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GENERAL THEORY OF CONICAL FLOWS AND ITS APPLICATION TO SUPERSONIC AERODYNAMICS

By Paul Germain

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By

Paul Germain

Preface

By

M. J. Peres

N O T I C E

This report deals with a method of studying the equation of cylindrical waves particularly indicated for the solution of certain problems in aerodynamics. One of the most remarkable aspects of this method is that it reduces problems of a hyperbolic equation to problems of harmonic functions. We have applied ourselves here to setting up the fundamental principles, to developing their investigation up to calculation of the pressures on the visualized obstacles, and to showing how the initial field of "conical flows" was considerably enlarged by a procedure of integral superposition.

Such an undertaking entails certain dangers. In France the existence of conical flows was not known before 1946. Abroad, this question has, for a long time, given rise to numerous reports which either were not published or were published only after a certain delay. Thus it must be pointed out that some of the results here obtained, original in France when found, doubtlessly were not original abroad. Nevertheless it seems possible to me to specify a certain number of points treated in this report which, even considering the lapse of time, appear as new: the parts concerning homogeneous flows, the general study of conical flows with infinitesimal cone angles, the numerical or analogous methods for the study of flows flattened in one direction, and a certain number of the results of chapter IV. Moreover, even where the results which we found independently were already known abroad, the employed methods are not always identical.

Another peculiarity should be noted. Since these questions actually are everywhere the object of numerous investigations, progress has made very rapid strides. This report edited at the beginning of 1948, risks appearing, in certain aspects, slightly outmoded in 1949. To extenuate this inconvenience we have indicated in a brief appendix placed at the end of this report the progress made in these questions during the last year. This appendix is followed by a supplementary bibliography which indicates recent reports concerning our subject, or older ones of which we had no previous knowledge.

I should not have been able to successfully terminate this report without the advice and support of my teacher, Mr. J. Peres, and it is very important to me to express here my great respect for and gratitude to him.

I should equally cite all those who directly or less directly have contributed to my intellectual development and to whom I owe so much: my teachers of special mathematics and of normal school, Mr. Bouligand who directed my first reports, Mr. Villat, promoter of the Study of the Mechanics of Fluids in France whose brilliant instruction has been of the greatest value to me.

I also feel obliged to thank the directors of the O.N.E.R.A. who have facilitated my task, and especially Mr. Girerd, director of aerodynamic research.

PREFACE

With his research on conical flows and their application, Mr. Paul Germain has made a major contribution to the very timely study of supersonic aerodynamics. The present volume offers a comprehensive exposé which had been still lacking, an exposé of elegance and solid construction containing a number of original developments. The author has furthermore considered very thoroughly the applications and has shown how one may solve within the scope of linear theory, by combinations of conical flows, the general problems of the supersonic wing, taking into account dihedral and sweepback, and also fuselage and control surface effects. The analysis he develops in this respect leads him to methods which permit, either by calculation alone or with the support of electrolytic-tank experimentation, complete and accurate numerical determinations.

After a few preliminary developments (particularly on the validity of the hypothesis of linearization), chapter I is devoted to the generalities concerning conical flows. In such flows the velocity components depend only on two variables and their determination makes use of harmonic functions or of functions which verify the wave equation with two variables according to whether one is inside or outside of the Mach cone. Mr. Germain specifies the conditions of agreement between functions defined in one domain or in the other and shows that the study of conical flows amounts in general to boundary problems relative to three analytical functions connected by differential relationships. He studies, on the other hand, homogeneous flows which generalize the cone flows and are no less useful in the applications.

From the viewpoint of the linear theory of supersonic flows one must maintain two principal types of conical flows, bounded respectively by an obstacle in the form of a cone with infinitesimal cone angle, and by an obstacle in the form of a cone flattened in one direction.

The general investigation of the flows of the first type is entirely Mr. Germain's own and forms the object of chapter II of his book. By a subtle analysis of the approximations which may be legitimate Mr. Germain succeeds in simplifying the rather complex boundary problem he had to deal with; he replaces it by an external Hilbert problem. He shows how it is possible, after having obtained the solution for an orientation of the cone in the relative air stream, to pass, in a manner as simple as it is elegant, to the calculation of the effect of a change in incidence. He gives general formulas for the forces, treats completely diverse noteworthy special cases and finally applies the method of trigonometric operators which is also his own to the practical numerical calculation of the flow about an arbitrary cone.

The determination of movements about infinitely flattened cones has formed the object of numerous reports. The analysis which Mr. Germain develops for this question (chapter III) contributes simplifications, specifications, and important supplements. Thus he evolves, in the case of an obstacle inside the Mach cone, a principle of minimum singularity which enters into the determination of the solution. Mr. Germain gives two original methods for treatment of the general case: one utilizes the electrolytic-tank analogy, surmounting the difficulty arising from the experimental application of the principle of minimum singularity; the other, purely numerical, involves the trigonometric operators quoted above.

In the last chapter, finally, Mr. Germain visualizes the composition of conical flows with regard to aerodynamic calculation of a supersonic aircraft. Concerning this subject he develops a complete theory which covers most of the known results and incorporates new ones. He concludes with an outline of the flows past a flat dihedral, with application to the fins and control surfaces.

The creation of the National Office for Aeronautical Study and Research has already made possible the setting up of groups of investigators which do excellent work in several domains that are of interest to modern aviation and put us on the level of the best research centers abroad. Mr. Paul Germain inspirits and directs one of those groups in the most efficient manner. He is one of those, and the present report will suffice to bear out this statement, on whom we can count for the development of the study of aerodynamics in France.

Joseph Peres
Member of the Academy of Sciences

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CHAPTER I - GENERALITIES ON CONICAL FLOWS

1.1 - Equations of Supersonic Linearized Flows

1.1.1 - General Equation for the Velocity Potential

Let us visualize the permanent irrotational flow of a compressible perfect fluid for which the pressure p and the density ρ are mutual functions. The space in which the flow takes place will be fixed by three trirectangular axes Ox_1, Ox_2, Ox_3, the coordinates of a fluid molecule will be x_1, x_2, x_3, the projections on Ox_i of the velocity $\vec{V}$ and of the acceleration $\vec{A}$ of a molecule will be denoted by u_i and a_i, respectively.

The fundamental equations which permit determination of the flow are the Euler equations

$$\vec{A} = -\frac{1}{\rho}\,\mathrm{grad}\,\vec{p}$$

or

$$a_i = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} \tag{I.1}$$

the equation of continuity[1]

*"La théorie générale des mouvements coniques et ses applications a l'aérodynamique supersonique." Office National d'Études et de Recherches Aéronautiques, no. 34, 1949.

[1]We employ the classic convention of the silent index: $\frac{\partial}{\partial x_i}(\rho u_i)$ is to be read: $\frac{\partial}{\partial x_1}(\rho u_1) + \frac{\partial}{\partial x_2}(\rho u_2) + \frac{\partial}{\partial x_3}(\rho u_3)$.

$$\operatorname{div} \rho\vec{V} = 0 \qquad \text{or} \qquad \frac{\partial}{\partial x_i}(\rho u_i) = 0 \tag{I.2}$$

and the equation of compressibility

$$p = f(\rho)$$

If one notes that

$$a_i = u_k \frac{\partial u_i}{\partial x_k} \tag{I.3}$$

and introduces the sonic velocity[2]

$$c^2 = \frac{dp}{d\rho} \tag{I.4}$$

the equation (I.1) assumes the form

$$u_k \frac{\partial u_i}{\partial x_k} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} = -\frac{1}{\rho}\frac{dp}{d\rho}\frac{\partial \rho}{\partial x_i} = -\frac{c^2}{\rho}\frac{\partial \rho}{\partial x_i} \tag{I.5}$$

We introduce the velocity potential $\Phi(x_1, x_2, x_3)$, defined with the exception of one constant, by

$$\vec{V} = \overrightarrow{\operatorname{grad} \Phi} \qquad u_i = \frac{\partial \Phi}{\partial x_i}$$

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[2]The velocity of sound, introduced here by the symbol $\frac{dp}{d\rho}$ has a well-known physical significance; it is the velocity of propagation of small disturbances. This significance frequently permits an intuitive interpretation of certain results which we shall encounter later on (see section 1.1.4).

which is legitimate since we shall assume the flow to be irrotational. If we make the combination

$$u_i u_k \frac{\partial u_i}{\partial u_k} = \frac{\partial \Phi}{\partial x_i} \frac{\partial \Phi}{\partial x_k} \frac{\partial^2 \Phi}{\partial x_i \ \partial x_k}$$

one sees, taking into account equations (I.5) and (I.2), that

$$\frac{\partial \Phi}{\partial x_i} \frac{\partial \Phi}{\partial x_k} \frac{\partial^2 \Phi}{\partial x_i \ \partial x_k} = c^2 \frac{\partial^2 \Phi}{\partial {x_i}^2} \tag{I.6}$$

This equation is the general equation for the velocity potential. One may show, besides, that c is a function of the velocity modulus; thus one obtains an equation with partial derivatives of the second order, linear with respect to the second derivatives, but not completely linear.

The nonlinear character of the equation for the velocity potential makes the rigorous investigation of compressible flows rather difficult, at least in the three-dimensional case.

In order to be able to study, at least approximately, the behavior of wings, fuselages, and other elements of aeronautical structures, at velocities due to the compressibility, one has been led to introduce simplifying hypothesis which permit "linearization" of the equation for the velocity potential.

1.1.2 - The Hypotheses of Linearization and Their Consequences

For aerodynamic calculation, one may assume that the body around which the flow occurs has a position fixed in space and that the fluid at infinity upstream is moving with a velocity $\vec{U}$, $\vec{U}$ being a constant vector, the modulus of which will be taken as velocity unit. We shall always assume that the axis Ox_1 has the same direction as $\vec{U}$; the hypotheses of linearization amount to assuming that at every point of the fluid the velocity is reasonably equivalent to $\vec{U}$.

We put in a more precise manner

$$u_1 = 1 + u \qquad u_2 = v \qquad u_3 = w$$

u, v, w are, according to definition, the components of the "perturbation velocity."

(1) u, v, w are quantities which are very small referred to unity; if one considers these quantities as infinitesimals of the first order, one makes it at least permissible to neglect[3] in the equations all infinitesimals of the second order such as u^2, v^2, uv, etc.

(2) All partial derivatives of u, v, w with respect to the coordinates are equally infinitesimals at least of the first order so that one is justified in neglecting terms such as $u \frac{\partial u}{\partial x_1}$, $\left(\frac{\partial v}{\partial x_2}\right)^2$, etc.

One may deduce from these hypotheses a few immediate consequences:

(a) At every point of the field, the angle of the velocity vector with the axis Ox_1 is an infinitesimal of the first order at least. Hence there results a condition imposed on the body about which the flow is to be investigated; at every point the tangent plane must make a small angle with the direction of the nondisturbed flow (this is what one calls the uniform motion, defined by the velocity $\vec{U}$).

If one designates by q the velocity modulus, one has, taking the hypotheses setup into account

$$q^2 = (1 + u)^2 + v^2 + w^2 = 1 + 2u$$

whence

$$q = 1 + u$$

(b) The pressure p and the density ρ differ from the values p_1 and ρ_1 which these magnitudes assume at infinity upstream only by an infinitesimal of the first order; the equation (I.5) is written in effect

$$\frac{\partial u}{\partial x_1} = -\frac{{c_1}^2}{\rho_1}\frac{\partial \rho}{\partial x_1}$$

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[3]This signifies that u, v, w may very well not be infinitesimals of the same order; in this case one takes as the principal infinitesimal the perturbation velocity component which has the lowest order.

with c_1 denoting the sonic velocity at infinity upstream; thus

$$u = -\frac{c_1^2}{\rho_1}\left(\rho - \rho_1\right) \tag{I.7}$$

On the other hand, according to equation (I.4)

$$p - p_1 = c_1^2\left(\rho - \rho_1\right) = -\rho_1 u$$

If one defines the pressure coefficient C_p by

$$C_p = \frac{p - p_1}{\rho_1/2\left|\vec{U}\right|^2}$$

one has

$$C_p = -2u \tag{I.8}$$

(c) Finally, an examination of what becomes of the equation for the velocity potential (equation (I.6)) under these hypotheses shows that it is reduced to

$$\frac{\partial^2\Phi}{\partial x_1^2} = c_1^2\left(\frac{\partial^2\Phi}{\partial x_1^2} + \frac{\partial^2\Phi}{\partial x_2^2} + \frac{\partial^2\Phi}{\partial x_3^2}\right)$$

Let $\varphi\left(x_1,x_2,x_3\right)$ be the "disturbance potential," that is, the potential the gradient of which is identical with the disturbance-velocity vector; $\varphi\left(x_1,x_2,x_3\right)$ is the solution of the equation with partial derivatives of the second order

$$\frac{1 - c_1^2}{c_1^2}\frac{\partial^2\varphi}{\partial x_1^2} = \frac{\partial^2\varphi}{\partial x_2^2} + \frac{\partial^2\varphi}{\partial x_3^2} \tag{I.9}$$

a completely linear equation.

The Mach number of the flow is called the dimensionless constant $M = \frac{|\vec{U}|}{c_1}$ which, with the velocity unit to be chosen arbitrarily, is written here $M = 1/c_1$.

We put: $\epsilon(M^2 - 1) = \beta^2$, with ϵ being equal +1 or -1 according to whether M is larger or smaller than unity.

(1) If $M < 1$, equation (I.9) is written

$$\beta^2 \frac{\partial^2\varphi}{\partial x_1{}^2} + \frac{\partial^2\varphi}{\partial x_2{}^2} + \frac{\partial^2\varphi}{\partial x_3{}^2} = 0$$

an equation which may be easily reduced to the Laplace equation.

This equation applies to flows called "subsonic" because the velocity of the nondisturbed flow is smaller than the sonic velocity at infinity upstream. These flows will not be investigated in the course of this report[4].

(2) If $M > 1$, equation (I.9) is thus written

$$\beta^2 \frac{\partial^2\varphi}{\partial x_1{}^2} = \frac{\partial^2\varphi}{\partial x_2{}^2} + \frac{\partial^2\varphi}{\partial x_3{}^2} \tag{I.10}$$

This equation applies to "supersonic" flows; if one interprets x_1 as representing the time t, this equation is identical with the equation for cylindrical waves, well-known in mathematical physics. Investigation of this equation will form the object of this report.

Remarks.

(1) It should be noted that, in order to write the preceding equation, it was not necessary to specify the form of the equation for the state of the fluid. In particular, the formulas written above do not introduce the value of the exponent γ of the adiabatic relation $p = k\rho\gamma$ which is the form usually assumed by the equation of compressibility.

[4]Investigation of linear subsonic flows has formed the object of numerous reports. See references 1 and 2.

(2) The preceding analysis shows clearly the very different character of subsonic flows which lead to an elliptic equation, and of supersonic flows which are represented by a hyperbolic equation.

(3) When we wrote equation (I.9), we supposed implicitly that $M^2 - 1$ was not infinitely small, that is, that the flow was not "transonic," according to the expression of Von Kármán[5]. Thus it is impossible to make M tend toward unity in the results we shall obtain, in the hope to acquire information on the transonic case[6].

(4) It may happen, in agreement with the statement made in footnote 3, that u is an infinitesimal of an order higher than first. In this case, one will take up again the analysis made in paragraph (b) of section 1.1.2, which leads to a formula yielding the C_p, more adequate than the formula (I.8)

$$C_p = -2u - \left(v^2 + w^2\right) \tag{I.11}$$

1.1.3 - Validity of the Hypotheses of Linearization

Any simplifying hypothesis leads necessarily to results different from those which one would obtain with a rigorous method. Nevertheless, it was shown in certain numerical investigations on profiles (two-dimensional flows) where the rigorous method and the method of linearization were applied simultaneously that the approximation method provided a very good approximation for the calculation of forces. Besides, it is well-known that the classic Prandtl equation for the investigation of

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[5]Study of the transonic flows, with simplifying hypotheses analogous to those that have been made, requires a more compact analysis of the phenomena. It leads to a nonlinear equation, described for the first time by Oswatitsch and Wieghart (ref. 3). From it one may very easily deduce interesting relations of similitude for the transonic flows (ref. 4). One may find these relations also, in a very simple manner, by utilizing the hodograph plane.

[6]In a general manner, according to the values of M, one may be led to neglect certain terms in the final formulas found for the pressure coefficient C_p. This requires an evaluation, in every particular case, of the order of magnitude of the terms occurring in the formulas when M varies. In this report, we shall never enter into such a discussion. We shall limit ourselves voluntarily to the general formulas. An interesting example of such a discussion may be found in the recent memorandum of E. Laitone (ref. 5).

wings of finite span in an incompressible fluid furnishes very acceptable results, and the Prandtl equation results from a linearization of the rigorous problem.

It happens frequently, we shall have occasion several times to point it out, that the solution found for u, v, w will not satisfy the hypotheses of section 1.1.2 in certain regions (for example in the neighborhood of a leading edge); eventually certain ones among these magnitudes could even become infinite.

Under rigorous conditions such a solution should not be retained. Anyhow, if the regions where the hypotheses of linearization are not satisfied are "sufficiently small," it is permissible to assume that the expressions found for the forces (obtained by integration of the pressures) will still remain valid. This constitutes a justification a posteriori for the linearization method so frequently utilized in numerous aerodynamic problems[7]. Therefore, we shall not systematically discard the solutions found which will not wholly satisfy the hypotheses we set up.

1.1.4 - Limiting Conditions. Existence Theorem

Physically, the definition of sonic velocity leads to the rule which has been called the "rule of forbidden signals" (see footnote 2 of section 1.1.1) and which can be stated as follows:

A disturbance in a uniform supersonic flow, of the velocity U produced at a point P, takes effect only inside of a half-cone of revolution of the axis U and of the apex half-angle $\alpha = \text{Arc}\sin(1/M)$; $(\beta \cot \alpha)$ α is called the Mach angle, the half-cone in question "Mach after-cone at P."

Correlatively, one may state that the condition of the fluid at a point M (pressure, velocity, etc.) depends only on the character of the disturbances produced in the nondisturbed flow at points situated inside of the "Mach fore-cone at M;" the Mach fore-cone at a point is obviously the symmetrical counterpart of the Mach after-cone with respect to its apex.

If one wants to justify this rule from the mathematical viewpoint, one must start out from the formulas solving the problem of Cauchy and take into account the boundary conditions particular to the problem. Along the obstacle one must write that the velocity is tangent to the obstacle which gives the value $d\varphi/dn$. Moreover, at infinity

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[7]For instance, in the investigation of vibratory motions of infinitely small amplitude about slender profiles.

upstream $(x_1 = -\infty)$ the first derivatives of φ must be zero, since φ is, from the aerodynamic viewpoint, only determined to within a constant, it will be assumed zero.

The characteristic surfaces of the equation (I.10) are the Mach cones. If one of the Mach cones of the point P cuts off a region (R) on a surface (Σ), the classic study of the problem of Cauchy[8] shows that the value of φ at P is a continuous linear function of the values of φ and of $d\varphi/dn$ on R.

Let us therefore consider a point M of a supersonic flow such that its fore-cone does not intersect the obstacle. We take as the surface Σ a plane $x_1 = -A$, with A being of arbitrary magnitude. On Σ, φ and $d\varphi/dn$, which are continuous functions, will be arbitrarily small. Consequently the value of φ at M is zero. Thus one aspect of the rule of "forbidden signal" is justified.

Let us suppose that the forward-cone of M cuts off a region r(M) on the obstacle; on r(M), $d\varphi/dn$ is given by the boundary conditions; thus $\varphi(M)$ is a linear function of the values of φ on r(M).

One sees therefore that, if one makes M tend toward a point M_0 of the obstacle, one will obtain a functional equation permitting the determination of φ on the obstacle, at least in the case where the existence and uniqueness of the solution will be insured[9]. Consequently, $\varphi(M)$ depends only on the values of $d\varphi/dn$ in the region r(M); this justifies the fundamental result of the rule of "forbidden signals."[10]

1.1.5 - General Methods for Investigation of Linearized Supersonic Flows

In a recent article[11] dealing with the study of linear supersonic flows, Von Kármán indicates that two major general procedures exist for

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[8]For the problem of Cauchy, relative to the equation for cylindrical waves, see for instance references 6 and 7.

[9]Such a method has been utilized by G. Temple and H. A. Jahn, in their study of a partial differential equation with two variables (ref. 8).

[10]A more exact investigation of this question may be found in appendix 1, at the end of this report.

[11]See reference 4. A quick exposé of the methods in question may also be found in the text, in reference 2.

the study of these flows, one called "the source method," the other "the acoustic analogy."

The first is an old method and its theoretical application is fairly simple. It consists in placing on the outer surface of the obstacle a continuous distribution of singularities, called sources, the superposition of which gives at every point of the space the desired potential; the local strength of the sources may, in general, easily be determined with the aid of the boundary conditions. The second method utilizes a fundamental solution of the equation (I.10), the composition of which permits one to obtain the desired potential; this procedure is interesting in that it permits utilization of the Fourier integrals and thus furnishes, at least in certain particular cases, rather simple expressions for the total energy.

Von Kármán also indicates, at the end of his report, a third general procedure, that of conical flows.

We intend to investigate in this report the conical flows and the development of this third procedure which utilizes systematically the composition of the "conical flows" and, more generally, of the flows which we shall call "homogeneous flows of the order n." We shall see that this procedure permits one to find very easily, and frequently with less expenditure, a great number of the results previously obtained by other methods, and to bring to a successful end the investigation of certain problems which, to our knowledge, have not yet been solved.

1.2 - Generalities on Conical Flows

1.2.1 - History and Definition

Conical flows have been introduced by A. Busemann (ref. 9) who has given the principal characteristics of these flows and has indicated briefly in what ways they could be utilized in the investigation of supersonic flows. Busemann gives as examples some results, frequently without proof. Several authors have supplemented the investigation of Busemann: Stewart (ref. 10) has studied the case of the lifting wing Δ to which we shall come back later on; L. Beschkine (ref. 11) has furnished a certain number of results but generally without demonstration. We thought it of interest to attempt a summary of the entire problem.

One calls "conical flows" (more precisely, "infinitesimal conical flows")[12] the flows in which there exists a point O such that along

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[12]The adjective "infinitesimal" is remindful of the fact that the flows have been "linearized;" we shall henceforward omit this qualification since no confusion can arise in this report.

every straight line issuing toward one side of O, the velocity vector remains of the same value.

Let (π) be a plane not containing O, normal to the vector $\vec{U}$; let us suppose only that the velocity vector at every point of (π) is not normal to (π); the projection of these velocity vectors on (π) determines a field of vectors, the lines of force of which we shall call (γ): the cones (σ) of vertex O and directrix (γ) are "stream cones" for the flow.

More generally, let (S) be a stream surface of the flow, passing through O; every surface deduced from (S) by homothety of the center O and of k, k being an arbitrary positive number, is a stream surface. (S) is not necessarily a conical surface of apex O, but having (S) given as an obstacle does not permit one to foresee the existence of such a flow. It is different if a conical obstacle of apex O is given; the designation "conical flow" is thus justified.

Conversely, let us consider a cone of the apex O, situated entirely in the region $x_1 \geqslant 0$, and suppose that a linearized supersonic flow exists around this cone; this flow is necessarily a conical flow such as has just been defined; in fact, if $\vec{V}(x_1,x_2,x_3)$ denotes this velocity field, $\vec{V}(\lambda x_1,\lambda x_2,\lambda x_3)$ (λ being any arbitrary positive number) is equally a velocity field satisfying all conditions of the problem; consequently, if the uniqueness of the desired flow is admitted, $\vec{V}$ must be constant along every half-straight line from O [13].

Let us also point out that according to equations (I.8) or (I.11), the surfaces of equal pressure are also cones of the apex O.

1.2.2 - Partial Differential Equations Satisfied by the Velocity Components

According to definition, the velocity components of a conical flow depend only on two variables; on the other hand, as functions of x_1,

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[13]It should be noted that this argument will no longer be valid without restriction in the case of a real supersonic flow around a cone because in this case the principle of "forbidden signals" is no longer valid in the rigorous form stated. Among other possibilities, a detached shock wave may form upstream from the cone behind which the motion is no longer irrotational.

x_2, x_3, they are naturally the solution of the equation

$$\beta^2 \frac{\partial^2 f}{\partial {x_1}^2} = \frac{\partial^2 f}{\partial {x_2}^2} + \frac{\partial^2 f}{\partial {x_3}^2}$$

Let us first put

$$x_2 = r \cos \theta \qquad x_3 = r \sin \theta$$

the equation then assumes the form

$$\beta^2 \frac{\partial^2 f}{\partial {x_1}^2} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r} \frac{\partial f}{\partial r} \tag{I.12}$$

The second term of equation (I.12) is actually nothing else but the Laplacian of $f(x_1,x_2,x_3)$ in the plane x_2, x_3 (x_1 being considered as parameter); naturally $f(x_1,r,\theta)$ is periodic in θ, the period being equal to 2π.

To make the conical character of the flow evident, let us put

$$x_1 = \beta r \chi \tag{I.13}$$

χ is a new variable; $\chi < 1$ characterizes the exterior of the Mach cone with the apex 0, $\chi > 1$ characterizes the interior of the cone. Under these conditions, the disturbance-velocity components are functions only of χ and θ. Since f is a function of χ and θ only

$$d^2 f = \frac{\partial^2 f}{\partial x^2} dx^2 + \frac{2\partial^2 f}{\partial \chi \, \partial \theta} d\chi \, d\theta + \frac{\partial^2 f}{\partial \theta^2} \partial\theta^2 + \frac{\partial f}{\partial \chi} d^2\chi + \frac{\partial f}{\partial \theta} d^2\theta$$

but

$$d\chi = \frac{1}{\beta r}\left(dx_1 - \beta\chi \, dr\right)$$

$$d^2\chi = \frac{1}{\beta r}\left(d^2 x_1 - \beta\chi \, d^2 r - 2 \frac{dr \, dx}{r} 1 + 2\beta \frac{\chi}{r} dr^2\right)$$

$\dfrac{\partial^2 f}{\partial x_1{}^2}$, $\dfrac{\partial^2 f}{\partial r^2}$, $\dfrac{\partial^2 f}{\partial \theta^2}$, $\dfrac{\partial f}{\partial r}$ are the respective coefficients of $dx_1{}^2$, dr^2, $d\theta^2$, d^2r in the expression of d^2f as a function of the variables x_1, r, θ.

As a consequence, the equation (I.12) becomes under these conditions

$$\left(\chi^2 - 1\right)\frac{\partial^2 f}{\partial \chi^2} + \frac{\partial^2 f}{\partial \theta^2} + \chi \frac{\partial f}{\partial \chi} = 0 \tag{I.14}$$

One may try to simplify this equation further by replacing the variable χ by the variable ξ, χ and ξ being connected by a relationship $\chi = \chi(\xi)$, and by making a judicious choice for the function $\chi(\xi)$. The first operation gives

$$\left(\chi^2 - 1\right)\frac{\partial^2 f}{\partial \xi^2} + \chi'^2 \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial f}{\partial \xi}\left[\chi\chi' - \frac{\left(\chi^2 - 1\right)\chi''}{\chi'}\right] = 0$$

with the primes denoting derivatives with respect to ξ. For simplifying this equation, one may make the term in $\dfrac{\partial f}{\partial \xi}$ disappear. This will be realized by putting

(1) If $\chi > 1$,

$$\chi = \operatorname{ch} \xi \tag{I.15}$$

one obtains for f Laplace's equation

$$\frac{\partial^2 f}{\partial \xi^2} + \frac{\partial^2 f}{\partial \theta^2} = 0 \tag{I.16}$$

(2) If $\chi < 1$,

$$\chi = \cos \eta \tag{I.17}$$

in this case, one obtains the equation for waves with two variables

$$\frac{\partial^2 f}{\partial \eta^2} - \frac{\partial^2 f}{\partial \theta^2} = 0 \tag{I.18}$$

Geometrical interpretation.- $\chi > 1$ corresponds to the interior of the Mach rearward cone (Γ) of the point O; every semi-infinite line, issuing from O, inside of this cone, has as image a point θ, ξ. One will assume, for instance, $-\pi < \theta \leqslant \pi$; $\xi = 0$ corresponds to the cone (Γ), $\xi = \infty$ corresponds to the cone axis (it will always be possible to assume ξ as positive). The image of the interior of (Γ) forms therefore on the region (A) of the plane (θ,ξ) (fig. 1), limited by the semi-infinite lines AT, A'T' and by the segment AA'. The correspondence is double valued in the sense that to a semi-infinite line issuing from O there corresponds one point and one only (θ,ξ) in the bounded region and conversely, to one point of this region there corresponds one semi-infinite line, and one only, issuing from O, inside of (Γ).

Since we shall suppose, in general, that the cone investigated is entirely in the region $x_1 \geqslant 0$, only this region will be of interest (φ then being identically zero for $x_1 < 0$). The semi-infinite lines of this region issuing from O, outside of (Γ), correspond to $0 < \chi < 1$ (fig. 2), that is, according to equation (I.17), $0 < \eta < \frac{\pi}{2}$; $\eta = 0$ corresponds to the cone (Γ), $\eta = \frac{\pi}{2}$ to the plane $x_1 = 0$; the semi-infinite lines issuing from O correspond biunivocally to the points of the region (A'), inside of the rectangle AA'B'B in the plane (θ,η).

Summing up, the velocity components satisfy the simple equations (I.16) and (I.18), the first of which is relative to the region (A), the second to the region (A').

1.2.3 - Fundamental Theorem

The equation (I.14) which represents the fundamental equation of our problem is an equation of mixed type; it is elliptic or hyperbolic according to whether χ is larger or smaller than unity. In order to study this equation in a simpler manner, we have been led to divide the domain of the variables into two parts and to represent them on two different planes. How an agreement will be reached between the solutions obtained for f in the two planes - that is the question which will be completely elucidated by the following theorem which will be fundamental in the course of our investigation.

Theorem: There exists "agreement" as to $\chi = 1$ for all derivatives of f, defined in either the region (A) or (A'), provided that there is "agreement" for the function itself.

In fact, let us take two functions $f_1(\theta,\xi)$, $f_2(\theta,\eta)$, the first satisfying the equation (I.16) in the region (A), the second the equation (I.18) in the region (A'), both assuming the same values $\varphi(\theta)$ on the respective segments $(\xi = 0, -\pi < \theta < \pi)$ $(\eta = 0, -\pi < \theta < \pi)$. Let θ_0 be the abcissa of a point of AA'. If $\frac{\partial^n f_1}{\partial\theta^n}(\theta_0,0)$ exists, $\frac{\partial^n f_1}{\partial\theta^n}(\theta_0,0) = \frac{d^n\varphi}{d\theta^n}$; consequently $\frac{\partial^n f_2}{\partial\theta^n}(\theta_0,0)$ exists and

$$\frac{\partial^n f_1}{\partial\theta^n}(\theta_0,0) = \frac{\partial^n f_2}{\partial\theta^n}(\theta_0,0)$$

Let us now pass to the investigation of the derivatives of the order n of the form $\frac{\partial^n f}{\partial\chi\partial\theta^{n-1}}$: the equation (I.14) shows first that

$$\frac{\partial f}{\partial\chi}(\theta,1) = -\frac{\partial^2 f}{\partial\theta^2}(\theta,1)$$

which proves that all partial derivatives of the order 1 with respect to χ have the same value on (Γ), whether they are calculated starting from f_1 or from f_2. The argument develops without difficulty through recurrence. By deriving equation (I.14) n times with respect to χ and making $\chi = 1$, one obtains

$$(2n + 1)\frac{\partial^{n+1} f}{\partial\chi^{n+1}} + n^2\frac{\partial^n f}{\partial\chi^n} + \frac{\partial^{n+2} f}{\partial\theta^2\partial\chi^n} = 0$$

which finally shows that the values $\frac{\partial^{n+p} f}{\partial\theta^p\partial\chi^n}$ can be uniquely expressed as a function of the derivatives of $\varphi(\theta)$ with respect to θ and that they, consequently, have the same value, whether calculated starting from f_1 or from f_2.

Summing up, one may say that it is sufficient for the establishment of the "agreement" between two solutions defined in (A) and (A'), if these solutions assume the same value on the segment AA'.

1.2.4 - Mode of Dependence of the Semi-Infinite Lines Issuing From 0

If one puts in the plane (θ,η)

$$\theta + \eta = 2\lambda \qquad \theta - \eta = 2\mu \tag{I.19}$$

one sees that the characteristics of the equation (I.18) are the parallels to the bisectrices $\lambda = c^{te}$, $\mu = c^{te}$. These characteristics are, in the plane (η,θ), the images of the planes

$$x_1 = \beta r \cos(2\lambda - \theta) \quad \text{and} \quad x_1 = \beta r \cos(\theta - 2\mu)$$

which are the planes tangent to the cone (Γ). The characteristics passing through a point $\delta_0(\theta_0,\eta_0)$ are the images of two planes tangent to the cone (Γ) which one may lay through the semi-infinite Δ_0 corresponding to the point δ_0 of the plane (θ,η) (fig. 3). The generatrices of contact are characterized on the cone by the values θ_1 and θ_2 of the angle θ. One encounters here a result which seems to contradict indications of section 1.1.4. This apparent contradiction is immediately explained if one notes that, since all points of a semi-infinite Δ_0 issued from 0 are equivalent, one must consider at the same time all Mach cones, the apexes of which are situated on Δ_0; the group of these cones admits as envelope precisely the two planes tangent to the cone (Γ) passing through Δ_0. We shall call "Mach dihedron posterior" to the semi-infinite Δ_0 that one of the dihedra formed by the two planes which contains the group of the Mach cones to the rear of the points of Δ_0. The region inside of this dihedron and outside of the cone (Γ) has as image in the plane (θ,η) the triangle $\theta_1\,\delta_0\theta_2$. A semi-infinite Δ_1 will be said to be dependent on or independent of Δ_0 according to whether the image of Δ_1 will be inside or outside of the triangle $\theta_1\,\delta_0\theta_2$. This argument also explains why the equation (I.14) shows elliptic character inside of (Γ). More precisely, two semi-infinite lines Δ_1 and Δ_2, inside of (Γ), are in a state of neutral dependence (ref. 9). In fact, let M_1 be a point of Δ_1, M_2 a point of Δ_2; let us suppose that M_1 is outside of the Mach forward cone of M_2; according to the argument of section 1.1.4 the point M_2 seems to be independent of M_1; but on the other hand, if one assumes M_1'

to be a point of Δ_1, inside of the Mach forward cone of M_2, M_1' . and M_1 are equivalent which explains that M_2 is actually not independent of M_1 (fig. 4).

1.2.5 - The Conditions of Compatibility

Thus one may foresee how the solution of a problem of conical flow will unfold itself. One will attempt to solve this problem in the region (A') which will generally be fairly easy since the general solution of the equation (I.18) is written immediately by adjoining an arbitrary function of the variable $\theta + \eta$ to an arbitrary function of the variable $\theta - \eta$. This will have the effect of "transporting" onto the segment AA' the boundary conditions relative to the region (A'). Applying the fundamental theorem, one will be led to a problem of harmonic functions in the region (A). But taking as unknown functions the components u, v, w, of the disturbance velocity, we have introduced three unknown functions (while there was only one when we dealt with the function φ). One must therefore write certain relationships of compatibility which express finally that the motion is indeed irrotational.

The motion will be irrotational if $u\,dx_1 + v\,dx_2 + w\,dx_3$ is an exact differential which will be the case when, and only when

$$x_1\,du + x_2\,dv + x_3\,dw = r(\beta\chi\,du + \cos\theta\,dv + \sin\theta\,dw)$$

is an exact differential. This can occur only if this expression is identically zero, with u, v, w being functions uniquely of θ and of χ (the total differential not containing a term in dr must be independent of r):

In a conical flow the potential is written

$$\varphi = ux_1 + vx_2 + wx_3 = r(\beta u\chi + v\cos\theta + w\sin\theta)$$

with u, v, w being the disturbance-velocity components.

One will note that φ is proportional to r.

Moreover

$$\beta\chi\,du + \cos\theta\,dv + \sin\theta\,dw = 0 \tag{I.20}$$

This is the relationship which is to be written, and this is the point in question, on one hand in the plane (θ,η), on the other in the plane (θ,ξ).

(a) Relations in the plane (θ,η). One may write

$$u = u_1(\lambda) + u_2(\mu)$$

and analogous formulas for v and w, λ and μ being defined by the relations (I.19). One has in particular

$$\frac{du_1}{d\lambda} = \frac{\partial u}{\partial \eta} + \frac{\partial u}{\partial \theta} \qquad \frac{du_2}{d\mu} = \frac{\partial u}{\partial \theta} - \frac{\partial u}{\partial \eta}$$

Besides, according to equation (I.20)

$$\begin{aligned} &\beta \cos \eta \; du_1 + \cos \theta \; dv_1 + \sin \theta \; dw_1 = 0 \\ &\beta \cos \eta \; du_2 + \cos \theta \; dv_2 + \sin \theta \; dw_2 = 0 \end{aligned} \qquad \text{(I.21)}$$

however: $\theta = \lambda + \mu$, $\eta = \lambda - \mu$; and consequently the first equation (I.21) is written

$$\cos \mu \left[\beta \cos \lambda \; du_1 + \cos \lambda \; dv_1 + \sin \lambda \; dw_1\right] +$$
$$\sin \mu \left[\beta \sin \lambda \; du_1 - \sin \lambda \; dv_1 + \cos \lambda \; dw_1\right] = 0$$

since the two quantities between brackets are unique functions of λ, the preceding equality causes

$$\beta \cos \lambda \; du_1 + \cos \lambda \; dv_1 + \sin \lambda \; dw_1 = 0$$

$$\beta \sin \lambda \; du_1 - \sin \lambda \; dv_1 + \cos \lambda \; dw_1 = 0$$

or

$$-\beta \; du_1 = \frac{dv_1}{\cos 2\lambda} = \frac{dw_1}{\sin 2\lambda} \qquad \text{(I.22)}$$

In the same manner one will show that

$$-\beta \, du_2 = \frac{dv_2}{\cos 2\mu} = \frac{dw_2}{\sin 2\mu} \qquad \text{(I.23)}$$

(b) Relations in the plane (θ,ξ).

The calculation is perfectly analogous. The equation (I.16) causes us to introduce the complex variable $\zeta = \theta + i\xi$ and the functions $U(\zeta)$, $V(\zeta)$, $W(\zeta)$, defined with the exception of an imaginary additive constant, the real parts of which in (A) are, respectively, identical to $u(\theta,\xi)$, $v(\theta,\xi)$, $w(\theta,\xi)$.

The equation (I.20) permits one to write

$$\beta \operatorname{ch} \xi \, dU + \cos \theta \, dV + \sin \theta \, dW = 0$$

If one puts

$$\theta + i\xi = \zeta \qquad \theta - i\xi = \bar{\zeta}$$

one obtains

$$\cos \frac{\bar{\zeta}}{2}\left[\beta \cos \frac{\zeta}{2} \, dU + \cos \frac{\zeta}{2} \, dV + \sin \frac{\zeta}{2} \, dW\right] +$$

$$\sin \frac{\bar{\zeta}}{2}\left[\beta \sin \frac{\zeta}{2} \, dU - \sin \frac{\zeta}{2} \, dV + \cos \frac{\zeta}{2} \, dW\right] = 0$$

thence one concludes as previously

$$-\beta \, dU = \frac{dV}{\cos \zeta} = \frac{dW}{\sin \zeta} \qquad \text{(I.24)}$$

The formulas (I.22), (I.23), (I.24) express the relationships of compatibility which we had in mind.

Remark.

We shall utilize frequently the conformal representation for studying the problems relative to the domain (A). If one puts, in particular

$$Z = e^{i\zeta} = e^{-\xi}e^{i\theta}$$

one sees that (A) becomes in the plane Z the interior area of the circle (C_0) with the center O [14] and the radius 1 (fig. 5).

If one puts $Z = \rho e^{i\theta}$, the point Z is the image of a semi-infinite line, issuing from the origin of the space (x_1,x_2,x_3), characterized by the angle θ and the relationship

$$\frac{x_1}{\beta r} = \chi = \frac{1 + \rho^2}{2\rho}$$

The origin of the plane Z corresponds to the axis of the cone (Γ), the circle (C_0) to the cone (Γ) itself. A problem of conical flow appears in a more intuitive manner in the plane Z than in the plane ζ. In the plane Z, the formulas (I.24) are written

$$-\beta\, dU = \frac{2Z\, dV}{Z^2 + 1} = 2iZ\,\frac{dW}{Z^2 - 1} \qquad (I.25)$$

We shall moreover utilize the plane z defined by

$$z = \frac{2Z}{Z^2 + 1}$$

The domain (A) corresponds conformably to the plane z notched by the semi-infinite lines Ax, A'x' (fig. 6), the cone (Γ) at the edges of the cuts thus determined, and the axis of the cone (Γ) at the origin

---

[14] No confusion is possible between the point O, origin of the system of axes x_1, x_2, x_3 and the point O, here introduced as the origin of the plane Z.

of the plane z. The relations of compatibility in the plane z then assume the form

$$-\beta\ dU = z\ dV = -\frac{iz\ dW}{\sqrt{1 - z^2}} \tag{I.26}$$

1.2.6 - Boundary Conditions

The Two Main Types of Conical Flows

The boundary conditions are obtained by writing that the velocity vector is tangent to the cone obstacle. Let, for instance, $x_2(t)$, $x_3(t)$ be a parametric representation of the section $x_1 = \beta$ of the cone; $x_3x_2' - x_2x_3'$, $\beta x_3'$, $-\beta x_2'$ constitute a system of direction parameters of the normal to the cone obstacle, and the boundary condition reads

$$wx_2' - vx_3' = \frac{1}{\beta}\left(x_3x_2' - x_2x_3'\right)(1 + u) \tag{I.27}$$

It will be possible to simplify this condition according to the cases. However, the simplification will have to be treated in a different manner according to the conical flows investigated. As set forth in section 1.1.2, two main types of conical flows may exist.

(1) The flow about cones with infinitesimal cone angles, that is, cones where every generatrix forms with the vector $\vec{U}$ an angle which remains small. Naturally, the cone section may, under these conditions, be of any arbitrary form; since the flow outside of (Γ) is undisturbed (velocity equivalent to $\vec{U}$), on the cone (Γ) u, v, w are zero.

The problem may have to be treated in the plane Z; $U(Z)$, $V(Z)$, $W(Z)$ will have real parts of zero on (C_0). The image (C) of the obstacle, in the plane Z, is defined by a relation $\rho = f(\theta)$; consequently, a parametric representation of the section $x_1 = \beta$ will be obtained by means of the formulas

$$x_2 = \frac{2\rho}{1 + \rho^2}\cos\theta \qquad x_3 = \frac{2\rho}{1 + \rho^2}\sin\Theta$$

Thus the condition (I.27) becomes

$$w\left[\rho \sin \theta - \rho' \cos \Theta + \rho^2(\rho \sin \theta + \rho' \cos \theta)\right] +$$
$$v\left[\rho \cos \theta + \rho' \sin \theta + \rho^2(\rho \cos \theta - \rho' \sin \theta)\right] = \frac{2\rho^2}{\beta}(1 + u) \qquad (I.28)$$

with θ taken as parameter, and ρ' denoting the derivative of ρ with respect to θ. The investigation of conical flows with infinitesimal cone angles will form the object of chapter II.

