{M}{\beta^{r+1}} \frac{1}{j+\nu+1} \sum_{j=0}^{M} \cos^{-1} (\frac{M x_0}{\beta}) \sin^{r+1} \theta \, d\theta + \]  

\[ \frac{M t_0^{r+1}}{j+\nu+2} \sum_{j=0}^{M} \left( \frac{j+1}{r} \right) \frac{M^{j+1}}{r} \int_{0}^{\cos^{-1} (-\nu \phi) \sin \phi \cos \phi} \sin^{r+1} \theta \, d\theta \]  
(B23)
APPENDIX C

DERIVATION OF RECIPROCITY RELATIONS

According to reference 5, the reciprocity relation for general three-dimensional unsteady motion can be written

\[ \int \int \int \frac{\Delta p}{\rho_{0}} (z_i, y_i, t_i) W_i(z_i, y_i, t_i) \ dz_i \ dy_i \ dt_i \]

\[ = \int \int \int \frac{\Delta p}{\rho_{0}} (z_i, y_i, t_i) W_i(z_i, y_i, t_i) \ dz_i \ dy_i \ dt_i \]  
\[ \text{ (C1) } \]

where the volume of integration \( V \) is that swept out in \( z, y, t \) space by the wing. The subscript 1 refers to the wing moving in the forward direction and subscript 2 refers to the wing moving in the opposite direction in the same manner. The coordinate systems are related by

\[ x_i = -x_2 + c - MT \]
\[ y_i = -y_2 + 2s \]
\[ t_i = -t_2 + T \]

where \( s \) is wing semispan and \( c \) is chord, respectively, and \( T \) is some fixed value of time. These quantities are elucidated in figure 18.

Figure 18.—Coordinate system in forward and reversed flow.

Now let the wing associated with the subscript 1 have the vertical velocity distribution

\[ w_1(z_i, y_i, t_i) = \left( \frac{z_i + M t_i}{c} \right)^m \left( \frac{y_i - s}{c} \right)^n \]

and that associated with the subscript 2 have

\[ w_2(z_2, y_2, t_2) = \left( \frac{z_2 + M t_2}{c} \right)^m \left( \frac{y_2 - s}{c} \right)^n \]

Then

\[ w_1(z_2, y_2, t_2) = \left( \frac{z_2 + M t_2}{c} \right)^m \left( \frac{y_2 - s}{c} \right)^n \]

\[ w_2(z_2, y_2, t_2) = \left( 1 - \frac{z_2 + M t_2}{c} \right)^m \left( \frac{y_2 - s}{c} \right)^n \]

\[ w_2(z_2, y_2, t_2) = \left( 1 - \frac{z_2 + M t_2}{c} \right)^m \left( \frac{y_2 - s}{c} \right)^n \]

Substitution of these results into equation (C1) yields

\[ \int_{0}^{T} dt \int_{-M T}^{c-M T} dz_1 \left( 1 - \frac{z_1 + M t_1}{c} \right)^m \int_{0}^{2s} dy_1 \left( \frac{y_1 - s}{c} \right)^n \frac{\Delta p_{1s}}{\rho_{0}} \]

\[ = \int_{0}^{T} dt \int_{-M T}^{c-M T} dz_2 \left( 1 - \frac{z_2 + M t_2}{c} \right)^m \int_{0}^{2s} dy_2 \left( \frac{y_2 - s}{c} \right)^n \frac{\Delta p_{2s}}{\rho_{0}} \]  
\[ \text{ (C2) } \]

Equation (C2) can be differentiated with respect to \( T \), yielding

\[ \int_{-M T}^{c-M T} dz \left( 1 - \frac{z + M T}{c} \right)^m \int_{0}^{2s} dy \left( \frac{y - s}{c} \right)^n \frac{\Delta p_{1s}}{\rho_{0}} \]

The binomial expansion is now performed:

\[ \sum_{n=0}^{\infty} (-1)^n \binom{n}{\mu} \binom{(1-\mu)}{\mu} \int_{-M T}^{c-M T} dz \left( \frac{z + M T}{c} \right)^m \int_{0}^{2s} dy \left( \frac{y - s}{c} \right)^n \frac{\Delta p_{1s}}{\rho_{0}} \]

\[ = \sum_{n=0}^{\infty} (-1)^n \binom{n}{\mu} \binom{(1-\mu)}{\mu} \int_{-M T}^{c-M T} dz \left( \frac{z + M T}{c} \right)^m \int_{0}^{2s} dy \left( \frac{y - s}{c} \right)^n \frac{\Delta p_{2s}}{\rho_{0}} \]  
\[ \text{ (C3) } \]

In equation (C3) the spanwise integration is carried over the whole wing, but it can easily be reduced to integration over, say, the left panel by use of the factor \( 1 + (1-\mu)^{1+n} \). Thus, equation (C3) can be written

\[ (-1)^n \sum_{\mu=0}^{\infty} (-1)^\mu \binom{n}{\mu} \binom{(1-\mu)}{\mu} \int_{-M T}^{c-M T} dz \left( \frac{z + M T}{c} \right)^m \int_{0}^{2s} dy \left( \frac{y - s}{c} \right)^n \frac{\Delta p_{1s}}{\rho_{0}} \]

\[ \int_{0}^{2s} dy \left( \frac{y - s}{c} \right)^n \frac{\Delta p_{1s}}{\rho_{0}} \]

By comparison with equations (36) and (37), it is seen that the integral terms in the last equation correspond to the generalized indicial forces \( f_{1s}^{\mu} \) and \( f_{2s}^{\mu} \), so that the summations can be written

\[ \sum_{\mu=0}^{\infty} (-1)^\mu \binom{n}{\mu} f_{1s}^{\mu} = \sum_{\mu=0}^{\infty} (-1)^\mu \binom{n}{\mu} f_{2s}^{\mu} \]

where the quantity \( (g+n) \) must be an even number.

REFERENCES

TABLE I.—VALUES OF GENERALIZED INDICIAL FORCES, $F_n$

The generalized indicial force coefficient $F_n$ is defined by equation (36). It is the response for a mode shape having a unit amplitude

$$\Psi_{A0} = -\left(\frac{x - M t}{c}\right)^n \phi_n$$

and a loading induced by a unit value of $w/U_0$,

$$\frac{w}{U_0} = -\left(\frac{x - M t}{c}\right)^n \phi_n$$

The table gives values of $F_n$ against time (actually chord lengths traveled) for $l=0; j=0; M=1.1$

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<th>$M=1.2$</th>
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TABLE I.—VALUES OF GENERALIZED INDICIAL FORCES, $F_n$

(a) $l=0; j=0; M=1.1$

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### TABLE I.—VALUES OF GENERALIZED INDICIAL FORCES, $F_n$—Continued

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| Table I.—Values of Generalized Indicial Forces, $P_e$—Continued
| (d) $1=1$; $j=1$; $M=1.1$

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| Table I.—Values of Generalized Indicial Forces, $P_e$—Continued
| (e) $1=0$; $j=0$; $M=1.2$

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**Note:** The table continues with more entries, but the provided excerpt is limited to the first few rows and columns for clarity. For a full dataset, please refer to the original source or a complete digital version of the document.
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**TABLE I.—VALUES OF GENERALIZED INDICIAL FORCES, $F_{k}$—Continued**

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**TABLE I.** VALUES OF GENERALIZED INDICIAL FORCES, \(F_{\xi}\) - Concluded