(2) The flow about flattened cones, that is, cones, the generatrices of which deviate only little from a plane containing $\vec{U}$. Let us remember that (section 1.1.2) the tangent plane is to form a small angle with $\vec{U}$; consequently, rigorously speaking, the section of such a cone cannot be a regular closed curve, an ellipse for instance; it must present a lenticular profile (fig. 7). In chapter III we shall study the flows about such cones.

Remark.

Actually, we have, therewith, not exhausted all types of conical flows, that is, those for which linearization is legitimate. One may, for instance, obtain flows about cones, the section of which presents the form shown in figure 8; the axis of such a cone has infinitely small inclination toward $\vec{U}$.

Before beginning the study of these flows we shall, in order to terminate these generalities, introduce a generalization of the flows, the possible utilization of which we shall see in a final chapter.

1.3 - Homogeneous Flows

1.3.1 - Definition and Properties

The conical flows are flows for which the velocity potential is of the form

$$\varphi = rf(\theta,\chi)$$

as we had seen in section 1.2.5. One may visualize flows for which

$$\varphi = r^n f(\theta,\chi)$$

We shall call them homogeneous flows of the nth order[15]. The conical flows defined in section 1.2 are, therefore, homogeneous flows of the order I. However, we shall maintain the expression "conical flow" to designate these flows since this term has been used by numerous authors and gives a good picture.

The derivatives of the velocity potential with respect to the variables x_1, x_2, x_3 all satisfy the equation (I.10). If one then considers the derivatives of the nth order of the potential of an homogeneous flow of the nth order, one finds that they depend only on X and θ and satisfy the equation (I.14); the analysis made in section 1.2.2 remains entirely valid. One may make the changes in variables (I.15) and (I.17) which lead to the equations (I.16) and (I.18). Thus one has here a method sufficiently general to obtain solutions of the equation (I.10) which may prove useful.

The simplest flows are the homogeneous flows of the order 0 which do not give rise to any particular condition of compatibility. For the flows of nth order, in contrast, one has to write a certain number of conditions connecting the derivatives of nth order. We shall examine[16] as an example the case of homogeneous flows of 2nd order.

There are six second derivatives which we shall denote φ_{ij} (i and j may assume independently the values 1, 2, 3), φ_{ij} designating $\frac{\partial^2\varphi}{\partial x_i\,\partial x_j}$. Outside of (Γ) we shall put

$$\varphi_{ij} = \varphi_{ij}{}^1 + \varphi_{ij}{}^2$$

with $\varphi_{ij}{}^1$ being a function of λ only, $\varphi_{ij}{}^2$ of μ only (see formula I.19). Inside of (Γ), φ_{ij} is the real part of a function $\Phi_{ij}(\zeta)$.

In order to obtain the desired relations, it is sufficient to note that

---

[15]The definition for homogeneous flows of the nth order has been given for the first time by L. Beshkine (ref. 11); this author, by the way, calls them conical flows of the nth order. One may also connect this question with the article of Hayes (ref. 12).

[16]See appendix 2.

$$\varphi_{ij}\ dx_j = d\varphi_i$$

and to apply the results of section 1.2.5; thus one may write the following six relations between the $\varphi_{ij}^{\ 1}$

$$-\beta\ d\varphi_{i1}^{\ 1} = \frac{1}{\cos 2\lambda}\ d\varphi_{i2}^{\ 1} = \frac{1}{\sin 2\lambda}\ d\varphi_{i3}^{\ 1} \qquad (i = 1,2,3)$$

which, besides, are reduced to five as one sees immediately. One will have analogous relations for the functions $\varphi_{ij}^{\ 2}$ (it is sufficient to exchange the role of λ and of μ).

Finally, one has for the analytic functions $\Phi_{ij}(\zeta)$

$$-\beta\ d\Phi_{i1} = \frac{1}{\cos \zeta}\ d\Phi_{i2} = \frac{1}{\sin \zeta}\ d\Phi_{i3}$$

namely six relations which as before are reduced to five. The written conditions are not only necessary but also sufficient since the functions φ_i necessarily are the components of a gradient. Thus one sees that there is no difficulty in writing the conditions of compatibility for a homogeneous flow of nth order.

1.3.2 - Relations Between the Homogeneous Flows of nth and of (n-1)th Order

We shall establish a theorem which can be useful in certain problems and which specifies the relations existing between homogeneous flows of nth and of (n-1)th order; we shall examine the case where $n = 1$.

1.3.2.1.- Let us consider inside of the cone (Γ) a homogeneous flow of the order 0 defined by

$$\varphi = \underline{R}\left[\Phi(Z)\right]$$

We shall first of all seek the components u, v, w of the disturbance velocity

$$d\varphi = u\, dx_1 + v\, dx_2 + w\, dx_3 = \underline{R}\left[\Phi'(Z)\, dZ\right] = \underline{R}\left[Z\Phi'(Z)\, \frac{dZ}{Z}\right]$$

then

$$\frac{dZ}{Z} = \frac{d\rho}{\rho} + i\, d\theta$$

$$x_1 = \beta r\, \frac{1 + \rho^2}{2\rho} \qquad x_2 = r \cos\theta \qquad x_3 = r \sin\theta$$

thus

$$\frac{d\rho}{\rho} = \frac{\rho^2 + 1}{\rho^2 - 1}\left[\frac{dx_1}{x_1} - \frac{x_2\, dx_2 + x_3\, dx_3}{r^2}\right]$$

$$d\theta = \frac{x_2\, dx_3 - x_3\, dx_2}{r^2}$$

whence one deduces

$$u = \frac{\rho^2 + 1}{\rho^2 - 1}\, \frac{1}{x_1}\, \underline{R}\left[Z\Phi'(Z)\right]$$

$$v = -\frac{\cos\theta}{r}\, \frac{\rho^2 + 1}{\rho^2 - 1}\, \underline{R}\left[Z\Phi'(Z)\right] + \frac{\sin\Theta}{r}\, \underline{T}\left[Z\Phi'(Z)\right]$$

$$w = -\frac{\sin\theta}{r}\, \frac{\rho^2 + 1}{\rho^2 - 1}\, \underline{R}\left[Z\Phi'(Z)\right] - \frac{\cos\theta}{r}\, \underline{T}\left[Z\Phi'(Z)\right]$$

however

$$Z + \frac{1}{Z} = \frac{\rho^2 + 1}{\rho} \cos\theta + i\, \frac{\rho^2 + 1}{\rho} \sin\theta$$

$$Z - \frac{1}{Z} = \frac{\rho^2 - 1}{\rho} \cos\theta + i\, \frac{\rho^2 - 1}{\rho} \sin\theta$$

hence the result

$$\left.\begin{aligned}
u &= \frac{1}{x_1}\,\frac{\rho^2 + 1}{\rho^2 - 1}\,\underline{R}\left[Z\Phi'(Z)\right] \\
v &= \frac{1}{x_1}\,\frac{\rho^2 + 1}{\rho^2 - 1}\,\underline{R}\left[-\frac{\beta}{2}\left(Z^2 + 1\right)\Phi'(Z)\right] \\
w &= \frac{1}{x_1}\,\frac{\rho^2 + 1}{\rho^2 - 1}\,\underline{R}\left[\frac{i\beta}{2}\left(Z^2 - 1\right)\Phi'(Z)\right]
\end{aligned}\right\} \tag{I.29}$$

1.3.2.2.- Let us now consider a point O' $\left(x_1 = \epsilon_1,\ x_2 = 0,\ x_3 = 0\right)$, ϵ_1 being a very small quantity. Let M be a point with the coordinates $\left(x_1,r,\theta\right)$ with respect to O, inside of (Γ), and with the parameters (ρ,θ) in the plane Z. For the conical flow (homogeneous of 1st order) with the vertex O', its coordinates are: $(x_1 - \epsilon_1,\ r,\ \theta)$ and its parameters in the Z-plane:

$$\rho\left(1 - \frac{\rho^2 + 1}{\rho^2 - 1}\,\frac{\epsilon_1}{x_1}\right),\theta$$

since

$$dx_1 = -\epsilon_1 = \beta r\,\frac{\rho^2 - 1}{2\rho^2}\,d\rho = x_1\,\frac{\rho^2 - 1}{\rho^2 + 1}\,\frac{d\rho}{\rho}$$

Let us then consider two identical conical fields but with the apexes O and O', and form their difference. We shall obtain a velocity field which, due to the linear character of the equation (I.10), will satisfy this equation. If

$$u_0 = \underline{R}\left[F(Z)\right]$$

denotes the component u of the field with the vertex O, one has as component u in the "difference field"

$$u = +\underline{R}\left[F(Z)\right] - \underline{R}\left[F\left(Z - \frac{\rho^2 + 1}{\rho^2 - 1}\,\frac{\epsilon_1}{x_1}\,Z\right)\right] = \frac{\rho^2 + 1}{\rho^2 - 1}\,\frac{\epsilon_1}{x_1}\,\underline{R}\left[ZF'(Z)\right] \tag{I.30}$$

ϵ_1 being considered as infinitely small. Moreover, according to the relations (I.25), the components v and w are written

$$\left.\begin{aligned} v &= \frac{\epsilon_1}{x_1}\frac{\rho^2+1}{\rho^2-1}\,\underline{R}\left[-\frac{\beta}{2}\left(Z^2+1\right)F'(Z)\right] \\ w &= \frac{\epsilon_1}{x_1}\frac{\rho^2+1}{\rho^2-1}\,\underline{R}\left[i\,\frac{\beta}{2}\left(Z^2-1\right)F'(Z)\right] \end{aligned}\right\} \tag{I.31}$$

1.3.2.3.- Let us consider the point $O''(0,\epsilon_2,0)$, with ϵ_2 being a small quantity. Let M be a point with the coordinates $(x_1,r,0)$ with respect to O, inside of (Γ), with the parameters (ρ,θ) in the plane Z. For the flow with apex O'', the coordinates of M are $\left(x_1,\ r-\epsilon_2\cos\theta,\ \theta+\epsilon_2\frac{\sin\theta}{r}\right)$ as can be easily stated by projecting M in m on the plane x_2x_3 (fig. 9). But on the other hand

$$dr = \frac{2x_1}{\beta}\frac{1-\rho^2}{\left(1+\rho^2\right)^2}\,d\rho = -\epsilon_2\cos\theta$$

$$r\,d\theta = \frac{2x_1}{\beta}\frac{\rho}{1+\rho^2}\,d\theta = \epsilon_2\sin\theta$$

thus

$$dZ = e^{i\theta}\left[d\rho + i\rho\,d\theta\right] = \epsilon_2\,\frac{1+\rho^2}{2x_1}\,\beta\left[i\sin\theta - \cos\theta\,\frac{1+\rho^2}{1-\rho^2}\right]e^{i\theta}$$

with Z + dZ representing the point M in the conical field with the vertex O''.

Let us then consider two identical conical flows, but with the apexes O and O'', and form their difference. We shall obtain a velocity field which due to the linear character of the equation (I.10) will satisfy this equation. If

$$v_0 = \underline{R}\left[G(Z)\right]$$

denotes the component v in the field of the vertex O, one has a component v in the "difference field"

$$
\begin{aligned}
v &= +\underline{R}\left[G(Z)\right] - \underline{R}\left[G(Z + dZ)\right] = -\underline{R}\left[G'(Z)dZ\right] \\
&= -\frac{\epsilon_2\beta}{2x_1}\,\frac{\rho^2 + 1}{\rho^2 - 1}\,\underline{R}\left[G'(Z)\left[\cos\theta\left(1 + \rho^2\right) + i\sin\theta\left(\rho^2 - 1\right)\right]e^{i\theta}\right] \\
&= -\frac{\epsilon_2\beta}{2x_1}\,\frac{\rho^2 + 1}{\rho^2 - 1}\,\underline{R}\left[ZG'(Z)\left(Z + \frac{1}{Z}\right)\right] \\
&= \frac{\epsilon_2}{x_1}\,\frac{\rho^2 + 1}{\rho^2 - 1}\cdot\underline{R}\left[-\frac{\beta}{2}\left(Z^2 + 1\right)G'(Z)\right]
\end{aligned}
\tag{I.32}
$$

besides, according to equation (I.25), the components u and w are written

$$
\left.
\begin{aligned}
u &= \frac{\epsilon_2}{x_1}\,\frac{\rho^2 + 1}{\rho^2 - 1}\,\underline{R}\left[ZG'(Z)\right] \\
w &= \frac{\epsilon_2}{x_1}\,\frac{\rho^2 + 1}{\rho^2 - 1}\,\underline{R}\left[\frac{i\beta}{2}\left(Z^2 - 1\right)G'(Z)\right]
\end{aligned}
\right\}
\tag{I.33}
$$

<u>1.3.2.4</u>.- With these three lemmas established, it is easy to demonstrate the property we have in mind. Let us call "complex potential" of a homogeneous flow of zero order the function $\Phi(Z)$ (section 1.3.2.1) so that

$$
\varphi = \underline{R}\left[\Phi(Z)\right]
$$

so that the function of complex variable, the real part of which gives inside of (Γ) the projection of the disturbance velocity in the direction $\vec{l}$, is the "complex velocity" of a conical field in the direction $\vec{l}$; so that, finally, the velocity field obtained by the difference of two identical conical fields, the vertices of which are infinitely close and ranged on a line parallel to $\vec{l}$, is the "field derived from a conical flow" in the direction $\vec{l}$; then we may state:

<u>Theorem: The field derived from a conical flow in the direction $\vec{l}$ is the velocity field of a homogeneous flow of zero order; the complex potential of that flow of zero order is proportional to the complex velocity of the conical field given in the direction $\vec{l}$, since the proportionality factor is real.</u>

The proof follows immediately. According to sections 1.1.2 and 1.1.3, one may be satisfied with considering, for definition of a homogeneous flow, the inside of the cone (Γ); comparison of the formulas (I.29), (I.30), (I.31), (I.32), (I.33) entails the validity of the above theorem if $\vec{l}$ is parallel or orthogonal to $\vec{U}$. Hence the general case where $\vec{l}$ is arbitrary may be deduced immediately; if $F(Z)$, $G(Z)$, $H(Z)$ are the complex velocities in projection on Ox_1, Ox_2, Ox_3, the expression for the component u of the field derived in the direction $\vec{l}(\epsilon_1,\epsilon_2,\epsilon_3)$ is

$$u = \frac{1}{x_1}\frac{\rho^2 + 1}{\rho^2 - 1}\,\underline{R}\left[Z\left[\epsilon_1 F'(Z) + \epsilon_2 G'(Z) + \epsilon_3 H'(Z)\right]\right]$$

Thus, with $\epsilon_1 F(Z) + \epsilon_2 G(Z) + \epsilon_3 H(Z)$ being the complex velocity in projection on $\vec{l}$, comparison of this formula with the first formula (I.29) completely demonstrates the theorem.

<u>Corollary: The field derived in the direction $\vec{l}$ of a conical flow, the complex velocity of which in the direction $\vec{l}$ is $K(Z)$, is a velocity field of a homogeneous flow dependent only on $K(Z)$ (not on the direction $\vec{l}$).</u>

The theorem just demonstrated may be extended without difficulty to the homogeneous flows of nth and (n-1)th order. A statement of this general theorem would require only specification of a few definitions; however, since we shall not have to utilize it later on, we shall not formulate this statement.

CHAPTER II - CONICAL FLOWS WITH INFINITESIMAL CONE ANGLES*

2.1 - Solution of the Problem

2.1.1 - Generalities

We shall now treat the first problem set up in section 1.2.6. We shall operate in the plane Z. Let us recall that the image of the cone (Γ) is the circle (C_0) of radius unity centered at the origin, and that the image of the obstacle is a curve (C), defined by its polar equation $\rho(\theta)$. We shall denote by (D) the annular domain comprised between (C) and (C_0); we shall call (γ_0) the circle of smallest radius centered at the origin and containing (A) in its interior, and we shall call k the radius of the circle (γ_0). In this entire chapter, k will be considered as the principal infinitesimal.

The problem then consists in finding three functions $U(Z)$, $V(Z)$, $W(Z)$ defined inside of (D) except for an additive imaginary constant, so that

(1)

$$-\beta\, dU = \frac{2Z}{Z^2 + 1}\, dV = \frac{2iZ}{Z^2 - 1}\, dW \tag{I.25}$$

(2) the real parts u, v, w, which are uniform become zero on (C_0),

(3) on (C), one has the relation

$$\begin{aligned} &v\left[\rho \cos\theta + \rho' \sin\theta + \rho^2(\rho\cos\theta - \rho'\sin\theta)\right] + \\ &w\left[\rho\sin\theta - \rho'\cos\theta + \rho^2(\rho\sin\theta + \rho'\cos\theta)\right] = \frac{2\rho^2}{\beta}(1 + u) \end{aligned}$$

Put in this manner, the problem is obviously very hard to solve in its whole generality; however, an analysis of the permissible approximations will simplify it considerably.

2.1.2 - Investigation of the Functions $U(Z)$, $V(Z)$, $W(Z)$

2.1.2.1.- An analytical function of Z will be the said function (A) if its real part becomes zero on (C_0). Let us designate by

*NACA editor's note: Some minor inconsistencies appear in the numbering of equations in this chapter and subsequently in chapters III and IV, but no attempt was made to change the numbering as given in the original text.

(γ_0') the circle with the radius $1/k$, centered at the origin, and by (D') the annulus limited by (γ_0) and (γ_0') (fig. 10).

Lemma I.- A uniform function (A), defined inside the annulus limited by (γ_0) and (C_0) may be continued over the entire domain (D').

This results immediately from Schwartz' principle. Let M and M' be two symmetrical points with respect to (C_0), M being inside of (C_0); one defines the function (A) at the point (M') as having, respectively, an opposite real and an equal imaginary part compared to the real and the imaginary part of the function given at the point M.

Lemma II.- A holomorphic function (A) inside of (D') has a Laurent development of the form[17]

$$i\beta + \sum_1^\infty \left(\frac{K_n}{Z^n} - \bar{K}_n Z^n\right)$$

Let $H(Z) = h + ih'$ be such a function (A). Let us write its Laurent development in (D') provisorily in the form

$$H(Z) = \sum_0^\infty J_n Z^n + \sum_1^\infty \frac{K_n}{Z^n}$$

It is an immediate demonstration and yields the formulas defining J_n and K_n

$$K_n = \frac{k^n}{2\pi}\int_0^{2\pi} (h + ih')_{\gamma_0} e^{in\theta} d\theta$$

[17]We remember that $\bar{K}_n$ denotes the conjugate imaginary of K_n.

$(h + ih')_{\gamma_0}$ denoting the value of H on (γ_0); likewise

$$J_n = \frac{k^n}{2\pi}\int_0^{2\pi} (h + ih')_{\gamma_0'} e^{-in\theta} d\theta$$

Consequently, according to the lemma I:

$$\overline{K}_n = -J_n$$

moreover

$$J_0 = \frac{1}{2i\pi}\int_{c_0} H(Z)\frac{dZ}{Z} = \frac{1}{2\pi}\int_0^{2\pi} (h + ih')_{c_0} d\theta = \frac{i}{2\pi}\int_0^{2\pi} h_{c_0}' d\theta$$

is purely imaginary, and the lemma II is therewith demonstrated.

We shall note that, if $H(Z)$ is limited by M on (γ_0) or (γ_0'), one has the inequality

$$|K_n| < Mk^n \tag{II.1}$$

Lemma III.- A function (A) with a real and uniform part defined in (D) can be developed inside of (D') in the form

$$B \log Z + i\beta + \sum_1^{\infty}\left(\frac{K_n}{Z^n} - \overline{K}_n Z^n\right) \tag{II.2}$$

with B being real.

Actually, the derivative of the function (A) is necessarily uniform. Thus one knows (see for instance ref. 13) that one may consider the given function as the sum of a uniform function $H(Z)$ and a logarithmic term; since the critical point of the logarithm is arbitrary inside of (γ_0), it is particularly indicated to choose this point at the origin; since the real part of the function is uniform, the coefficient of $\log Z$ is real. Besides, since $\log Z$ has a real part zero on (C_0), $H(Z)$ is itself a function (A). The given function may therefore be continued inside of (D') and the development (II.2) is thus justified.

Remark.

If one chooses as pole of the logarithmic term a point inside of (γ_0) but different from the origin, one obtains a development of the form

$$B' \log\left(\frac{Z - a}{\bar{a}Z - 1}\frac{\bar{a} - 1}{1 - a}\right) + i\beta + \sum_{1}^{\infty} \left(\frac{K_n'}{Z^n} - \bar{K}'_n Z^n\right)$$

2.1.2.2.- The functions U, V, W of the variable Z are all three functions (A) with a real uniform part and, consequently, can be developed in the form (II.2). We shall write henceforward

$$\left.\begin{aligned} -\frac{\beta}{2} U(Z) &= A \log Z + i\alpha + \sum_{1}^{\infty} \left(\frac{J_n}{Z^n} - \bar{J}_n Z^n\right) \\ V(Z) &= B \log Z + i\beta + \sum_{1}^{\infty} \left(\frac{K_n}{Z^n} - \bar{K}_n Z^n\right) \\ W(Z) &= C \log Z - i\gamma + \sum_{1}^{\infty} \left(\frac{L_n}{Z^n} - \bar{L}_n Z^n\right) \end{aligned}\right\} \qquad \text{(II.3)}$$

A, B, C are real, α, β, γ are real and also arbitrary; but these developments are not independent since the relations (I.25) must be taken into account. For instance, $Z\,dV/dZ$ must be divisible by $Z^2 + 1$; otherwise we would have for U logarithmic singularities on the cone (Γ) which is inadmissible. Now

$$Z \frac{dV}{dZ} = B - \sum_{1}^{\infty} n\left(\frac{K_n}{Z^n} + \bar{K}_n Z^n\right)$$

Hence one deduces the relations

$$\left.\begin{aligned} B &= \sum_{1}^{\infty} (-1)^{p} 2p\left[K_{2p} + \bar{K}_{2p}\right] \\ 0 &= \sum_{0}^{\infty} (-1)^{p}(2p+1)\left[K_{2p+1} - \bar{K}_{2p+1}\right] \end{aligned}\right\} \qquad (II.4)$$

obtained by putting in the preceding equality $Z = i$ and $Z = -i$.

Likewise, $Z\, dW/dZ$ must be divisible by $Z^2 - 1$ which gives

$$\left.\begin{aligned} C &= \sum_{1}^{\infty} 2p\left(L_{2p} + \bar{L}_{2p}\right) \\ 0 &= \sum_{0}^{\infty} (2p+1)\left(L_{2p+1} + \bar{L}_{2p+1}\right) \end{aligned}\right\} \qquad (II.5)$$

Finally, the equalities (I.25) lead, in addition, to relationships connecting the coefficients of the developments (II.3) among themselves; thus one may write the relations

$$B + 2K_2 = -i\left[C - 2L_2\right] \qquad \bar{K}_1 - K_1 = -i\left[\bar{L}_1 + L_1\right] \qquad (II.6)$$

$$nK_n - (n-2)K_{n-2} = i\left[(n-2)L_{n-2} + nL_n\right] \qquad (n \geqslant 2)$$

and on the other hand

$$\left.\begin{aligned} B &= -\left(\bar{J}_1 + J_1\right) \\ K_1 &= -A + 2J_2 \\ nK_n &= (n-1)J_{n-1} + (n+1)J_{n+1} \qquad (n \geqslant 2) \end{aligned}\right\} \qquad (II.7)$$

2.1.2.3. Approximations for the developments (II.3).- Moreover, the hypotheses of linearization must be taken into account which, as we shall see, will permit us to simplify the developments (II.3) considerably and will lead us in a very simple manner to the solution of the problem posed in section 2.1.1.

The equalities (II.6) make $V(Z)$ and $W(Z)$ seem of the same order. We shall denote by M an upper limit of their modulus on the circle (γ_0). M will be equally an upper limit of their modulus on (γ_0') and hence in the entire domain (D').

If one utilizes the inequality (II.1), (II.4) shows that[18]

$$B = O\left(Mk^2\right) \qquad K_1 - \overline{K}_1 = O\left(Mk^2\right)$$

If one assumes α, β, γ zero in what follows, which does not at all impair the generality, one may write the second formula (II.3) in the form

$$V(Z) - \underline{R}\left(K_1\right)\left(\frac{1}{Z} - Z\right) - \sum_2^{\infty} \frac{K_n}{Z^n} = B \log Z - \sum_2^{\infty} \overline{K}_n Z^n + i\underline{T}\left(K_1\right)\left(\frac{1}{Z} + Z\right)$$

and consequently:

In the annulus limited by (γ_0) and (C_0), the second term of this equality is

$$O\left(Mk^2 \log k\right)$$

Likewise according to equation (II.5)

$$C = O\left(Mk^2\right) \qquad L_1 + \overline{L}_1 = O\left(Mk^3\right)$$

$$W(Z) - i\underline{T}\left(L_1\right)\left(\frac{1}{Z} + Z\right) - \sum_2^{\infty} \frac{L_n}{Z^n} = C \log Z + \underline{R}\left(L_1\right)\left(\frac{1}{Z} - Z\right) - \sum_2^{\infty} \overline{L}_n Z^n$$

---

[18] O denotes Landau's symbol, $A = O\left(Mk^2\right)$ signifies that $\dfrac{A}{Mk^2}$ is limited when k tends toward zero.

In the annulus comprised between (γ_0) and (C_0), the second term of this equality is also

$$O\left(Mk^2 \log k\right)$$

Furthermore, according to equation (II.6)

$$K_{n-2} + iL_{n-2} = O\left(Mk^n\right) \qquad (n > 2)$$

Thus

$$W(Z) - iV(Z) = O\left(Mk^2 \log k\right) + 2iK_1 Z$$

in the annulus (γ_0, C_0).

Finally, according to equation (II.7)

$$A = -K_1 + O\left(Mk^3\right) \qquad J_n = \frac{n+1}{n} K_{n+1} + O\left(Mk^{n+2}\right)$$

Thus

$$-\frac{\beta}{2} U(Z) = -\underline{R}\left(K_1\right) \log Z - 2\bar{K}_2 Z + \sum_1^{\infty} \frac{n+1}{n} \frac{K_{n+1}}{Z^n} + O\left(Mk^3 \log k\right)$$

Summing up: If one is satisfied with defining $V(Z)$ and $W(Z)$ except for $O\left(Mk^2 \log k\right)$ and $U(Z)$ except for $O\left(Mk^3 \log k\right)$, one may write in the corona (γ_0, C_0)

$$W(Z) = iV(Z) + 2iK_1 Z \tag{II.8}$$

$$V(Z) = H(Z) - K_1 Z \tag{II.9}$$

with

$$H(Z) = \sum_{1}^{\infty} \frac{K_n}{Z^n} \tag{II.10}$$

and

$$U(Z) = -\frac{2}{\beta}\left[\int Z \frac{dH}{dZ}\, dZ - 2\overline{K}_2 Z\right] \tag{II.11}$$

The coefficient K_1 may be supposed to be real, and the integration occurring in equation (II.11) must be made in such a manner that $\underline{R}\left[U(Z)\right]$ will be an infinitely small quantity of the third order at least on $|Z| = 1$.

2.1.2.4 - Remarks.

(1) The formula (II.8) which is the most important may be established immediately from the second formula (I.25). However, the method followed in the text, even though a little lengthy, seems to us more natural; also, it shows more clearly the developments of the functions U, V, W.

(2) Strictly speaking, the hypotheses set forth in the course of this study must be verified by the solutions found in each particular case. We shall, however, omit this verification which in the usual cases is automatically satisfactory.

(3) The results obtained by the preceding analysis and condensed in the formulas (II.8), (II.9), (II.11) are in all strictness valid only in the annulus (γ_0, C_0), but not in the domain (D). However, it is very easy to extend, by analytical continuation, the definition of H to (D). Let us now first suppose that (C) contains 0 in its interior; since one may write V(Z) in the form

$$V(Z) = H(Z) - \sum_{1}^{\infty} \overline{K}_n Z^n + B \log Z$$

one sees that, since V(Z) is defined by hypothesis in (D), and one can extend $\sum_{1}^{\infty} \overline{K}_n Z^n$ and B log Z inside of (γ_0) up to (C), H(Z) may itself be defined without difficulty inside of (D). The case where (C) does not contain the origin offers no difficulty; it is then sufficient to utilize the development given at the end of section 2.1.2.1.

As to the order of the terms neglected when one writes the equality (II.9) in the domain (D), they are found to be $O(Mk^2 \log k)$ in (D) in the case where there exists inside of (C) a circle of the radius λk (λ and $1/\lambda$ may be considered as $O(1)$). Besides, if that is not the case, one may justify the validity of the results of the formulas (II.8), (II.9), (II.10), (II.11) by making a conformal representation of the domain (D) on an annulus; the radius of the image circle of (C_0) may be assumed equal to unity; the image circle of (C) has a radius infinitely small of first order with respect to k and the study may be carried out in the new plane of complex variable thus introduced, without essential complication.

2.1.3 - Reduction of the Problem to a Hilbert Problem

If one puts, according to the formula (II.8)

$$V = v + iv'$$

with v' denoting the imaginary part of V, one may write on (C) the relation

$$w = -v'$$

Since one may, of course, with the accepted approximations, neglect u compared to 1 in the second term of the formula (I.28), one sees that this boundary condition (I.28) affects now only one single analytical function, the function V(Z); this is a first fundamental consequence of the preceding study. Formula (II.9) shows that this condition consists in posing a linear relation between the real and the imaginary part of H(Z) on the obstacle. Now according to equation (II.10) the function H(Z) is a holomorphic function outside of (C), regular at infinity; the problem stated which initially referred to an annular area (D) is thus reduced to a Hilbert problem for the function H defined in a simply connected region; exactly speaking, one has to solve an exterior Hilbert problem. This is the second fundamental consequence of the results of section 2.1.2.

Since we attempt to calculate V(Z) and W(Z) not further than within $O(Mk^2 \log k)$, and U(Z) within $O(Mk^3 \log k)$, the relation (I.28) which is written

$$\underline{R}\left[(v - iw)\left[2Z\rho^2 d\theta - i\ dZ\left(1 - \rho^2\right)\right]\right] = \frac{2\rho^2}{\beta}\ d\theta$$

may be simplified and reduced to

$$\underline{R}\left[-\ i\ dZ(v - iw)\right] = \frac{2\rho^2}{\beta}\ d\theta$$

On (C), $K_1 Z$ is, according to equation (II.1), of the order of Mk^2, and therefore

$$H = V = v + iv' = v - iw$$

consequently, H satisfies, on (C), the Hilbert condition

$$\underline{R}\left[-\ iH(Z)\ dZ\right] = \frac{2\rho^2}{\beta}\ d\theta \tag{II.12}$$

2.1.4 - Solution of the Hilbert Problem

A function $H(Z)$, holomorphic outside (C), regular and zero at infinity, satisfying on (C) the relation (II.12) must be found. Let

$$z = Z + a_0 + \frac{a_1}{Z} + \ . \ . \ . \tag{II.13}$$

be the conformal canonical representation of the outside of (C) on the outside of a circle (γ) centered at the origin of the plane z; the adjective canonical simply signifies that z and Z are equivalent at infinity.

On (γ) we shall put

$$z = re^{i\varphi}$$

r being constant and well determined. Let us put

$$F(Z) = i \log \frac{Z}{r} \tag{II.14}$$

One has on (C) or on (γ)

$$F'(Z)\ dZ = i\ \frac{dz}{z} = -d\varphi = f(\theta)\ d\theta \tag{II.15}$$

with f being real; consequently

$$\underline{R}\left[-\ iH\ \frac{dZ}{d\theta}\right] = \underline{R}\left[iH\ \frac{d\varphi}{d\theta}\ \frac{1}{F'(Z)}\right] = \frac{d\varphi}{d\theta}\ \underline{R}\left[i\ \frac{H(Z)}{F'(Z)}\right]$$

and therefore equation (II.12) is written

$$\underline{R}\left[i\ \frac{H(Z)}{F'(Z)}\right] = 2\ \frac{\rho^2}{\beta}\ \frac{d\theta}{d\varphi} \tag{II.16}$$

$H(Z)/F'(Z)$ is a holomorphic function outside of (C) and regular at infinity. Following a classical procedure, we thus have reduced the Hilbert problem to an exterior problem of Dirichlet.

Let $G(Z)$ be the holomorphic function outside of (C), real at infinity; its real part assumes on (C) the values $\frac{2\rho^2}{\beta}\ \frac{d\theta}{d\varphi}$. $G(Z)$ is determined in a unique manner. According to equation (II.12)

$$H(Z) = -iG(Z)F'(Z) + i\epsilon F'(Z) \tag{II.17}$$

with ϵ being a real constant.

However, we have seen (section 2.1.2.3) that the coefficient of $1/Z$ in the development of $H(Z)$ around the point at infinity (coefficient K_1) was real; now, around the point at infinity

$$iF(Z) = -\ \frac{1}{Z} + \frac{a_0}{Z^2} + \ .\ .\ .$$

In order to have the development of the second term of the formula (II.17) admit a real coefficient of $1/Z$, ϵ must be zero since $G(Z)$ is real at infinity. Thus the desired solution is

$$H(Z) = -iG(Z)F'(Z) \tag{II.18}$$

With the function $H(Z)$ thus determined, the formulas (II.8), (II.9), (II.11) permit calculation of the complex velocities $U(Z)$, $V(Z)$, $W(Z)$ within the scope of the accepted approximations. Thus the problem posed in section 2.1.1 is solved.

Remarks.

(1) Uniqueness of the solution.- The preceding reasoning shows the solution of the Hilbert problem satisfying the conditions (II.16) to be unique. This result will be valid for our problem if one shows that every function satisfying the condition (II.16) is a solution of the initially posed problem (condition (II.14)) which is immediate since it suffices to repeat the calculation.

(2) Calculation of the coefficient K_1.- According to what has been said above, the coefficient K_1 is equal to the (real) value assumed by $G(Z)$ at infinity. In order to find $G(Z)$, we may solve the Dirichlet problem in the plane z; according to a classic result of the study of harmonic functions, K_1 is equal to the mean value of $2\rho^2 \frac{d\theta}{d\varphi}$ on the circle (γ). Hence

$$K_1 = \frac{1}{2\pi}\int_0^{2\pi} \frac{2\rho^2}{\beta}\frac{d\theta}{d\varphi}\,d\varphi = \frac{1}{\pi\beta}\int_{(C)} \rho^2 d\theta = \frac{2S}{\pi\beta}$$

wherein S represents the area inside the contour (C).

2.2 - Applications

2.2.1 - General Remark

Let us consider a cone of the apex O in the space (Ox_1,x_2,x_3), the image of which in the plane Z is the curve (C), defined by its polar equation $\rho(\theta)$. According to the definition of ρ (see the remark of section 1.2.5) the sections of this cone made by planes parallel to Ox_2x_3 are homothetic to the curve

$$x_2 = \frac{2\rho}{1 + \rho^2} \cos \theta \qquad x_3 = \frac{2\rho}{1 + \rho^2} \sin \theta \tag{II.19}$$

In the case of the linear approximations, with $\overrightarrow{\text{grad}}\ u$, $\overrightarrow{\text{grad}}\ v$, $\overrightarrow{\text{grad}}\ w$ being infinitely small (it would even be sufficient that they should be limited), one sees that one may, within the scope of the approximations of section 2.1, simplify the formulas (II.19) without inconvenience and write them

$$x_2 = 2\rho \cos \theta \qquad x_3 = 2\rho \sin \theta$$

hence the result, essential for the applications.

The curve (C) in the plane Z is homothetic to the sections of the cone obstacle made by planes normal to the nondisturbed velocity.

Let us likewise consider a cone with variable but small incidences so that the flow about the cone should always be a flow in accordance with the hypotheses of this chapter. One sees that if the orientation of the cone varies with respect to the wind, the curve (C) in the plane Z undergoes a translation.

2.2.2 - Study of a Cone of Variable Incidence

This last remark allows us to foresee that when a thorough investigation of a cone has been made for a certain orientation with respect to the velocity it will not be necessary to repeat all the work for any other orientation. This we shall specify after having demonstrated the following lemma.

2.2.2.1 - Lemma.- One may write on (C) that

$$\frac{2\rho^2}{\beta} \frac{d\theta}{d\varphi} = \frac{2}{\beta} \underline{R}\left[z\overline{Z} \frac{dZ}{dz}\right] \tag{II.20}$$

Actually, let us put

$$Z = \rho \cos \theta + i\rho \sin \theta = X + iY$$

X and Y may be considered as functions of φ.

Hence one deduces that

$$\tan\theta = \frac{Y}{X} \qquad \frac{d\theta}{\cos^2\theta} = \frac{Y'_\varphi X - X'_\varphi Y}{X^2}\, d\varphi$$

and consequently

$$\frac{2}{\beta}\rho^2 \frac{d\theta}{d\varphi} = \frac{2}{\beta}\left(Y'_\varphi X - X'_\varphi Y\right) = \frac{2}{\beta}\,\underline{R}\left[-i\overline{Z}\,\frac{dZ}{d\varphi}\right] = \frac{2}{\beta}\,\underline{R}\left[z\overline{Z}\,\frac{dZ}{dz}\right]$$

which establishes the formula (II.20).

<u>2.2.2.2</u>.- Let us now consider two contours (C^0) and (C^1) defined in the plane Z by two functions $Z^{(0)}(\varphi)$ and $Z^{(1)}(\varphi)$ such that $Z^{(0)} = Z^{(1)} + \alpha$, α being a complex constant determining the change in orientation. In the development (II.13) which gives the conformal representation, only the coefficient a_0 varies when one passes from the contour (C^0) to the contour (C^1). Consequently

$$z\,\frac{dZ^{(1)}}{dz} = z\,\frac{dZ^{(0)}}{dz}$$

and the Dirichlet condition determining the function $G^{(1)}(z)$ is written in the plane z

$$\underline{R}\left[G^{(1)}(z)\right] = \frac{2}{\beta}\,\underline{R}\left[\overline{Z}^{(1)} z\,\frac{dZ}{dz}\right] = \frac{2}{\beta}\left[\underline{R}\left[\overline{Z}^{(0)} z\,\frac{dZ}{dz}\right] + \underline{R}\left[\overline{\alpha} z\,\frac{dZ}{dz}\right]\right]$$

(we have omitted superscripts for the quantities which retain the same value, affected by the index 0 or 1). Consequently

$$G^{(1)}(z) = G^{(0)}(z) + \frac{2}{\beta}\left[g(z)\right]$$

since $g(z)$ is a regular function and real at infinity, holomorphic outside of (γ), the real part of which on (γ) assumes the values $\underline{R}(\bar{\alpha}z\ dZ/dz)$, $g(z)$ is then very easily determined. One has exactly

$$g(z) = \bar{\alpha}z\left(\frac{dZ}{dz} - 1\right) + \frac{\alpha r^2}{z}$$

Thence for the function $H^{(1)}(z)$, (since $F'(Z) = i/z\ dz/dZ$)

$$H^{(1)}(z) = H^{(0)}(z) + \frac{2}{\beta}\left[\bar{\alpha}\left(1 - \frac{dz}{dZ}\right) + \alpha\,\frac{r^2}{z^2}\,\frac{dz}{dZ}\right] \tag{II.21}$$

The formula (II.21) gives immediately the solution of the problem of change in orientation with respect to the nondisturbed flow.

2.2.3 - Cone of Revolution

We shall study first of all the case of the cone of zero incidence. One may then do without the preceding analysis and obtain the solution directly; that is what we shall do here. The curve (C) is a circle of the radius $\rho = c^{te} = r$; the relation (I.28) is written

$$v \cos\theta + w \sin\theta = \frac{2r_0}{\beta\left(1 + r_0^{\,2}\right)}$$

On the other hand, for reasons of symmetry

$$v \sin\theta - w \cos\theta = 0$$

Hence one deduces immediately the values of v and w on (C)

$$v = \frac{2r_0 \cos\theta}{\beta\left(1 + r_0^{\,2}\right)} \qquad w = \frac{2r_0 \sin\theta}{\beta\left(1 + r_0^{\,2}\right)}$$

whence

$$V(Z) = \frac{2r_0^2}{\beta\left(1 - r_0^4\right)}\left(\frac{1}{Z} - Z\right) \qquad W(Z) = i\,\frac{2r_0^2}{\beta\left(1 - r_0^4\right)}\left(\frac{1}{Z} + Z\right) \tag{II.22}$$

Finally the relations (I.25) permit the calculation of U

$$-\beta\ dU = -\frac{2r_0^2}{\beta\left(1 - r_0^4\right)}\ \frac{2Z}{Z^2 + 1}\left(\frac{1 + Z^2}{Z^2}\right) = -\frac{4}{\beta}\ \frac{r_0^2}{1 - r_0^4}\ \frac{1}{Z}$$

whence

$$U(Z) = \frac{4}{\beta^2}\ \frac{r_0^2}{1 - r_0^4}\log Z \tag{II.23}$$

We shall now study, returning to the method of section 2.1, the case of a cone of revolution with incidence.

The formula (II.13) is written

$$z = Z - a$$

a being a constant which may be supposed to be real.

Consequently

$$F'(Z) = \frac{i}{Z - a}$$

On the other hand, an immediate calculation shows that

$$\frac{d\theta}{d\varphi} = \frac{r(r + a\cos\varphi)}{\rho^2}$$

and consequently

$$\frac{2\rho^2}{\beta}\frac{d\theta}{d\varphi} = \frac{2}{\beta}\left(r^2 + ar \cos \varphi\right)$$

whence

$$G(Z) = \frac{2}{\beta}\left(r^2 + \frac{ar^2}{Z - a}\right)$$

According to equation (II.18)

$$H(Z) = \frac{2}{\beta}\left(r^2 + \frac{ar^2}{Z - a}\right)\frac{1}{Z - a} = 2\frac{r^2}{\beta}\frac{Z}{(Z - a)^2}$$

the calculation is easily accomplished; one finds

$$V(Z) = \frac{2r^2}{\beta} Z\left[\frac{1}{(Z - a)^2} - 1\right] \tag{II.24}$$

and

$$U(Z) = 4\cdot\frac{r^2}{\beta^2}\left[\log(Z - a) - \frac{3a}{Z - a} - \frac{a^2}{(Z - a)^2} + 4aZ\right] \tag{II.25}$$

since

$$K_2 = +4a\frac{r^2}{\beta}$$

In particular, one finds, if $a = 0$, by means of the approximate formulas (II.24) and (II.25), the same result as by the formulas (II.22) and (II.23) under the condition of neglecting in these formulas the term in r_0^4 of the denominator.

In order to give to these formulas a directly applicable form it suffices to again connect the quantities a, r with the geometrical data; for this purpose, one must use the formula defining ρ (p. 42).

Figure 11 represents the cone section made by the aerodynamic plane of symmetry; α is the semiangle at the apex, γ denotes the angle of the cone axis with the nondisturbed velocity.

One has immediately

$$2r = \beta\alpha \qquad 2a = \beta\gamma$$

Finally, we shall utilize for the calculation of C_p the formula (I.11) since the velocity component u is infinitely small compared to the components v and w. This formula is here written

$$C_p = -2\underline{R}\left[U(Z)\right] - \left|V(Z)\right|^2 \tag{II.26}$$

According to equations (II.24) and (II.25) one has

$$C_p = 2\alpha^2\log\frac{2}{\beta\alpha} - \alpha^2 - \gamma^2 + 4\alpha\gamma\cos\theta + 2\gamma^2\cos 2\theta \tag{II.27}$$

The case of the cone of revolution of zero incidence is obtained by making $\gamma = 0$. One finds then again a known result. The formula (II.27) had already been given by Busemann (see ref. 9) without demonstration.

2.2.4 - Elliptic Cone

We assume first of all the simplest hypotheses where the planes Ox_1x_2, Ox_1x_3 are symmetry planes of the flow ($\vec{U}$ is in the direction of the cone axis), with the cone flattened out on Ox_1x_2. The formula (II.13) may be written in the form

$$Z = z + \frac{a^2}{z}$$

or

$$\rho\cos\theta + i\rho\sin\theta = \left(r + \frac{a^2}{r}\right)\cos\varphi + i\left(r - \frac{a^2}{r}\right)\sin\varphi$$

Hence one deduces successively

$$\tan\theta = \frac{r^2 - a^2}{r^2 + a^2}\tan\varphi$$

$$\frac{d\theta}{d\varphi} = \frac{\cos^2\theta}{\cos^2\varphi}\,\frac{r^2 - a^2}{r^2 + a^2} = \left(r^2 - \frac{a^4}{r^2}\right)\frac{1}{\rho^2}$$

and

$$\frac{2}{\beta}\,\rho^2\,\frac{d\theta}{d\varphi} = \frac{2}{\beta}\left(r^2 - \frac{a^4}{r^2}\right)$$

The Dirichlet problem, which permits calculation of $G(Z)$, is readily formulated; since $G(Z)$ has a constant real part on the contour (C), $G(Z)$ is constant:

$$\frac{dZ}{dz} = 1 - \frac{a^2}{z^2} \qquad F'(Z) = i\,\frac{1}{z\left(1 - \frac{a^2}{z^2}\right)}$$

whence

$$H(z) = \frac{2}{\beta}\left(r^2 - \frac{a^4}{r^2}\right)\frac{1}{z - \frac{a^2}{z}}$$

and

$$H(Z) = \frac{2}{\beta}\left(r^2 - \frac{a^4}{r^2}\right)\frac{1}{\sqrt{Z^2 - 4a^2}}$$

We note besides that $K_2 = 0$.

One calculates $V(Z)$ by the formula (II.9)

$$V(Z) = \frac{2}{\beta}\left(r^2 - \frac{a^4}{r^2}\right)\left(\frac{1}{\sqrt{Z^2 - 4a^2}} - Z\right) \tag{II.28}$$

and $U(Z)$ by the formula (II.11) which may also be written

$$U(Z) = -\frac{2}{\beta}\left[ZH - K_1 - \int G \frac{dz}{z} - 2\bar{K}_2 Z\right] \tag{II.29}$$

whence

$$U(z) = \frac{4}{\beta^2}\left(r^2 - \frac{a^4}{r^2}\right)\left(\log z - \frac{2a^2}{z^2 - a^2}\right)$$

or

$$U(Z) = \frac{4}{\beta^2}\left(r^2 - \frac{a^4}{r^2}\right)\left[\log \frac{Z + \sqrt{Z^2 - 4a^2}}{2} + \frac{\sqrt{Z^2 - 4a^2} - Z}{\sqrt{Z^2 - 4a^2}}\right] \tag{II.30}$$

If one makes $a = 0$, one will find again the expressions already obtained for $U(Z)$ and $V(Z)$ in the case of a cone of revolution of zero incidence (formulas (II.24) and (II.25) in which one makes $a = 0$).

We shall denote by ϵ and by η the principal angles of the elliptic cone (see fig. 12). One has

$$\epsilon\beta = 2\left(r + \frac{a^2}{r}\right)$$

$$\eta\beta = 2\left(r - \frac{a^2}{r}\right)$$

whence

$$r = \frac{\beta}{4}(\epsilon + \eta) \qquad a^2 = \frac{\beta^2}{16}\left(\epsilon^2 - \eta^2\right)$$

The pressure distribution on the cone circumference is easily calculated. It is sufficient to apply the formula (II.26); besides

$$\left|V(Z)\right|^2 = \frac{\epsilon^2\eta^2}{\eta^2\cos^2\varphi + \epsilon^2\sin^2\varphi}$$

and

$$\underline{R}\left[U(Z)\right] = \epsilon\eta\left[\log\left|\frac{\beta(\epsilon + \eta)}{4}\right| + 1\right] - \frac{\epsilon^2\eta^2}{\eta^2\cos^2\varphi + \epsilon^2\sin^2\varphi}$$

hence the final formula

$$C_p = 2\epsilon\eta\left[-\log\left|\frac{\beta(\epsilon + \eta)}{4}\right| - 1 + \frac{\epsilon\eta}{2\left(\eta^2\cos^2\varphi + \epsilon^2\sin^2\varphi\right)}\right] \qquad \text{(II.31)}$$

The case where the velocity is not in the direction of the axis may be treated equally by utilizing the formula (II.21). In this formula one must put

$$H^0(z) = \frac{2}{\beta}\left(r^2 - \frac{a^4}{r^2}\right)\frac{1}{z - \frac{a^2}{z}} \qquad \frac{dZ}{dz} = 1 - \frac{a^2}{z^2}$$

One then obtains

$$H^{(1)}(z) = \frac{2}{\beta}\left(r^2 - \frac{a^4}{r^2}\right)\frac{z}{z^2 - a^2} + \frac{2}{\beta}\left[\bar{\alpha}\left(1 - \frac{z^2}{z^2 - a^2}\right) + \frac{\alpha r^2}{z^2}\,\frac{z^2}{z^2 - a^2}\right]$$

$$= \frac{2}{\beta\left(z^2 - a^2\right)}\left[\left(r^2 - \frac{a^4}{r^2}\right)z + \alpha r^2 - \bar{\alpha}a^2\right]$$

hence, remarking that

$$Z = z + \frac{a^2}{z} + \alpha$$

$$H(Z) = \frac{2}{\beta}\left[\left(r^2 - \frac{a^4}{r^2}\right)\left(\frac{1}{\sqrt{(Z - \alpha)^2 - 4a^2}}\right) + \frac{\left(\alpha r^2 - \bar{\alpha} a^2\right)\left(Z - \alpha - \sqrt{(Z - \alpha)^2 - 4a^2}\right)}{2a^2\sqrt{(Z - \alpha)^2 - 4a^2}}\right]$$

and

$$V(Z) = H(Z) - \frac{2}{\beta}\left(r^2 - \frac{a^4}{r^2}\right)Z$$

On the other hand, we shall calculate U by utilizing the variable z and the formula (II.20). The coefficient K_2 is equal to

$$K_2 = \frac{2}{\beta}\left[\left(r^2 - \frac{a^4}{r^2}\right)\alpha + \alpha r^2 - \bar{\alpha} a^2\right]$$

and $U(z)$ is then given by the formula

$$U(z) = \frac{4}{\beta^2}\left[\left(r^2 - \frac{a^4}{r^2}\right)\left(\log z - \frac{2a^2 + \alpha z}{z^2 - a^2}\right) + \frac{\bar{\alpha} a^2 - \alpha r^2}{z\left(z^2 - a^2\right)}\left(2z^2 + \alpha z\right)\right] + 4\,\frac{\bar{K}_2}{\beta}\,Z \tag{II.32}$$

One will note that, if one puts $\alpha = 0$, one finds again the formula (II.30), and that, for $a = 0$, one finds again the formula (II.25), except for the notations.

Thus one can, without any difficulty other than the lengthy writing expenditure, calculate the pressure distribution coefficient on the elliptic cone of any arbitrary orientation with respect to the wind.

2.2.5 - Calculation of the Total Forces

We have already seen in section 1.2.6 that the normal to the conical obstacle directed toward the outside has as direction parameters

$$\frac{1}{\beta}\left(x_3x_2' - x_2x_3'\right), \quad x_3', \quad -x_2'$$

Let $\vec{n}$ be the unit vector coincidental with this normal, s be the area of the section with the abscissa x_1, L the length of this section; one may make correspond to the resultant of the forces acting on a section $\left(x_1, \; x_1 + dx_1\right)$ a (dimensionless) vector

$$\vec{C}_z = - \frac{1}{L}\int C_p\vec{n}\, ds \tag{II.33}$$

situated in the plane x_2x_3, and a dimensionless number

$$C_x = - \frac{1}{L}\int C_p(\vec{n}\vec{U})ds \tag{II.34}$$

the vector $\vec{C}_z$ characterizes the lift, the number C_x the drag.

The integrals appearing in the formulas (33) and (34) are taken along the section. Naturally $\vec{C}_z$ and C_x are independent of this section. One may also replace $\vec{C}_z$ by a complex number C_z, the real and imaginary parts of which are equal to the components of the vector $\vec{C}_z$ on Ox_2 and Ox_3. For calculating equations (II.33) and (II.34) one may utilize the section $x_1 = \beta$. If we assume l to be the length of the contour (C) in the plane Z, we may write, taking into account the habitual approximations

$$C_z = \frac{1}{l}\int_C C_p \; dZ \tag{II.35}$$

and

$$C_x = - \frac{2}{\beta l}\, \underline{R}\left[i\int_C C_p\overline{Z}\; dZ\right] \tag{II.36}$$

with the integrals appearing in equations (II.35) and (II.36) taken in the plane Z. These integrals present a certain analogy to the Blasius integrals (ref. 13); C_p is given by the formula (II.26); unfortunately, it is not possible to give simple formulas for the total forces since the integrals (II.35) and (II.36) make use of all coefficients of the conformal representation[19].

We shall apply the formulas (II.35) and (II.36) to the case of the circular cone; C_p is given by equation (II.27)

$$dZ = i\,\frac{\beta\alpha}{2}\,e^{i\theta}d\theta \qquad \overline{Z}\,dZ = i\,\frac{\beta^2\alpha^2}{4}\,d\theta \qquad l = \pi\beta\alpha$$

One obtains

$$C_z = -2\alpha\gamma \qquad C_x = 2\alpha^3\log\frac{2}{\beta\alpha} - \alpha^3 - \alpha\gamma^2 \tag{II.37}$$

In the case of the elliptic cone of zero incidence, C_z is obviously zero

$$\overline{Z} = re^{-i\varphi} + \frac{a^2}{r}\,e^{i\varphi} \qquad dZ = i\left[re^{i\varphi} - \frac{a^2}{r}\,e^{-i\varphi}\right]d\varphi$$

whence

$$C_x = \frac{2}{\beta l}\left(r^2 - \frac{a^4}{r^2}\right)\int_0^{2\pi} C_p\,d\varphi$$

with C_p being given by formula (II.31). Now

$$\int_{-\pi}^{\pi}\frac{\epsilon\eta\,d\varphi}{2\left(\eta^2\cos^2\varphi + \epsilon^2\sin^2\varphi\right)} = \int_{-\infty}^{\infty}\frac{\epsilon\eta\,dt}{\eta^2 + \epsilon^2t^2}$$

---

[19]See appendix No. 7.

As one can see immediately by putting

$$t = \tan \varphi$$

the calculation of this last integral is immediate.

Thus one obtains

$$C_x = \frac{2\pi\beta}{l} \epsilon^2\eta^2\left[\log \frac{4}{\beta(\epsilon + \eta)} - \frac{1}{2}\right] \tag{II.38}$$

with l being the length of the ellipse with the semiaxes $\frac{\epsilon\beta}{2}$, $\frac{\eta\beta}{2}$.

2.2.6 - Approximate Formula for the Calculation of C_x

Let us consider the function $U(z)$; according to formula (II.11) and the remark 2 of section 2.1.4 one may say that the principal term for $U(z)$ is

$$U(z) = 4 \frac{S}{\pi\beta^2} \log z$$

Consequently, in first approximation

$$C_p = - \frac{8S}{\pi\beta^2} \log r$$

with S being the area inside of the contour (C), and r the radius of the circle (γ) on which one makes the conformal canonical representation of (C). If one now calculates C_x, taking into account this approximate formula, one has, according to equation (II.36)

$$C_x = - \frac{18S}{\pi\beta^3 l} \log r \underline{R}\left[i \int_c \overline{Z}\, dZ\right]$$

whence

$$C_x = + \frac{32S^2}{\pi\beta^3 l} \log \frac{1}{r} \tag{II.39}$$

We shall state: In every first approximation the value of the drag coefficient C_x is given by the formula (II.39).

2.2.7 - Case Where the Cone Presents an Exterior Generatrix

If the contour (C) shows an exterior angular point, the various functions introduced in the course of the study (first paragraph of this chapter) present certain singularities. These singularities we shall specify. Let Z_0 be the designation angular point of (C), and $\delta\pi$ the angle of the two semitangents to (C) at the point $Z_0 (0 < \delta < 1)$ (see fig. 13); if z_0 is the image of the point Z_0 in the plane z, one may write, according to a well-known result, in the neighborhood of z_0

$$\left(\frac{dZ}{dz}\right)_0 = K\left(z - z_0\right)^k$$

with K being a complex constant and $k = 1 - \delta$; consequently

$$\left[F'(Z)\right]_0 = K_1\left(z - z_0\right)^{-k} = K_2\left(Z - Z_0\right)^{-\frac{k}{1+k}}$$

with K_1 and K_2 being complex constants. $F'(Z)$ thus becomes infinite at the point $Z = Z_0$.

In contrast, the function $G(z)$ has, according to definition, a real part which assumes on the circle (γ) the values

$$\frac{2}{\beta}\, \underline{R}\left[z\bar{Z}\,\frac{dZ}{dz}\right]$$

This real part thus remains finite on the circle (γ) (and it satisfies there a condition of Hölder). According to a known theorem, its imaginary part likewise remains continuous on (γ) (and likewise satisfies a condition of Hölder). Consequently, one sees, if one refers to formula (II.18) that

$$H(Z) = K_3\left(Z - Z_0\right)^{-\frac{k}{1+k}}$$

in the neighborhood of Z_0; likewise, U, V, W will, in the proximity of this point, be of the order $\frac{k}{1 + k}$ with respect to $\frac{1}{Z - Z_0}$.

Thus the analysis made in section 2.1 is no longer applicable to this case. However, the formulas (II.35) and (II.36) show that if the pressure coefficient assumes very high values in the neighborhood of $Z = Z_0$, the total energy remains finite. According to what we have indicated in section 1.1.3 we consider the solution still valid, with the understanding that the values of C_p in the surroundings of $Z = Z_0$ are not reliable.

2.2.8 - Delta (Δ) Wing of Small Apex Angle at an Infinitely Small Incidence

If one puts in the formulas $r^2 = a^2$, at the end of section 2.2.4, one obtains the pressure distribution on a delta wing with small apex angle. Let us recall that a delta wing is an infinitely small angle. Its angle, according to definition, is the half-angle ω at the vertex (compare fig. 14). Thus one has

$$\omega\beta = 4a$$

The formulas (II.31) and (II.32) are applicable to a delta wing of small angle placed at an incidence also rather small.

Let us moreover assume that this opening is infinitely small with respect to the incidence. Under these conditions, the formulas yielding U(Z) and V(Z) are written

$$V(Z) = \frac{2}{\beta^2}\,\frac{\alpha - \overline{\alpha}}{2}\,\frac{Z - \sqrt{Z^2 - 4a^2}}{\sqrt{Z^2 - 4a^2}}$$

$$U(Z) = -\frac{4}{\beta^2}\,\frac{\alpha - \overline{\alpha}}{2}\,\frac{4a^2}{\sqrt{Z^2 - 4a^2}} + \frac{8a^2}{\beta^2}(\overline{\alpha} - \alpha)Z \qquad \text{(II.40)}$$

Actually one is justified in omitting the second-order terms with respect to α. For calculating C_p it suffices to apply the formula (II.8); the second term of the second formula (II.40) may be neglected.

With the incidence γ, the delta wing being parallel to Ox_2, one has

$$\gamma\beta = 2i\alpha$$

Finally, one may put along the Δ

$$Z = 2a \cos \varphi = \frac{\omega\beta}{2} \cos \varphi$$

One then finds

$$C_p = \frac{2\omega\gamma}{\sin \varphi} \tag{II.41}$$

We remark further that φ is related to the angle ψ of figure 14 by

$$2\psi\beta = \omega\beta \cos \varphi \qquad \psi = \frac{\omega \cos \varphi}{2}$$

One may state: the pressure coefficient on a delta wing of infinitely small opening angle is independent of the Mach number of the flow.

One has

$$C_p = \frac{2\omega\gamma}{\sqrt{1 - t^2}} \quad \text{if} \quad t = \frac{2\psi}{\omega}$$

if one applies formula (II.35), one finds

$$C_z = i\pi\omega\gamma$$

This coefficient C_z has not the same significance as the one utilized in the theory of the lifting wing. Actually, it is, according to the very manner in which it was obtained, relative to the total area of the Δ (pressure side and suction side); if one takes only one of these areas into account, one must write (neglecting the factor $-i$)

$$C_z = 2\pi\omega\gamma$$

This formula has been found by other methods by R. T. Jones (ref. 14). We shall find it again in chapter III, section 3.1.2.4, when studying the general problem of the delta wing which is here only touched on incidentally and for the particular case of a Δ with infinitely small opening angle.

2.2.9 - Study of a Cone With Semicircular Section

As the last application, we shall treat the case of a cone with semicircular section, with the velocity $\vec{U}$ being directed along the intersection of the symmetry plane and of the face plane of the cone[20] (fig. 15).

The contour (C) in the plane Z then is a semicircle, centered at the origin, of the radius a (fig. 16).

One obtains very easily the conformal canonical representation of the exterior of this contour, on the outside of a circle (γ) of the radius r, centered at the origin of the plane z, by means of a particular Kármán-Trefftz transformation (ref. 13, p. 128) which is written

$$\frac{Z - a}{Z + a} = \left[\frac{z - re^{-i\frac{\pi}{6}}}{z - re^{-i\frac{5\pi}{6}}}\right]^{\frac{3}{2}} \tag{II.42}$$

a and r are connected by the relationship

$$4a = 3r\sqrt{3}$$

In order to obtain the correspondence between the circle (γ) and the contour (C), one must distinguish two cases. Let us put

$$z = re^{i\varphi}$$

[20]Such a cone formed the front of supersonic models planned by German engineers.

(1) $-\frac{\pi}{6} < \varphi < \frac{7\pi}{6}$, the corresponding point of (C) is on the arc of the circle.

Let us put under these conditions

$$Z = ae^{i\psi}$$

and we shall find according to formula (II.42):

$$\tan\frac{\psi}{2} = \left[\frac{\sin\left(\frac{\varphi}{2} + \frac{\pi}{12}\right)}{\sin\left(\frac{\varphi}{2} + \frac{5\pi}{12}\right)}\right]^{\frac{3}{2}} \tag{II.43}$$

(2) $\frac{7\pi}{6} < \varphi < \frac{11\pi}{6}$, the corresponding point of (C) is on the segment AA'; let us put under these conditions

$$Z = a\cos\chi$$

The formula (II.42) shows that

$$\tan\frac{\chi}{2} = \left[\frac{\sin\left(\frac{\varphi}{2} + \frac{\pi}{12}\right)}{\sin\left(\frac{\varphi}{2} + \frac{5\pi}{12}\right)}\right]^{\frac{3}{4}} \tag{II.44}$$

The two last formulas define completely the desired conformal representation. Figures (17) and (18) give the variations of ψ and χ as functions of φ.

We shall have to utilize equally the value of dz/dZ. The simplest method for obtaining this value consists in logarithmic differentiation of the two terms of formula (II.42). One thus obtains the result

$$\frac{dz}{dZ} = \frac{z^2 + irz - r^2}{Z^2 - r^2} \tag{II.45}$$

If one has $-\frac{\pi}{6} < \varphi < \frac{7\pi}{6}$, one must put in the preceding formula

$$z = re^{i\varphi} \qquad Z = ae^{i\psi}$$

whence

$$\frac{dz}{dZ} = \frac{r^2}{2a^2} \frac{1 + 2 \sin \varphi}{\sin \psi} e^{i(\varphi-\psi)} = \frac{8}{27} \frac{1 + 2 \sin \varphi}{\sin \psi} e^{i(\varphi-\psi)} \tag{II.46}$$

If φ is comprised between $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$, one puts $z = re^{i\varphi}$, $Z = a \cos \chi$. Thus one obtains

$$\frac{dz}{dZ} = \frac{16}{27} \frac{1 + 2 \sin \varphi}{\sin^2 \chi} e^{i\left(\frac{\pi}{2} - \varphi\right)} \tag{II.47}$$

The function $G(Z)$ has as its real part $\underline{R}\left[z\overline{Z} \frac{dZ}{dz}\right]$, that is

$$\left.\begin{matrix} \frac{27}{8} ar \frac{\sin \psi}{1 + 2 \sin \varphi} & \text{if} \quad -\frac{\pi}{6} < \varphi < \frac{7\pi}{6} \\ 0 & \text{if} \quad \frac{7\pi}{6} < \varphi < \frac{11\pi}{6} \end{matrix}\right\} \tag{II.48}$$

The analytic function

$$a^2 \frac{z}{Z} \frac{dZ}{dz}$$

has a real part which, on (γ), assumes these same values. This function is regular at infinity, holomorphic outside of (γ), but with a pole $z = -ir$, with the corresponding residue being equal to $-ia^2$.

Let us then consider the function

$$a^2\left(\frac{z}{Z}\frac{dZ}{dz} - \frac{1}{2}\frac{z - ir}{z + ir}\right)$$

This function is holomorphic outside of (γ). It is regular at infinity; its value at infinity is equal to $a^2/2$. On (γ), these real and imaginary parts satisfy Hölder conditions. This function is therefore identical with the desired function $G(z)$.

Hence one deduces according to equation (II.18)

$$H(Z) = \left(\frac{z}{Z}\frac{dZ}{dz} - \frac{1}{2}\frac{z - ir}{z + ir}\right)\frac{a^2}{z}\frac{dz}{dZ} = \frac{a^2}{Z} - \frac{a^2}{2z}\frac{z - ir}{z + ir}\frac{dz}{dZ}$$

and according to equation (II.19)

$$V(Z) = a^2\left(\frac{1}{Z} - \frac{Z}{2} - \frac{1}{2z}\frac{z - ir}{z + ir}\frac{dz}{dZ}\right)$$

Finally, the calculation of $U(Z)$ may be carried out with the aid of formula (II.29)

$$\int G\frac{dz}{z} = a^2\log Z - \frac{a^2}{2}\int\frac{z - ir}{z + ir}\frac{dz}{z} = a^2\left(\log\frac{Z}{z + ir} + \frac{1}{2}\log z\right)$$

and

$$ZH - K_1 = a^2\left(1 - \frac{1}{2}\frac{Z}{z}\frac{dz}{dZ}\frac{z - ir}{z + ir} - \frac{1}{2}\right) = \frac{a^2}{2}\left(1 - \frac{z - ir}{z + ir}\frac{Z}{z}\frac{dz}{dZ}\right)$$

whence

$$U(Z) = \frac{a^2}{\beta}\left(\frac{z - ir}{z + ir}\frac{Z}{z}\frac{dz}{dZ} - 1 + \frac{4\overline{K}_2 Z}{a^2} + 2\log\frac{Z}{z + ir} + \log z\right)$$

The calculation of the coefficients $\overline{K}_2$ offers no difficulty whatsoever; however, as one had already opportunity to note, the term $\overline{K}_2 Z$ does not occur in the calculation of the pressures along the cone.

This pressure distribution along the cone calculated with the aid of equation (II.26) is represented in figure 19.

2.3 - Numerical Calculation of Conical Flows With Infinitesimal Cone Angles

2.3.1 - General Remarks

In the preceding paragraph, we have studied a certain number of particularly simple cases. However, if the cone (C) is arbitrary, it will be necessary to carry out various operations leading to the solution by purely numerical procedures.

Let us analyze the various operations necessary for the calculation:

(1) The conformal canonical representation of the exterior of (C) on the outside of the circle (γ) must be made; this calculation permits, in particular, determination of the radius r of (γ), correspondence of the points of (C) and of (γ), and calculation of the expression dZ on the contour (γ).

(2) The function $G(z)$, holomorphic outside of (γ), regular and real at infinity must be determined, the real part on (γ) of which is known; we shall designate it by $g(\varphi)$. In fact, it suffices to know, on (γ), only the imaginary part of $G(z)$, for instance $g'(\varphi)$; $g'(\varphi)$ is the conjugate function of $g(\varphi)$ and is given by the formula

$$g'(\varphi) = \frac{1}{2\pi}\int_0^{2\pi} g(\varphi')\cot\frac{\varphi' - \varphi}{2}\,d\varphi'$$

This formula is called "Poisson's integral."

(3) With these two operations accomplished, the values of $H(z)$ on the circle (γ) (formula (II.18)) are known which provides the values of v and w on the cone; u is obtained by the formula (II.29). The only new calculation to be made is that of the expression:

$$\underline{R}\left[-\int G\,\frac{dz}{z}\right] = \int g'\,d\varphi$$

the constant of integration being determined so that u should have a mean value zero on (γ).

All these operations always amount to the following numerical problems:

(a) With a function given, to calculate its conjugate function (Poisson integral)

(b) With a function prescribed, to calculate the derivative of the conjugated function

(c) With a function prescribed, to calculate its derivative[21].

We shall justify this result in the following paragraph by showing that the operation (1) may be performed by applying the calculations (a), (b), (c). We shall then indicate a general method, relatively simple and accurate, which permits solution of these problems. We shall terminate this chapter by giving an application.

2.3.2 - Conformal Canonical Representation of a Contour (C) on a Circle (γ)

The numerical problem of determination of the conformal canonical representation of a contour (C) on a circle (γ) has been solved for the first time by Theodorsen[22]. We shall briefly summarize the principle of this method, simplifying, however, the initial exposé of that author.

Let us suppose, first of all, that the contour (C) is neighboring on a circle of the radius a, centered at the origin (fig. 20); in a more accurate manner, putting on (C)

$$Z = ae^{\psi+i\theta} \tag{II.49}$$

with ψ being a function of θ, $\psi = \psi(\theta)$, we shall suppose that $\psi(\theta)$ and $\frac{d\psi}{d\theta}$ are functions which assume small values. We shall then call

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[21]If the conformal representation of the exterior of (C) on the outside of (γ) is known in explicit form, it will naturally be sufficient to apply operation (a).

[22]Compare references 15 and 16. One may achieve this conformal representation also by the elegant method of electrical analogies (ref. 17); the time expenditure required by the experimental method and by the purely numerical methods here described as well as the accuracy of these procedures are of the same order of magnitude.

(C) "quasicircular." Let φ be the angular abscissa of the point of (γ) which corresponds to the point of (C), the polar angle of which is θ; we put

$$\theta = \varphi + \bar{\epsilon}(\varphi) \qquad \varphi = \theta - \epsilon(\theta) \tag{II.50}$$

$\epsilon(\theta)$ and $\bar{\epsilon}(\varphi)$ representing the same function but expressed as a function of θ or as a function of φ; we shall put likewise

$$\bar{\psi}(\varphi) = \psi(\theta)$$

The desired conformal transformation may be written

$$Z = ze^{h(z)}$$

with $h(z)$ being a holomorphic function outside of (γ), regular and zero at infinity. The equality (II.50) becomes, if one writes it on the circle (γ),

$$ae^{\bar{\psi}(\varphi)+i\left[\varphi+\bar{\epsilon}(\varphi)\right]} = re^{i\varphi}e^{h(z)}$$

whence

$$h(z) = \frac{a}{r}\bar{\psi}(\varphi) + i\bar{\epsilon}(\varphi) + \log\frac{a}{r} \tag{II.51}$$

Finding the conformal representation of (C) on (φ) amounts to calculating the functions $\bar{\psi}(\varphi)$ and $\bar{\epsilon}(\varphi)$. First of all, one knows (equation of (C)) that

$$\bar{\psi}(\varphi) = \psi\left[\varphi + \bar{\epsilon}(\varphi)\right] \tag{II.52}$$

On the other hand, according to equation (II.51), $\bar{\epsilon}(\varphi)$ is the conjugate function of $\bar{\psi}(\varphi)$, and consequently

$$\bar{\epsilon}(\varphi) = \frac{1}{2\pi}\int_0^{2\pi} \bar{\psi}(\varphi')\cot\left(\frac{\varphi' - \varphi}{2}\right)d\varphi' \tag{II.53}$$

the integral being taken at its principal value. There is no constant to add to the second term of equation (II.53), for $\overline{\epsilon}(\varphi)$ has a mean value zero since $h(z)$ is zero at infinity. For the same reason, if $\overline{\Psi}_0$ denotes the mean value of $\overline{\Psi}(\varphi)$ in an interval of the amplitude 2π

$$r = ae^{\overline{\Psi}_0} \tag{II.54}$$

an equality which will permit calculation of r if $\overline{\Psi}(\varphi)$ is known. In order to calculate $\overline{\epsilon}(\varphi)$ and $\overline{\Psi}(\varphi)$, one disposes therefore of the relations (II.52) and (II.53); one can solve this system by a procedure of successive approximations.

We shall put first

$$\epsilon_0(\theta) = \overline{\epsilon}_0(\varphi) = 0$$

According to equation (II.52)

$$\overline{\Psi}(\theta) = \Psi(\theta)$$

and according to equation (II.53)

$$\epsilon_1(\theta) = \frac{1}{2\pi}\int_0^{2\pi} \Psi(\theta')\cot\frac{\theta' - \theta}{2}\,d\theta'$$

Thence a first approximation for φ

$$\varphi_1 = \theta - \epsilon_1(\theta) \qquad \theta = \varphi_1 + \overline{\epsilon}_1(\varphi_1)$$

From it one deduces, according to equation (II.52), a first approximation for $\overline{\Psi}(\varphi)$

$$\overline{\Psi}_1(\varphi_1) = \Psi\left[\varphi_1 + \overline{\epsilon}_1(\varphi_1)\right]$$

whence a second approximation for the function ϵ

$$\epsilon_2(\varphi_1) = \frac{1}{2\pi}\int_0^{2\pi} \overline{\Psi}_1(\varphi_1')\cot\frac{\varphi_1' - \varphi_1}{2}\,d\varphi_1'$$

$$\epsilon_2(\theta) = \epsilon_2\left[\theta - \epsilon_1(\theta)\right]$$

whence

$$\varphi_2 = \theta - \epsilon_2(\theta) \qquad \theta = \varphi_2 + \overline{\epsilon}_2(\varphi_2)$$

The procedure can be followed indefinitely.

The convergence of the successive approximations forms the subject of a memorandum by S. E. Warschawski (ref. 18). We refer the reader who wants to go more deeply into that question to this meritorious report.

From the practical point of view one may say that the convergence is very rapid; two approximations suffice very amply in the majority of cases; the different changes in variables which encumber the preceding exposé are very easily made by graphic method. Thus one sees that the numerical work essentially consists in calculating twice the integral (II.53). This calculation is precisely the object of the problem (a) stated at the end of section 2.3.1.

If the contour (C) is not "quasicircular," one may make, first of all, a conformal representation which transforms it into the "quasicircular" contour (C'); one will then apply the preceding analysis to the contour (C'). For certain cases it will be quicker to use a direct method. Let us assume, for instance, that (C) is a contour flattened on the axis of the X (compare fig. 21) and for simplification that X'OX is permissible as the axis of symmetry.

Let us suppose that X varies along (C) from -a to +a while $|Y|$ remains bounded by ma (with m being, for instance, of the order of 1/10); it will then be indicated to operate as follows:

We put along (γ)

$$Z = \frac{z}{r}\left[f(\varphi) + ig(\varphi)\right]$$

One has

$$\left.\begin{aligned} X(\varphi) &= f \cos \varphi - g \sin \varphi \\ Y(\varphi) &= f \sin \varphi + g \cos \varphi \end{aligned}\right\} \tag{II.55}$$

or also

$$\left.\begin{aligned} f &= X \cos \varphi + Y \sin \varphi \\ g &= Y \cos \varphi - X \sin \varphi \end{aligned}\right\} \tag{II.56}$$

$f(\varphi)$ is an even function of φ, $g(\varphi)$ is an odd function

$$f(0) = +f(\pi) = a \qquad g(0) = g(\pi) = 0$$

The functions $X(\varphi)$ and $Y(\varphi)$ have to be found. Let us take as starting point

$$X_0(\varphi) = a \cos \varphi$$

an approximation which would be definitive if (C) were an ellipse.

On the contour (C) one reads the corresponding value $Y_0(\varphi)$, and by means of the second formula (II.56) one obtains a first approximation

$$g_1(\varphi) = Y_0(\varphi)\cos \varphi - X_0(\varphi)\sin \varphi$$

$f_1(\varphi)$ will be given by a Poisson integral

$$f_1(\varphi) = \frac{1}{2\pi}\int_0^{2\pi} g_1(\varphi)\cot \frac{\varphi' - \varphi}{2}\, d\varphi' + \lambda_1$$

with λ_1 being a constant, such as $f_1(0) = a$.

Owing to the formulas (II.55), one has a first approximation $X_1(\varphi)$, $Y_1(\varphi)$ for the functions $X(\varphi)$, $Y(\varphi)$. One proceeds in the same manner, reading off on (C) the functions $Y_1(\varphi)$ corresponding to $X_1(\varphi)$, then calculating

$$g_2(\varphi) = Y_1(\varphi)\cos\varphi - X_1(\varphi)\sin\varphi$$

and

$$f_2(\varphi) = \frac{1}{2\pi}\int_0^{1\pi} g_2(\varphi')\cot\frac{\varphi' - \varphi}{2}\,d\varphi' + \lambda_2$$

etc.

When one has obtained a pair $f_n(\varphi)$, $g_n(\varphi)$ providing a sufficient approximation $X_n(\varphi)$, $Y_n(\varphi)$ of $X(\varphi)$, $Y(\varphi)$, one stops the calculations; then

$$r = \lambda_n$$

In practice[23] it suffices to take $n = 2$; the same method (averaging of very slight adaptations) will apply to the case where (C), although being flattened on OX, will no longer admit of OX as the symmetry axis.

Finally, for a complete solution of the problem (1) posed at the beginning of the preceding paragraph, only dZ/dz remains to be calculated, which will obviously be possible with the aid of the problems (b) or (c).

2.3.3 - Calculation of the Trigonometric Operators[24]

The method we shall summarize permits calculation of the linear operators A, transforming a function $P(\theta)$ into a function $Q(\theta)$

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[23]The principle of this method is the one we applied for the study of profiles in an incompressible fluid. But in the case of the profiles a few complications (which can, however, easily be eliminated) arise due to the fact of the "tip."

[24]We gave the principle of this method for the first time in March 1945 (ref. 19). Compare also reference 20. In continuation of this report, M. Watson provided a demonstration of the formulas which we obtained by a different method (ref. 21). We also point out a "Wartime Report" of Irven Naiman, of September 1945, proposing this same method of calculation for the Poisson integral (ref. 22).

$$Q(\theta) = A\left[P(\theta)\right]$$

and re-entering one or the other of the following categories:

First category: The operator possesses the following properties

$$\left.\begin{aligned} A(\cos m\theta) &= a_m \sin m\theta \\ A(\sin m\theta) &= -a_m \cos m\theta \\ A(1) &= 0 \end{aligned}\right\} \qquad \text{(II.57)}$$

with a_m being a nonzero constant, m any arbitrary integral different from zero.

Second category: A possesses the properties

$$\left.\begin{aligned} A(\cos m\theta) &= b_m \cos m\theta \\ A(\sin m\theta) &= b_m \sin m\theta \\ A(1) &= b_0 \end{aligned}\right\}$$

with b_m being a nonzero constant, m any arbitrary integral.

We shall call these operators "trigonometric operators." The operations which form the subject of the problems (a), (b), (c) are, precisely, particular cases of "trigonometric operators."

With the function $P(\theta)$ known, one now has to calculate the function $Q(\theta)$; the functions $P(\theta)$ and $Q(\theta)$ are assumed as periodic, of the period 2π. $P(\theta)$ and $Q(\theta)$ are determined approximately by knowledge of their values for $2n$ particular values of θ, uniformly distributed in the interval $0, 2\pi$. One knows that the unknown $2n$ values of Q are linear functions of the known $2n$ values of P. The entire problem consists in calculating the coefficients of these linear equations. We shall do this, examining two possible modes of calculation.

2.3.3.1 - First mode of calculation.- After having divided the circle into $2n$ equal parts, we shall put

$$f_i = f\left(\frac{i\pi}{n}\right)$$

(1) Operators of the first category.- Obvious considerations of parity show that the Q_i are expressed as functions of the P_j by equations of the form

$$Q_i = \sum_{1}^{n-1} K_p\left(P_{i+p} - P_{i-p}\right) \tag{II.58}$$

We shall apply the relations (II.57), that is, carry into the $2n$ equations (II.58)

$$P(\theta) = \cos m\theta \qquad Q(\theta) = a_m \sin m\theta$$

and

$$P(\theta) = \sin m\theta \qquad Q(\theta) = -a_m \cos m\theta$$

We thus obtain $4n$ equations which are all reduced to the unique equation

$$\sum_{1}^{n-1} K_p \sin p \frac{m\pi}{n} = \frac{a_m}{2} \tag{II.59}$$

This reduction is the explanation for the success of the method. We have to determine $(n - 1)$ unknown K_p. For this purpose, we shall write the equation (II.59), for the values of p varying from 1 to $n - 1$. The system remains to be solved. If one multiplies the first equation by $\sin \frac{\mu\pi}{n}$, the second by $\sin \frac{2\mu\pi}{n}$, the $(n - 1)^{th}$ by $\sin(n - 1)\frac{\mu\pi}{n}$, and if one adds term by term, one obtains a linear relation between the K_p, with the following coefficients of K_p

$$\sum_{m=1}^{n-1} \sin m \frac{p\pi}{n} \sin m \frac{\mu\pi}{n} = \frac{1}{2} \sum_{m=1}^{n-1} \left[\cos m \frac{(p - \mu)\pi}{n} - \cos m \frac{(p + \mu)\pi}{n}\right]$$

$$= \frac{1}{2}\left[C_n\left[\frac{(p - \mu)\pi}{n}\right] - C_n\left[\frac{(p + \mu)\pi}{n}\right]\right]$$

with

$$C_n(x) = \sum_{m=0}^{n-1} \cos mx = \cos \frac{(n - 1)}{2} x \frac{\sin \frac{n}{2} x}{\sin \frac{x}{2}}$$

Thus the coefficient of K_p is zero if $p \neq \mu$, and equal to $\frac{n}{2}$ if $p = \mu$.

Thence the desired value of K_p

$$K_p = - \frac{1}{n} \sum_{m=1}^{n-1} a_m \sin \frac{mp\pi}{n} \tag{II.60}$$

Let us apply this result to the calculation of the Poisson integral.

This integral defines an operator $Q = A(P)$ of the first category for which $a_m = -1$.

Consequently, the formula (II.60) is written

$$K_p = \frac{1}{n} \sum_{1}^{n-1} \sin \frac{mp\pi}{n} = \frac{1}{n} S_n\left(\frac{p\pi}{n}\right)$$

if one puts

$$S_n(x) = \sum_{1}^{n-1} \sin mx = \sin \frac{(n - 1)x}{2} \frac{\sin \frac{nx}{2}}{\sin \frac{x}{2}}$$

Thus

$$\left.\begin{aligned} K_p &= 0 & &\text{if} \quad p \quad \text{even} \\ K_p &= \frac{1}{n} \cot \frac{p\pi}{2n} & &\text{if} \quad p \quad \text{odd} \end{aligned}\right\} \tag{II.61}$$

(2) Operators of the second category.- The considerations of parity permit one to write the general formula

$$Q_i = K_0 P_i + \sum_{1}^{n-1} K_p \left(P_{i+p} + P_{i-p}\right) + K_n P_{i+n} \tag{II.62}$$

Using the same reasoning as before, one is led to determine the coefficients K_p by the system

$$K_0 + \sum_{p=1}^{n-1} 2K_p \cos m \frac{p\pi}{n} + (-1)^m K_m = b_m \tag{II.63}$$

with m assuming the values $0, 1, 2, \ldots n$.

Multiplying the first value by $1/2$, the second by $\cos \mu\pi/n$, the third by $\cos \frac{2\mu\pi}{n}$, the nth by $\cos \frac{(n-1)\mu\pi}{n}$, and the last by $(-1)^{\mu/2}$, and adding them, one obtains a linear relation between the K_p, with the coefficient of K_p being $(p \neq 0, \ p \neq n)$

$$2\left[\frac{1}{2} + \frac{(-1)^{p+\mu}}{2} + \frac{1}{2}\left[C_n\left[\frac{(p+\mu)\pi}{n}\right] + C_n\left[\frac{(p-\mu)\pi}{n}\right] - 2\right]\right]$$

that is, n if $\mu = p$, and 0 if $\mu \neq p$.

The coefficient of K_0 is

$$\frac{1}{2} + \frac{(-1)^{\mu}}{2} + \frac{1}{2}\left[C_n\left(\frac{\mu\pi}{n}\right) + C_n\left(-\frac{\mu\pi}{n}\right) - 2\right]$$

The preceding conclusions remain valid, it is zero for $\mu \neq 0$ and equal to n if $\mu = 0$; the same result is valid for K_n. Finally, one obtains the general formula of solution

$$K_p = \frac{1}{n}\left[\frac{b_0}{2} + \sum_{m=1}^{n-1} b_m \cos\frac{mp\pi}{n} + (-1)^p \frac{b_n}{2}\right] \tag{II.64}$$

Let us consider, for instance, the operator transforming the function $P(\theta)$ into the function $dQ/d\theta$, with Q being the conjugate function of P; it is an operator of the second category for which

$$b_m = -m$$

Applying formula (II.64), one obtains

$$K_0 = -\frac{m}{2}$$

$$K_p = -\frac{1}{n}\left[\sum_{1}^{n-1} m \cos\frac{pm\pi}{n} + (-1)^p \frac{b_n}{2}\right] \qquad p \neq 0$$

If one notes that

$$\sum_n (x) = \sum_0^{n-1} m \cos mx = \frac{1}{2\sin^2\frac{x}{2}}\left[n \sin\left(n - \frac{1}{2}\right)x - \sin^2\frac{nx}{2}\right]$$

one sees that

$$\left.\begin{aligned} K_p &= 0 && \text{if } p \text{ even} \\ K_p &= \frac{1}{n\left(1 - \cos\frac{p\pi}{n}\right)} && \text{if } p \text{ odd} \end{aligned}\right\}$$

2.3.3.2 - Second mode of calculation.- Examination of an important particular case will show us that in certain cases it will be advantageous to consider a second mode of calculation.

The method consists in replacing the function $P(\theta)$ by a function of the form

$$\Phi(\theta) = \sum_{0}^{n} a_n \cos n\theta + b_n \sin n\theta \qquad \text{(II.66)}$$

for which the method is applied with the strictest exactness; the constants a_n and b_n are such that $P_i = \Phi_i$. One operator of the first category, one of the most important ones, is the operator of derivation which makes the function $dP/d\theta$ correspond to the function $P(\theta)$. If we apply the first type of calculation, we shall replace $\left(\frac{dP}{d\theta}\right)_i$ by $\left(\frac{d\Phi}{d\theta}\right)_i$; now, it is precisely at the points $\theta = \frac{i\pi}{n}$ that the derivatives $\frac{dP}{d\theta}$ and $\frac{d\Phi}{d\theta}$ show the greatest deviation. In contrast, we shall obtain a good approximation of the desired function by replacing

$$\frac{dP}{d\theta}\left[\frac{(2i + 1)\pi}{2n}\right] \qquad \text{by} \qquad \frac{d\Phi}{d\theta}\left[\frac{(2i + 1)\pi}{2n}\right]$$

We are thus led to the following mode of calculation: the circle is divided into $4n$ equal parts; we shall put

$$f_i = f\left(\frac{i\pi}{2n}\right)$$

and we shall express the $2n$ values Q_{2i} as a function of the $2n$ values P_{2j+1}.

We shall limit ourselves to the operators of the first category. The formula expressing the Q_{2i} as a function of P_{2j+1} is written

$$Q_{2i} = \sum_{p=1}^{n} K_p\left(P_{2i+2p-1} - P_{2i-2p+1}\right)$$

and we obtain for determination of the K_p the system

$$\sum_{p=1}^{n} K_p \sin \frac{(2p - 1)m\pi}{2n} = -\frac{a_m}{2}$$

with m varying from 1 to n.

Multiplying the first equation by $\sin(2\mu - 1)\frac{\pi}{2n}$, the second by $\sin \frac{(2\mu - 1)2\pi}{2n}$, . . ., the $(n - 1)^{th}$ by $\sin \frac{(2\mu - 1)(n - 1)\pi}{2n}$, the last by $\frac{(-1)^{\mu-1}}{2}$ and adding them, one obtains a linear relation in which the coefficient of K_p is

$$\sum_{m=1}^{n-1} \sin(2p - 1)\frac{m\pi}{2n} \sin(2\mu - 1)\frac{m\pi}{2n} + \frac{(-1)^{\mu+p}}{2} =$$

$$\sum_{1}^{n-1} \frac{1}{2}\left[\cos(p - \mu)\frac{m\pi}{n} - \cos(p + \mu - 1)\frac{m\pi}{n}\right] + \frac{(-1)^{\mu-p}}{2} =$$

$$\frac{1}{2}\left[C_n\left[(p - \mu)\frac{\pi}{n}\right] - C_n\left[(p + \mu - 1)\frac{\pi}{n}\right] + (-1)^{\mu+p}\right]$$

The coefficient is zero if $p \neq \mu$, and equal to $\frac{n}{2}$ if $p = \mu$. Hence

$$K_p = -\frac{1}{n}\left[\sum_{m=1}^{n-1} a_m \sin \frac{(2p - 1)m\pi}{2n} + \frac{(-1)^{p-1}}{2} a_n\right] \qquad \text{(II.67)}$$

This procedure may be applied to the calculation of the derivative of a periodic function. In this case, $a_m = -m$. Applying formula (II.67), one obtains

$$K_p = (-1)^{p-1} \frac{1}{2n\left[1 - \cos \frac{(2p - 1)\pi}{2n}\right]} \tag{II.68}$$

2.3.3.4 - Remarks on the Employment of the Suggested Methods.- In order to convey some idea of the accuracy of the proposed methods we shall give first of all a few examples where the desired results are theoretically known.

Let us take as the pair of functions $P(\theta)$, $Q(\theta)$, the functions

$$P(\theta) = \frac{4 \cos 2\theta - 4 \cos \theta + 1}{(5 - 4 \cos \theta)^2} \qquad Q(\theta) = \frac{-4 \sin \theta(2 \cos \theta - 1)}{(5 - 4 \cos \theta)^2}$$

which are the real and imaginary parts, respectively, on the circle of radius 1 of the function

$$f(z) = \frac{1}{(2z - 1)^2} \qquad \left(z = e^{i\theta}\right)$$

One will find in figure 22 the graphic representation of the functions $P(\theta)$, $Q(\theta)$ and of the derivative $Q'(\theta)$ of this function, and also the values of these functions for $\theta = \frac{p\pi}{12}$ (with p ranging between 0 and 12). Furthermore, one will find in figure 23 the values of $Q(\theta)$, calculated from $P(\theta)$ as starting point, by the method just explained (coefficients K_p, defined by equation (II.61)), and in figure 24 on one hand the values of $Q'(\theta)$, calculated from $P(\theta)$ as starting point (from coefficients K_p defined by equation (II.65)), and, on the other, these same values calculated from $Q(\theta)$ as starting point (coefficients K_p defined by equation (II.68)). One will see that the accuracy obtained is excellent although the selected functions show rather rapid variations. Such calculations by means of customary calculation methods are a delicate matter; this is particularly obvious in the case of the Poisson integral which is an integral "of principal value." Systematic comparisons of the method of trigonometric operators with those used so far have been made by M. Thwaites (ref. 23); they have shown that this method gives, in certain calculations, an accuracy largely superior to any attained before.

The calculation procedure, with the aid of tables like the one represented (fig. 25) is very easy. One sees that one fills out this table parallel to the main diagonal of the table. With such a table, about one and a half hours suffice for a Poisson integral if one has a calculating machine at his disposal.

We have had occasion to point out that the accuracy of the method obviously increases to the same degree as the functions one operates with are "regular" and present "rather slight" variations. This leads in practice to two remarks which are based on the "difference method" and reasonably improve the result in certain cases. We shall, for instance, discuss the case of the Poisson integral.

(1) If the function $P(\theta)$ presents singularities (for instance discontinuities of the derivative for certain values of θ), it will be of interest to seek a function $P_1(\theta)$, presenting the same singularities as the function $P(\theta)$, for which one knows explicitly the conjugate function $Q_1(\theta)$. One will make the calculation by means of the function $P(\theta) - P_1(\theta)$; this function no longer presents a singularity.

(2) If the function $P(\theta)$ has a very extended range of variations, one will seek a function $P_1(\theta)$ for which one knows explicitly the function $Q_1(\theta)$ so that the difference $P(\theta) - P_1(\theta)$ remains of small value, and one will operate with this difference.

Finally we note that, if the calculation of the derivative of a function $P(\theta)$ as described above necessitates that $P(\theta)$ be periodic, one can always return to this case, applying, precisely, the "difference method."

2.3.4 - Example: Numerical Calculation of a Flow about a Semicircular Cone

As an application, we have taken up again the case of the semicircular cone studied in section 2.2.9. The function $g(\varphi)$ is given by the formula (II.48), and $g'(\varphi)$ will be calculated by a Poisson integral. Figure 26 shows the value $g'(\varphi)$ thus calculated compared to the theoretical value[25].

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[25]We wanted to test the accuracy of the proposed method by assuming an extremely unfavorable case, without taking into account the singularities presented by the function $g(\varphi)$. For a numerical operation of great exactness, this particular case would have required application of the lemma of Schwartz, with the contour (C) completed symmetrically with respect to OX.

It is then possible to calculate the representation of the pressures, by calculating successively the function H, ZH, and the integral $g'(\varphi)$.

One will find the pressure distribution thus calculated in figure 19; one may then compare the result obtained by the calculation method (for a very unfavorable case) with the result obtained theoretically.

CHAPTER III - CONICAL FLOWS INFINITELY FLATTENED IN ONE DIRECTION

The purpose of this chapter will be the study of conical flows of the second type (see chapter I, section 1.2.6). Before starting this study proper, we shall make a few remarks concerning the boundary conditions. The conical obstacle is flattened in the direction Ox_1x_2. Under these conditions, reassuming the formula (I.27)

$$wx_2' - vx_3' = \frac{1}{\beta}\left(x_3x_2' - x_2x_3'\right)(1 + u) \tag{I.27}$$

one may say that it reduces itself, in first approximation, to

$$wx_2' = \frac{1}{\beta}\left(x_3x_2' - x_2x_3'\right) \tag{III.1}$$

since x_3, x_3', v, u are infinitesimals of first order, while x_2 and x_2' are not infinitesimals. Under these conditions, one may say that one knows the function w on the contour (C). On the other hand, one may write, within the scope of the approximations made, this boundary condition on the surface (d) of the plane Ox_1x_2, projection of the cone obstacle on the plane. Let us designate, provisionally, the value w by $w^{(1)}\left(x_1x_2x_3\right)$ if one operates as follows

$$w^{(1)}\left[x_1,x_2(t),x_3(t)\right] = w^{(1)}\left[x_1,x_2(t),0\right] + x_3(t)\,\frac{\partial w^{(1)}}{\partial x_3}\left[x_1,x_2(t),0\right]$$

With the derivatives of w being, by hypothesis, supposed to be of first order, and the boundary equation written with neglect of the terms of second order, the intended simplification is justified.

Various cases may arise, according to whether the cone obstacle is entirely comprised inside the Mach cone (fig. 27), whether it entirely bisects the Mach cone (fig. 28), whether the entire obstacle is completely outside the Mach cone (fig. 29), or whether it is partly inside and partly outside the Mach cone (fig. 30). In each of these cases there are two elementary problems, the solution of which is particularly interesting: the first, where the relation (III.1) is reduced to

$$w = \text{constant} = w_0 .$$

which we shall call the elementary lifting problem (the corresponding flow is the flow about a delta wing placed at a certain incidence); the second, where the relation (III.1) is reduced to

$$\left.\begin{aligned} w &= w_0 \qquad \text{for} \qquad x_3 = +0 \\ w &= -w_0 \qquad \text{for} \qquad x_3 = -0 \end{aligned}\right\}$$

which we shall call the elementary symmetrical problem. This is the case of, for instance, the flow about a body consisting essentially of two delta wings, symmetrical with respect to Ox_1x_2 and forming an infinitely small angle with this plane. It is also the case that will be obtained, the section of which, produced by a plane parallel to Ox_2x_3, would be an infinitely flattened rhombus. The fact that one obtains the same mathematical formulation for two different cases indicates the relative character of the results which will be obtained. In the case of the symmetrical problem one may naturally assume that w is zero on the plane Ox_1x_2 at every point situated outside of (d).

Let us finally point out that very frequently the obtained results do not satisfy the conditions of linearized flows; in particular, the velocity components and their derivatives will frequently be infinite along the semi-infinite lines bounding the area (d). However, we admit once more that the results obtained provide a first approximation of the problem posed above, in accordance with the remarks made in section 1.1.3 of chapter I.

3.1 - Cone Obstacle Entirely Inside the Mach Cone

3.1.1 - Study of the Elementary Problems

The case of the lifting cone has already formed the subject of a memorandum by Stewart (ref. 10); however, the demonstration we are going to give is more elementary and will permit us to treat simultaneously the lifting and the symmetrical case.

3.1.1.1 - Definition of the function $F(Z)$.- We shall make our study in the plane Z. Let $A'A(-a,+a)$ be the image of the cut of the surface (d)[26], (C_0), as usual, the circle of radius 1 (fig. 31).

Naturally, we shall operate with the function $W(Z)$. One of the conditions to be realized which we shall find again everywhere below is that dW/dZ must be divisible by $(Z^2 - 1)$, unless the compatibility relations show that $U(Z)$ would admit the points $Z = \pm 1$ as singular points which is inadmissible. Thus we introduce the function

$$F(Z) = \frac{Z^2}{Z^2 - 1}\frac{dW}{dZ} \tag{III.2}$$

and we shall attempt to determine $F(Z)$ for the symmetrical as well as for the lifting problem.

$F(Z)$ is a holomorphic function inside of the domain (D), bounded by the cut and the circle (C_0); the only singular points this function can present on the boundary of (D), are A and A'; on the other hand, $F(Z)$ must be divisible by Z^2, unless U, V, W have singularities at the origin. On the two edges of the cut $F(Z)$ must have a real zero part. On the circle (C_0)

$$\frac{Z}{Z^2 - 1} = \frac{1}{Z - \frac{1}{Z}} = \frac{1}{2i \sin \theta}$$

$$Z\frac{dW}{dZ} = e^{i\theta}\frac{dW}{dZ} = -i\frac{dW}{d\theta}$$

Consequently, $F(Z)$ has a real zero part on (C_0) as well. The fact that $F(Z)$ cannot be identically zero, and that its real part is zero on the boundary of (D), admits A and A' as singular points. We shall study the nature of these singularities.

3.1.1.2 - Singularities of $F(Z)$.- Physically, it is clear that A and A' cannot be essential singular points. Let us therefore suppose that, in the neighborhood of $Z = a$, one has

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[26]One assumes, as a start, that the problem permits the use of the plane Ox_1x_3 as the plane of symmetry.

$$F(Z) \sim K_{m_0}(Z - a)^{m_0}$$

m_0 being arbitrary, $K_{m_0} \neq 0$; let us put

$$Z - a = re^{i\varphi}$$

with φ being equal to $+\pi$ on the upper edge of the cut, to $-\pi$ on the lower edge; for sufficiently small values of r

$$K_{m_0}r^{m_0}e^{im_0\pi} \qquad \text{and} \qquad K_{m_0}r^{m_0}e^{-im_0\pi}$$

must be purely imaginary quantities; thus the same will hold true for

$$K_{m_0} \cos m_0\pi \qquad \text{and for} \qquad iK_{m_0} \sin m_0\pi;$$

$$K_{m_0}{}^2 = K_{m_0}{}^2\cos^2 m\pi - \left(iK_{m_0} \sin m\pi\right)^2$$

is therefore real. On the other hand

$$iK_{m_0}{}^2 \frac{\sin 2m_0\pi}{2} = \left(K_{m_0} \cos m_0\pi\right)\left(iK_{m_0} \sin n_0\pi\right)$$

must be real which entails

$$\sin 2m_0\pi = 0$$

Thus there are two possibilities; let us denote by k an arbitrary integral; either

$$m_0 = k, \qquad K_{m_0} \text{ is purely imaginary}$$

or else

$$m_0 = k + \frac{1}{2}, \quad K_{m_0} \text{ is real.}$$

Let us now consider

$$F_1(Z) = F(Z) - K_{m_0}(Z - a)^{m_0}$$

In the neighborhood of $Z = a$

$$F_1(Z) \sim K_{m_1}(Z - a)^{m_1}$$

and the same argument shows that $2m_1$ must be an integral. Finally, one may state the following theorem:

Theorem: Inside of (C_0) the function $F(Z)$ may assume the form

$$F(Z) = \Phi(Z) + \frac{1}{\sqrt{a^2 - Z^2}} \Psi(Z) \tag{III.3}$$

with $\Phi(Z)$ and $\Psi(Z)$ admitting no singularities other than the poles at A and A'.

The analysis we shall make will be simplified owing to certain symmetry conditions which $F(Z)$ satisfies. Let us put

$$W = w + iw'$$

Obviously, X in $w(X,Y)$ is even (when Y is constant).

Consequently, $F(Z)$ has a real part zero on OY. Applying Schwartz' principle one may write

$$F(Z) = -\overline{F}(-\overline{Z}) \tag{III.4}$$

This equation shows that knowledge of the development of $F(Z)$ around $Z = a$ immediately entails knowledge of $F(Z)$ around $Z = -a$.

<u>3.1.1.3 - Study of the case where $F(Z)$ is uniform $\left[\Psi(Z) = 0\right]$</u>.- Let us consider the function

$$A_p(Z) = \frac{iZ^{2p}}{\left[\left(a^2 - Z^2\right)\left(1 - a^2Z^2\right)\right]^p} \tag{III.5}$$

with p an integral and $\geqslant 1$.

This function satisfies all conditions imposed on $F(Z)$.

Indeed, it satisfies equation (III.4); inside of $\left(C_0\right)$ it does not admit singularities other than a and $-a$ which are poles of the order p_1. Its real part is zero on the cut as well as on $\left(C_0\right)$, as one can see when writing

$$A_p(Z) = \frac{i}{\left[a^2\left(Z^2 + \frac{1}{Z^2}\right) - \left(1 + a^4\right)\right]^p}$$

Finally, the origin should be double zero (at least).

Let us assume $F(Z)$ to be the general solution of the problem stated; we shall then demonstrate the following theorem:

<u>Theorem: If $F(Z)$ is uniform, one has</u>

$$F(Z) = \sum_1^n \lambda_p A_p(Z) = i\sum_1^n \frac{\lambda_p Z^{2p}}{\left[\left(a^2 - Z^2\right)\left(1 - a^2Z^2\right)\right]^p} \tag{III.6}$$

<u>with n being an integral, and the λ_p being real coefficients.</u>

In case $F(Z)$ is assumed to be a solution of the problem having a pole of the order n, one can determine a number λ_n so that

$$F_1(Z) = F(Z) - \lambda_n A_n(Z)$$

will be a solution admitting the pole $Z = a$ only of an order not higher than $(n - 1)$ at most. But in consequence of equation (III.4),

$F_1(Z)$ will allow of $Z = -a$ as pole of, at most, the order $(n - 1)$. Proceeding by recurrence, one finally defines a function

$$F_n(Z) = F(Z) - \sum_1^n \lambda_p A_p(Z)$$

which must satisfy all conditions of the problem and be holomorphic inside of (C_0). The boundary conditions on the circle and on the cut entail $F_n(Z)$ to be a constant which must be zero because $F_n(Z)$ must become zero at the origin.

3.1.1.4 - Case where $\Phi(Z) = 0$.- We shall study the case where $\Phi(Z) = 0$ in a perfectly analogous manner.

Let us put

$$f(Z) = \frac{\sqrt{(a^2 - Z^2)(1 - a^2 Z^2)}}{Z} F(Z)$$

$f(Z)$ is a uniform function inside of (C_0) which admits as poles only the points $(Z = -a,\ Z = a)$. Actually, the origin is not a pole since, according to hypothesis, $F(Z)$ is divisible by Z^2. The function $f(Z)$ possesses the following properties: It is imaginary on the cut, real on (C_0), and real on OY (which entails properties of symmetry if one changes Z to $-\bar{Z}$). Moreover, $f(Z)$ admits the origin as zero of, at least, the order 1. All these properties appertain equally to the functions

$$B_p(Z) = A_p\left(Z - \frac{1}{Z}\right) = \frac{iZ^{2p-1}(Z^2 - 1)}{\left[(a^2 - Z^2)(1 - a^2 Z^2)\right]^p}$$

p is an integral $\geqslant 1$.

Thus one deduces, as before, the theorem:

Theorem: In the case where $\Phi(Z) = 0$, one may write

$$F(Z) = i \sum_{1}^{n} \lambda_p \frac{Z^{2p}\left(Z^2 - 1\right)}{\left[\left(a^2 - Z^2\right)\left(1 - a^2Z^2\right)\right]^{p+\frac{1}{2}}} \tag{III.7}$$

with n being an integral, the λ_p being real.

3.1.1.5 - The principle of "minimum singularities".- The formulas (III.6) and (III.7) depend on an arbitrary number of coefficients. The only datum we know is the w_0, the value w assumes on the upper edge of the cut. Thus we have to introduce a principle which will guarantee the uniqueness of the solution of the problems we have set ourselves. This principle which we shall call principle of the "minimum singularities" may be formulated in the following manner (it is constantly being employed in mathematical physics):

When the conditions of a problem require the introduction of functions presenting singularities, one will, in a case of indeterminiteness, be satisfied with introducing the singularities of the lowest possible order permitting a complete solution of the posed problem.

In the case which is of interest to us, this amounts to assuming $n = 1$ in the formulas (III.6) and (III.7). For the problem of interest to us, this principle has immediate significance; it amounts to stating that $F(Z)$ and hence dW/dZ must be of an order lower than 2 in $1/Z - a$, or $W(Z)$ must be of an order lower than 1 with respect to that same infinity; the considerations set forth in section 2.2.7 show that these conditions entail the total energy to remain finite.

3.1.1.6 - Solution of the elementary symmetrical problem.- Let us turn again to formula (III.6); one deduces from it, according to equation (III.2), that in the case where $F(Z)$ is uniform

$$\frac{dW}{dZ} = i\lambda_1 \frac{Z^2 - 1}{\left(a^2 - Z^2\right)\left(1 - a^2Z^2\right)}$$

and hence

$$W(Z) = \frac{i\lambda_1}{2a\left(1 + a^2\right)} \log \frac{(a - Z)(1 - aZ)}{(a + Z)(1 + aZ)} + w_0$$

The determination of the logarithm is just that the real part of $W(Z)$ is zero on (C_0). Besides

$$\lambda_1 = -\frac{2a\left(1 + a^2\right)w_0}{\pi}$$

On the upper edge of the cut

$$w = w_0$$

and on the lower edge w assumes the opposite value. This shows us that the case investigated is that of the symmetrical problem. The value $W(Z)$ for this problem is therefore

$$W(Z) = -\frac{iw_0}{\pi} \log\left[\frac{(a - Z)(1 - aZ)}{(a + Z)(1 + aZ)}\right] + w_0 \qquad \text{(III.8)}$$

The calculation of the functions $U(Z)$ and $V(Z)$ offers no difficulty whatsoever. It suffices to apply the relationships of compatibility (I.25) and to integrate; the only precaution to be taken consists in choosing the constant of integration in such a manner that the real parts of U and V on (C_0) become zero; one then finds

$$V(Z) = \frac{w_0}{\pi} \frac{\left(1 + a^2\right)}{\left(1 - a^2\right)} \log\left[\frac{(a + Z)(1 - aZ)}{(Z - a)(1 + aZ)}\right] \qquad \text{(III.9)}$$

and

$$U(Z) = \frac{2w_0 a}{\pi\beta\left(1 - a^2\right)} \log\left[\frac{Z^2 - a^2}{1 - a^2Z^2}\right] \qquad \text{(III.10)}$$

This last formula is the most interesting one since it permits calculation of the pressure coefficient (see formula (I.8)). One finds

$$C_p = -\frac{4w_0}{\pi\beta} \frac{a}{1 - a^2} \log\left[\frac{a^2 - X^2}{1 - a^2X^2}\right] \qquad \text{(III.11)}$$

In order to interpret this formula, one must connect the quantities a, X, to geometrical quantities, related to the given cone. First of all

$$w_0 = \alpha$$

α being the constant inclination of the cone on Ox. On the other hand

$$\frac{2X}{1 + X^2} = \beta \frac{r}{x_1} = \beta \tan \omega$$

whence

$$X = \frac{\cos \omega - \sqrt{1 - M^2 \sin^2 \omega}}{\beta \sin \omega}$$

(see fig. 32) and

$$a = \frac{\cos \omega_0 - \sqrt{1 - M^2 \sin^2 \omega_0}}{\beta \sin \omega_0} \qquad \text{(III.12)}$$

In figure 33 one will find the curves giving the values of C_p as functions of ω, for various Mach numbers and various values of ω_0.

3.1.1.7 - Solution of the elementary lifting problem.- If one starts from the formula (III.7), one obtains

$$\frac{dW}{dZ} = i\lambda_1 \frac{\left(Z^2 - 1\right)^2}{\left[\left(a^2 - Z^2\right)\left(1 - a^2 Z^2\right)\right]^{\frac{3}{2}}}$$

The integration which yields $W(Z)$ introduces elliptic functions (see section 3.1.1.8); on the other hand, it will (now) be possible to calculate $U(Z)$. We note beforehand that, according to the preceding formula, $W(Z)$ assumes the same value on the two edges of the cut and that, consequently, this solution corresponds to the lifting problem.

The relationships of compatibility show that

$$\frac{dU}{dZ} = \frac{2\lambda_1}{\beta} \frac{Z\left(Z^2 - 1\right)}{\left[\left(a^2 - Z^2\right)\left(1 - a^2Z^2\right)\right]^{\frac{3}{2}}}$$

and hence

$$U(Z) = -\frac{2\lambda_1}{\beta\left(a^2 + 1\right)^2} \frac{Z^2 + 1}{\left[\left(a^2 - Z^2\right)\left(1 - a^2Z^2\right)\right]^{\frac{1}{2}}} \tag{III.13}$$

We still have to calculate λ_1 as a function of w_0. For this purpose, one may write

$$-w_0 = \int_0^i \frac{dW}{dZ}\, dZ = i\lambda_1 \int_0^i \frac{\left(Z^2 - 1\right)^2 dZ}{\left[\left(a^2 - Z^2\right)\left(1 - a^2Z^2\right)\right]^{\frac{3}{2}}}$$

We put in this integral $Z = iu$

$$w_0 = \lambda_1 \int_0^1 \frac{\left(1 + u^2\right)^2 du}{\left[\left(a^2 + u^2\right)\left(1 + a^2u^2\right)\right]^{\frac{3}{2}}} = \lambda_1 I(a)$$

The calculation of $I(a)$ can be made with the aid of the function E (see ref. 24). We shall put

$$u + \frac{1}{u} = \frac{2}{t}$$

After a few calculations one obtains

$$I(a) = 4 \int_0^1 \frac{dt}{\sqrt{1 - t^2}\left[4a^2 + \left(a^2 - 1\right)^2 t^2\right]^{\frac{3}{2}}}$$

Finally, the change in variable

$$\sin \varphi = \frac{t\left(a^2 + 1\right)}{\sqrt{4a^2 + \left(a^2 - 1\right)^2 t^2}}$$

shows that if one puts

$$k = \frac{1 - a^2}{1 + a^2}$$

$$I = \frac{1}{a^2\left(a^2 + 1\right)} \int_0^1 \sqrt{1 - k^2\sin^2\varphi}\, d\varphi = \frac{1}{a^2\left(a^2 + 1\right)} E\left(\frac{1 - a^2}{1 + a^2}\right)$$

Hence the new formula for $U(Z)$

$$U(Z) = -\frac{2}{\beta} \frac{a^2 w_0}{\left(a^2 + 1\right) E\left(\frac{1 - a^2}{1 + a^2}\right)} \frac{Z^2 + 1}{\left[\left(a^2 - Z^2\right)\left(1 - a^2 Z^2\right)\right]^{\frac{1}{2}}} \qquad \text{(III.14)}$$

We still have to connect a and Z to the geometrical quantities. One has (fig. 32)

$$\frac{2a}{1 + a^2} = \beta \tan \omega_0 \qquad \frac{2X}{1 + X^2} = \beta \tan \omega$$

One puts

$$t = \frac{\tan \omega}{\tan \omega_0}$$

and obtains

$$u = -\frac{w_0 \tan \omega_0}{E\left[\sqrt{1 - \beta^2 \tan^2 \omega_0}\right]} \frac{1}{\sqrt{1 - t^2}}$$

and

$$C_p = \frac{2\alpha \tan \omega_0}{E\left[\sqrt{1 - \beta^2\tan^2\omega_0}\right]} \frac{1}{\sqrt{1 - t^2}} \tag{III.15}$$

if one puts, as usual

$$w_0 = \alpha$$

If ω_0 is small, $E\left[\sqrt{1 - \beta^2\tan^2\omega_0}\right]$ is close to 1, and the formula (III.15), except for the notations, again gives a result found before (formula (II.33)).

On the other hand, if $\beta \tan \omega_0 \longrightarrow 1$

$$E\left[\sqrt{1 - \beta^2\tan^2\omega_0}\right] \longrightarrow \frac{\pi}{2}$$

and the formula (III.15) is written

$$C_p = \frac{4\alpha}{\beta\pi} \frac{1}{\sqrt{1 - t^2}}$$

Remark.

Thus one sees that the elliptic functions need not be used in an essential manner in order to obtain the pressure coefficient. Actually they appear only in the multiplicative coefficient. (In contrast, Stewart, in his demonstration (ref. 10), makes essential use of the elliptic functions.) However, these functions are indispensable in the explicit calculation of $W(Z)$ and $V(Z)$.

3.1.1.8 - Calculation of $W(Z)$ and $V(Z)$.- There exist several simple methods for calculating $W(Z)$; the first consists in putting[27]

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[27]For all the properties of the elliptic functions made use of in this report, see for instance reference 24. In this paragraph, u will be a complex variable and will have no relation to the velocity component along Ox_1.

$$Z = a\ \mathrm{sn}\left(u, a^2\right) \qquad \left(k = a^2\right) \tag{III.16}$$

This transformation achieves the conformal representation of the domain (D) on a strip of the plane u (see fig. 34); the values written inside of small circles indicate the values of Z taken for the corresponding value of u.

One has actually

$$\mathrm{sn}\ 0 = 0 \qquad \mathrm{sn}\ K = 1$$

$$\mathrm{sn}\left(\frac{i}{2}\ K'\right) = i\ \mathrm{sc}\left(\frac{K'}{2}, k'\right) = \frac{i}{\sqrt{k}} = \frac{i}{a}$$

$$\mathrm{sn}\left(K + \frac{iK'}{2}\right) = \mathrm{cd}\left(\frac{iK'}{2}\right) = \frac{1}{\mathrm{dn}\left(\frac{K'}{2}, k'\right)} = \frac{1}{\sqrt{k}} = \frac{1}{a}$$

Under these conditions

$$\frac{dW}{du} = \frac{dW}{dZ}\frac{dZ}{du} = \frac{i\lambda_1\left(Z^2 - 1\right)^2}{\left(a^2 - Z^2\right)\left(1 - a^2Z^2\right)} =$$

$$i\lambda_1\left[\frac{1}{a^2} - \frac{1 - a^2}{1 + a^2}\left[\frac{1}{Z^2 - a^2} - \frac{1}{a^2\left(a^2Z^2 - 1\right)}\right]\right] = \frac{i\lambda_1}{a^2}\left[1 + \frac{1}{\mathrm{dn}^2 u} - \frac{1}{\mathrm{cn}^2 u}\right]$$

whence

$$W(u) = w_0 + \frac{i\lambda_1}{a^2\left(a^2 + 1\right)^2}\left[2\left(a^2 + 1\right)u - 2E(u) + \frac{a^4\mathrm{sn}\ u\ \mathrm{cn}\ u}{\mathrm{dn}\ u} + \frac{\mathrm{dn}\ u\ \mathrm{sn}\ u}{\mathrm{cn}\ u}\right] \tag{III.17}$$

For determination of λ_1, it suffices to write, for instance, that

$$W\left(\frac{iK'}{2}\right) = 0$$

Now

$$W\left(\frac{iK'}{2}\right) = \frac{i\lambda_1}{a^2\left(a^2 + 1\right)^2}\left[\left(a^2 + 1\right)i(1 + K') - 2E\left(\frac{iK'}{2}\right)\right] + w_0 = 0$$

However,

$$2E\left(\frac{iK'}{2}\right) = iK' + 2i\,\mathrm{dn}\left(\frac{K'}{2},k'\right)\mathrm{sc}\left(\frac{K'}{2},k'\right) - 2iE\left(\frac{K'}{2},k'\right)$$

$$\mathrm{dn}\left(\frac{K'}{2},k'\right)\mathrm{sc}\left(\frac{K'}{2},k'\right) = 1$$

$$2E\left(\frac{K}{2},k'\right) = E(K',k') + \frac{k'^2}{1 + k}$$

whence the value of λ_1

$$\lambda_1 = \frac{w_0 a^2\left(a^2 + 1\right)^2}{a^2K' + E(k')}$$

This expression differs from the formula given for λ_1 in the course of section 3.1.7; besides, one may, in a general manner, put the formula (III.17) in another form (using a modulus $k_1 = \dfrac{1 - a^2}{1 + a^2}$ which is different from the modulus $k = a^2$ utilized so far) by applying the Landen transformation.

This transformation permits, in particular, establishment of the following formula

$$E\left[(1 + k)u,k_1\right] = \frac{1}{1 + k}\left[2E(u,k') + 2ku - k'^2\mathrm{sn}\ u\ \mathrm{cd}\ u\right]$$

with the functions of the term at the right of the preceding equality being relative to the modulus $k' = \sqrt{1 - a^4}$.

If one puts

$$u = iy$$

this formula is written

$$E\left[(1 + k)iy, k_1\right] = \frac{i}{1 + k}\left[2(1 + k)y - 2E(y,k) + \frac{2\ \text{dn}\ y\ \text{sn}\ y}{\text{cn}\ y} - \frac{k'^2\text{sn}\ y}{\text{cn}\ y\ \text{dn}\ y}\right]$$

These last functions are relative to the modulus $k = a^2$.

However,

$$\frac{2\ \text{dn}\ y\ \text{sn}\ y}{\text{cn}\ y} - k^2 \frac{\text{sn}\ y}{\text{cn}\ y\ \text{dn}\ y} = \frac{\text{sn}\ y}{\text{cn}\ y\ \text{dn}\ y}\left[2\ \text{dn}^2 y - k'^2\right] =$$

$$\frac{\text{sn}\ y}{\text{cn}\ y\ \text{dn}\ y}\left(k^2\text{cn}^2 y + \text{dn}^2 y\right) = a^4 \frac{\text{sn}\ y\ \text{cn}\ y}{\text{dn}\ y} + \frac{\text{dn}\ y\ \text{sn}\ y}{\text{cn}\ y}$$

If one now refers to the formula (III.17), one sees that it may also be written

$$W(u) = w_0 + \frac{\lambda_1}{a^2\left(a^2 + 1\right)} E\left[\left(1 + a^2\right)iu, k_1\right]$$

and that under these conditions

$$W\left(\frac{iK'}{2}\right) = w_0 - \frac{\lambda_1}{a^2\left(a^2 + 1\right)} E\left[\frac{(1 + k)K'}{2}, k_1\right] = 0$$

However, $K_1 = \dfrac{(1 + k)K'}{2}$ is precisely such that

$$\text{sn}\left(K_1, k_1\right) = 1$$

Consequently

$$\lambda_1 = \frac{w_0 a^2 \left(a^2 + 1\right)}{E\left(\frac{1 - a^2}{1 + a^2}\right)}$$

which is, of course, the formula found previously. Hence

$$W(u) = w_0 \left[1 + \frac{E\left[\left(1 + a^2\right) iu, k_1\right]}{E\left(k_1\right)}\right] \qquad \text{(III.18)}$$

One may also proceed in another manner, introducing a variable other than the variable u.

We put

$$t = \frac{2iZ}{Z^2 - 1}$$

The integration of $\frac{dW}{dZ}$ leads to

$$W(t) = w_0 - 4\lambda_1 \int_0^t \frac{dt}{\sqrt{1 - t^2}\left[4a^2 + \left(a^2 - 1\right)^2 t^2\right]^{\frac{3}{2}}}$$

We put

$$k_1 = \frac{1 - a^2}{1 + a^2}$$

The complementary modulus is $\frac{2a}{1 + a^2}$.

If one puts, therefore

$$t = \mathrm{cn}\left(\tau,k_1\right)$$

$$W(\tau) = w_0 + \frac{4\lambda_1}{\left(a^2 + 1\right)^3}\int_{\kappa_1}^{\tau}\frac{d\tau}{\mathrm{dn}^2\tau} = w_0 -$$

$$\frac{\lambda_1}{a^2\left(a^2 + 1\right)}\left[E\left(k_1\right) - E\left(\tau,k_1\right) + \frac{\left(1 - a^2\right)^2}{\left(1 + a^2\right)^2}\frac{\mathrm{sn}\ \tau\ \mathrm{cn}\ \tau}{\mathrm{dn}\ \tau}\right]$$

If $Z = i$, $t = 1$, $\tau = 0$, one always still finds the same value for λ_1

$$\lambda_1 = \frac{w_0a^2\left(a^2 + 1\right)}{E\left(\frac{1 - a^2}{1 + a^2}\right)}$$

and

$$W(\tau) = w_0\left[\frac{E\left(\tau,k_1\right)}{E\left(k_1\right)} - \left(\frac{1 - a^2}{1 + a^2}\right)^2\frac{1}{E\left(k_1\right)}\frac{\mathrm{sn}\ \tau\ \mathrm{cn}\ \tau}{\mathrm{dn}\ \tau}\right] \tag{III.19}$$

The formulas (III.18) and (III.19) are indicated for the calculation of W along the axis OY, whereas equation (III.17) is more suitable for the calculation of W along the axis OX. We now turn to the calculation of $V(Z)$. The calculation with the aid of the variable u is particularly simple. dV/dZ is calculated with the aid of the relationships of compatibility

$$\frac{dV}{dZ} = -\frac{w_0a^2\left(a^2 + 1\right)}{E\left(k_1\right)}\frac{Z^4 - 1}{\left[\left(a^2 - Z^2\right)\left(1 - a^2Z^2\right)\right]^{\frac{3}{2}}}$$

Let us recall that

$$k_1 = \frac{1 - a^2}{1 + a^2}$$

and perform the change in variable (III.16). We obtain immediately

$$\frac{dV}{du} = \frac{w_0(a^2 + 1)}{E(k_1)} \frac{1 - a^4 sn^4 u}{cn^2 u \, dn^2 u}$$

but V must be zero for $u = 0$. The integration of dV/du then gives

$$V(u) = \frac{w_0(a^2 + 1)}{E(k_1)} \frac{sn\ u}{cn\ u\ dn\ u} \tag{III.20}$$

We verify, for instance, that for $Z = i$, V has a real part zero,

$$Z = i \quad \text{corresponds to} \quad u = \frac{iK'}{2}$$

$$sn\left(\frac{iK'}{2}\right) = \frac{i}{a}$$

$$cn\left(\frac{iK'}{2}\right) = nc\left(\frac{K'}{2}, k'\right) = \sqrt{\frac{1 + k}{k}}$$

$$dn\left(\frac{iK'}{2}\right) = dc\left(\frac{K'}{2}, k'\right) = \sqrt{k(1 + k)} \qquad \left(k = a^2\right)$$

One can state that $V\left(\frac{iK'}{2}\right)$ is purely imaginary. We shall not give another formula for the calculation of $V(Z)$; the formula (III.20) which is particularly simple (it does not make use of the function E) permits the calculation of v on the axis OX; on the other hand, v is zero on OY.

3.1.2 - Study of the Case Where the Cut is Not Symmetrical With Respect to OY

<u>3.1.2.1 - General Principle</u>.- The case where the cone investigated does not admit the plane Ox_1,x_3 as the symmetry plane is easily led back to the preceding by a conformal representation, maintaining the circle (C_0).

Let us suppose, for instance, that in the plane Z the obstacle is represented by a cut along the segment (b,c) of the real axis (see fig. 35); the conformal transformation

$$Z_1 = \frac{Z - \alpha_1}{1 - \alpha_1 Z} \tag{III.21}$$

where α_1 is a real number $\left(|\alpha_1| < 1\right)$ maintains definitely the real axis and the circle $\left(C_0\right)$. We shall attempt to determine the numbers a_1 and α_1 in such a manner that $Z = c$ corresponds to $Z_1 = a_1$, $Z = b$ to $Z_1 = -a_1$. One must write

$$a_1 = \frac{c - \alpha_1}{1 - \alpha_1 c} \qquad -a_1 = \frac{b - \alpha_1}{1 - \alpha_1 b}$$

α_1 is determined by the equation

$$\frac{c - \alpha_1}{1 - \alpha_1 c} + \frac{b - \alpha_1}{1 - \alpha_1 b} = 0$$

which gives

$$\alpha_1 = \frac{1 + bc - \sqrt{\left(1 - b^2\right)\left(1 - c^2\right)}}{b + c}$$

(we note that, if $b + c = 0$, $\alpha_1 = 0$).

One will then determine a_1 by one of the two formulas described above or by the formula symmetrical with respect to b and c

$$a_1 = \frac{\sqrt{1 - b^2} - \sqrt{1 - c^2}}{b\sqrt{1 - c^2} + c\sqrt{1 - b^2}} = \frac{bc - 1 + \sqrt{\left(1 - b^2\right)\left(1 - c^2\right)}}{b - c}$$

a relationship which one may find directly by writing

$$\left(1,-1,a_1,-a_1\right) = (1,-1,c,b)$$

In particular

$$\frac{1 - a_1^2}{1 + a_1^2} = \frac{\sqrt{\left(1 - b^2\right)\left(1 - c^2\right)}}{1 - bc}$$

3.1.2.2 - Symmetrical problem.- It will now be very easy for us to study the case of the symmetrical problem (that is, the case where w assumes the value w_0 on the upper edge, and the value $-w_0$ on the lower edge of the cut).

The formula which gives W as a function of Z_1 is written (formula (III.8))

$$W\left(Z_1\right) = -\frac{iw_0}{\pi}\log\left[\frac{a_1 - Z_1}{a_1 + Z_1}\,\frac{1 - a_1Z_1}{1 + a_1Z_1}\right] + w_0$$

whence

$$W(Z) = -\frac{iw_0}{\pi}\left[\log\left[(c - Z)(1 - Zc)\right] - \log\left[(b - Z)(1 - Zb)\right]\right] + w_0 \tag{III.22}$$

$V(Z)$ and $U(Z)$ are obtained by the compatibility formulas

$$\frac{dV}{dZ} = -\frac{w_0}{\pi}\,\frac{\left(Z^2 + 1\right)(b - c)(1 - bc)}{(c - Z)(1 - Zc)(Z - b)(1 - Zb)}$$

whence

$$V(Z) = -\frac{w_0}{\pi}\left[\frac{1 + c^2}{1 - c^2}\log\frac{c - Z}{1 - Zc} - \frac{1 + b^2}{1 - b^2}\log\frac{Z - b}{1 - Zb}\right]$$

Finally

$$\frac{dU}{dZ} = \frac{2w_0}{\beta\pi}\,\frac{(b - c)(1 - bc)Z}{(c - Z)(1 - Zc)(Z - b)(1 - Zb)}$$

whence

$$U(Z) = \frac{2w_0}{\pi\beta}\left[\frac{c}{\left(1 - c^2\right)} \log \frac{c - Z}{1 - cZ} - \frac{b}{1 - b^2} \log \frac{Z - b}{1 - bZ}\right] \qquad \text{(III.23)}$$

Naturally, one could have obtained these expressions directly, by a reasoning analogous to the one made before in the sections above (3.1.4, 3.1.5, 3.1.6).

We remark that this problem possesses a property of "additivity" which is, besides, evident from the outset but is entirely obvious in the formulas (III.21), (III.22), (III.23). This means that, if one knows the solution of the problem for a segment bc and the one relative to a segment cd, one obtains the solution relative to the segment bd by adding the given solutions. Also, we point out that in the preceding formulas the manner of determination of the logarithms should be conveniently chosen.

3.1.2.3 - Lifting problem.- We shall be satisfied with the calculation of the function $U(Z)$. Let us put in this paragraph

$$k_1 = \frac{1 - a_1{}^2}{1 + a_1{}^2} = \frac{\sqrt{\left(1 - b^2\right)\left(1 - c^2\right)}}{1 - bc}$$

One has

$$\frac{dW}{dZ_1} = \frac{i w_0 a_1{}^2\left(a_1{}^2 + 1\right)}{E\left(k_1\right)} \frac{\left(Z_1{}^2 - 1\right)^2}{\left[\left(a_1{}^2 - Z_1{}^2\right)\left(1 - a_1{}^2 Z_1{}^2\right)\right]^{\frac{3}{2}}}$$

$$\frac{dZ_1}{dZ} = \frac{1 - \alpha_1{}^2}{\left(1 - \alpha_1 Z\right)^2}$$

whence one obtains very easily

$$\frac{dU}{dZ} = \frac{2w_0}{\beta E\left(k_1\right)} \frac{a_1{}^2\left(1 + a_1{}^2\right)(1 + c)^3(1 - b)^3}{\left(1 + a_1\right)^6} \frac{Z\left(Z^2 - 1\right)}{\left[(c - Z)(Z - b)(1 - bZ)(1 - cZ)\right]^{\frac{3}{2}}}$$

The equality

$$\left(b,\frac{1}{b},c,1\right) = \left(-a_1,-\frac{1}{a_1},a_1,1\right)$$

is written

$$F = \frac{c - b}{1 - bc} = \frac{2a_1}{1 + a_1^2}$$

and if one forms the combination $\frac{(F + 1)^3}{F^2}$, one may deduce from it the identity

$$\frac{(1 + c)^3(1 - b)^3}{(c - b)^2(1 - bc)} = \frac{\left(1 + a_1\right)^6}{4a_1^2\left(1 + a_1^2\right)}$$

which permits one to write

$$\frac{dU}{dZ} = \frac{w_0}{2\beta E\left(k_1\right)}(b - c)^2(1 - bc)\frac{Z\left(Z^2 - 1\right)}{\left[(c - Z)(Z - b)(1 - bZ)(1 - cZ)\right]^{\frac{3}{2}}}$$

The integration is easily made, with the aid of the elementary functions

$$U(Z) = \frac{w_0}{\beta E\left(k_1\right)(1 - bc)}\frac{2bc\left(Z^2 + 1\right) - (b + c)(1 + bc)Z}{\sqrt{(c - Z)(Z - b)(1 - bZ)(1 - cZ)}} \qquad \text{(III.24)}$$

<u>3.1.2.4 - Lift of a delta wing</u>.- The total energy on an obstacle will be obtained, in a general manner, by integration of the pressures. However, the lift may be calculated by means of a very simple general formula which we shall set up.

We shall start from the formula

$$C_p = -2\underline{R}\left[U(Z)\right]$$

Let us consider an elementary triangle OMM' (see fig. 36), with M having the coordinates $(\beta, x_2, 0)$; its area is equal to $\frac{\beta\, dx_2}{2}$. One has, by definition of C_z

$$C_z = \frac{-2\int_{M_1M_2} C_p\, dx_2}{\int_{M_1M_2} dx_2}$$

which in the plane z is written

$$C_z = \frac{\int_{\underline{L}} C_p\, dz}{\mu - \lambda}$$

In z, λ and μ are the images of the limiting generatrices of the obstacle, $\underline{L}$ is the loop surrounding the cut (λ,μ). If one denotes by (L) the loop surrounding the corresponding cut bc in the plane Z, one has, since

$$x = \frac{2X}{1 + X^2} \qquad (Z = X + iY)$$

$$\int_{\underline{L}} C_p\, dx = \int_L C_p\, dx_2 = 2\int_L C_p \frac{1 - X^2}{\left(1 + X^2\right)^2} dX =$$

$$-4\underline{R}\left[\int_L U(Z) \frac{1 - Z^2}{\left(1 + Z^2\right)^2} dZ\right] = -4\underline{R}\left[\int_{C_0'} U(Z) \frac{1 - Z^2}{\left(1 + Z^2\right)^2} dZ\right]$$

with (C_0') denoting the circle of the radius 1, modified in the neighborhood of i and -i by two small arcs ll', mm', in order to avoid the singular points (see fig. 37); the arrows indicate the direction of the course. Along the circle (C_0), $\left(Z = e^{i\theta}\right)$

$$\frac{1 - Z^2}{\left(1 + Z^2\right)^2} dZ = - \frac{2i \sin \theta}{4 \cos^2\theta} i \, d\theta = \frac{\sin \theta}{2 \cos^2\theta} d\theta$$

and since

$$\underline{R}\left[U(Z)\right] = 0$$

one deduces that the integral is zero along the arcs $l'm'$, ml; the points $Z = \pm i$ are double poles of the quantity that must be integrated; but one can easily see that the integral remains finite along the circular arcs ll' and $m'm$. Exactly speaking: if one denotes by R_i and R_{-i} the remainders of the function

$$U(Z) \frac{1 - Z^2}{\left(1 + Z^2\right)^2}$$

at the points $Z = i$ and $Z = -i$, one has, since

$$\mu - \lambda = \frac{2c}{1 + c^2} - \frac{2b}{1 - b^2} = \frac{2(c - b)(1 - bc)}{\left(1 + b^2\right)\left(1 + c^2\right)}$$

$$C_z = +2 \frac{\left(1 + b^2\right)\left(1 + c^2\right)}{(c - b)(1 - bc)} \underline{R}\left[i\pi\left(R_i + R_{-i}\right)\right]$$

However,

$$R_i = - \frac{1}{2} \frac{dU}{dZ}(Z = i) \qquad R_{-i} = - \frac{1}{2} \frac{dU}{dZ}(Z = -i)$$

whence

$$C_z = - \frac{\left(1 + b^2\right)\left(1 + c^2\right)}{(c - b)(1 - bc)} \underline{R}\left[i\left[\frac{dU}{dZ}_{(z=i)} + \frac{dU}{dZ}_{(z=-i)}\right]\pi\right] \qquad \text{(III.25)}$$

One can also express C_z as a function of the values of dW/dZ at the points i and $-i$

$$C_z = \frac{\pi\left(1 + b^2\right)\left(1 + c^2\right)}{\beta(c - b)(1 - bc)} \underline{R}\left[i\left[\frac{dW}{dZ}_{(z=i)} - \frac{dW}{dZ}_{(z=-i)}\right]\right] \tag{III.26}$$

One may finally remark that

$$\frac{dW}{dZ_{(Z=i)}} = -i \frac{\partial w}{\partial Y}(0,1)$$

whence

$$C_z = \frac{\pi\left(1 + b^2\right)\left(1 + c^2\right)}{\beta(c - b)(1 - bc)}\left[\frac{\partial w}{\partial Y}(0,+1) - \frac{\partial w}{\partial Y}(0,-1)\right] \tag{III.27}$$

We shall apply this result to the case of the lifting delta wing studied in section 3.1.2.3:

$$\frac{dU}{dZ_{(Z=i)}} = -i \frac{w_0}{\beta E\left(k_1\right)} \frac{(b - c)^2(1 - bc)}{\left[\left(1 + b^2\right)\left(1 + c^2\right)\right]^{\frac{3}{2}}}$$

whence

$$C_z = - \frac{2\alpha\pi}{\beta E\left(k_1\right)} \frac{c - b}{\sqrt{\left(1 + b^2\right)\left(1 + c^2\right)}}$$

with k_1 being equal to

$$\frac{\sqrt{\left(1 - b^2\right)\left(1 - c^2\right)}}{1 - bc}$$

In the wing theory, one designates the incidence by i; with the usual notations one has here

$$\alpha = -i$$

The desired formula $C_z(i)$ is

$$C_z = \frac{2\pi i}{\beta E(k_1)} \frac{c - b}{\sqrt{(1 + b^2)(1 + c^2)}} \tag{III.28}$$

In the case where $b = -a$, $c = a$, one finds

$$C_z = \frac{4\pi a i}{\beta(1 + a^2)E(k_1)}$$

or again with the notations of figure 32

$$C_z = \frac{2\pi \tan \omega_0}{E\left(\sqrt{1 - \beta^2 \tan^2 \omega_0}\right)} i \tag{III.29}$$

A few applications of this formula may be found in figure 38.

If ω_0 is small, one will find again the result obtained in section 2.2.4 (except for the notations)

$$C_z = 2\pi\omega_0 i$$

<u>3.1.3.1 - Study of the general case</u>.- So far, we have treated only the elementary cases, that is, those for which the function w assumed a constant value on each edge of the cut. We shall now treat the case where the function w assumes on the upper edge of the cut prescribed values

$$w = w_1(X)$$

and on the lower edge prescribed values which we shall note

$$w = w_2(X)$$

Let us note, first of all, that the solution of the general problem may always be considered as the result of superposition of the solutions of a purely lifting problem (with w assuming the same value $\frac{w_1(X) + w_2(X)}{2}$ on each edge of the cut) and of a purely symmetrical problem (with w assuming opposite values $\pm\frac{w_1(X) - w_2(X)}{2}$ on the two edges of the cut). Thus we shall be able to limit ourselves to these two types of problems. We shall note, in addition, that in the purely symmetrical problem u assumes the same values on the two edges of the cut, whereas it assumes, in contrast, opposite values in the case of a pure lifting problem.

A first idea for the treatment of this problem consists in utilizing the elementary solutions found before and in superposing them conveniently. Let us consider, for instance, for a symmetrical problem, an elementary wing of infinitely small span, the image of which in the plane Z is a segment of the real axis of the length ΔX, situated in the neighborhood of the point X, and let us assume $w = w(X)$ to be the value corresponding to w; the complex velocities of this flow are given by the formulas (III.21), (III.22), (III.23); using the hypotheses made, one may write, designating the complex velocities by $\Delta U(Z)$, $\Delta V(Z)$, $\Delta W(Z)$,

$$\Delta W(Z) = -\frac{iw(X)}{\pi}\frac{d}{dX}\left[\log(X - Z)(1 - ZX)\right]\Delta X$$

$$\Delta V(Z) = -\frac{w(X)}{\pi}\frac{d}{dX}\left[\frac{1 + X^2}{1 - X^2}\log\frac{X - Z}{1 - ZX}\right]\Delta X$$

$$\Delta U(Z) = \frac{2w(X)}{\pi\beta}\frac{d}{dX}\left[\frac{X}{1 - X^2}\log\frac{X - Z}{1 - ZX}\right]\Delta X$$

One arrives at writing the solution of the symmetrical problem in the form

$$\left.\begin{aligned}
W(Z) &= -\frac{i}{\pi}\int_b^c w(\xi)\frac{d}{d\xi}\Big[\log(\xi - Z)(1 - Z\xi)\Big]d\xi \\
V(Z) &= -\frac{1}{\pi}\int_b^c w(\xi)\frac{d}{d\xi}\left[\frac{1 - \xi^2}{1 + \xi^2}\log\frac{\xi - Z}{1 - Z\xi}\right]d\xi \\
U(Z) &= \frac{2}{\pi\beta}\int_b^c w(\xi)\frac{d}{d\xi}\left[\frac{\xi}{1 - \xi^2}\log\frac{\xi - Z}{1 - Z\xi}\right]d\xi
\end{aligned}\right\} \tag{III.30}$$

The integrals occurring in these formulas make sense only if Z is not on the segment bc. If Z is real and comprised between b and c, one has to take the "principal value" of these integrals. Furthermore, one must demonstrate, in order to justify these formulas, that the real part of the function $W(Z)$, defined by the first formula (III.30), actually assumes the value $w(X)$ when Z is real $(Z = X)$.

For this purpose, one calculates $W(Z)$ in a point of $Z = X + i\eta$ (with η being positive and small) by dividing the integral appearing in the first formula (III.30) into three parts

$$W(Z) = -\frac{i}{\pi}\left[\int_b^{X-\epsilon} + \int_{X+\epsilon}^{c} + \int_{X-\epsilon}^{X+\epsilon}\right]$$

After this has been done, one chooses ϵ and η in such a manner that the last integral is arbitrarily close to the value

$$I = w(X)\int_{X-\epsilon}^{X+\epsilon}\frac{d\xi}{\xi - Z}$$

which is possible since this integral may be written

$$\int_{X-\epsilon}^{X+\epsilon}\frac{w(\xi)\left(1 - Z^2\right)^2}{(\xi - Z)(1 - \xi Z)}\,d\xi$$

One may then, diminishing as necessary the upper limit fixed for η, choose that last number so that

$$\underline{R}\left[-\frac{1}{\pi}\left[\int_{b}^{X-\epsilon}+\int_{X-\epsilon}^{c}\right]\right]$$

should be arbitrarily small. There is no difficulty whatsoever since the quantity under the sign $\int$ is continued in Z. Finally, I may be made arbitrarily close to

$$i\pi w(x)$$

which shows that, if η is sufficiently small

$$\underline{R}\left[W(Z)\right] - w(X)$$

is arbitrarily small which had to be demonstrated.

This procedure, while theoretically simple, is rather delicate in practice since the calculations to be made affect the integrals, the principal value of which has to be taken. In the lifting case, on the other hand, the application of this method would require previous solution of an integral equation of a rather complicated type. For that reason we prefer to give the following calculation methods; the first utilizes the "electric analogies;" the second which is purely numerical will reduce the numerical calculation to that of a Poisson integral; in section 2.2.7 we have given a simple and accurate procedure for solving such a problem.

3.1.3.2 - Utilization of the "electric analogies"[28].- The analogy consists in identifying the harmonic function $w(X,Y)$ with an electric potential $\varphi(X,Y)$, through a conductor constituted by a liquid occupying a tank with horizontal bottom of half-circular shape (see fig. 39). On the circular boundary w is constant; consequently, the semicircumference will be brought to a constant potential; it will be possible to regard that potential as the zero of the scale of potentials. This circumference will, therefore, be conducting; (this half-circle is nothing else but the part of the circle (C_0) of the plane Z for which $Y > 0$).

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[28]For all questions concerning electric analogy, see the fundamental memoranda by M. Malavard (refs. 25 and 26).

On the cut bc which represents the conical obstacle, one distributes electrodes which will be brought, by means of adjustable potentiometers, to the given potential φ. For specification of the boundary conditions on the segments A'b and cA, one must distinguish between the symmetrical and the lifting problem.

<u>3.1.3.2.1 - Symmetrical problem</u>.- w must be zero on the portions of the axis outside of the cut; consequently, the corresponding boundaries of the tank are brought to the potential zero, that is, to the same potential as the semicircumference A'BA; w is given directly by a pure Dirichlet problem. However, the unknown of our problem is the value of the pressure along the segment bc, that is, u.

u is connected with w by the relationships of compatibility which permit one to write on the axis of the X

$$\beta \frac{\partial u}{\partial X} = \frac{2X}{1 - X^2} \frac{\partial w}{\partial Y}$$

with $\partial w/\partial Y$ being proportional to the intensity entering the tank through the electrodes; this quantity is easily measured with the aid of a convenient arrangement[29]. With the value of $\partial u/\partial X$ thus known, we must, in order to obtain the desired pressure distribution, determine, in addition, a value of u along bc, for instance the one at the point O[30]. On the axis OY one may write

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[29]One may, for instance, feed the electrodes of the cut through resistances R, insuring a drop of the potential from $\overline{\Phi}$ to φ (see fig. 39). Under these conditions, one has a relation of the form

$$\frac{\partial u}{\partial X} = k(X)(\overline{\varphi} - \varphi)$$

with k(X) being a function which depends on the chosen resistances and on the resistivity of the tank, but can always easily be obtained; the manipulation to be performed is then as follows: after the resistances R have been determined, one has to choose the values of $\overline{\varphi}$ in order to obtain at the electrodes the values of φ prescribed by the boundary conditions.

[30]We shall assume the point O to lie on the cut. In the opposite case the procedure indicated here may be very easily modified.

$$\beta \frac{\partial u}{\partial Y} = - \frac{2Y}{1 + Y^2} \frac{\partial w}{\partial Y}$$

Since $u(X,Y)$ is zero at the point $B(0,1)$,

$$\beta u(0) = \int_0^1 \frac{2t}{1 + t^2} \frac{\partial w}{\partial Y}(0,t)dt = \left[\frac{2Y}{1 + Y^2} w(0,t)\right]_0^1 -$$

$$2\int_0^1 w(0,t)\frac{1 - t^2}{\left(1 + t^2\right)^2} dt$$

Hence

$$\beta u(0) = -2\int_0^1 w(0,t)\frac{1 - t^2}{\left(1 + t^2\right)^2} dt \qquad \text{(III.31)}$$

One will know $u(0)$ by means of a simple integral if one knows the distribution of the w (the same as that of the φ) on the axis OY. Since this may very easily be determined, the problem is entirely solved.

3.1.3.2.2 - Lifting problem.- The boundary conditions to be realized for the lifting problem are the same as for the symmetrical problem as far as the semicircumference $A'BA$ and the cut b,c are concerned. On the segments $A'b$ and cA one must, of course, write

$$\frac{\partial w}{\partial Y} = \frac{dw}{dn} = 0$$

that is, the corresponding walls will be insulating walls.

However, this is not sufficient. If no precaution is taken, the harmonic function corresponding to the electric field thus realized will not be a solution of the aerodynamic problem posed. Actually, there is no reason whatsoever why the gradient of this potential should be zero at the points A and A', since the intensity at A and A' is, in general, not zero. Since the corresponding function dW/dZ is not zero at $Z = \pm 1$, we have already pointed out that this leads to singularities inadmissible for $U(Z)$ (see section 3.1.1.1).

The investigation of the elementary lifting problem, admitting OY as symmetry axis, will permit us to better understand the difficulty, and to solve it. If one realizes in the tank the preceding boundary conditions by bringing the electrodes from the cut $(-a,+a)$ to a constant potential, it is quite obvious that the potential thus realized in the tank will remain finite at every point of the field, even at A and A'. Thus one obtains a solution by taking for $\varphi(X,Y)$ the real part of the analytic function $F(Z)$, defined by

$$\frac{dF}{dZ} = \frac{i\lambda}{\left|\left(a^2 - Z^2\right)\left(1 - a^2Z^2\right)\right|^{1/2}}$$

with λ being a real constant.

This solution does not correspond to the solution of the aerodynamic problem (see section 3.1.7) which, in contrast, gives a singularity at $\left(a^2 - Z^2\right)^{-1/2}$ for the function $W(Z)$, in the neighborhood of $Z = \pm a$. As a consequence, $w(X,Y)$ must be infinitely large at points close to $+a$ and $-a$[31]. This particularity must, therefore, be taken into account in the circuit.

It is not the first time one encounters problems of analogy with singularities[32]. One knows that one must then realize in the neighborhood of the points $\pm a$, a material model, partly conducting, partly insulating, which schematizes the arrangement of an equipotential electric line and a current line.

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[31]One encounters there an interesting example of precautions to be taken in a given problem when one applies the principle of minimum singularities. This principle has led us to pose, for our aerodynamic problem, a solution for dW/dZ in $\left(a^2 - Z^2\right)^{-3/2}$. But if one makes the analogy, the electric tank has no reason to "know," a priori, that realization of other conditions than those directly concerning $W(Z)$ is desired. Thus it "applies" the principle of minimum singularities, realizing the solution for dW/dZ in $\left(a^2 - Z^2\right)^{-1/2}$.

[32]See for instance references 27 and 28. For several months, the laboratory of electric analogies of the O.N.E.R.A. has been utilizing singularities for the study of compressible subsonic flows in the hodograph plane.

In the case of interest to us, in the neighborhood of the point $X = +a$, one has

$$W(Z) = \frac{K}{\sqrt{Z - a}}$$

with K being a real constant; consequently, if one puts

$$W(Z) = w(X,Y) + iw'(X,Y)$$

$$Z - a = \zeta = se^{i\sigma}$$

and

$$w + iw' = \frac{K}{\sqrt{s}}\left[\cos \frac{\sigma}{2} - i \sin \frac{\sigma}{2}\right]$$

the lines $w = \text{constant}$ are determined by

$$s = s_0 \cos^2 \frac{\sigma}{2} = \frac{s_0(1 + \cos \sigma)}{2}$$

and the lines $w' = \text{constant}$ by

$$s = s_1 \sin^2 \frac{\sigma}{2} = \frac{s_1(1 - \cos \sigma)}{2}$$

s_0 and s_1 being two positive constants. They are, therefore, cardioids; their arrangement is given by figure 40. Also, one finds in this figure the scheme of the singularity which must be placed at b and c. Thus the manipulation is as follows: after the circumference ABA' has been brought to the potential zero and the boundary conditions have been realized along the cut bc, one brings the conductive part of the two singularities to rather high potentials which must be determined so that the intensity at the points A and A' is zero (of course, if the problem presents the axis OY as symmetry axis, the two singularities must be brought to the same potential, and the nullity of the intensity at A will insure that of the intensity at A'). This one will realize, from the practical point of view, by detaching at A (and eventually at A') on the semicircumference a small electrode which will not be fed and the potential of which will be made opposite to the potential of the rest of the circumference, through a zero apparatus. It is this condition which permits determination of the potential to which the conductive part of the singularity at c (and eventually at b) must be brought. The field $\varphi(X,Y)$ realized in the tank will then, in consequence of the principle of "minimum singularities," be proportional to the field $w(X,Y)$ of the velocity component following Ox_3.

After that, the manipulation unfolds as for the symmetrical case. One measures the intensities along the cut (b,c) which furnishes the values du/dX. One determines the value of u at the point 0 by restoring the field of values of w along OY and by applying the formula (III.31).

3.1.3.2.3 - Electric measurement of C_z in the case of the lifting problem.- In all cases, the total energy can be determined by integration. In the case of the lifting problem, one will yet have a supplementary verification by utilizing the formula (III.27) which we shall write

$$C_z = \frac{2\pi}{\beta} \frac{\left(1 + b^2\right)\left(1 + c^2\right)}{(c - b)(1 - cb)} \frac{\partial w}{\partial Y}(0,1)$$

Actually, this last formula permits to obtain directly the C_z, by a simple electric measurement which gives the intensity entering at the point B, since $dw/dY(0,1)$ is proportional to that intensity. For this purpose, it suffices to detach, in the neighborhood of B, a small electrode (fig. 41) and to feed it by the intermediary of an arbitrary resistance R. With all boundary conditions satisfied, it suffices to regulate $\overline{\varphi}$ to make the potential at B zero as on the rest of the semicircle. C_z is then proportional to $\overline{\varphi}$.

3.1.3.2.4 - Applications.- The scheme of the circuit used is given by figure 39. We do not intend to give here the details of operation, the precautions taken for increasing the accuracy, the determination of the scales, and the reduction of experiments. All this will form the subject of a later report.

Here we shall give simply the results of the first experiments made following these principles[33]. In every case studied, we have

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[33]There is every reason to assume that the satisfactory precision obtained could be further improved by employing a more suitable material than the one that was utilized. These tests were made frequently with utilization of chance setups with the material that happened to be at the laboratory.

treated the elementary lifting case and the elementary symmetrical case which permits a verification of the procedure.

(1) Elementary symmetrical problem. $a = 0.6$, $w_0 = 1$.- Let us designate the corresponding pressure coefficient by C_{p_0}:

X	0	0.06	0.12	0.18	0.24	0.30	0.36	0.42	0.48	0.54	0.585
$C_{p_{0_{experimental}}}$	1.204	1.214	1.248	1.306	1.400	1.518	1.692	1.946	2.340	3.140	5.160
$C_{p_{0_{theoretical}}}$	1.219	1.230	1.262	1.318	1.402	1.524	1.694	1.946	2.336	3.072	4.640

(2) Symmetrical problem: case of a lenticular cone.- If the section $x_1 = \beta$ of the conical obstacle given is formed by two parabolic arcs

$$x_3 = \pm\epsilon_0\left(k^2 - x_2^{\,2}\right)$$

with ϵ_0 being a positive small number (see fig. 42), the function $w(X,Y)$ will assume on the cut the values

$$w = \epsilon_0\left(k^2 + x^2\right)$$

Let us recall that

$$k = \frac{2a}{1 + a^2} \qquad x = \frac{2X}{1 + X^2}$$

One may write the pressure coefficient

$$C_{p_1} = \epsilon_0 P_1(X)$$

The results found have given, for $a = 0.6$

X	0	0.06	0.12	0.18	0.24	0.30	0.36	0.42	0.48	0.54	0.585
$P_1(X)$	1.111	1.112	1.117	1.133	1.181	1.285	1.491	1.849	2.472	3.778	8.02

For verification, one has studied also the case where the distribution of w along the axis of the X was given by

$$w = \epsilon_0\left(k^2 - x^2\right)$$

The values found for the corresponding C_p were as follows; one has put

$$C_p = \epsilon_0 P_2(X)$$

X	0	0.06	0.12	0.18	0.24	0.30	0.36	0.42	0.48	0.54	0.585
$P_2(X)$	0.777	0.793	0.843	0.917	1.204	1.088	1.157	1.195	1.183	1.099	0.934

Naturally, one must verify that $P_1(X) + P_2(X)$ and $2k^2\left(C_{p_0}\right)$ assume the same values. Now one has

X	0	0.06	0.12	0.18	0.24	0.30	0.36	0.42	0.48	0.54	0.585
$P_1 + P_2$	1.887	1.905	1.961	2.050	2.185	2.373	2.648	3.043	3.655	4.877	8.96
$2k^2C_{p_0}$	1.899	1.915	1.965	2.052	2.183	2.373	2.638	3.030	3.637	4.783	7.23

(3) Elementary lifting problem $w_0 = 1$, $a = 0.6$.-

X	0	0.06	0.12	0.18	0.24	0.30	0.36	0.42	0.48	0.54
$C_{p_{experimental}}$	1.196	1.208	1.242	1.304	1.398	1.536	1.742	2.062	2.536	3.776
$C_{p_{theoretical}}$	1.192	1.204	1.238	1.298	1.390	1.526	1.724	2.030	2.552	3.732

(4) Lifting problem: parabolic cone.- $w = \epsilon_0\left(k^2 + x^2\right)$; one will put $C_{p_3} = \epsilon_0 P_3(X)$

X	0	0.06	0.12	0.18	0.24	0.30	0.36	0.42	0.48	0.54
$P_3(X)$	1.068	1.070	1.074	1.094	1.152	1.278	1.520	1.956	2.732	4.636

3.1.3.3 - Purely numerical methods. Utilization of the plane z.- We have introduced this plane in section (1.2.5). Let us recall that z corresponds to Z by the conformal transformation

$$z = \frac{2Z}{Z^2 + 1}$$

and that in this plane the relations of compatibility are written

$$-\beta\ dU = z\ dV = \frac{-iz}{\sqrt{1 - z^2}}\ dW \tag{III.26}$$

One of the advantages of the plane which is of practical interest is that one has on the real axis (if $z = x + iy$)

$$x = x_2$$

x_2 being the ordinate of a point of the section $x_1 = \beta$, situated on $x_3 = 0$, in the axis system $Ox_1x_2x_3$.

Some of the formulas established before may be written more simply. If one denotes, for instance, the image of the cut (b,c) of the plane Z in z by (λ,μ), the formula (III.21) is written

$$W(z) = -i\frac{w_0}{\pi}\log\frac{\mu - z}{\lambda - z} \tag{III.32}$$

$W(z)$ thus appears as the complex potential due to two vortices placed at the points λ and μ and of opposite intensity. Likewise, the formula (III.24) may be written

$$U(z) = \frac{w_0}{\beta E(k_1)}\ \frac{\sqrt{(1 + b^2)(1 + c^2)}}{1 - bc}\ \frac{2\lambda\mu - z(\lambda + \mu)}{\sqrt{(\mu - z)(z - \lambda)}} \tag{III.33}$$

If one puts

$$\lambda = \cos\psi \qquad \mu = \cos\omega$$

ψ and ω lying between 0 and π

$$k_1 = \frac{\sqrt{\sin\psi \sin\omega}}{\sin\frac{\psi + \omega}{2}} \quad \text{and} \quad \frac{\sqrt{\left(1 + b^2\right)\left(1 + c^2\right)}}{1 - bc} = \frac{1}{\sin\frac{\psi + \omega}{2}}$$

In the case where $\mu = -\lambda = k$, one has, in particular

$$C_p = \frac{2k^2}{\beta E'(k)} \frac{w_0}{\sqrt{k^2 - x^2}}$$

Let us recall that

$$E'(k) = E\left(\sqrt{1 - k^2}\right)$$

3.1.3.3.1 - Case of the symmetrical problem.- Let us now assume that the problem corresponding to the boundary conditions $w = f(x)$ on the upper edge of the cut, $w = -f(x)$ on the lower edge has to be solved. The formula (III.32) leads us to represent $W(z)$ as the potential of a distribution of vortices carried by the segment $\lambda\mu$; consequently

$$W(z) = -\frac{i}{\pi}\int_{\lambda}^{\mu} \frac{f(u)}{u - z} du$$

At a point of the upper edge of the cut, one has actually

$$W(x) = -\frac{i}{\pi}\int_{\lambda}^{\mu} \frac{f(u)du}{u - x} + f(x) = w + iw'$$

with the integral taken at principal value.

Let us put on the cut

$$x = \frac{\lambda + \mu}{2} + \frac{\mu - \lambda}{2}\cos\varphi \qquad u = \frac{\lambda + \mu}{2} + \frac{\mu - \lambda}{2}\cos\theta$$

$$f(u) = F(\theta)$$

Let us assume, to begin with, that

$$F(0) = F(\pi) = 0$$

and that $F(\theta)$ can be developed in a Fourier series

$$F(\theta) = \sum_{0}^{\infty} A_n \sin n\theta \qquad 0 < \theta < \pi$$

Then

$$w'(\theta) = -\frac{1}{\pi}\int_0^{\pi} \frac{\left(\sum_{0}^{\infty} A_n \sin n\theta\right)\sin\theta}{\cos\theta - \cos\varphi}\, d\theta$$

We shall furthermore admit that the signs $\sum$ and $\int$ are interchangeable. According to a known result (ref. 13)

$$-\frac{1}{\pi}\int_0^{\pi} \frac{\sin n\theta \sin\theta \, d\theta}{\cos\theta - \cos\varphi} = -\frac{1}{2\pi}\int_0^{\pi} \frac{\left[\cos(n - 1)\theta - \cos(n + 1)\theta\right]}{\cos\theta - \cos\varphi}\, d\theta =$$

$$-\frac{\left[\sin(n - 1)\varphi - \sin(n + 1)\varphi\right]}{2 \sin\varphi} = \cos n\varphi$$

and consequently

$$w'(\varphi) = \sum_{0}^{\infty} A_n \cos n\varphi$$

Thus one sees that $w'(\theta)$ is the conjugate function of $F(\theta)$ which could have been easily established by other methods as well.

However, according to the relation of compatibility

$$-\beta \frac{\partial u}{\partial x} = - \frac{x}{\sqrt{1 - x^2}} \frac{\partial w}{\partial y} = \frac{x}{\sqrt{1 - x^2}} \frac{\partial w'}{\partial x}$$

and

$$\frac{\partial u}{\partial \varphi} = - \frac{x}{\beta\sqrt{1 - x^2}} \frac{\partial w'}{\partial \varphi} = \frac{x}{\beta\sqrt{1 - x^2}} \sum_0^\infty nA_n \sin n\varphi$$

We shall put

$$G(\varphi) = - \sum_0^n nA_n \sin n\varphi \tag{III.34}$$

$G(\varphi)$ is the derivative of the conjugate function of $F(\varphi)$. Thus one has

$$\frac{\partial u}{\partial \varphi} = \frac{xG(\varphi)}{\beta\sqrt{1 - x^2}} \tag{III.35}$$

Knowledge of $F(\varphi)$ entails that of $G(\varphi)$ by a calculation of trigonometric operator (section 2.3.3) and, consequently, that of $\frac{\partial u}{\partial \varphi}$. In order to set up formula (III.35), we have made a certain number of hypotheses. These hypotheses will be satisfied if the derivative of F with respect to φ satisfies a condition of Cauchy-Lipschitz.

In order to calculate the pressure at every point of the cone one must integrate the formula (III.35); for that, however, one must know the integration constant.

The exact determination of the function u will be easily obtained as soon as we have studied thoroughly the character of the function $U(z)$. We suppose first

$$-\lambda = \mu$$

In order to study the function U, we shall perform the conformal transformation of the plane z, provided with cuts $(-\infty,-1)$, $(-\mu,+\mu)$,

$(1,+\infty)$ traced on the real axis, on an annular corona. This is immediate (see, for instance, section 3.1.7.1). Let z_1 first be a complex variable defined by

$$\frac{dz_1}{dz} = \frac{1}{\sqrt{\left(\mu^2 - z^2\right)\left(1 - z^2\right)}}$$

or

$$z = \mu \, \mathrm{sn}\left(z_1,\mu\right), \quad (k = \mu), \text{ then}$$

$$z_2 = e^{\frac{i\pi}{2K}\left(iK'-z_1\right)}$$

The plane z provided with its cuts then is represented on a strip $0 < \underline{T}\left(z_1\right) < K'$ of the plane z_1, and on an annular area of the plane z_2 (see fig. 43) bounded by the circumferences (γ_1) of the radius 1 and $\left(\gamma_2\right)$ of the radius

$$q = e^{-\frac{\pi K'}{2K}}$$

In the plane z_2, U is of the form

$$U\left(z_2\right) = A \log z_2 + f\left(z_2\right)$$

with $f\left(z_2\right)$ being a uniform holomorphic function inside of the annulus (see for instance section 2.1.2.1), since $U\left(z_2\right)$ is finite, even at the image points of $z = \pm\mu$, because of the hypothesis

$$F(0) = F(\pi) = 0$$

We remark that $f\left(z_2\right)$ has a real part zero on the circle $\left(\gamma_1\right)$. We assume the value of the coefficient A to be known; on the circumference $\left(\gamma_1\right)$, $A \log z_2$ maintains as constant real part

$$A \log q = -\frac{\pi}{2}\frac{K'}{K} A \tag{III.36}$$

According to a well-known theorem of the theory of harmonic functions (see ref. 29) one now knows that, if a uniform harmonic function $H(x,y)$, defined inside of a circular annulus, assumes on the two limiting circles the values $\varphi_0(\theta)$ and $\varphi_1(\theta)$, (with θ being the angle at the center representing the running point on each circle), one has

$$\int_0^{2\pi} \varphi_0(\theta)d\theta = \int_0^{2\pi} \varphi_1(\theta)d\theta$$

This theorem will allow us to demonstrate the following theorem:

Theorem: If $\mu = -\lambda$, the function $u(\varphi)$ satisfies the equality

$$\int_0^{\pi} \frac{u(\varphi)d\varphi}{\sqrt{1 - \mu^2\cos^2\varphi}} = 2K(\mu)A \log q$$

$K(\mu)$ being the elliptic function of first kind relative to the modulus μ

$$K(\mu) = \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1 - \mu^2\cos^2\varphi}}$$

In fact, the mean value of the real part of $f(z_2)$ on the circle (γ_1) must be zero, but the mean value of u on (γ_2) reads

$$\frac{1}{2\pi}\int_0^{2\pi} u \frac{dz_2}{iz_2} = \frac{1}{2\pi}\frac{\pi}{2K}\int_0^{4K} u\, dz_1 = \frac{1}{4K}\int_{\underline{L}} u \frac{dz}{\sqrt{\left(\mu^2 - z^2\right)\left(1 - z^2\right)}}$$

with $\underline{L}$ designating the loop surrounding the cut $(-\mu,+\mu)$ in the positive direction. However, the function $u(\varphi)$ assumes the same values at points which have the same abscissa on the upper and on the lower edge of the cut; consequently, this mean value is equal to

$$\frac{1}{2K}\int_0^{\pi} \frac{u(\varphi)d\varphi}{\sqrt{1 - \mu^2\cos^2\varphi}}$$

In order to have a mean value of $f(z)$ on (γ_2) of zero, it is necessary and sufficient that the mean value of u should be equal to $A \log q$ which justifies the theorem. One utilizes this theorem in the following manner:

If $u_0(\varphi)$ is a primitive of $\frac{\partial u}{\partial \varphi}$, calculated by the formula (III.35), and if

$$\frac{1}{2K}\int_0^{\pi} \frac{u_0(\varphi)d\varphi}{\sqrt{1 - \mu^2\cos^2\varphi}} = C$$

the desired value of $u(\varphi)$ may be written

$$u(\varphi) = A \log q + u_0(\varphi) - C$$

To establish this result, we have assumed that the cut is extended on the segment $(-\mu,+\mu)$, symmetrical with respect to the origin. In order to reduce the general case to this particular case, it suffices to make a conformal representation, analogous to the one already made in section 3.1.2. Let

$$z' = \frac{z - \alpha}{1 - \alpha z}$$

be this conformal representation which makes the cut $(-k,+k)$ of the plane z' correspond to the cut (λ,μ) of the plane z. One has, in particular

$$k = \frac{1 - \lambda\mu - \sqrt{(1 - \lambda^2)(1 - \mu^2)}}{\mu - \lambda}$$

The mean value of u on the image circle of the cut $(-k,+k)$ of the plane z', in the conformal representation which transforms the plane z' into a ring, reads, according to what we have just learned

$$\frac{1}{4K(k)}\int_{\underline{L}'}\frac{u\,dz'}{\sqrt{\left(k^2 - z'^2\right)\left(1 - z'^2\right)}}$$

with $\underline{L}'$ designating the loop surrounding the cut $(-k,+k)$ in the plane z'.

However

$$dz' = \frac{dz\left(1 - \alpha^2\right)}{(1 - \alpha z)^2}$$

$$\sqrt{\left(k^2 - z'^2\right)\left(1 - z'^2\right)} = \frac{1 - \alpha^2}{(1 - \alpha z)^2}\,\frac{\sqrt{1 - \alpha^2}}{\sqrt{(1 - \alpha\lambda)(1 - \alpha\mu)}}\sqrt{\left(1 - z^2\right)(\mu - z)(z - \lambda)}$$

We remark that

$$\sqrt{\frac{(1 - \alpha\lambda)(1 - \alpha\mu)}{1 - \alpha^2}} = \sqrt{\frac{\mu - \lambda}{2k}}$$

The mean value is then written

$$\frac{1}{4K(k)}\sqrt{\frac{\mu - \lambda}{2k}}\int_{\underline{L}}\frac{u\,dz}{\sqrt{\left(1 - z^2\right)(\mu - z)(z - \lambda)}}$$

$\underline{L}$ being the loop surrounding the cut (λ,μ) in positive direction.

If we finally put

$$z = \frac{\mu + \lambda}{2} + \frac{\mu - \lambda}{2}\cos\varphi$$

the desired mean value on the upper edge of the cut is written

$$\frac{1}{2K(k)}\sqrt{\frac{\mu - \lambda}{2k}}\int_0^{\pi}\frac{u(\varphi)d\varphi}{\sqrt{1 - z^2}}$$

As previously, one draws the conclusion:

If $u_0(\varphi)$ is a primitive calculated from equation (III.35), and if

$$\frac{1}{2K(k)}\sqrt{\frac{\mu - \lambda}{2k}}\int_0^{\pi}\frac{u(\varphi)d\varphi}{\sqrt{1 - z^2}} = C$$

the desired value of $u(\varphi)$ is

$$u(\varphi) = u_0(\varphi) + A \log q - C \tag{III.37}$$

Thus the entire matter amounts to calculating the constant A. This constant is calculated very easily if one considers the imaginary part $u'(x,y)$ of $U(z)$.

In fact:

When, in the plane z, one circles once in the positive direction of the cut (λ,μ), the imaginary part of $U(z)$ increases by $-2\pi A$. If one circles the cut by the loop $\underline{L}$, one notices that $u'(x,y)$ assumes opposite values at the two points of the cut which have the same abscissa but are situated on different edges. Thus one may write

$$A = \frac{1}{\pi}\int_{\lambda}^{\mu}\frac{\partial u'}{\partial x}\,dx$$

However, according to the relations of compatibility, one may also write

$$A = \frac{1}{\beta\pi}\int_{\lambda}^{\mu}\frac{x}{\sqrt{1 - x^2}}\frac{\partial w}{\partial x}\,dx \tag{III.38}$$

which permits directly the calculation of A, (starting) from the function

$$w = f(x)$$

given on the cut.

The entire analysis above assumes that $f(x)$ becomes zero for $x = \lambda$ and $x = \mu$. We now still have to reduce the general case to this particular case. One may put

$$f(x) = f_0(x) + C_0 + C_1 x$$

with $f_0(x)$ becoming zero for $x = \lambda$ and $x = \mu$, and C_0 and C_1 being two suitably selected constants. The problem then may be reduced to the superposition of three problems, the first where

$$w(x) = f_0(x)$$

the second where

$$w(x) = C_0$$

the third where

$$w(x) = C_1 x$$

Since the two first problems already have been dealt with, we now only have to treat the last problem. Thus we put

$$f(x) = x$$

and seek the function $U(z)$

$$W(z) = -\frac{i}{\pi}\int_{\lambda}^{\mu} \frac{t}{t - z}\,dt = -\frac{i}{\pi}(\mu - \lambda) - \frac{i}{\pi} z \log \frac{\mu - z}{\lambda - z}$$

hence

$$\frac{dW}{dz} = -\frac{i}{\pi}\log\frac{\mu - z}{\lambda - z} - \frac{i}{\pi}(\mu - \lambda)\frac{z}{(\mu - z)(\lambda - z)}$$

and according to equation (I.26)

$$\frac{dU}{dz} = \frac{1}{\pi\beta}\frac{z}{\sqrt{1 - z^2}}\left[\log\frac{\mu - z}{\lambda - z} + \frac{(\mu - \lambda)z}{(\mu - z)(\lambda - z)}\right]$$

whence by integration (determining the integration constant so that $U(1)$ should have a real part zero)

$$U(z) = -\frac{1}{\beta\pi}\sqrt{1 - z^2}\log\frac{\mu - z}{\lambda - z} + \frac{1}{\sqrt{1 - \mu^2}}\log\frac{1 - \mu z + \sqrt{(1 - z^2)(1 - \mu^2)}}{\mu - z} - \frac{1}{\sqrt{1 - \lambda^2}}\log\frac{1 - \lambda z + \sqrt{(1 - z^2)(1 - \lambda^2)}}{\lambda - z} \qquad \text{(III.39)}$$

Summing up: In order to calculate numerically the pressures in a symmetrical problem, one has to perform the following operations:

(1) One turns to the case where $w(x)$ becomes zero for $x = \lambda$ and $x = \mu$, following the method just exposed.

(2) Calculate the constants A (formula (III.38)) and q (formula (III.36)).

(3) Calculate the function $G(\varphi)$ for a trigonometric operator.

(4) Calculate $\frac{\partial u}{\partial \varphi}$ (formula (III.35)) and a primitive $U_0(\varphi)$.

(5) Calculate

$$C = \frac{1}{2K(\mu)}\int_0^{\pi}\frac{u_0(\varphi)d\varphi}{\sqrt{1 - \mu^2\cos^2\varphi}}$$

$u(\varphi)$ is then given by the formula (III.37).

Example: In applying this method to the calculation of the case of the parabolic cone where

$$w = \pm\epsilon_0\left(k^2 + x^2\right) \qquad C_p = \epsilon_0 p_1(\varphi)$$

one has found the following distribution of the C_p

φ	0	15°	30°	45°	60°	75°	90°
$p_1(\varphi)$	∞	6.840	2.535	1.524	1.196	1.102	1.088

In order to compare this with the results of the electric analogy, one must recall that

$$x = \frac{2X}{1 + X^2}$$

The comparison is given by the figure 44.

3.1.3.3.2 - Study of the lifting problem.- For simplification, we shall limit ourselves to the case where the problem admits the plane Ox_1x_3 as symmetry plane.

Let us consider the function $W(z)$; one may put it in the form

$$W(z) = AW_0(z) + F(z)$$

$W_0(z)$ being the solution in W of the elementary lifting problem (for which $w_0 = 1$), A being a real constant and $F(z)$ a function which remains finite in the domain where $W(z)$ is defined. We shall put along the cut

$$F(z) = f(x) + if'(x)$$

Let us put likewise

$$U(z) = AU_0(z) + G(z)$$

$G(z)$ being the value of $U(z)$ corresponding to the case where $W(z) = F(z)$.

We shall designate the real and imaginary parts of the function $G(z)$ on the cut by $g(x)$ and $g'(x)$.

If one notes that along the cut

$$\frac{\partial f}{\partial x} = \frac{\partial w}{\partial x}$$

one sees that the relations of compatibility permit one to write

$$\beta \frac{\partial g}{\partial y} = -\beta \frac{\partial g'}{\partial x} = -\frac{x}{\sqrt{1 - x^2}} \frac{\partial f}{\partial x} = -\frac{x}{\sqrt{1 - x^2}} \frac{\partial w}{\partial x}$$

$w(x)$ is the function given by hypothesis; hence

$$\frac{\partial g'}{\partial x} = \frac{x}{\beta\sqrt{1 - x^2}} \frac{\partial w}{\partial x}$$

If we assume $\frac{\partial w}{\partial x}$ to be limited, one may visualize the development in trigonometric series of $dg'/d\varphi$ in the form

$$\frac{dg'}{d\varphi} = \sum A_n \sin n\varphi \tag{III.40}$$

Now $G(z)$ may be visualized as the potential of a vortex distribution carried by the cut (in particular, the real part of $U(z)$ is zero on the real axis outside of the cut).

Let us consider a vortex distribution of the intensity

$$g(\varphi) = \sum B_n \sin n\varphi$$

The value of $dg'/d\varphi$ will be identical to the one written in the formula (III.40), if, and only if

$$-nB_n = A_n$$

as it results from very simple calculations, already carried out in the preceding paragraph.

Hence one then deduces the value of $g(\varphi)$ corresponding to $dg'/d\varphi$, defined by the formula (III.40), by means of a trigonometric operator the numerical calculation of which results from the considerations developed in chapter II, section 2.3.3.

One can also simply first calculate

$$\frac{dg}{d\varphi} = -\sum_{1}^{\infty} A_n \cos n\varphi \tag{III.41}$$

by means of a Poisson integral, and then deduce from it $g(\varphi)$ by simple integration, noting that

$$g(\varphi) = 0 \quad \text{for} \quad \varphi = 0, \quad \varphi = \pi$$

Thus the problem will be completely solved as soon as we have calculated the constant A. One may put, as before

$$F(z_2) = B \log z_2 + \Phi(z_2)$$

$\Phi(z_2)$ being a uniform function inside of the annulus (γ_1, γ_2) of the previously defined plane z_2.

$\Phi(z_2)$ has a real part zero on the circle (γ_1) of the radius 1. Consequently, the mean value of $\underline{R}\left[\Phi(z_2)\right]$ on the circle (γ_2) is zero. Thus one deduces, as in the preceding paragraph, that

$$A + B \log q = \frac{1}{2K(k)} \int_0^{\pi} \frac{w(\varphi)\,d\varphi}{\sqrt{1 - k^2\cos^2\varphi}} \tag{III.42}$$

With w known, it is then easy to calculate $A + B \log q$. Thus the entire matter amounts to calculating B.

If one now describes in the plane z the loop $\underline{L}$ surrounding the cut $(-\mu, \mu)$ in positive direction, the imaginary value of $F(z)$ must increase by $-2\pi B$, according to definition. Now

$$-\beta \frac{\partial g}{\partial x} = \frac{x}{\sqrt{1 - x^2}} \frac{\partial f'}{\partial x}$$

However, $\frac{\partial g}{\partial x}$ is known (formula (III.41)), and consequently[34]

$$B = \frac{-\beta}{\pi} \int_{-\mu}^{+\mu} \frac{\sqrt{1 - x^2}}{x} \frac{\partial g}{\partial x} dx \tag{III.43}$$

Summarizing, one may say that the following operations have to be carried out:

(1) Calculation of $g'(\varphi)$.

(2) Calculation of $dg/d\varphi$, by a Poisson integral.

(3) Calculation of $g(\varphi)$, by an integration of $dg/d\varphi$.

(4) Calculation of B (formula (III.43)).

(5) Calculation of A (formula (III.42)).

The result reads

$$u = Au_0(\varphi) + g(\varphi)$$

with $u_0(\varphi)$ representing the value of u for the elementary lifting problem when $w_0 = 1$.

Application.- Lifting parabolic cone

$$w = \epsilon_0\left(k^2 + x^2\right)$$

$$C_p = \epsilon_0 p_3(\varphi)$$

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[34]One will easily ascertain that $\partial g/\partial x$ becomes zero for $x = 0$. The integral then does not present any difficulty.

φ	0	15°	30°	45°	60°	75°	90°
$p_3(\varphi)$	∞	5.844	2.472	1.472	1.144	1.070	1.062

One will find in figure 53 the pressure distribution compared to the one found by electric analogy.

3.2 - Case Where the Cone Is Not Inside the Mach Cone (Γ)

3.2.1 - Generalities

From the mathematical viewpoint, there is an essential difference between the case where the conical obstacle is entirely inside of (Γ) and the case where, in contrast, it is not entirely inside. The difference becomes very clear if one visualizes oneself in the plane Z. Whereas the flows studied in section 3.1 led to problems of complex variables relative to an annular area, the problems to be studied now will be relative to simply connected areas. This simplifies the investigation considerably. It can be foreseen that we shall no longer have to utilize the theory of elliptic functions, and in the numerical or analogical study of the problems we shall avoid the difficulties arising from the determination of the "integration constant" for the pressure (see sections 3.1.3.2 and 3.1.3.3).

If one places oneself in the plane Z, the functions $U(Z)$, $V(Z)$, $W(Z)$ will no longer be identically zero on $\left(C_0\right)$. We shall show that the relations of compatibility then take on a form particularly simple.

These relations may be written

$$-\beta Z \frac{dU}{dZ} = \frac{2Z}{Z^2 + 1} Z \frac{dV}{dZ} = \frac{2iZ}{Z^2 - 1} Z \frac{dW}{dZ} \tag{III.44}$$

and if one notes that on $\left(C_0\right)$

$$Z \frac{dU}{dZ} = -i \frac{dU}{d\theta}$$

one can deduce from the formulas (III.44) the following relations between the real parts u, v, w of U, V, W on C_0

$$-\beta \frac{du}{d\theta} = \frac{1}{\cos\theta} \frac{dv}{d\theta} = \frac{1}{\sin\theta} \frac{dw}{d\theta} \tag{III.45}$$

Knowledge of one function u, v, or w on an arc of the circle of (C_0) entails (except for an additive constant) knowledge of the two others.

It is easy to extend this result to the case where U, V, W present certain discontinuities. Let A_1 be a point of (C_0) of the argument θ_1, and let us suppose that the real part of W(Z) increases by Δw if θ passes from $\theta_1 - \epsilon$ to $\theta_1 + \epsilon$, with ϵ being positive and arbitrarily small. Let (γ) (see fig. 45) be a small arc of the circle centered at A_1 and lying inside of (C_0). One has

$$\Delta w = \underline{R}\left[\int_\gamma \frac{dW}{dZ}\, dZ\right]$$

However

$$\Delta v = \underline{R}\left[\int_\gamma \frac{dV}{dZ}\, dZ\right]$$

and

$$\Delta u = \underline{R}\left[\int_\gamma \frac{dU}{dZ}\, dZ\right]$$

Consequently, it suffices in the case where dU/dZ, dV/dZ, dW/dZ have a simple pole at A_1, to utilize the relations of compatibility in order to establish the formulas

$$-\beta\, \Delta u = \frac{1}{\cos\theta}\, \Delta v = \frac{1}{\sin\theta}\, \Delta w \tag{III.46}$$

Remark.

The formulas which we are going to set up below will be demonstrated in the case of the figure where the conical obstacle is in its entirety in the region $x_1 > 0$. But it suffices to return to the generalities of section 1.2.2 to recognize that the obtained results will be valid in more general cases. Under these conditions, one may have in the region (A') (see fig. 2) domains which encroach on one another. However, no difficulty arises since the relations of compatibility in the plane (η,θ), formula (I.22), show that the functions u, v, w in the plane (A') are perfectly known, owing to the boundary conditions. One will note the identity of the formulas (III.45) and (I.22).

3.2.2 - Cone Totally Bisecting the Mach Cone (Fig. 28)

If one utilizes the plane Z, the problem amounts to determining the functions $U(Z)$, $V(Z)$, $W(Z)$ in such a manner that u, v, w are zero on the circular arcs A_1A_2, $A_1'A_2'$ (see fig. 46), and that w assumes prescribed values, with one part on the line $A_1AA'A_2$, and the other part on the line $A_1'AA'A_2'$. In contrast to what happened in the preceding problem, the two half spaces, separated by the plane $x_3 = 0$, are independent of each other. From the mathematical viewpoint, it may for instance be a matter of determining the solution in one of the semicircles determined in (C_0) by the cut AA'. There follows that there is no theoretical distinction between the symmetrical and the lifting problem. Naturally, one may operate in the same manner in the plane z. There will then be occasion to determine the solution in a semiplane, the upper semiplane for instance; the function $w = f(x)$ is assumed to be known along a segment $\lambda\mu$, comprising in its interior the segment $-1,+1$ of the real axis. The function is zero on the rest of the real axis[35].

<u>3.2.2.1 - Elementary problem</u>.- As before, we shall start with the study of the elementary problem, that is, the one where $w = w_0$ on the part of a cone situated in the region $x_3 > 0$.

We shall operate, for instance, in the plane Z; the function $W(Z) - w_0$ has a real part zero on the segment AA' and the arcs AA_1 and $A'A_1'$, and equal to $-w_0$ on the arc A_1A_2. One can, by application of Schwartz principle, extend the definition of this function to a complete circle; its determination is then classical. (See, for instance, ref. 13, p. 162.)

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[35]See appendix 3.

This permits one to write immediately

$$W(Z) = w_0 - \frac{iw_0}{\pi} \log \frac{1 + Z^2 - 2Z \cos \theta_1}{1 + Z^2 - 2Z \cos \theta_2} \tag{III.47}$$

with the logarithm being real for a real Z, and with θ_1 and θ_2 being the respective angular abscissas of the points A_1 and A_2. The function $V(Z)$ may be determined, for instance, with the aid of the relations of compatibility

$$\frac{dV}{dZ} = \frac{w_0}{\pi} \frac{Z^2 + 1}{Z^2 - 1} \left[\frac{1}{Z - e^{i\theta_1}} + \frac{1}{Z - e^{-i\theta_1}} - \frac{1}{Z - e^{i\theta_2}} - \frac{1}{Z - e^{-i\theta_2}} \right]$$

In the integration it suffices to choose the integration constant in such a manner that the real part of $V(Z)$ becomes zero on the arc A_1A_2. Thus one obtains

$$V(Z) = \frac{iw_0}{\pi} \left[\cot \theta_1 \log \frac{e^{i\theta_1} - Z}{1 - Ze^{i\theta_1}} - \cot \theta_2 \log \frac{Z - e^{i\theta_2}}{1 - Ze^{i\theta_2}} \right] \tag{III.48}$$

with the logarithms having an argument zero on the arc A_1A_2. One finds for v the following values

$$v = w_0 \cot \theta_1, \text{ on the arc } A_1A_2$$

$$v = w_0 \cot \theta_2, \text{ on the arc } A'A_2$$

besides, one could have written these values directly by virtue of the relations[36] (III.45) and (III.46).

[36]This shows that one could have written the formula (III.48) directly, without writing the relations of compatibility.

In order to write the value of v on the axis AA', one must calculate the argument of

$$\frac{e^{i\theta_1} - Z}{1 - Ze^{i\theta_1}}$$

Now

$$\text{Arg}\left[\frac{e^{i\theta_1} - Z}{1 - Ze^{i\theta_1}}\right] = \text{Arg}\left[\left(e^{i\theta_1} - Z\right)\left(1 - Ze^{-i\theta_1}\right)\right]$$

For calculating this argument, for $Z = X$, one notes that the modulus of $\left(e^{i\theta_1} - Z\right)\left(1 - Ze^{-i\theta_1}\right)$ is the one of $\left(e^{i\theta_1} - X\right)^2$, under the assumption of $1 + X^2 - 2X \cos \theta_1$; on the other hand, its real part is written $\cos \theta_1\left(1 + X^2\right) - 2X$. If one puts, therefore

$$x = \frac{2X}{1 + X^2}$$

$$\text{Arg}\left[\frac{e^{i\theta_1} - Z}{1 - Ze^{i\theta_1}}\right] = \text{Arc cos}\ \frac{\cos \theta_1 - x}{1 - x \cos \theta_1} \qquad \text{(III.49)}$$

with the arc cosine having thus, besides, its principal value. One finds likewise

$$\text{Arg}\left[\frac{Z - e^{i\theta_2}}{1 - Ze^{i\theta_2}}\right] = -\text{Arc cos}\ \frac{x - \cos \theta_2}{1 - x \cos \theta_2} \qquad \text{(III.50)}$$

hence on the axis AA'

$$v = -\frac{w_0}{\pi}\left[\cot \theta_1\ \text{Arc cos}\ \frac{\cos \theta_1 - x}{1 - x \cos \theta_1} + \cot \theta_2\ \text{Arc cos}\ \frac{x - \cos \theta_2}{1 - x \cos \theta_2}\right]$$

The calculation of U(Z) is perfectly analogous. One finds

$$U(Z) = -\frac{iw_0}{\beta\pi}\left[\frac{1}{\sin\theta_2}\log\frac{Z - e^{i\theta_2}}{1 - Ze^{i\theta_2}} - \frac{1}{\sin\theta_1}\log\frac{e^{i\theta_1} - Z}{1 - Ze^{i\theta_1}}\right] \tag{III.51}$$

with the logarithms having the same value as in the formula (III.48). One finds as the value of the pressure coefficient $\left(w_0 = \alpha\right)$

$$C_p = \frac{2\alpha}{\beta}\frac{1}{\sin\theta_2}, \quad \text{on the arc } A'A_2$$

$$C_p = \frac{2\alpha}{\beta}\frac{1}{\sin\theta_1}, \quad \text{on the arc } AA_1 \tag{III.52}$$

$$C_p = \frac{2\alpha}{\beta\pi}\left[\frac{1}{\sin\theta_1}\text{Arc cos}\frac{\cos\theta_1 - x}{1 - x\cos\theta_1} + \frac{1}{\sin\theta_2}\text{Arc cos}\frac{x - \cos\theta_2}{1 - x\cos\theta_2}\right],$$

on the axis AA'

In the case where Ox_1x_3 is a symmetry plane

$$\theta_2 = \pi - \theta_1$$

and the last formula (III.52) may also be written

$$C_p = \frac{2\alpha}{\beta\pi\sin\theta_1}\left[\text{Arc cos}\frac{\cos\theta_1 - x}{1 - x\cos\theta_1} + \text{Arc cos}\frac{x + \cos\theta_1}{1 + x\cos\theta_1}\right] =$$

$$\frac{4\alpha}{\beta\pi\sin\theta_1}\text{Arc sin}\frac{\sin\theta_1}{\sqrt{1 - x^2\cos^2\theta_1}} \tag{III.53}$$

In order to utilize these formulas, it is sufficient to connect the angles θ_1 and θ_2 with the geometrical form of the given delta wing (fig. 47). One has, according to definition

$$\cos \theta_1 = 1/\beta \tan \omega_1 \qquad \cos \theta_2 = 1/\beta \tan \omega_2$$

Let us recall also that

$$x = \frac{\beta x_2}{x_1}$$

One will find in figure 48 a few applications of the formula (III.53).

<u>3.2.2.2 - Resultant of the normal forces on the upper region $(x_3 > 0)$</u>.- One can give, as in section 3.1.9, a simple formula permitting the calculation of the resultant of the normal forces. If we designate by C_z^+ the dimensionless coefficient characterizing this resultant, C_z^+ is defined by the equality

$$C_z^+ = -\frac{\int_\lambda^\mu C_p \, dx}{\int_\lambda^\mu dx}$$

Likewise we define the dimensionless number C_z^-, characterizing the forces normal to the lower region $(x_3 < 0)$, by the equality

$$C_z^- = \frac{\int_\lambda^\mu C_p \, dx}{\int_\lambda^\mu dx}$$

with the integrals taken in the plane z, the first on the upper edge of the cut (λ,μ), the second on the lower edge. This definition entails that the total C_z of a cone is written

$$C_z = C_z^+ + C_z^-$$

Now

$$\int_\lambda^\mu dx = \frac{1}{\cos\theta_1} - \frac{1}{\cos\theta_2}$$

On the other hand

$$\int_\lambda^\mu C_p\, dx = 2\underline{R}\left[\int_\lambda^\mu U(z)dz\right] = -4\underline{R}\left[\int_{A_2A'AA_1} U(Z)\,\frac{1 - Z^2}{\left(1 + Z^2\right)^2}\,dZ\right]$$

However, the integral of $U(Z)\,\frac{1 - Z^2}{\left(1 + Z^2\right)^2}$ along the closed contour $BA_2A'A_1B$ (fig. 46) is zero. On the other hand, with $U(Z)$ having a real part zero on the arc $A_2\mathring{A}_1$, one has

$$\underline{R}\left[\int_{A_2A'AA_1} U(Z)\,\frac{1 - Z^2}{\left(1 + Z^2\right)^2}\,dZ\right] = -\underline{R}\left[\int_{A_1A_2} U(Z)\,\frac{1 - Z^2}{\left(1 + Z^2\right)^2}\,dZ\right] = \underline{R}\left[i\pi R_i\right]$$

R_i denoting the residue of the function to be integrated, at the point $Z = i$

$$R_i = -\frac{1}{2}\,\frac{dU}{dZ}_{(z=i)} = \frac{1}{2\beta}\,\frac{dW}{dZ}_{(Z=i)}$$

Thus one obtains the general formula

$$C_z^+ = -\frac{2i\pi}{\beta}\,\frac{\cos\theta_1\cos\theta_2}{\cos\theta_1 - \cos\theta_2}\,\frac{dW}{dZ}_{(Z=i)} \tag{III.54}$$

In the case of the elementary problem, studied in section 3.2.2.1, one has

$$\frac{dW}{dZ}_{(Z=i)} = \frac{i\alpha}{\pi} \frac{\cos \theta_2 - \cos \theta_1}{\cos \theta_1 \cos \theta_2}$$

whence

$$C_z^{\,+} = - \frac{2\alpha}{\beta} = \frac{2i}{\beta} \tag{III.55}$$

if one puts $\alpha = -i$, following the notation customary in the wing theory.

Thus we shall find anew a remarkable result: the value of the coefficient $C_z^{\,+}$ is independent of the angles θ_1 and θ_2.

<u>3.2.2.3 - Study of the general case by means of the method of electric analogies</u>.- The method set forth above (section 3.1.3.3) may be applied in superposition. The electrodes must be disposed on the arcs AA_1, $A'A_2$, and on the segment AA'. These electrodes must be brought to prescribed potentials; the conductive arc A_1A_2 is brought to the potential 0. Finally, one will detach a small electrode at the point B with the purpose of measuring the resultant of the normal forces; this resultant, given by the formula (III.54) is, in fact, proportional to the intensity entering at B.

The value of u on the arcs AA_1 and $A'A_2$ is immediately known by simple integration.

In fact, if for instance w_1 designates the value of w given for

$$\theta = \theta_1 - \epsilon$$

(ϵ positive and arbitrarily small), one has[37], according to formula (III.46)

[37]Physically, the fact that the pressure on the bounding generatrices of the conical obstacle depends only on the inclination of the tangent plane along these generatrices is obvious. It expresses the independence (see section 1.2.4) of these bounding generatrices with respect to the other generatrices of the conical obstacle.

$$u_1 = -\frac{w_1}{\beta \sin \theta_1}$$

and the formulas (III.45) permit the calculation of u on the entire arc AA_1. Thus it is not necessary to measure the intensities leaving each of the electrodes except over the length of the segment AA'. As before, this intensity, proportional to $\partial w/\partial Y$, furnishes immediately the value of $\partial u/\partial X$ along the axis OX, owing to the formula

$$\beta \frac{\partial u}{\partial X} = \frac{2X}{1 - X^2} \frac{\partial w}{\partial Y}$$

Since one knows the value of u at the points A and A', one uses the superabundant data for calculation of the value of u on the axis AA'. Thus it is unnecessary to obtain the distribution of the potential, inside of the tank, as in the case described in section 3.1.3.2.

3.2.2.4 - Study of the general problem by purely numerical methods.- In order to simplify the exposition, we shall be content to examine the case where the given cone admits the plane Ox_1x_3 as symmetry plane. This amounts to stating that in the plane z the function $w(x)$ is even in x on the cut $(-\mu,\mu)$ representing the given cone.

We assume w_1 to be the value of w at the points $x = 1$ and $x = -1$, and put

$$f(x) = w(x) - w_1$$

If $1 < x < \mu$, one will put $x = \frac{1}{\cos \theta}$, $\mu = \frac{1}{\cos \theta_1}$ and $F(\theta) = f(x)$. One notes that $F(0) = 0$. After this statement, it is first of all evident, according to the foregoing, that one can immediately calculate the pressure outside of the cone (Γ).

In order to calculate the pressure inside of (Γ), one will consider the flow as the superposition,

1.- of an elementary flow $\left(w = -w_1,\text{ on the entire cut}\right)$,

2.- of an infinite number of elementary flows bisecting the cone (Γ) and symmetrical with respect to Ox_1x_3. These flows give at the point $x(x < 1)$ a pressure coefficient equal to

$$C_p = \frac{4}{\beta\pi}\int_0^{\theta_1} \frac{1}{\sin\theta}\frac{dF}{d\theta}\,\text{Arc sin}\,\frac{\sin\theta}{\sqrt{1 - x^2\cos^2\theta}}\,d\theta$$

3.- of a symmetric flow inside the Mach cone, defined by $w = f(x)$, on the cut $(-1,+1)$. One may apply the method described in section 3.1.3.3 for the calculation of this flow. We shall simply remark that it is not necessary to determine the integration constant since one knows that $u = 0$, for $x = \pm 1$.

3.2.3 - Cone Partially Inside and Partially Outside of the Mach Cone (Γ) (Fig. 30)

3.2.3.1 - Symmetrical elementary problem.- The circle bounded by (C_0) must be notched by a cut CA (see fig. 49), with the real part of $W(Z)$ assuming the constant value $w_0 = \alpha$ on the upper edge of the cut, and the value $-w_0$ on the lower edge. On the circle (C_0), w is zero, except on the arc AA_1 where $w = w_0$, and on the arc AA_1' where $w = -w_0$. One will designate the point C on the circle C_0 by $Z = a$, and the argument of A_1 on the circle (C_0) by θ_1.

The function $W(Z)$ can be written without difficulty

$$W(Z) = w_0 + i\,\frac{w_0}{\pi}\log\frac{(Z - a)(1 - aZ)}{\left(Z - e^{i\theta_1}\right)\left(Z - e^{-i\theta_1}\right)}$$

with the argument

$$\frac{(Z - a)(1 - aZ)}{\left(Z - e^{i\theta_1}\right)\left(Z - e^{-i\theta_1}\right)}$$

being chosen equal to zero at the point A on the upper edge of the cut. Since $W(Z)$ is defined with exception of an imaginary constant only, one may also write

$$W(Z) = w_0 + i\,\frac{w_0}{\pi}\log\frac{2Zt_0 - \left(1 + Z^2\right)}{1 + Z^2 - 2Z\cos\theta_1} \tag{III.56}$$

putting $t_0 = \frac{1}{2}\left(a + \frac{1}{a}\right)$.

We shall now seek $U(Z)$

$$-\beta\,\frac{dU}{dZ} = \frac{2iZ}{Z^2 - 1}\,\frac{dW}{dZ} = -\frac{2w_0}{\pi}\,\frac{Z}{Z^2 - 1}\left[\frac{1}{Z - a} - \frac{a}{1 - aZ} - \frac{1}{Z - e^{i\theta_1}} - \frac{1}{Z - e^{-i\theta_1}}\right]$$

whence

$$U(Z) = \frac{2w_0}{\beta\pi}\left[\frac{i}{2\sin\theta_1}\log\frac{e^{i\theta_1} - Z}{1 - Ze^{i\theta_1}} - \frac{a}{1 - a^2}\log\frac{a - Z}{1 - aZ}\right] \tag{III.57}$$

Consequently, on the arc AA_1

$$C_p = \frac{2\alpha}{\beta}\,\frac{1}{\sin\theta_1}$$

which is a result one could foresee immediately.

One obtains easily the value of C_p along the axis OX; it suffices to write the formula (III.49)

$$C_p = \frac{2\alpha}{\pi\beta}\left[\frac{1}{\sin\theta_1}\,\text{Arc cos}\,\frac{\cos\theta_1 - x}{1 - x\cos\theta_1} + \frac{2a}{1 - a^2}\log\left|\frac{a - X}{1 - aX}\right|\right] \tag{III.58}$$

Let us recall that $x = \dfrac{2X}{1 + X^2}$.

Particular case: Let us assume that $\theta_1 = \frac{\pi}{2}$, $a = 0$, under the following conditions

$$\left.\begin{aligned}
&\text{On the arc } AA_1, && C_p = \frac{2\alpha}{\beta} \\
&\text{On the segment } AA', && C_p = \frac{2\alpha}{\beta\pi}\left[\text{Arc }\cos(-x)\right] \\
& && \quad = \frac{2\alpha}{\beta\pi}\left(\frac{\pi}{2} + \text{Arc }\sin x\right)
\end{aligned}\right\} \tag{III.59}$$

Let us recall that in all these formulas $x = \beta \frac{r}{x_1}$, with $(x_1,r,0)$ being the semipolar coordinates of a point of the wing Δ in the system of axes $(0x_1,x_2,x_3)$, and that $\cos \theta_1 = 1/\beta \tan \omega_1$.

3.2.3.2 - Elementary lifting problem, in the case where $a = 0$.- The transformation $s = \sqrt{Z}$ transforms the circle (C_0) into a semicircle in the plane of the complex variable s. In this plane, A_1 and A_1' have as homologues M_1 and M_1' (see fig. 50). The function $W(s)$ has a real part zero on the arc M_1M_1' and equal to w_0 on the arcs AM_1, BM_1', and on the segment AB.

We shall determine directly the function $U(Z)$ or rather the function $U(s)$. In fact, $U(s)$ has its real part zero on M_1M_1' and one knows, according to the relations of compatibility, that as in the preceding paragraphs

$$u = \frac{w_0}{\beta} \frac{1}{\sin \theta_1}, \quad \text{on } \widehat{AM_1}$$

$$u = \frac{w_0}{\beta} \frac{1}{\sin \theta_1}, \quad \text{on } \widehat{BM_1}'$$

Moreover, the imaginary part of U(s) is constant on the real axis and may, consequently, be put equal to 0. Thus one may analytically continue the function U(s) across the real axis. U(s) is then determined as solution of a Dirichlet problem inside of the circle of radius unity. One has

$$U(s) = \frac{iw_0}{\beta\pi \sin \theta_1} \log \frac{\left(s - e^{i\theta_1/2}\right)\left(s + e^{-i\theta_1/2}\right)}{\left(s + e^{i\theta_1/2}\right)\left(s - e^{-i\theta_1/2}\right)}$$

with the logarithm having the value of $i\pi$ for $s = 1$.

It is then easy to calculate u on the real axis, that is, on the segment OA of the original plane Z. Let us put

$$x = \frac{2Z}{1 + Z^2} = \frac{2s^2}{1 + s^4}$$

The quantity under the logarithmic sign is written

$$\frac{s^2 - 1 - 2is \sin \frac{\theta_1}{2}}{s^2 - 1 + 2is \sin \frac{\theta_1}{2}}$$

Its argument is equal to that of

$$\left(s^2 - 1 - 2is \sin \frac{\theta_1}{2}\right)^2$$

Now, the real part and the modulus of this expression are, respectively, equal to

$$\left(s^2 - 1\right)^2 - 4s^2\sin^2 \frac{\theta_1}{2} = s^4 + 1 - 2s^2\left(2 - \cos \theta_1\right)$$

and

$$\left(s^2 - 1\right)^2 + 4s^2\sin^2 \frac{\theta_1}{2} = s^4 + 1 - 2s^2\cos \theta_1$$

Hence

$$u = - \frac{w_0}{\beta\pi \sin \theta_1} \text{Arc cos}\left[1 - \frac{2x\left(1 - \cos \theta_1\right)}{1 - x \cos \theta_1}\right]$$

$$C_p = \frac{2\alpha}{\beta\pi \sin \theta_1} \text{Arc cos}\left[1 - \frac{2x\left(1 - \cos \theta_1\right)}{1 - x \cos \theta_1}\right] \qquad \text{(III.60)}$$

Particular case.- Let us suppose that $\theta_1 = \frac{\pi}{2}$

$$C_p = \frac{2\alpha}{\pi\beta} \text{Arc cos}(1 - 2x)$$

3.2.3.3 - Elementary lifting problem in the case where $a \neq 0$.- The elegant demonstration which has just been made for $a = 0$ and the principle of which is to be found in the original memorandum by Busemann, conceals one difficulty; this has caused M. Beschkine (ref. 11) to give a formula in the case where $a \neq 0$ which, at least in certain cases, leads to difficulties. In working directly with the function U, one risks forgetting the supplementary conditions which, because of the relations of compatibility, must be applied if one does not want singularities for the functions U, V, W at points other than the ends of the cut.

In fact, if U(Z) is regular inside of the circle $\left(C_0\right)$, V(Z) and W(Z) will have a logarithmic singularity at the point $Z = 0$. We shall study the case where $a \neq 0$, by studying directly the function W and limiting ourselves to not having singularities outside of the boundary generatrices of the cone. Besides, we shall again take up this important problem in section 3.3.

Thus it is a matter of studying the case where $w = w_0$ on the arc AA_1 and on the upper and lower edges of the cut CA (see fig. 49) and on the arc AA_1'; the transformation

$$\sigma = \frac{Z - a}{1 - aZ}$$

which maintains the circle of radius unity, leads us to the case where the cut is arranged following a radius. Finally the transformation

$$\sigma = s^2$$

leads, in the plane s, to search for a function $W(s)$ the real part of which assumes the value w_0 on the arcs BM_1, $B'M_1'$ of the semicircle of radius unity of the positive plane and on the segment BB', and becomes zero on the arc M_1M_1' by application of Schwartz' principle; one may continue the function $W(s) - w_0$ to the lower semicircle of the plane s. This function is defined by the values of its real part on the circumference of radius 1 of the plane s. However, since dW/dZ must become zero at the point $Z = -1$, dW/ds must become zero for $s = \pm i$.

In order to satisfy this condition, one decides to admit, for $W(s)$, singular points at the points M_1, M_1', M_2, M_2', and at the point $s = 0$. According to the investigation of section 3.1.1.2, this point may be a pole of the order one, with the residue being necessarily purely imaginary.

If ik is the residue of this pole, one may therefore write

$$W(s) = w_0 + G(s) + ik\frac{1 + s^2}{s}$$

with $G(s)$ being a holomorphic function inside of the circle of radius 1.

However, on the circle $|s| = 1$, $ik\dfrac{1 + s^2}{s}$ is purely imaginary. One deduces from it immediately the function $G(s)$; consequently, $W(s)$ is of the form

$$W(s) = w_0 - \frac{iw_0}{\pi}\log\frac{1 + s^2 - 2s\cos\dfrac{\varphi_1}{2}}{1 + s^2 + 2s\cos\dfrac{\varphi_1}{2}} + ik\frac{1 + s^2}{s}$$

$\dfrac{\varphi_1}{2}$ being the argument of the point M_1.

One calculates $U(s)$, owing to the relations of compatibility

$$\frac{dU}{ds} = -\frac{2w_0}{\beta\pi\left(1 - a^2\right)} \frac{\left(s^2 + a\right)\left(1 + s^2 a\right)}{s^4 - 1}\left(\frac{2e^{i\frac{\varphi_1}{2}}}{s^2 - e^{i\varphi_1}} + \frac{2e^{-i\frac{\varphi_1}{2}}}{s^2 - e^{-i\varphi_1}}\right) +$$

$$\frac{2k}{\beta} \frac{\left(s^2 + a\right)\left(1 + s^2 a\right)}{\left(1 + a^2\right)s^2\left(s^2 + 1\right)}$$

because

$$\frac{2Z}{Z^2 - 1} = 2\,\frac{(\sigma + a)(1 + \sigma a)}{\left(\sigma^2 - 1\right)\left(1 - a^2\right)}$$

One verifies immediately that the points $s = \pm i$ are not poles (and that, consequently, the points $Z = \pm 1$ are not singular points), if

$$k = -\frac{w_0}{\pi \cos\frac{\varphi_1}{2}}$$

Hence, for $U(s)$

$$U(s) = \frac{2w_0}{\beta\pi \cos\frac{\varphi_1}{2}} \frac{a\left(1 - s^2\right)}{\left(1 - a^2\right)s} +$$

$$\frac{iw_0}{\beta\pi} \frac{\left(1 + 2a \cos \varphi_1 + a^2\right)}{\left(1 - a^2\right)\sin \varphi_1} \log \frac{\left(s - e^{i\frac{\varphi_1}{2}}\right)\left(s + e^{-i\frac{\varphi_1}{2}}\right)}{\left(s - e^{-i\frac{\varphi}{2}}\right)\left(s + e^{i\frac{\varphi_1}{2}}\right)} \tag{III.61}$$

It is easy to relate the angle φ_1 to the given angle θ_1, fixing the point A_1 on $\left(C_0\right)$ in the plane Z

$$e^{i\theta_1} = \frac{\cos \varphi_1 (1 + a^2) + 2a + i(1 - a^2)\sin \varphi_1}{1 + 2a \cos \varphi_1 + a^2}$$

The calculation of C_p is then simple; it suffices to resume the calculation at the end of the preceding paragraph

$$\frac{2s^2}{s^4 + 1} = \frac{2(1 + a^2) - 2a(1 + Z^2)}{(1 + a^2)(1 + Z^2)4aZ} = \frac{x - x_0}{1 - xx_0}$$

if one puts, on the real axis

$$x = \frac{2Z}{1 + Z^2} \qquad x_0 = \frac{2a}{1 + a^2}$$

Under these conditions

$$\cos \varphi_1 = \frac{\cos \theta_1 - x_0}{1 - x_0 \cos \theta_1} \tag{III.62}$$

and, consequently,

$$C_p = \frac{2\alpha}{\pi\beta \sin \theta_1}, \quad \text{on the arc } AA_1$$

$$C_p = - \frac{4a\alpha}{\pi\beta(1 - a)\cos \frac{\varphi_1}{2}} \frac{1 - X}{\sqrt{(X - a)(1 - aX)}} +$$

$$\frac{2\alpha}{\pi\beta \sin \theta_1} \operatorname{Arc} \cos\left[1 - \frac{2(x - x_0)(1 - \cos \varphi_1)}{1 - xx_0 - (x - x_0)\cos \varphi_1}\right] \tag{III.63}$$

on the upper edge of the cut AC.

If $a = x_0 = 0$, one arrives again at the formula (III.60); the formula given by L. Beschkine (ref. 11) does not contain the first term.

3.2.3.4 - Calculation of C_z in the lifting case.- C_z in the plane z is always defined by the equality

$$C_z = \frac{\int_{\underline{L}} C_p \, dx}{\int_{\lambda}^{\mu} dx}$$

with the first integral being taken in the positive sense on the loop surrounding the cut.

However, with the adopted notations

$$\lambda = x_0 \qquad \mu = \frac{1}{\cos \theta_1}$$

and

$$\int_{\lambda}^{\mu} dx = \frac{1}{\cos \theta_1} - x_0 = \frac{1 - x_0 \cos \theta_1}{\cos \theta_1}$$

On the other hand

$$\int_{\underline{L}} C_p \, dz = 2\int_{A_1ACAA_1'} C_p \frac{1 - Z^2}{\left(1 + Z^2\right)^2} dZ$$

But

$$\int_{A_1ACAA_1'} \underline{R}\left[U(Z)\right] \frac{1 - Z^2}{\left(1 + Z^2\right)^2} dZ = \underline{R}\left[\int_{A_1ACAA_1'} U(Z)\frac{1 - Z^2}{\left(1 + Z^2\right)^2} dZ\right] =$$

$$\underline{R}\left[\int_{A_1A'A_1'} U(Z)\frac{1 - Z^2}{\left(1 + Z^2\right)^2} dZ\right] = \underline{R}\left[i\pi\left(R_i + R_{-i}\right)\right]$$

R_i and R_{-i} being the residues of the function to be integrated at the points $Z = i$ and $Z = -i$; finally

$$C_z = \frac{2i\pi \cos \theta_1}{\beta\left(1 - x_0 \cos \theta_1\right)}\left[\left(\frac{dW}{dZ}\right)_{(z=i)} - \left(\frac{dW}{dZ}\right)_{(z=-i)}\right]$$

In the purely lifting case

$$C_z = \frac{4i\pi \cos \theta_1}{\beta\left(1 - x_0 \cos \theta_1\right)}\left(\frac{dW}{dZ}\right)_{(z=i)} \tag{III.64}$$

We apply this formula to the elementary case

$$\frac{dW}{dZ} = \frac{dW}{ds}\frac{ds}{d\sigma}\frac{d\sigma}{dZ} = \frac{dW}{ds}\frac{1}{2\sqrt{\sigma}}\frac{1 - a^2}{(1 - aZ)^2}$$

One will put for simplification for $Z = i$

$$a = \tan\left(\frac{\beta}{2} - \frac{\pi}{4}\right) \qquad x_0 = -\cos \beta \qquad \sigma = e^{i\beta}$$

One then finds that

$$\left(\frac{dW}{dZ}\right)_{(z=i)} = \frac{iw_0}{\pi}\frac{\sin^2\beta \cos \frac{\beta}{2}}{\left(\cos \varphi_1 - \cos \beta\right)\cos \frac{\varphi_1}{2}}$$

Hence $\left(w_0 = \alpha\right)$

$$C_z = -\frac{4\alpha}{\beta}\frac{\cos \theta_1 \sin \beta}{\left(1 + \cos \beta \cos \theta_1\right)\left(\cos \varphi_1 - \cos \beta\right)\cos \frac{\varphi_1}{2}}$$

If one utilizes the equality (III.62) in defining φ_1 and if one puts $\alpha = -i$, with i designating the incidence, one finds the very simple formula

$$C_z = \frac{4i}{\beta} \frac{\cos \frac{\beta}{2}}{\cos \frac{\varphi_1}{2}} \tag{III.65}$$

3.2.3.5 - General case.- The investigation of the general case may be made either by electric analogy or by calculation. The methods to be employed result from what has been seen in sections 3.1.3.2, 3.1.3.3, 3.2.2.3, and 3.2.2.4.

Let us only indicate that, in the solution of the lifting problem by electric analogy, one must arrange a singularity at the point C like the one defined in section 3.1.3.2.3. The adjustment of the potential to which the conductive part of this singularity must be brought is obtained by the condition that no intensity enters at the point A'. To verify this condition, one will use the method already indicated in the section noted.

Naturally, the total C_z will be very easily determined by measurement of the intensity entering at the point B and application of the formula (III.64).

3.2.4 - Cone Entirely Outside of the Cone (Γ) (Fig. 29)

3.2.4.1 - Elementary symmetrical problem.- The problem consists in determining $U(Z)$, $V(Z)$, $W(Z)$ by means of the following conditions: the real part of $W(Z)$ assumes on the arc A_1A_2 (see fig. 51) of the circle (C_0) the constant value $w_0 = \alpha$, and on the arc $A_1'A_2'$ the value $-w_0$. On the other portions of (C_0) this real part is zero. Thus one may write immediately the value of the real part of $U(Z)$ on the circle (C_0) (formulas (III.45), (III.46)). It is an even function of the argument θ. One has

$$u = 0, \qquad \text{on the arc } A'A_2$$

$$u = -\frac{w_0}{\beta}\frac{1}{\sin\theta_2}, \qquad \text{on the arc } A_2A_1$$

$$u = -\frac{w_0}{\beta}\left(\frac{1}{\sin\theta_2} - \frac{1}{\sin\theta_1}\right), \qquad \text{on the arc } A_1A$$

whence for the function $U(Z)$, the formula

$$U(Z) = \frac{iw_0}{\beta\pi}\left[\frac{1}{\sin\theta_2}\log\frac{e^{i\theta_2} - Z}{1 - Ze^{i\theta_2}} - \frac{1}{\sin\theta_1}\log\frac{e^{i\theta_1} - Z}{1 - Ze^{i\theta_1}}\right] \qquad \text{(III.66)}$$

the logarithms assuming the value $i\pi$ at the point $Z = 1$.

The complete calculation of $V(Z)$ and $W(Z)$, likewise, does not offer any difficulties.

One deduces from this formula the calculation of the pressures on the obstacle and outside of the obstacle.

In the plane x_1Ox_2 the pressure coefficient has the value

$$C_p = \frac{2\alpha}{\beta}\frac{1}{\sin\theta_2}, \qquad \text{on the obstacle}$$

$$C_p = \frac{2\alpha}{\beta}\left(\frac{1}{\sin\theta_2} - \frac{1}{\sin\theta_1}\right), \qquad \text{in the region comprised between the obstacle and the Mach cone of the point } O$$

Let us recall that if ω_1 and ω_2 designate the angles formed by the bounding generatrices of the obstacle with Ox_1, one has according to definition (see fig. 52):

$$\cos\theta_1 = 1/\beta\tan\omega_1 \qquad \cos\theta_2 = 1/\beta\tan\omega_2$$

Inside of the Mach cone, finally, at the point x_1, x_2, one has

$$C_p = \frac{2\alpha}{\beta\pi}\left[\frac{1}{\sin\theta_2}\text{Arc cos}\frac{\cos\theta_2 - x}{1 - x\cos\theta_2} - \frac{1}{\sin\theta_1}\text{Arc cos}\frac{\cos\theta_1 - x}{1 - x\cos\theta_1}\right]$$

if $x = \beta\frac{x_2}{x_1}$, the arc cosines having their principle values.

3.2.4.2 - General symmetrical problem.- The general symmetrical problem does not present any difficulty, since one may operate by means of superposition; let $w = \alpha(\theta)$ be the given value of velocity component following Ox_3 over the length of the obstacle $\left(\theta_1 < \theta < \theta_2\right)$. The formulas giving the C_p may be written immediately

$$\left.\begin{aligned}
C_p &= -\frac{2}{\beta}\int_{\theta}^{\theta_2+0}\frac{d\alpha}{\sin\theta}, && \text{at the point of the obstacle of parameter } \theta\\
C_p &= -\frac{2}{\beta}\int_{\theta_1-0}^{\theta_2+0}\frac{d\alpha}{\sin\theta}, && \text{behind the obstacle, outside of the cone } (\Gamma)\\
C_p &= -\frac{2}{\beta\pi}\int_{\theta_1-0}^{\theta_2+0}\text{Arc cos}\frac{\cos\theta - t}{1 - t\cos\theta}\frac{d\alpha}{\sin\theta}, && \text{inside of the Mach cone}
\end{aligned}\right\}\qquad\text{(III.67)}$$

The integrals of the preceding formulas must be taken according to the signification of Stieljes; this is a fundamental condition for the case where $\alpha(\theta)$ presents discontinuities. In particular, one will have to take account of two discontinuities: the discontinuity $+\alpha\left(\theta_1\right)$ for $\theta = \theta_1$, and the discontinuity $-\alpha\left(\theta_2\right)$ for $\theta = \theta_2$. Not to forget these discontinuities was the reason that we wrote certain limits of the integrals $\theta_1 - 0$, $\theta_2 + 0$.

3.2.4.3 - Elementary lifting problem.- The solution obtained for the symmetrical problem (formula (III.66)) is valid, since dW/dZ necessarily becomes zero at the points $Z = \pm 1$; also, dU/dZ becomes zero at the point $Z = 0$; thus the relations of compatibility do not entail any singularity other than the points A_1 and A_2. We shall see that in case of the lifting problem a few precautions must be taken if this condition is again to be satisfied.

Let us first assume that the points A_1, A_2 and A_1', A_2' are simple poles for $\frac{dU}{dZ}$, $\frac{dV}{dZ}$, and $\frac{dW}{dZ}$. One may then write the values of u, v, w on the circle (C_0) utilizing the relations (III.45) and (III.46) as well as the boundary conditions. These latter let us know that w assumes the value w_0 on the arcs A_1A_2, $A_1'A_2'$ (fig. 51). On the other hand, the component u necessarily continues outside of the cone (since u represents the pressure except for one constant) and, being odd in x_3, must become zero in the plane Ox_1x_2 outside of the given delta wing. Consequently, $u = 0$ on the circle (C_0), outside of the arcs A_1A_2, $A_1'A_2'$. Hence one deduces, as before, that on A_1A_2

$$v = \alpha \cot \theta_2 \qquad u = -\frac{\alpha}{\beta} \frac{1}{\sin \theta_2}$$

but on the arc AA_1

$$w = \alpha \frac{\sin \theta_2 - \sin \theta_1}{\sin \theta_2} \qquad v = \alpha \frac{\cos \theta_2 - \cos \theta_1}{\sin \theta_2}$$

We note therefore that w assumes on the arc AA_1' the same values as on the arc AA_1, whereas v assumes opposite values. Hence one deduces that the region of the plane Ox_1x_2, comprised between the trailing edge Δ_1 and the Mach cone (see fig. 52), is thus a region of discontinuity for the velocity.

One sees therefore that the hypothesis set up before $\left(\text{simple poles for } \frac{dU}{dZ}, \frac{dV}{dZ}, \frac{dW}{dZ}\right)$ is incompatible with the fact that U, V, W do not admit singularities other than the points A_1, A_2, A_1', A_2'. One may realize this, besides, in another manner; in order to satisfy in the simplest possible way the boundary conditions imposed on $U(Z)$, it suffices to write $U(Z)$ in the form

$$U_1(Z) = -\frac{i\alpha}{\beta\pi \sin\theta_2} \log \frac{1 - 2Z\cos\theta_1 + Z^2}{1 - 2Z\cos\theta_2 + Z^2}$$

since this function $U_1(Z)$ well fulfills the boundary conditions required for the function $U(Z)$ on the circumference (C_0). However,

$$\frac{dU_1}{dZ} = -\frac{2i\alpha}{\beta\pi \sin\theta_2}\left[\frac{Z - \cos\theta_1}{1 - 2Z\cos\theta_1 + Z^2} - \frac{Z - \cos\theta_2}{1 - 2Z\cos\theta_2 + Z^2}\right]$$

and for $Z = 0$

$$\left(\frac{dU_1}{dZ}\right)_{Z=0} = \frac{2i\alpha}{\beta\pi \sin\theta_2}\left(\cos\theta_1 - \cos\theta_2\right)$$

If, therefore, the functions $U(Z)$, $V(Z)$, $W(Z)$ are not to admit singularities inside of (C_0), the solution $U_1(Z)$ cannot be retained just as it is because the corresponding functions $V_1(Z)$ and $W_1(Z)$ would have a critical logarithmic point at the origin[38].

Thus we are led to modify the solution $U_1(Z)$ by introducing a singularity at one of the points A_1 or A_2 (and, by symmetry, at A_1' or A_2'). Physically, by virtue of the rule of forbidden signals, this singularity must be placed at the pair of points A_1, A_1', because the bounding generatrix Δ_2 (fig. 52) which takes the place of the leading edge (having as image the pair of points A_2, A_2' in the plane Z) is independent (see section 1.2.4) of the trailing edge (pair of points A_1, A_1', in the plane Z). One then sees that, by putting

$$U(Z) = -\frac{i\alpha}{\pi\beta \sin\theta_2} \log \frac{1 - 2Z\cos\theta_1 + Z^2}{1 - 2Z\cos\theta_2 + Z^2} - \frac{2i\alpha}{\pi\beta \sin\theta_2} \frac{\left(\cos\theta_1 - \cos\theta_2\right)Z}{1 - 2Z\cos\theta_1 + Z^2} \tag{III.68}$$

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[38]L. Beschkine (ref. 11) took the function $U_1(Z)$ as the value of $U(Z)$; see further on, in section 3.3.2, the discussion of this question.

one has, for $U(Z)$, a function satisfying the boundary conditions on (C_0), holomorphic inside of (C_0), the derivative of which becomes zero at the point $Z = 0$ and consequently leads to functions $W(Z)$ and $V(Z)$ which do not present singularities inside of (C_0). Besides, this solution is unique if one takes account of the principle of minimum singularities.

One may then calculate the functions $V(Z)$ and $W(Z)$. Thus one finds for $W(Z)$

$$W(Z) = -\frac{i\alpha}{\pi}\log\frac{e^{i\theta_2} - Z}{1 - Ze^{i\theta_2}} + \frac{i\alpha}{\pi \sin\theta_2}\frac{\left(1 - \cos\theta_1\cos\theta_2\right)}{\sin\theta_1}\log\frac{e^{i\theta_1} - Z}{1 - Ze^{i\theta_1}} +$$

$$\frac{\alpha}{\pi}\frac{\cos\theta_1 - \cos\theta_2}{\sin\theta_2}\frac{Z^2 - 1}{1 + Z^2 - 2Z\cos\theta_1} \tag{III.69}$$

and

$$V(Z) = -\frac{i\alpha}{\pi}\cot\theta_2\log\frac{\left(Z - e^{i\theta_2}\right)\left(Z - e^{-i\theta_2}\right)}{\left(Z - e^{i\theta_1}\right)\left(Z - e^{-i\theta_1}\right)} +$$

$$\frac{2i\alpha}{\pi}\frac{\cos\theta_1}{\sin\theta_2}\frac{Z\left(\cos\theta_1 - \cos\theta_2\right)}{1 + Z^2 - 2Z\cos\theta_1} \tag{III.70}$$

Thus one finds that on the wing $\left(\text{arc } A_1A_2\right)$ the component v has the value

$$v = \alpha\cot\theta_2$$

In the region of the plane Ox_1x_2 outside of the wing, the component v is always zero; whereas w assumes a constant value in the part comprised between the trailing edge and the cone (Γ):

$$w = \alpha\left(1 - \frac{1 - \cos\theta_1\cos\theta_2}{\sin\theta_2\sin\theta_1}\right) = -\alpha\frac{1 - \cos\left(\theta_1 - \theta_2\right)}{\sin\theta_1\sin\theta_2}$$

Finally, in the part of the plane Ox_1x_2 inside of (Γ) (segment AA') $v = 0$, and w is given by the formula

$$w = \frac{\alpha}{\pi} \text{Arc cos}\left(\frac{\cos \theta_2 - x}{1 - x \cos \theta_2}\right) -$$

$$\frac{\alpha}{\pi} \frac{\left(1 - \cos \theta_1 \cos \theta_2\right)}{\sin \theta_1 \sin \theta_2} \text{Arc cos}\left(\frac{\cos \theta_1 - x}{1 - x \cos \theta_1}\right) +$$

$$\frac{\alpha}{\pi} \frac{\cos \theta_1 - \cos \theta_2}{\sin \theta_2} \frac{\sqrt{1 - x^2}}{1 - x \cos \theta_1} \tag{III.71}$$

3.2.4.4 - General lifting problem.- One sees immediately that, if one wants to uniquely calculate the pressure on the obstacle, one may utilize the same formula as for the general symmetrical problem (formula (III.67)). Besides, the study of the values $U(Z)$, $V(Z)$, and $W(Z)$ in the general case will also be very simple with the aid of superposition. One will easily verify that, if $w = \alpha(\theta)$ is the prescribed value of the normal component along the obstacle $\left(\theta_1 < \theta < \theta_2\right)$, one has, for instance

$$U(Z) = \frac{i}{\beta\pi} \int_{\theta_1}^{\theta_2+0} \log \frac{1 + Z^2 - 2Z \cos \theta_1}{1 + Z^2 - 2Z \cos \theta} \frac{d\alpha(\theta)}{\sin \theta} +$$

$$\frac{2iZ}{\beta\pi\left(1 - 2Z \cos \theta_1 + Z^2\right)} \int_{\theta_1}^{\theta_2+0} \frac{\cos \theta_1 - \cos \theta}{\sin \theta} d\alpha(\theta)$$

Analogous formulas could be written for $V(Z)$ and $W(Z)$.

Thus the electric analogy is less interesting in this case, since there is a way of solving the problem explicitly. We shall simply note that the singularity to be placed at the tank at the image point of the trailing edge is a doublet.

3.3 - Supplementary Remarks on the Infinitely Flattened Conical Flows

3.3.1 - Continuity of the Results

At the end of this investigation, it will not be unnecessary to state briefly the continuity of the obtained results.

If one takes for instance an elementary flow bisecting the Mach cone for which one makes θ_1 tend toward 0, θ_2 toward π, one finds, passing to the corresponding limit in the formula (III.52) as limiting value of the pressure coefficient

$$C_p = \frac{2\alpha}{\beta\pi}\left[\lim_{\theta_1 \to 0} \frac{1}{\sin\theta_1} \text{Arc cos} \frac{\cos\theta_1 - x}{1 - x\cos\theta_1} + \right.$$

$$\left. \lim_{\theta_2 \to \pi} \frac{1}{\sin\theta_2} \text{Arc cos} \frac{x - \cos\theta_2}{1 - x\cos\theta_2}\right] =$$

$$\frac{2\alpha}{\beta\pi}\left[\sqrt{\frac{1 + x}{1 - x}} + \sqrt{\frac{1 - x}{1 + x}}\right] = \frac{4\alpha}{\beta\pi} \frac{1}{\sqrt{1 - x^2}} \qquad \text{(III.72)}$$

If one now makes, in an elementary flow, symmetrical or lifting (see sections 2.1.2.2 and 3.1.2.3), b and c, respectively, tend toward -1 and 1, one again arrives at the formula (III.72). Besides, the formula (III.72) has already been written, at the end of section 3.1.1.7. One finds, finally, the same result by transferring likewise results from section 3.2.3. If one makes, for instance, in the formula (III.58), θ_1 tend toward zero and a toward -1, one obtains

$$C_p = \frac{2\alpha}{\beta\pi}\left[\lim_{\theta_1 \to \theta} \frac{1}{\sin\theta_1} \text{Arc} \frac{\cos\theta_1 - x}{1 - x\cos\theta_1} + \right.$$

$$\left. \lim_{a \to -1} \frac{1}{1 - a^2} \log\left[\frac{a - X}{1 - aX}\right]\right] = \frac{2\alpha}{\beta\pi}\left[\sqrt{\frac{1 + x}{1 - x}} + \frac{1 - X}{1 + X}\right] = \frac{4\alpha}{\beta\pi} \frac{1}{\sqrt{1 - x^2}}$$

Likewise, starting from equation (III.63) and making a tend toward -1, θ_1 toward zero $\left(\varphi_1\right.$ tend toward zero$\left.\right)$

$$C_p = \frac{4\alpha}{2\beta\pi}\frac{1 - X}{1 + X} + \frac{2\alpha}{\beta\pi}\lim_{\substack{\theta_1 \to 0 \\ x_0 \to -1}} \frac{1}{\sin\theta_1} \text{Arc cos}\left(1 - \frac{2\left(x - x_0\right)\left(1 - \cos\varphi_1\right)}{1 - xx_0 - \left(x - x_0\right)\cos\varphi_1}\right) = \frac{2\alpha}{\beta\pi}\left[\frac{1 - X}{1 + X} + \right.$$

$$\left.\lim_{x_0 \to -1}\sqrt{\frac{2\left(x - x_0\right)}{\left(1 + x_0\right)\left(1 - x\right)}}\sqrt{\frac{1 + x_0}{1 - x_0}}\right] = \frac{2\alpha}{\beta\pi}\left[\frac{1 - X}{1 + X} + \sqrt{\frac{1 + x}{1 - x}}\right] = \frac{4\alpha}{\beta\pi}\frac{1}{\sqrt{1 - x^2}}$$

Likewise, one may verify the continuity of the results under the hypothesis where a single one of the generatrices of the conical obstacle is situated on the Mach cone. One thus obtains a limiting case between the flows studied in section 3.1.2 and those studied in section 3.2.3. If one supposes, for instance, that one of the bounding generatrices has as image the point $Z = 1$, the second the point $Z = a$, $-1 < a < 1$, one finds, whatever the manner of making the passage to the limit, for the symmetrical problem

$$C_p = \frac{2\alpha}{\pi\beta}\left[\sqrt{\frac{1 + x}{1 - x}} + \frac{2a}{1 - a^2}\log\left[\frac{X - a}{1 - aX}\right]\right]$$

and for the lifting problem

$$C_p = -\frac{4a\alpha}{\beta\pi(1 - a)}\frac{1 - x}{\sqrt{(x - a)(1 - aX)}} +$$

$$\frac{2\alpha}{\beta\pi}\sqrt{\frac{2\left(x - x_0\right)}{(1 - x)\left(1 - x_0\right)}}\frac{4\alpha\left[X(1 + a)^2 - 2a\left(X^2 + 1\right)\right]}{\beta\pi(1 - X)(1 - a)\sqrt{(X - a)(1 - aX)}}$$

In the same manner one can verify the continuity between the flows studied in sections 3.2.3 and 3.2.4.

3.3.2 - Discussion on the Possible Singularities of Lifting Problems

In this entire chapter, we have limited ourselves to giving the solutions which satisfy the condition, stated frequently: To admit as singularities in the plane Z only the bounding generatrices of the Δ, and to choose from among all possible solutions the solution which satisfies the principle of minimum singularities. This is a hypothesis which is justified by its simplicity and which we have set up here without using the experimental results apt to guide our choice for placing the singularities[39].

A first theoretical possibility would consist in admitting singularities possible on the generatrices of the Mach cone, having as image the points $Z = \pm 1$ in the plane Z. This seems to us not easily admissible from the physical point of view. Besides, to our knowledge, the various authors who have treated problems of infinitely flattened conical flows have always eliminated this possibility (see in particular refs. 10 and 11). In fact, it is hard to understand how the pressure could become infinite in the neighborhood of these generatrices.

In contrast, one has a means of obtaining solutions different from those obtained in the course of this investigation, in tolerating, as possible singular point, the point $Z = 0$.

We shall first make the following general remark: Let us take the case of a cone where one of the bounding generatrices has as image the point $Z = 0$ in the plane Z; in this case the pressure remains finite in the neighborhood of the corresponding bounding generatrix. This results from the formulas (III.23) and (III.24) for the case of a cone entirely inside of (Γ) (section 3.1.2), and from formulas (III.58) and (III.60) for the case of a cone partially outside, partially inside of (Γ) (section 3.2.3). We shall show that, utilizing conformal representations and maintaining the circle (C_0), it will be possible, even in the case where Ox_1 is not a bounding generatrix, to define a solution of the lifting problem in such a manner that the pressure remains finite along a bounding generatrix inside of (Γ), under the condition of admitting the point $Z = 0$ as singular point.

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[39]The theoretical study of flows (movements) in incompressible fluid has been rendered possible and effective only owing to the famous hypothesis of Joukowsky which indicates the choice to be made among the singularities which are possible for the flow. The study of the problems treated here shows uncertainty in the state of our actual knowledge concerning the conditions which the theoretical solution must satisfy in order to represent best the real phenomena.

We return to the investigation of section 3.2.3.3 where $a \neq 0$: One may in fact come back to the case where $a = 0$, by the transformations utilized before

$$\sigma = \frac{Z - a}{1 - aZ} \qquad s^2 = \sigma$$

The function $U(s)$ the determination of which was the problem is then defined inside of the semicircle, and it satisfies exactly the same conditions as the function $U(s)$ studied in section 3.2.3.2. Thus one will have

$$U(s) = \frac{iw_0}{\beta\pi \sin \theta_1} \log \frac{\left(s - e^{i\frac{\varphi_1}{2}}\right)\left(s + e^{-i\frac{\varphi_1}{2}}\right)}{\left(s + e^{i\frac{\varphi_1}{2}}\right)\left(s - e^{-i\frac{\varphi_1}{2}}\right)} \tag{III.73}$$

φ_1 being defined starting from θ_1 by the equality (III.62). This leads us to a value of the pressure coefficient

$$C_p = \frac{2\alpha}{\beta\pi \sin \theta_1} \operatorname{Arc}\cos\left[1 - \frac{2(x - x_0)(1 - \cos \varphi_1)}{1 - xx_0 - (x - x_0)\cos \varphi_1}\right]$$

a value already given by Beschkine which is deduced from the formula (III.63) by suppression of the term in logarithm. This pressure coefficient remains finite along the bounding generatrix inside of (Γ): $x = x_0$.

However, if one calculates the functions $W(Z)$ and $V(Z)$, corresponding to the function U defined by equation (III.73), one finds the following results

$$W(s) = w_0 - i\frac{w_0}{\pi}\log\left|\frac{1 + s^2 - 2s\cos\frac{\varphi_1}{2}}{1 + s^2 + 2s\cos\frac{\varphi_1}{2}}\right| +$$

$$\frac{w_0}{\pi}\cos\frac{\varphi_1}{2}\left[\frac{1}{\sqrt{a}}\log\frac{\left(s - i\sqrt{a}\right)\left(1 + \sqrt{a}\right)}{\left(s + i\sqrt{a}\right)\left(1 - \sqrt{a}\right)} + \sqrt{a}\log\left|\frac{s\sqrt{a} - i}{s\sqrt{a} + i}\frac{\sqrt{a} + 1}{\sqrt{a} - 1}\right|\right]$$

$$V(s) = -\frac{iw_0}{\pi}\cot\theta_1\log\left(\frac{s^2 - 1 - 2s\sin\frac{\varphi_1}{2}}{s^2 - 1 + 2is\sin\frac{\varphi_1}{2}}\right) -$$

$$\frac{iw_0}{\pi}\cos\frac{\varphi_1}{2}\left[\frac{1}{\sqrt{a}}\log\frac{\left(s - i\sqrt{a}\right)\left(1 + \sqrt{a}\right)}{\left(s + i\sqrt{a}\right)\left(1 - \sqrt{a}\right)} - \sqrt{a}\log\left|\frac{s\sqrt{a} - i}{s\sqrt{a} + i}\frac{\sqrt{a} + 1}{\sqrt{a} - 1}\right|\right] \tag{III.74}$$

These formulas call for the following remarks (see fig. 54). We assume $a > 0$:

1. On the region of the obstacle comprised between OD and $O\Delta_2$ $\left(\left|\operatorname{Arg} s\right| < \frac{\varphi_1}{2}\right)$ one has

$$w = w_0$$

$$v = \pm w\cot\theta_1$$

a result which is quite conformal to the formulas (III.44) and (III.46).

2. On the region of the obstacle comprised between OD and $O\Delta_1$, s is real $\left[-1 < s < 0, \text{ for the surface } x_3 < 0\right]$; one sees that $w = w_0$ on every surface, whereas v assumes the opposite values $\pm w_0\frac{\cos\frac{\varphi_1}{2}}{\sqrt{a}}$.

3. On the region of the plane Ox_1x_2 comprised between $O\,\Delta_1$ and Ox_1 (s is purely imaginary and varies on the segment $O\omega$), v maintains constant opposite values, equal to $a \pm w \dfrac{\cos \dfrac{\varphi_1}{2}}{\sqrt{a}}$, whereas w increases infinitely in absolute value.

4. On the region of the plane comprised between Ox_1 and OD' (s, which is purely imaginary, describes the segment ωB), v is zero; w, infinite on Ox_1, becomes zero on OD'.

Behind the generatrix $O\,\Delta_1$ which one may consider as the trailing edge of the wing Δ studied, this solution furnishes therefore a zone of discontinuity of velocity (the discontinuity being in the direction of Ox_2) which occupies the region $O\,\Delta_1$, Ox_1. Moreover, the axis Ox_1 is a singular straight line for the flow. Thus one encounters a scheme which seems at first rather tempting and reminds one of the study of the wing in subsonic flow; behind the wing there appears a zone of discontinuity of velocity produced by vortices following the direction of Ox_1, and the singularity encountered along the axis Ox_1 reminds one of the "marginal vortex" of the wing theory. As in the case of subsonic flows, this flow scheme appears linked to the condition of having a finite pressure along the trailing edge.

The formulas (III.74) likewise show us that the flow found does not satisfy the boundary conditions if a is negative, that is, if the obstacle is not situated on the same side of the plane Ox_1x_3. In fact, in this case w would admit on the obstacle a discontinuity in the neighborhood of the axis Ox_1; but this is incompatible with the boundary data[40].

If one wants to apply a similar method in the case of a symmetrical flow, one likewise notices immediately that the result is incompatible with the given boundary conditions since one obtains a discontinuity for w.

Let us now visualize the case of a flow around a cone entirely inside of the Mach cone, with the bounding generatrices on the same

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[40]This solution which has been suggested by Beschkine must, therefore, certainly be rejected in the case where a is negative; the figure 6 given by Beschkine (ref. 11) seems to show that this author has not seen this fundamental restriction. In this case one must certainly adopt the solution set forth in section 3.2.3.3.

side as Ox_1 (fig. 55) and the remaining finite on the trailing edge $0\,\Delta_1$. The function $U(Z)$ then has the form

$$U(Z) = \frac{w_0}{\beta} C(b,c) \sqrt{\frac{(Z - b)(1 - Zb)}{(c - Z)(1 - Zc)}} \tag{III.75}$$

with $C(b,c)$ being a function of b and of c.

One then sees that in calculating $V(Z)$ and $W(Z)$ one will find the same particularities as previously: the point $Z = 0$ will be a singular point. In the region comprised between Ox_1 and $0\,\Delta_1$ one states a discontinuity of the component v whereas the velocity w becomes infinite along Ox_1.

The following problem arises: Should one adopt in the case where the two bounding generatrices $0\,\Delta_1$ and $0\,\Delta_2$ are on the same side as Ox_1 the solutions exposed in the course of this chapter, which we shall call solutions of type I (singularities on $0\,\Delta_1$ and $0\,\Delta_2$), or the solutions we just indicated, which we shall call solutions of type II (singularities along $0\,\Delta_2$ and Ox_1)?

Let us note first of all that, for reasons of continuity, it is absolutely necessary to adopt completely one or the other viewpoint; one cannot admit a solution of the type I for the flows entirely inside the Mach cone, and a solution of the type II for the flows partly inside, partly outside.

Under this presupposition, the solutions of the type II are, at a first glance, rather tempting; perhaps certain authors were thinking of these solutions when they exposed the condition of the subsonic trailing edge which could be stated in the following manner:

Since the tangent to the trailing edge forms with the flow an angle which is smaller than the Mach angle of the flow, one must write on the corresponding trailing edge the condition of Joukowsky in order to be sure that the velocity remains finite (see for instance ref. 4).

Now the solutions corresponding to the formulas (III.73) and (III.75) seem to satisfy these conditions. And as we remarked before, these flows show, behind the trailing edge, actually a character which reminds one of subsonic flows.

We do not want to definitely reject these flows; however, we have to make three remarks.

1. As we have stated that the solutions with finite pressure along the trailing edge are not possible for the symmetrical problems, the pressure cannot remain finite in the case of a flow of the type II around a cone having thickness.

2. It would be dangerous to link the solutions of the type II to the "subsonic trailing edge" since, if the wing is entirely outside of the cone (Γ), there exists still another solution which yields a finite pressure on $O\Delta_1$ and gives rise to a surface of discontinuity between Ox_1 and $O\Delta_1$: It is the solution $U_1(Z)$ visualized at the beginning of section 3.2.4.3. One has, in fact, under this hypothesis

$$V_1(Z) = \frac{i\alpha}{\pi \sin \theta_2}\left(\cos \theta_2 - \cos \theta_1\right)\log(-Z) +$$

$$\frac{i\alpha}{\pi \sin \theta_2}\left[\cos \theta_1 \log\left(1 + Z^2 - 2Z \cos \theta_1\right) - \right.$$

$$\left.\cos \theta_2 \log\left(1 + Z^2 - 2Z \cos \theta_2\right)\right]$$

which gives in the region comprised between Ox_1 and $O\Delta_1$ equal values of v

$$\pm \frac{\alpha}{\sin \theta_2}\left(\cos \theta_2 - \cos \theta_1\right)$$

If one adopts for such a cone the lifting solution of the type II, one finds that the velocity remains finite at the trailing edge, even under the hypothesis of a cone of nonzero thickness.

3. Adopting, still by virtue of the principle of continuity, the type II for the lifting solutions in the case where the bounding generatrices are on the same side as Ox_1 would lead us to a restriction of the range of the study of the flows with infinitely small cone angle made in chapter II; for this problem, such as it has been posed, would no longer be valid in the case where the contour (C) in the plane Z no longer contains O in its interior. In contrast, we already have had occasion to state that the results of chapter III are in complete

agreement with those of chapter II (see section 2.2.8); this statement is valid for the case of any figure whatsoever.

We may conclude that, according to the actual state of our knowledge, it does not seem imperative to adopt the viewpoint of the solutions of type II. In our opinion, only an experimental study can indicate where the theorist must place the singularities; the viewpoint adopted in this chapter seems to us to be the most natural one. It becomes required in the case where Ox_1 is comprised in the angle $O\,\Delta_1$ and $O\,\Delta_2$; in the opposite case, if in one way or another our knowledge of the physical phenomenon should widen and lead us to a change in our hypotheses on the singularities, it will still be easy to obtain the desired solutions, provided the conical character of the flow is maintained[41].

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[41]See Appendix No. 4.

CHAPTER IV - THE COMPOSITION OF CONICAL FLOWS AND ITS APPLICATION TO THE AERODYNAMIC CALCULATION OF SUPERSONIC AIRCRAFT

We shall show in this chapter how the conical flows studied in the previous chapter and possibly the homogeneous flows defined in section 1.3 of chapter I permit to study, at least in certain particular cases, the various elements of a supersonic airplane (fuselages, wings, controls, etc.) by "superposition" if one can apply the general method of linear approximations. Our aim is not to furnish all possible applications nor to give all the formulas the constructor may need. We shall, rather, insist on the principles of such a composition; we shall give the simplest and most significant results and, more specially, those which, at least to our knowledge, have a character of newness. We shall voluntarily reserve the results of technical character for a later publication.

Such a superposition is justified by the linear character of the fundamental equation (I.10). The simplicity of the following arguments frequently results from the rule of "forbidden signals" which we have stressed already in section 1.1.4.

4.1 - Application of Conical Flows to the Calculation of the Wings

In his fundamental memorandum, often quoted above (ref. 4), Th. Von Kármán indicates that the theory of conical flows permits the investigation of wings the profiles of which are formed by straight lines[42]. We intend to show in this paragraph that one can investigate a wing of finite span and with a curvilinear profile by means of composition of conical flows. Like the problems of conical flows (compare chapter III), a wing problem may be divided into a symmetrical and a lifting problem.

We shall note $\delta^+\left(x_1,x_2\right)$ and $\delta^-\left(x_1,x_2\right)$, the inclinations of the top surface profiles $\left(x_3 = +0\right)$ and bottom surface profiles $\left(x_3 = -0\right)$

[42]The subject of a certain number of memoranda is the study of wings with polygonal profile. One must then superpose a finite number of conical flows. The most recent and most complete investigation of this problem is the one by A. E. Pukett and H. J. Stewart (ref. 30).

of the wing investigated, and we shall put (compare fig. 56)

$$\delta^+ = -i + \alpha^+ \qquad \delta^- = -i + \alpha^-$$

with i representing the general incidence of the wing (one will define it as the incidence of the chord of one of the sections). We shall then put

$$j_0(x_1, x_2) = \frac{\alpha^+ + \alpha^-}{2} \qquad \alpha = \frac{\alpha^+ - \alpha^-}{2}$$

In the case of a purely symmetrical problem

$$i = 0 \qquad j_0 = 0$$

In the case of a purely lifting problem

$$\alpha = 0$$

Let C_p^- and C_p^+ be the pressure coefficients on the upper side and lower side of the wing. The local c_z and the local c_x of a section parallel to Ox_1,x_3 will be defined by (compare fig. 56)

$$c_z = \int_{mm'} \left(C_p^- - C_p^+\right) dx_1$$

$$c_x = \int_{mm'} \left(C_p^+ \delta^+ - C_p^- \delta^-\right) dx_1$$

Designating by $C_p^{(1)}$ and $C_p^{(2)}$ the pressure coefficients obtained in the study of the symmetrical and lifting problems, the superposition of which gives the general problem investigated, one has

$$C_p^+ = C_p^{(1)} + C_p^{(2)} \qquad C_p^- = C_p^{(1)} - C_p^{(2)}$$

and consequently

$$c_z = c_z^{(2)}$$

$c_z^{(2)}$ being the local c_z of the lifting problem and

$$\begin{aligned} c_x &= \int_{mm'} C_p^{(1)}(\alpha^+ - \alpha^-)dx_1 + \int_{mm'} C_p^{(2)}(-2i + \alpha^+ + \alpha^-)dx_1 \\ &= 2\int_{mm'} C_p^{(1)}\alpha\, dx + 2\int_{mm'} C_p^{(2)}\left(-i + j_0\right)dx_1 \\ &= c_x^{(1)} + c_x^{(2)} \end{aligned}$$

$c_x^{(1)}$ and $c_x^{(2)}$ designating the local c_x of the symmetrical and lifting problems[43].

Designating by C_z and C_x the total-lift and drag coefficients, one will, of course, have

$$C_z = C_z^{(2)} \qquad C_x = C_x^{(1)} + C_x^{(2)}$$

One sees thus very clearly how a general problem is divided into a symmetrical and a lifting problem. One may say, figuratively speaking, that the symmetrical problem investigates "the effect of thickness" and that the lifting problem investigates "the effect of curvature and incidence." We shall treat these two problems successively.

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[43]One could put: $c_x^{(2)} = c'_x{}^{(2)} + ic_z$, noting that $c'_x{}^{(2)} = 2\int_{mm'} c_p^{(2)} j_0\, dx_1$. The local c_x is, therefore, the sum of $c_x^{(1)}$, drag due to the thickness, $c'_x{}^{(2)}$, drag due to the curvature, and of ic_z, drag due to the incidence (induced drag).

4.1.1 - Symmetrical Problem

4.1.1.1 - Rectangular wing with symmetrical profile and zero lift

4.1.1.1.1 - General remarks.- The projection of the wing is a rectangle (R): ABA'B' (compare fig. 57). We shall put

$$AA' = BB' = l \qquad AB = A'B' = \lambda l$$

The problem is to find a flow such as to make the value of the normal component ω zero at every point of the plane $x_3 = 0$, except in (R). Furthermore we shall, for a start, assume that the wing cross section is constant for the entire span. This profile, symmetrical according to hypothesis, will be defined by the function $\alpha(x_1)$ which gives the value of the inclination of the profile (supposed to be small) toward the axis of the x_1; ω will therefore assume the value $\omega^+ = \alpha(x_1)$ on the upper side $(x_3 > 0)$ of the rectangle ABB'A', and the opposite value $\omega^- = -\alpha(x_1)$ on the lower side $(x_3 < 0)$.

In order to solve the problem, we shall compose conical flows the vertices of which are situated on the sides AA' and BB'.

In order to simplify the notation, we shall call $\overrightarrow{C_s}(M,\alpha)$ the elementary symmetrical conical flow which has its vertex at a point M of the plane Ox_1x_2 (compare fig. 58) for which w is zero outside of the quadrant limited by the semi-infinite lines parallel to Ox_1 and Ox_2 issuing from M; w is equal to the constant α on the upper part of this quadrant and to $-\alpha$ on the lower part. $\overleftarrow{C_s}(M,\alpha)$ will designate an analogous flow where the axis Ox_2 will have been replaced by its symmetrical counterpart. Such a flow has been investigated in section 3.2.3.1. If one designates the angle $\widehat{x_1MP}$ by φ, the formulas (III.59) show that the pressure coefficient C_p is given by

$$C_p = \frac{2\alpha}{\beta}\left[\frac{1}{2} + \frac{1}{\pi}\,\text{Arc}\sin(\beta\tan\varphi)\right] \qquad \left|\beta\tan\varphi\right| < 1 \qquad \text{(IV.1)}$$

$$C_p = 0 \text{ if } \beta\tan\varphi < -1 \qquad C_p = \frac{2\alpha}{\beta} \text{ if } \beta\tan\varphi > 1$$

4.1.1.1.2 - General principle of the superposition.- Let us visualize, first of all, the superposition of the following flows

$$\vec{C}_s\left[A,\alpha(0)\right] \quad \text{and} \quad \overleftarrow{C}_s\left[B,\alpha(0)\right]$$

The resultant flow gives in the plane $x_3 = 0$ the values of w indicated by the figure 59(a). If we now subtract the two-dimensional flow about a dihedron of the angle $2\alpha(0)$, it is disposed symmetrically with respect to the plane Ox_1x_2 and has Ox_2 as edge; the semi-infinite Ox_1 is inside of the dihedron, and one obtains in the plane $x_3 = +0$ the values of w indicated by the figure 59(b). This gives us the principle of the composition. One will obtain the desired flow by superposing conical flows of the type $\vec{C}_s$ the vertices M of which will be situated on AA', conical flows of the type $\overleftarrow{C}_s$ the vertices of which will be situated on BB', and by subtracting suitable two-dimensional flows. It will be possible to schematize the flow in a precise manner as follows

$$\int_{AA'} \vec{C}_s(M,\ d\alpha) + \int_{BB'} \overleftarrow{C}_s(M,\ d\alpha) - E\left[\alpha(x_1)\right] \tag{IV.2}$$

with $E\left|\alpha(x_1)\right|$ designating the two-dimensional flow about a wing of infinite span the profile of which is identical with the profile of the given rectangular wing.

In fact, one verifies immediately that the flow, symbolically defined by the formula (IV.2), satisfies the given boundary conditions. We want, nevertheless, to specify that the integrals of this formula ought to be understood in the sense of Stieljes, in order to understand the case where the function $\alpha(x_1)$ will represent discontinuities of the first kind. Such discontinuities exist, in general, at the leading edge AB and at the trailing edge $A'B'$.

4.1.1.1.3 - Study of the flow $\int_{AA'} \vec{C}_s(M,\ d\alpha)$.- In order to make this investigation, we introduce the axes Axy, Ax parallel to Ox_1, Ay coinciding with Ox_2, and put

$$\frac{x}{l} = x^x \qquad \frac{\beta y}{l} = y^x \qquad \alpha^x\left(x^x\right) = \alpha(x)$$

The section of the Mach cone behind the point A is formed by two semi-infinites which have as equations

$$x^x \pm y^x = 0$$

Let (x^x,y^x) be the reduced coordinates of a point P of the plane Axy (fig. 60). We shall suppose $x^x < 1$. If $0 < x^x < y^x$, the point P is outside of the Mach cones behind all points M of the segment AA'; consequently, according to equation (IV.1)

$$C_p(x^x,y^x) = \frac{2}{\beta}\int_0^x d\alpha = \frac{2}{\beta}\int_0^{x^x} d\alpha^x = \frac{2}{\beta}\,\alpha^x(x^x)$$

If now $0 < y^x < x^x$, the point P is outside of the Mach cones of the points of the segment P_1P_0, but inside of the Mach cones of the points situated on AP_1, P_1 being the point of AA' of the abscissa $x^x - y^x$. Besides, the conical flows, the vertex of which is on P_0A', have no influence on the point P. Consequently, the pressure at the point P is written, according to equation (IV.1)

$$C_p = \frac{2}{\beta\pi}\int_0^{x^x-y^x}\left(\frac{\pi}{2} + \text{Arc sin}\,\frac{y^x}{x^x - \xi}\right)d\alpha^x(\xi) + \frac{2}{\beta}\int_{x^x-y^x}^{x^x} d\alpha^x(\xi)$$

or

$$\left.\begin{aligned} C_p &= \frac{1}{\beta}\left[2\alpha^x(x^x) - P(x^x,y^x)\right] \\ P(x^x,y^x) &= \alpha^x(x^x - y^x) - \frac{2}{\pi}\int_0^{x^x-y^x} \text{Arc sin}\,\frac{y^x}{x^x - \xi}\,d\alpha^x(\xi) \end{aligned}\right\} \qquad \text{(IV.3)}$$

This formula, set up for the case where $0 < y^x < x^x$, may be extended to the case already studied $0 < x^x < y^x$ since α may be considered zero for the negative values of the abscissa.

One can now calculate the drag of the section y^x

$$c_x\left(y^x\right) = 2\int_0^1 C_p\left(\xi, y^x\right)\alpha^x(\xi)d\xi$$

Consequently

$$c_x\left(y^x\right) = \frac{4}{\beta}\int_0^1 \alpha^{x2}(\xi)d\xi - \frac{2}{\beta}\int_{y^x}^1 \alpha^x(\xi)\alpha^x\left(\xi - y^x\right)d\xi +$$

$$\frac{4}{\beta\pi}\int_{y^x}^1 \alpha^x(\xi)d\xi\int_0^{\xi - y^x} \text{Arc sin}\,\frac{y^x}{\xi - \eta}\,d\alpha^x(\eta)$$

or, changing the order of integration in the last term and putting

$$\int_0^1 \alpha^{x2}(\xi)d\xi = \overline{\alpha}^2$$

$$\left.\begin{aligned} F\left(y^x\right) &= 2\int_{y^x}^1 \alpha^x(\xi)\alpha^x\left(\xi - y^x\right)d\xi - \\ &\quad \frac{4}{\pi}\int_0^{1-y^x} d\alpha^x(\eta)\int_{y^x+\eta}^1 \text{Arc sin}\,\frac{y^x}{\xi - \eta}\,\alpha^x(\xi)d\xi \\ c_x\left(y^x\right) &= \frac{4\overline{\alpha}^2}{\beta} - \frac{1}{\beta}F\left(y^x\right) \end{aligned}\right\} \qquad \text{(IV.4)}$$

According to our conventions, if $y^x \geqslant 1$

$$F\left(y^x\right) = 0 \qquad c_x\left(y^x\right) = \frac{4\overline{\alpha}^2}{\beta}$$

Such a section actually behaves like the section of a wing of infinite span which is quite obvious according to the rule of forbidden signals. We note in addition that

$$c_x(0) = \frac{2\overline{\alpha}^2}{\beta}$$

thus the drag of the section $y^x = 0$ is half the drag of the same section at infinite aspect ratio.

We want to point out another remarkable result

$$\int_0^1 c_x\left(y^x\right)dy^x = \frac{4\overline{\alpha}^2}{\beta} \qquad \text{(IV.5)}$$

that is, the mean value of the drag in the region $0 < y^x < 1$ where the $c_x\left(y^x\right)$ is not constant is equal to the value of the drag in infinite flow.

In fact, first of all

$$\int_0^1 dy^x \int_{y^x}^1 \alpha^x(\xi)\alpha^x\left(\xi - y^x\right)d\xi = \int_0^1 \alpha^x(\xi)d\xi \int_0^\xi \alpha^x\left(\xi - y^x\right)dy^x =$$

$$\int_0^1 \alpha^x(\xi)e^x(\xi)d\xi = \frac{1}{2}\left[e^x(\xi)\right]^2\Big|_0^1 = 0$$

if one puts

$$e^x(x) = \int_0^x \alpha^x(\xi)d\xi$$

On the other hand

$$\int_0^1 dy^x \int_{y^x}^1 \alpha^x(\xi)d\xi \int_0^{\xi - y^x} \text{Arc sin } \frac{y^x}{\xi - \eta}\, d\alpha^x(\eta) =$$

$$\int_0^1 \alpha^x(\xi)d\xi \int_0^{\xi} d\alpha^x(\eta) \int_0^{\xi - \eta} \text{Arc sin } \frac{y^x}{\xi - \eta}\, dy^x$$

If we put in the last integral

$$y^x = (\xi - \eta)\sin t$$

the preceding expression becomes equal to

$$\left(\frac{\pi}{2} - 1\right) \int_0^1 \alpha^x(\xi)d\xi \int_0^{\xi} (\xi - \eta)d\alpha^x(\eta) = \left(\frac{\pi}{2} - 1\right)\left[\int_0^1 \xi\alpha^{x2}(\xi)d\xi - \right.$$

$$\left.\int_0^1 \alpha^x(\xi)\left[\xi\alpha^x(\xi) - e^x(\xi)\right]d\xi\right] = 0$$

The formula (IV.5) is thus justified.

<u>4.1.1.1.4 - Study of the rectangular wing with constant profile</u>.- We shall call the quantity $\beta\lambda$, which we shall note $2\eta_0$, $(\beta\lambda = 2\eta_0)$, the "reduced aspect ratio" of a rectangular wing.

We shall designate by t the "reduced chord"

$$t = \frac{x_1}{l}$$

and we shall put (compare fig. 56)

$$\eta = \frac{\beta x_2}{l}$$

In applying the formula of composition (formula (IV.2)) one sees that the pressure coefficient at a point of the rectangular wing is given by

$$C_p(t,\eta) = \frac{2\alpha^x(t)}{\beta} - \frac{1}{\beta}\left[P\left(t,\eta_0 + \eta\right) + P\left(t,\eta_0 - \eta\right)\right] \tag{IV.6}$$

P is the function defined by the formula (IV.3). The drag of the section η is given by

$$c_x(\eta) = 4\,\frac{\overline{\alpha}^2}{\beta} - \frac{2}{\beta}\int_0^1 \left[P\left(t,\eta_0 + \eta\right) + P\left(t,\eta_0 - \eta\right)\right]\alpha^x(t)dt$$

However, by definition

$$2\int_0^1 P(t,u)\alpha^x(t)dt = F(u)$$

F being, besides, the function defined by equation (IV.4). Consequently

$$c_x(\eta) = \frac{4\overline{\alpha}^2}{\beta} - \frac{1}{\beta}\left[F\left(\eta_0 + \eta\right) + F\left(\eta_0 - \eta\right)\right] \tag{IV.7}$$

we remark that if $\eta_0 > 1$, that is, if $\lambda > \frac{2}{\beta}$, there is always at least one of the functions F zero; in this case the c_x of the sections close to the center is equal to $\frac{4\overline{\alpha}^2}{\beta}$. This is an immediate consequence of the principle of forbidden signals.

Now we can finally calculate the total drag which we shall fix by the coefficient

$$C_x = \frac{1}{2\eta_0}\int_{-\eta_0}^{\eta_0} c_x(\eta)d\eta$$

If one puts

$$\Phi(v) = \int_0^v F(u)du$$

one sees immediately that

$$C_x = \frac{4\bar{\alpha}^2}{\beta} - \frac{1}{\beta\eta_0}\Phi\left(2\eta_0\right) \tag{IV.8}$$

However, the result obtained by the formula (IV.5) amounts to stating that

$$\Phi(v) = 0 \quad \text{if} \quad v \geqslant 1$$

Consequently, the drag of a rectangular wing has a value independent of the aspect ratio and equal to $\frac{4\bar{\alpha}^2}{\beta}$, for geometrical aspect ratios λ greater than $\frac{1}{\beta}$.

Summarizing, one may say that the complete investigation of a symmetrical rectangular wing of zero lift amounts to calculating the functions P, F, Φ which are all calculated by quadratures.

<u>4.1.1.1.5 - Applications</u>.- 1. The profile is a rhomb; in this case

$$\alpha^x(t) = \alpha_0 \quad \text{if} \quad t < \frac{1}{2}$$

$$\alpha^x(t) = -\alpha_0 \quad \text{if} \quad t > \frac{1}{2} \qquad \bar{\alpha} = \alpha_0$$

We shall now calculate the function $F(y^x)$, defined by equation (IV.4). For this purpose we remark first that

$$\int_{y^x}^1 \alpha^x(\xi)\alpha\left(\xi - y^x\right)d\xi = \begin{cases} \alpha_0^2\left(1 - 3y^x\right) & \text{if} \quad 0 \leqslant y^x \leqslant \frac{1}{2} \\ -\alpha_0^2\left(1 - y^x\right) & \text{if} \quad \frac{1}{2} \leqslant y^x \leqslant 1 \end{cases}$$

There remains to be calculated

$$\int_0^{1-y^x} d\alpha^x(\eta) \int_{y^x+\eta}^{1} \text{Arc sin } \frac{y^x}{\xi - \eta} \alpha^x(\xi)d\xi$$

However,

$$\int_a^b \text{Arc sin } \frac{y^x}{\xi - \eta} d\xi = (b - \eta)\text{Arc sin } \frac{y^x}{b - \eta} - (a - \eta)\text{Arc sin } \frac{y^x}{a - \eta} +$$

$$y^x\left(\text{Arg ch } \frac{b - \eta}{y^x} - \text{Arg ch } \frac{a - \eta}{y^x}\right)$$

as one sees immediately, integrating by parts.

If $0 \leqslant y^x \leqslant \frac{1}{2}$, $\alpha^x(\eta)$ is subjected to two discontinuities, the first for $\eta = 0$, the contribution of which is

$$\alpha_0^{\,2}\left[\left(\frac{1}{2} \text{ Arc sin } 2y^x + y^x\text{Arg ch } \frac{1}{2y^x} - \frac{\pi}{2} y^x\right) - \left(\text{Arc sin } y^x - \right.\right.$$

$$\left.\left.\frac{1}{2} \text{ Arc sin } 2y^x\right) - y^x\left(\text{Arg ch } \frac{1}{y^x} - \text{Arg ch } \frac{1}{2y^x}\right)\right]$$

the second for $\eta = \frac{1}{2}$, the contribution of which is

$$2\alpha_0^{\,2}\left[\frac{1}{2} \text{ Arc sin } 2y^x - y^x \frac{\pi}{2} + y^x\text{Arg ch } \frac{1}{2y^x}\right]$$

If $y^x > \frac{1}{2}$, only the discontinuity for $\eta = 0$ comes into play, the contribution of which is

$$-\alpha_0^{\,2}\left[\text{Arc sin } y^x - y^x \frac{\pi}{2} + y^x\text{Arg ch } \frac{1}{y^x}\right]$$

If one assembles these partial results and puts

$$\left.\begin{aligned} Y\left(y^x\right) &= \frac{1}{\pi}\left(\text{Arc sin } y^x + y^x\text{Arg ch }\frac{1}{y^x}\right) && \text{for} \quad 0 < y^x < 1 \\ Y\left(y^x\right) &= \frac{1}{2} && \text{for} \quad y^x > 1 \end{aligned}\right\} \tag{IV.9}$$

one sees that one may write in a general manner

$$F\left(y^x\right) = 4\alpha_0^2\left[\frac{1}{2} + Y\left(y^x\right) - 2Y\left(2y^x\right)\right]$$

and consequently

$$c_x(\eta) = \frac{4\alpha_0^2}{\beta}\left[\frac{1}{2} - Y\left(\eta_0 + \eta\right) - Y\left(\eta_0 - \eta\right) + 2Y\left(2\eta_0 + 2\eta\right) + 2Y\left(2\eta_0 - 2\eta\right)\right] \tag{IV.10}$$

Figure 61 gives the variation of $c_x(\eta)$ for two values of η_0.

For knowing, finally, the total drag it suffices to calculate the function $\Phi(u)$.

Now

$$\int_0^u Y\left(y^x\right)dy^x = \frac{1}{2}\, u\, D(u)$$

with

$$\begin{aligned} D(u) &= \frac{1}{2}\left[\frac{\sqrt{1 - u^2}}{u} + 2\text{ Arc sin } u + u\text{ Arg ch }\frac{1}{u}\right] && \text{if} \quad 0 < u \leqslant 1 \\ D(u) &= 1 && \text{if} \quad u \geqslant 1 \end{aligned}$$

Hence

$$\Phi(u) = 4\alpha_0^2\left[\frac{u}{2}\left[1 + D(u)\right] - u\,D(2u)\right] \qquad (IV.11)$$

Consequently, applying equation (IV.8), one obtains

$$C_x = \frac{4\alpha_0^2}{\beta}\left[2\,D\left(4\eta_0\right) - D\left(2\eta_0\right)\right] \qquad (IV.12)$$

One will find in figure 62 the curve giving C_x as a function of the reduced aspect ratio.

The curves of the figures 61 and 62 have already been given by Th. Von Kármán (ref. 4), but this author does not give any analytical formula. Moreover it seems as if the results Th. Von Kármán's had been obtained by application of the method of "acoustic analogy." The curve given in figure 61 may also be found in a memorandum of Lighthill (ref. 31) who utilized the method of sources.

2. The profile is formed by two symmetrical parabolic arcs; in this case one must put

$$\alpha^x(t) = \epsilon_0(1 - 2t)$$

ϵ_0 characterizes the thickness of the profile.

The problem consists in calculating the functions $F\left(y^x\right)$ and $\Phi(v)$ defined in the previous paragraph. One finds after a few integrations of elementary character

$$F\left(y^x\right) = \frac{4\epsilon_0^2}{3\pi}\left[\text{Arc cos } y^x + y^x\left(y^{x2} - 3\right)\text{Arg ch } \frac{1}{y^x} + y^x\sqrt{1 - y^{x2}}\right] \qquad (IV.13)$$

and

$$\Phi(v) = \frac{4\epsilon_0^2}{3\pi}\left[\left(\frac{y^{x4}}{4} - \frac{3y^{x2}}{2}\right)\text{Arg ch } \frac{1}{y^x} + y^x\text{Arc cos } y^x + \frac{y^{x2}\sqrt{1 - y^{x2}}}{4}\right] \qquad (IV.14)$$

On the other hand

$$\bar{\alpha}^2 = \frac{\epsilon_0^{\ 2}}{3}$$

One can clearly verify that

$$F(0) = \frac{2\epsilon_0^{\ 2}}{3} = 2\bar{\alpha}^2 \qquad F(1) = 0 \qquad \Phi(1) = 0$$

In figures 63 and 64 one will find the distribution of c_x over the span for a wing of reduced aspect ratio $2\eta_0 = 2$, and the variation of C_x (total-drag coefficient) as a function of the aspect ratio.

4.1.1.1.6 - Case where the profile is variable in span.- It is possible to calculate the symmetrical rectangular wing at zero lift in the general case where the profile is variable in span. We shall here be satisfied to examine the relatively simple case where the profiles along the span are deduced from one another by affinity; the ratio of the affinity varies with the span. We shall assume that the wing of reduced span $2\eta_0$ has a local inclination of the form $k(\eta)\alpha^x(t)$ at a point of reduced coordinates η, t.

The function $k(\eta)$ must of course satisfy the usual limitations so that the problem posed can be treated by means of linear approximations. Finally, we shall assume the function $k(\eta)$ to be even in η.

Let us first of all remark that the wing of reduced span 2η, the profile of which (which is constant along the entire span) is defined by the function $\alpha^x(t)$, causes outside of the wing, at a point of reduced coordinates t, $y^x\left(y^x > \eta\right)$, a pressure coefficient

$$C_p\left(t,y^x\right) = \frac{1}{\beta}\left[P\left(t,y^x - \eta\right) - P\left(t,y^x + \eta\right)\right] \tag{IV.15}$$

P is the function defined by equation (IV.3) as one sees reassuming the arguments of the sections 4.1.1.1.3 and 4.1.1.1.4.

One will now obtain the desired boundary conditions by superposing a succession of rectangular wings which are symmetrical with respect to O_1x_1, of equal chord reduced to 1 and of variable reduced span $2\eta(0 < \eta < \eta_0)$ for which the profile remains constant in span. This is justified since $k(\eta)$ had been assumed to be even.

At a point (t,y^x) the pressure coefficient is written

$$C_p(t,y^x) = \frac{1}{\beta}\left[2\alpha^x(t)k(y^x) + \int_0^{\eta_0} P(t,\eta + y^x)dk(\eta) + \int_{y^x}^{\eta_0} P(t,\eta - y^x)dk(\eta) - \int_0^{y^x} P(t,y^x - \eta)dk(\eta)\right]$$

All these integrals are taken in the sense of Stieljes.

One will obtain a simpler formula by putting

$$\underline{P}(t,v,\eta_0) = -\int_0^{\eta_0} P\left[t,\epsilon(\eta - v)\right]\epsilon\ dk(\eta) \qquad \text{(IV.16)}$$

ϵ being defined by the equality

$$\epsilon(\eta - v) = \left|\eta - v\right|$$

In this case

$$C_p(t,y^x) = \frac{2}{\beta}\alpha^x(t)k(y^x) - \frac{1}{\beta}\left[\underline{P}(t,y^x,\eta_0) + \underline{P}(t,-y^x,\eta_0)\right] \qquad \text{(IV.17)}$$

This formula is reduced to the formula (IV.6) in the case where $k(\eta) = 1$ over the entire span.

The drag of the section y^x is easily obtained

$$c_x(y^x) = \frac{4}{\beta}k^2(y^x)\overline{\alpha}^2 - \frac{k(y^x)}{\beta}\left[\underline{F}(y^x,\eta_0) + \underline{F}(-y^x,\eta_0)\right] \qquad \text{(IV.18)}$$

by putting

$$\underline{F}(v,\eta_0) = 2\int_0^1 \underline{P}(t,v,\eta_0)\,\alpha^x(t)dt$$

$$= -\int_0^1 \alpha^x(t)dt\int_0^{\eta_0} P\left[t,\epsilon(\eta - v)\right]\epsilon\, dk(\eta)$$

$$= -\int_0^{\eta_0} \epsilon\, dk(\eta)\int_0^1 P\left[t,\epsilon(\eta - v)\right]\alpha^x(t)dt$$

whence the formula

$$\underline{F}(v,\eta_0) = -\int_0^{\eta_0} F\left[\epsilon(\eta - v)\right]\epsilon\, dk(\eta) \qquad \text{(IV.19)}$$

F is the function defined by the equality (IV.4).

Thus one can see that the pressure coefficient and the local-drag coefficient are expressed by formulas analogous to those obtained in the case where the profile is constant under the condition that the functions $P\left(t,y^x\right)$ and $F\left(y^x\right)$ are replaced by weighted averages, $\underline{P}\left(t,v,\eta_0\right)$ and $\underline{F}\left(v,\eta_0\right)$, defined by the formulas (IV.16) and (IV.19).

Finally, the total-drag coefficient is obtained immediately

$$C_x = \frac{1}{2\eta_0}\int_{-\eta_0}^{\eta_0} c_x\left(y^x\right)dy^x = \frac{1}{\eta_0}\int_0^{\eta_0} c_x\left(y^x\right)dy^x$$

whence

$$C_x = \frac{4}{\beta}\,\overline{K}^2\overline{\alpha}^2 - \frac{1}{\beta}\int_0^{\eta_0}\left[\underline{F}\left(y^x,\eta_0\right) + \underline{F}\left(-y^x,\eta_0\right)\right]k\left(y^x\right)dy^x$$

As an example, we shall suppose $k(\eta)$ to be defined by

$$k(\eta) = 1 - \frac{\eta}{\eta_0} \qquad \left(0 \leqslant \eta \leqslant \eta_0\right)$$

One will then have

$$\underline{F}(v,\eta_0) = -\int_0^{\eta_0} F\left[\epsilon(\eta - v)\right] \epsilon \, dk(\eta) = +\int_0^{\eta_0} F\left[\epsilon(\eta - v)\right] \frac{\epsilon}{\eta_0} \, d\eta$$

If v is positive

$$\underline{F}(v,\eta_0) = -\frac{1}{\eta_0}\int_0^{v} F(v - \eta)d\eta + \frac{1}{\eta_0}\int_v^{\eta_0} F(\eta - v)d\eta$$

$$= -\frac{1}{\eta_0}\int_0^{v} F(u)du + \frac{1}{\eta_0}\int_0^{\eta_0 - v} F(u)du$$

$$= \frac{1}{\eta_0}\left[\Phi\left(\eta_0 - v\right) - \Phi(v)\right]$$

$\Phi(v) = \int_0^{v} F(u)du$ being the function introduced before in section 4.1.1.1.4.

If v is negative: $v = -v'$

$$\underline{F}(v,\eta_0) = \frac{1}{\eta_0}\int_0^{\eta_0} F(\eta + v')d\eta$$

$$= \frac{1}{\eta_0}\int_{v'}^{\eta_0 + v'} F(u)du$$

$$= \frac{1}{\eta_0}\left[\Phi\left(\eta_0 - v\right) - \Phi(-v)\right]$$

whence

$$c_x(y^x) = \frac{4}{\beta}\left(1 \frac{y^x}{\eta_0}\right)^2 \bar{\alpha}^2 - \left(1 - \frac{y^x}{\eta_0}\right)\frac{1}{\beta\eta_0}\left[\Phi\left(\eta_0 - y^x\right) + \Phi\left(\eta_0 + y^x\right) - 2\Phi\left(y^x\right)\right] \tag{IV.20}$$

Let us recall that $\Phi(v) = 0$, if $v \geqslant 1$.

It is then easy to make applications of this formula in the case where the profile is a rhomb or lenticular formed by zero parabolic arcs.

One will find the curve which gives in the first case the variation of c_x as a function of y^x, for a reduced aspect ratio $\eta_0 = 2$, in figure 65.

4.1.1.2 - Study of the sweptback wing with constant profile

Without investigating the sweptback wing as thoroughly as the rectangular wing, we shall show that one may, without essential difficulty, apply the method used for study of the rectangular wing for the sweptback wing of constant profile the plan-form of which is schematized in figure 66. We shall suppose that the plane Ox_1x_3 is a symmetry plane for the wing. With γ designating the angle of sweepback, it is obvious that we shall have flows of different type according to whether the leading edge AOB will be outside or inside the Mach cone of O. One has become accustomed to say that in the first case the leading edge is "supersonic" while it is "subsonic" in the second case, thus recalling that the velocity component normal to the leading edge is higher than sonic velocity in the first case, lower in the second. The number ν, defined by: $\beta \cot \gamma = \frac{1}{\nu}$, ($\nu < 1$ characterizes the case where the leading edge is outside of the Mach cone, $\nu > 1$, in contrast, the case where it is inside) will, therefore, be an essential parameter in the investigation of sweptback wings.

4.1.1.2.1 - Case where $\nu < 1$.- We shall put in this case $\nu = \cos \theta$. We shall define, as for the rectangular wing, "the reduced aspect ratio" $2\eta_0$ (compare figure 66) by the relation

$$2\eta_0 = \beta\lambda$$

if λ designates the span of the wing taken along Ox_2.

For simplification we shall assume that the profile chord is taken as length unit, and that the profile is defined by the function $\alpha(x_1)$, with x_1 varying from 0 to 1. It is obvious that the desired flow will be obtained by a superposition of elementary conical flows which one may note schematically

$$\int_{OO'} C_s(M,d\alpha,\theta) - \int_{BB'} \overrightarrow{C_s}(M,d\alpha,\theta) - \int_{AA'} \overleftarrow{C_s}(M,d\alpha,\theta)$$

$C_s(M,d\alpha,\theta)$ designates a flow completely bisecting the Mach cone, admitting the plane Ox_1x_3 as symmetry plane (section 3.2.2); $\overrightarrow{C_s}(M,d\alpha,\theta)$ designates a flow partially inside of the Mach cone; the sign $\longrightarrow$ indicates the direction of the bounding generatrix which forms with Ox_2 the angle γ; the other bounding generatrix is supposed to be parallel to the wind. Because of the symmetry it will be sufficient to study the region of the wing where $x_2 > 0$.

It will be convenient to put

$$y^x = \beta x_2$$

$$x_1 = x + y^x \cos \theta$$

A conical flow with the vertex $M_0\left(x_1 = \xi, \quad x_2 = 0\right)$, of the type

$$C_s\left(M_0,\alpha,\theta\right)$$

causes (compare formula III.53) the following pressure field in the region $y^x > 0$:

$$\left.\begin{aligned}
&C_p = \frac{4\alpha}{\beta\pi}\,\frac{1}{\sin\theta}\,\text{Arc sin}\left[\frac{\sin\theta}{\sqrt{1 - t^2\cos^2\theta}}\right] && \text{if} \quad 0 < t < 1 \\
&t \text{ being defined by } t = \frac{y^x}{x - \xi + y^x\cos\theta} && \\
&C_p = \frac{2\alpha}{\beta\sin\theta} && \text{if} \quad 1 < t < \frac{1}{\cos\theta} \\
&C_p = 0 && \text{if} \quad t > \frac{1}{\cos\theta}
\end{aligned}\right\}$$

At a point (x,y^x) the pressure coefficient due to the flows $\int_{OO'} C_s(M,d\alpha,\theta)$ is equal to

$$\frac{2}{\beta\sin\theta}\left[\alpha(x) - \alpha\left[x - y^x(1 - \cos\theta)\right] + \right.$$

$$\left.\frac{2}{\pi}\int_0^{x-y^x(1-\cos\theta)} \text{Arc sin}\,\frac{\sin\theta\left(x - \xi + y^x\cos\theta\right)}{\sqrt{(x - \xi)\left(x - \xi + 2y^x\cos\theta\right)}}\,d\alpha(\xi)\right] =$$

$$\frac{2}{\beta\sin\theta}\left[\alpha(x) - Q\left(x,y^x,\theta\right)\right]$$

putting

$$Q\left(x,y^{x},\theta\right) = \alpha\left[x - y^{x}(1 - \cos\theta)\right] -$$

$$\frac{2}{\pi}\int_{0}^{x-y^{x}(1-\cos\theta)} \text{Arc sin}\ \frac{\sin\theta\left(x - \xi + y^{x}\cos\theta\right)}{\sqrt{(x - \xi)\left(x - \xi + 2y^{x}\cos\theta\right)}}\, d\alpha(\xi) =$$

$$\frac{2}{\pi}\int_{0}^{x-y^{x}(1-\cos\theta)} \text{Arc cos}\ \frac{\left[\sin\theta\left(x - \xi + y^{x}\cos\theta\right)\right]}{\sqrt{(x - \xi)\left(x - \xi + 2y^{x}\cos\theta\right)}}\, d\alpha(\xi)$$

Let us note that

$$Q\left(x,y^{x},\theta\right) = 0 \quad \text{if} \quad y^{x}(1 - \cos\theta) > x$$

and that the same holds true also in the case where the sweepback is zero $\left(\theta = \frac{\pi}{2}\right)$.

For simplification, we shall henceforward assume $\eta_0 > \frac{1}{1 + \cos\theta}$ (which will always be verified if $\eta_0 > 1$), that is, that the edge AA' has no influence whatsoever on the wing region $x_2 > 0$.

The contribution due to the flows $\int_{BB'} \vec{C}(M,d\alpha,\theta)$ is very easily obtained from the formula (III.58). The pressure coefficient due to these flows may immediately be written

$$C_p = P\left(x,\eta_0 - y^{x},\theta\right)\frac{1}{\beta\sin\theta}$$

if one puts

$$P\left(x,y^x,\theta\right) = \alpha\left[x - y^x(1 + \cos\,\theta)\right] -$$

$$\frac{2}{\pi}\int_0^{x^x-y^x(1+\cos\theta)} \text{Arc sin}\left[\frac{y^x\sin^2\theta}{x - \xi} - \cos\,\theta\right]d\alpha(\xi) \tag{IV.22}$$

If $\theta = \frac{\pi}{2}$, one falls back on the function P defined by the formula (IV.3); on the other hand, $P\left(x,y^x,\theta\right)$ is obviously zero if

$$y^x(1 + \cos\,\theta) > x$$

Finally, with the reservation that

$$\eta_0 > \frac{1}{1 + \cos\,\theta}$$

one has at a point of the wing

$$C_p = \frac{1}{\beta\,\sin\,\theta}\left[2\alpha^x(x) - 2Q\left(x,y^x,\theta\right) - P\left(x,\eta_0 - y^x,\theta\right)\right] \tag{IV.23}$$

The local-drag coefficient is immediately obtained

$$c_x\left(y^x\right) = \frac{4\overline{\alpha}^2}{\beta\,\sin\,\theta} - \frac{4}{\beta\,\sin\,\theta}\,G\left(y^x,\theta\right) - \frac{1}{\beta\,\sin\,\theta}\,F\left(\eta_0 - y^x,\theta\right) \tag{IV.24}$$

putting

$$G\left(y^x,\theta\right) = \frac{2}{\pi}\int_{y^x(1-\cos\theta)}^{1} \alpha(x)dx \int_{0}^{x-y^x(1-\cos\theta)} \text{Arc cos}\ \frac{\sin\theta\left(x - \xi + y^x\cos\theta\right)}{\sqrt{(x - \xi)\left(x - \xi + 2y^x\cos\theta\right)}}\, d\alpha(\xi) \tag{IV.25}$$

$$F\left(y^x,\theta\right) = 2\int_{y^x(1+\cos\theta)}^{1} \alpha(x)\alpha\left[x - y^x(1 + \cos\theta)\right]dx -$$

$$\frac{4}{\pi}\int_{y^x(1+\cos\theta)}^{1} \alpha(x)dx \int_{0}^{x-y^x(1+\cos\theta)} \text{Arc sin}\left[\frac{y^x\sin^2\theta}{x - \xi} - \cos\theta\right]d\alpha(\xi) \tag{IV.26}$$

The calculation of the total drag offers no difficulties. We shall content ourselves with the following remarks:

1.

$$\int_{0}^{\frac{1}{1+\cos\theta}} F\left(y^x,\theta\right)dy^x = 0$$

This result is established in the same manner as in the case where $\theta = \frac{\pi}{2}$ (compare section 4.1.1.1.3).

It signifies that the effect of limitation of the span does not modify the total drag.

2.

$$\int_0^{\frac{1}{1-\cos\theta}} G\left(y^x,\theta\right)dy^x = 0$$

In fact, this expression is equal to

$$\int_0^1 \alpha(x)dx \int_0^x d\alpha(\xi) \int_0^{\frac{x-\xi}{1-\cos\theta}} \text{Arc cos } \frac{\sin\theta\left(x - \xi + y^x\cos\theta\right)}{\sqrt{(x - \xi)\left(x - \xi + 2y^x\cos\theta\right)}} dy^x$$

The last integral is written

$$(x - \xi)\int_0^1 \text{Arc cos } \frac{\sin\theta}{\sqrt{1 - t^2\cos^2\theta}} \frac{dt}{(1 - t\cos\theta)^2}$$

If one puts

$$t = \frac{y^x}{x - \xi + y^x\cos\theta}$$

the result is then immediate. It signifies that if $\eta_0 > \frac{1}{1 - \cos\theta}$, the drag of the investigated wing is identical with that of the yawed wing of infinite span

$$C_x = \frac{4\bar{\alpha}^2}{\beta\sin\theta} \tag{IV.27}$$

3. If $\theta = 0$ (the leading edge is situated on the Mach cone of 0), the given formulas present an indeterminate form. Nevertheless it is very easy to eliminate the indetermination. We shall, in particular, calculate the total drag. The value we shall obtain is very interesting

because it corresponds for a given sweptback wing to the maximum of the total drag when the Mach number varies.

If θ tends toward zero,

$$\frac{1}{\sin\theta} \text{Arc sin} \frac{\sin\theta}{\sqrt{1 - t^2\cos^2\theta}}$$

has as a limit

$$\frac{1}{\sqrt{1 - t^2}} = \frac{x - \xi + y^x}{\sqrt{(x - \xi)(x - \xi + 2y^x)}}$$

We assume $\eta_0 > \frac{1}{2}$; since our purpose is calculation of the total drag, the edge BB' may be neglected. The desired total drag which we shall denote by $C_{x_{max}}$ is written

$$C_{x_{max}} = \frac{8}{\beta\pi\eta_0}\int_0^{\eta_0} dy^x \int_0^1 \alpha(x)dx \int_0^x \frac{x - \xi + y^x}{\sqrt{(x - \xi)(x - \xi + 2y^x)}} d\alpha(\xi)$$

$$= \frac{4}{\beta\pi\eta_0}\int_0^1 \alpha(x)dx \int_0^x d\alpha(\xi) \int_0^{\eta_0} \frac{x - \xi + y^x}{\sqrt{(x - \xi)(x - \xi + 2y^x)}} dy^x$$

whence, carrying out the last integration

$$C_{x_{max}} = \frac{4}{3\beta\pi\eta_0}\int_0^1 \alpha(x)dx \int_0^x \sqrt{\frac{x - \xi + 2\eta_0}{x - \xi}}\left[2(x - \xi) + \eta_0\right]d\alpha(\xi) \tag{IV.28}$$

One thus obtains a very simple formula giving the value of the total C_x when $\theta = 0$.

If the profile is a rhomb, one sees, writing

$$C_{x_{max}} = \frac{8}{3\pi\beta\eta_0}\int_0^1 d\alpha(\xi)\int_\xi^1 \sqrt{\frac{x - \xi + 2\eta_0}{x - \xi}}\left[2(x - \xi) + \eta_0\right]\alpha(x)dx$$

that

$$C_{x_{max}} = \frac{4\alpha_0{}^2}{\beta}\,\frac{2}{3\pi\eta_0}\left[4\Phi\left(\frac{1}{2}\right) - \Phi(1)\right]$$

putting

$$\Phi(u) = \int_0^u \sqrt{\frac{x + 2\eta_0}{x}}\left(2x + \eta_0\right)dx = u^{\frac{1}{2}}\left(u + 2\eta_0\right)^{\frac{3}{2}}$$

whence

$$C_{x_{max}} = \left(C_x\right)_\infty \frac{2}{3\pi}\left[\frac{\left(1 + 4\eta_0\right)^{\frac{3}{2}} - \left(1 + 2\eta_0\right)^{\frac{3}{2}}}{\eta_0}\right] \tag{IV.29}$$

In figure 67 one will find the variation of $\left(C_x\right)_{max}$ as a function of η_0.

If the profile is formed by two parabolic arcs,

$$\alpha(x) = \epsilon_0(1 - 2x)$$

and

$$C_{x_{max}} = \frac{8\epsilon_0{}^2}{3\beta\pi\eta_0}\int_0^1 (1 - 2x)\left[\sqrt{\frac{x + 2\eta_0}{x}}\left(2x + \eta_0\right) - 2\Phi(x)\right]dx$$

or

$$C_{x_{max}} = \frac{8\epsilon_0^2}{3\beta\pi\eta_0}\left[4I(1) - 6J(1) + 6\eta_0 K(1) + \Phi(1)\right]$$

putting

$$I(x) = \int_0^x x^{\frac{3}{2}}(x + 2\eta_0)^{\frac{3}{2}}dx = \frac{1}{4}\sqrt{x(x + 2\eta_0)}\left[(x + 2\eta_0)^2(x - \eta_0) + \frac{\eta_0^2}{2}(x + 5\eta_0)\right] + \frac{3\eta_0^4}{8}\log\frac{x + \eta_0 + \sqrt{x(x + 2\eta_0)}}{\eta_0}$$

$$J(x) = \int_0^x x^{\frac{1}{2}}(x + 2\eta_0)^{\frac{3}{2}}dx = \frac{1}{3}\sqrt{x(x + 2\eta_0)}\left[(x + 2\eta_0)^2 - \frac{\eta_0}{2}(x + 2\eta_0) - \frac{3\eta_0^2}{2}\right] - \frac{\eta_0^3}{2}\log\frac{\left(\eta_0 + x + \sqrt{(x + 2\eta_0)x}\right)}{\eta_0}$$

$$K(x) = \int_0^x x^{\frac{1}{2}}(x + 2\eta_0)^{\frac{1}{2}}dx = \frac{1}{2}\sqrt{x(x + 2\eta_0)}(x + \eta_0) - \frac{\eta_0^2}{2}\log\frac{\eta_0 + x + \sqrt{x(x + 2\eta_0)}}{\eta_0}$$

Hence

$$C_{x_{max}} = \frac{4\epsilon_0^2}{\beta\pi}\left[\sqrt{1 + 2\eta_0}\left(\eta_0 + \frac{2}{3}\right)(1 - \eta_0) + \eta_0^3\log\frac{1 + \eta_0 + \sqrt{1 + 2\eta_0}}{\eta_0}\right] \qquad \text{(IV.30)}$$

One will find the corresponding curve in figure 68.

4.1.1.2.2 - Case where $\nu > 1$.- We shall begin by examining the case of an infinite half-wing inside slip (compare figure 69).

It is convenient to put

$$\nu = \frac{1 + c^2}{2c}$$

The flow is obtained by a superposition of conical flows symbolized by

$$\int_{00'} \vec{C}_s(M,d\alpha,c) \tag{IV.31}$$

with $\vec{C}_s(M,d\alpha,c)$ designating the elementary flow investigated in section 3.1.2.2. in the case where $b = 0$. If one puts

$$t = \frac{y^x}{x - \xi + \nu y^x} = \frac{2\rho}{1 + \rho^2} \tag{IV.32}$$

the pressure coefficient is given by the formula (III.23) which may also be written

$$C_p = \frac{2\alpha}{\beta\pi\sqrt{\nu^2 - 1}} \log\left|\frac{1 - c\rho}{c - \rho}\right|$$

This formula is valid for $|\rho| < 1$. If $|\rho| > 1$, one has $C_p = 0$. One sees immediately the essential difference compared to the cases investigated before: a conical flow with the vertex $(\xi 0)$ can influence a point (x,y^x) for which $x < \xi$. In particular, the trailing edge will play a role in the calculation of the pressure. Finally, if $x = \xi$, $\rho = c$, the C_p of the corresponding conical flow becomes infinite. If the method remains exactly the same, one must also expect a few additional difficulties.

The pressure coefficient at a point of the wing will therefore be written

$$C_p(x,y^x) = \frac{2}{\beta\pi\sqrt{\nu^2 - 1}} \int_0^{x+(\nu-1)y^x} \log\left|\frac{1 - c\rho}{c - \rho}\right| d\alpha(\xi)$$

ρ is of course defined by the equality (IV.32).

The c_x of the section y^x is then written

$$c_x(y^x) = \frac{4}{\beta\pi\sqrt{\nu^2 - 1}} \int_0^1 \alpha(t)dt \int_0^{t+(\nu-1)y^x} \log\left|\frac{1 - c\rho}{c - \rho}\right| d\alpha(\xi)$$

One will notice that, for $y^x = 0$, $\rho = 0$; and consequently

$$C_p(x,0) = \frac{2\alpha(x)}{\beta\pi\sqrt{\nu^2 - 1}} \log\frac{1}{c} = \frac{2\alpha(x)}{\beta\pi\sqrt{\nu^2 - 1}} \log\left[\nu + \sqrt{\nu^2 - 1}\right]$$

As in the case of a wing of infinite span, the c_x depends only on the local inclination of the profile. Likewise

$$c_x(0) = \frac{4\overline{\alpha}^2}{\beta} \frac{\log\left(\nu + \sqrt{\nu^2 - 1}\right)}{\pi\sqrt{\nu^2 - 1}} \tag{IV.33}$$

The calculation of the function $c_x(y^x)$, for $y^x \neq 0$, presents no theoretical difficulty whatsoever. We shall now calculate the drag of the infinite half-wing, and shall show that it is finite in spite of the infinite dimensions of the wing. Assuming X to be this total drag, we shall put

$$X = \frac{1}{2}\rho\left|\vec{U}\right|^2 C_x$$

Our purpose is the calculation of C_x.

The desired value of C_x will be the limit, if it exists, of the integral

$$I\left(y_0^x\right) = \frac{4}{\pi\beta^2 \sqrt{\nu^2 - 1}} \int_0^{y_0^x} dy^x \int_0^1 \alpha(x)dx \int_0^{x+(\nu-1)y^x} \log\left|\frac{1 - c\rho}{c - \rho}\right| d\alpha(\xi)$$

when y_0^x increases indefinitely.

In order to calculate this triple integral, we shall replace the ensemble of the variables y^x,x,ξ by the variables x,ξ,ρ; the functional determinant $\frac{D\left(y^x,x,\xi\right)}{D(x,\xi,\rho)}$ is equal to

$$\frac{dy^x}{d\rho} = \frac{2c^2(x - \xi)\left(1 - \rho^2\right)}{(\rho - c)^2(1 - \rho c)^2}$$

This expression one obtains from equation (IV.32) if one writes this equality in the form

$$y^x = \frac{2\rho c(x - \xi)}{(c - \rho)(1 - \rho c)}$$

The volume in which the triple integral must be calculated is represented in figure 70. One can write

$$I\left(y_0{}^x\right) = \frac{4}{\beta^2\pi\sqrt{\nu^2 - 1}}\int_0^1 \alpha(x)dx \int_0^x (x - \xi)d\alpha(\xi) \int_0^{\rho_0(x-\xi)} \log\left|\frac{1 - c\rho}{c - \rho}\right| \frac{\left(1 - \rho^2\right)2c^2}{(\rho - c)^2(1 - c\rho)^2}\, d\rho \; +$$

$$\int_0^1 \alpha(x)dx \int_x^1 (\xi - x)d\alpha(\xi) \int_{\rho_1(\xi-x)}^1 \log\left|\frac{1 - c\rho}{c - \rho}\right| \frac{\left(1 - \rho^2\right)2c^2}{(\rho - c)^2(1 - c\rho)^2}\, d\rho \qquad \text{(IV.34)}$$

$\rho_0(x - \xi)$ and $\rho_1(\xi - x)$ are defined, respectively, by the equalities

$$\left.\begin{aligned} \frac{2\rho_0}{1 + \rho_0{}^2} &= \frac{y_0{}^x}{x - \xi + \nu y_0{}^x} \qquad x > \xi \\ \frac{2\rho_1}{1 + \rho_1{}^2} &= \frac{y_0{}^x}{x - \xi + \nu y_0{}^x} \qquad \xi > x \end{aligned}\right\}$$

Then one will have to make $y_0{}^x$ tend toward infinity. Under these conditions, ρ_0 and ρ_1 tend toward c; but it is impossible to make the transition to the limit brusquely because the triple integral then assumes indeterminate values. We remark likewise that, if the two limits ρ_0 and ρ_1 are replaced by two constant numbers, one smaller than c and the other larger than c, the triple integrals of equation (IV.34) will be zero because

$$\int_0^1 \alpha(x)dx \int_0^x (x - \xi)d\alpha(\xi) = 0$$

$$\int_0^1 \alpha(x)dx \int_x^1 (\xi - x)d\alpha(\xi) = 0$$

Since one wants to calculate the limit, if it exists, of $I\left(y_0{}^x\right)$, one will utilize the limited developments. Let us put

$$\rho = c(1 + r)$$

$$\log\left|\frac{1 - c\rho}{c - \rho}\right| = \log\left|\frac{1 - c^2}{cr}\right| - \frac{c^2 r}{1 - c^2} + \ . \ . \ . \ \frac{2c^2\left(1 - \rho^2\right)}{\left[(c - \rho)(1 - \rho c)\right]^2}$$

$$= \frac{2}{1 - c^2}\,\frac{1}{r^2}\left(1 + kr^2 + \ . \ . \ .\right)$$

We designate the values of r corresponding to ρ_0 and ρ_1 by r_0 and r_1; in the integrals

$$\int^{\rho_0} \log\left|\frac{1 - c\rho}{c - \rho}\right| \frac{2c^2\left(1 - \rho^2\right)}{(\rho - c)^2(1 - \rho c)^2}\, d\rho$$

and

$$\int_{\rho_1} \log\left|\frac{1 - c\rho}{c - \rho}\right| \frac{2c^2\left(1 - \rho^2\right)}{(\rho - c)^2(1 - \rho c)^2}\, d\rho$$

one may neglect the terms which are constant with respect to x and ξ, or infinitely small with respect to r_0 and r_1. Thus there is every reason to maintain only

$$\int^{r_0}\left[\frac{2}{1 - c^2}\frac{1}{r^2}\log\left|\frac{1 - c^2}{cr}\right| - \frac{2c^2}{\left(1 - c^2\right)^2}\frac{1}{r}\right]c\ dr$$

and

$$\int_{r_1}\left[\frac{2}{1 - c^2}\frac{1}{r^2}\log\left|\frac{1 - c^2}{cr}\right| - \frac{2c^2}{\left(1 - c^2\right)^2}\frac{1}{r}\right]c\ dr$$

which gives for the first

$$\frac{2c}{1 - c^2}\left[\frac{1}{r_0}\log\left|\frac{cr_0}{1 - c^2}\right| + \frac{1}{r_0} - \frac{2c^3}{\left(1 - c^2\right)^2}\log\left|r_0\right|\right]$$

and an analogous expression for the integral $\int_{r_1}$.

But if one puts

$$\frac{x - \xi}{y_0{}^x} = \epsilon$$

$$r = \frac{\rho - c}{c} = -\frac{\epsilon}{\sqrt{\nu^2 - 1}}\left(1 + \frac{\epsilon}{2c}\frac{1}{\nu^2 - 1} + \ldots\right)$$

and one obtains an expression of the form

$$\frac{A}{\epsilon}\log\epsilon + \frac{B}{\epsilon} + C\log\epsilon + \ldots$$

the dots indicate terms not infinitely large which may be neglected, according to a remark made before.

The term in $\frac{1}{\epsilon}\log\epsilon$ gives in the first triple integral

$$y_0^x\int_0^1 \alpha(x)dx\int_0^x d\alpha(\xi)\log\left|x - \xi\right|$$

and in the second (the one which corresponds to ρ_1)

$$y_0^x\int_0^1 \alpha(x)dx\int_x^1 d\alpha(\xi)\log\left|x - \xi\right|$$

Hence, summing up

$$y_0^x\int_0^1 \alpha(x)dx\int_0^1 \log\left|x - \xi\right| d\alpha(\xi) = y_0^x\int_0^1\int_0^1 \frac{\alpha(x)\alpha(\xi)}{x - \xi}\,dx\,d\xi = 0$$

The term in $\frac{1}{\epsilon}$ brings into the first integral the contribution

$$y_0^x\int_0^1 \alpha(x)dx\int_0^x d\alpha(\xi) = \bar{\alpha}^2 y_0^x$$

and into the second

$$y_0^x\int_0^1 \alpha(x)dx\int_x^1 d\alpha(\xi) = -\bar{\alpha}^2 y_0^x$$

Finally, only the term in $\log\epsilon$ gives a result which is nonzero. Now

$$C = -\frac{2c}{1 - c^2}\left[\frac{c^2}{1 - c^2} + \frac{1}{2c\sqrt{\nu^2 - 1}}\right]$$

and since

$$\frac{1}{\sqrt{\nu^2 - 1}} = \frac{2c}{1 - c^2}$$

one sees that one obtains

$$C_x = \frac{4}{\beta^2 \pi} \frac{4c^2 \left(1 + c^2\right)}{\left(1 - c^2\right)^3} \int_0^1 \alpha(x)dx \int_0^1 \log\left|x - \xi\right| (\xi - x)d\alpha(\xi)$$

One may replace c by its value as a function of γ which gives the simple expression

$$C_x = \frac{4}{\pi} \frac{\cos^2\gamma \sin\gamma}{\left(1 - M^2\cos^2\gamma\right)^{3/2}} \int_0^1 \alpha(x)dx \int_0^1 (\xi - x)\log\left|x - \xi\right| d\alpha(\xi) \tag{IV.35}$$

Let us take, for instance, the case of the rhombic profile. One then finds immediately that the double integral is equal to $\alpha_0^2 \log 2$. This result, in the special case of a rhombic profile, has been given by Th. von Karman (ref. 4).

If one takes the profile formed by two parabolic arcs

$$\alpha(x) = \epsilon(1 - 2x)$$

one finds as value of the double integral $\epsilon^2/4 = e^2$, with e designating the relative thickness of the profile. With an equal relative thickness and equal sweepback, the drags are in the ratio $\log 2 = 0.69$ whereas one obtains for an infinite wing, straight or oblique, the ratio 0.75. Thus one deduces that the rhombic profile is even more advantageous for a sweptback half-wing[44].

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[44]If one compares the drags, at infinite aspect ratio, of a profile formed by two parabolic arcs and of a rhombic profile, of equal area, one finds that the first is 3/4 of the second. With a pronounced sweepback, this ratio is equal to 0.92.

If one wants to investigate a bounded wing, like the one represented in figure 71, one must add the end effect due to the edge BB'. It suffices to subtract the flow symbolized by

$$-\int_{BB'} \vec{C}_s(M,d\alpha,c) \tag{IV.36}$$

from the flow defined by the formula (IV.31). The pressure coefficient due to the flow symbolized by equation (IV.36) is written

$$C_p\left(x,y^x\right) = -\frac{2}{\beta\pi\sqrt{\nu^2 - 1}}\int_0^{x-(1+\nu)\left(\nu_0-y^x\right)} \log\left|\frac{1 - c\rho}{c - \rho}\right| d\alpha(\xi)$$

with

$$\frac{2\rho}{1 + \rho^2} = \frac{y^x - \eta_0}{x - \xi + \nu\left(y^x - \eta_0\right)}$$

with C_p being zero if $x < (1 + \nu)\left(\eta_0 - y^x\right)$.

If $\eta_0 > \frac{1}{1 + \nu}$, the edge BB' does not influence the point O'. In this case it may be easily shown that the contribution of the flow (equation (IV.36)) to the total drag is zero. In fact, this contribution is proportional to

$$\int_0^{\frac{1}{1+\nu}} dy'^x \int_{(1+\nu)y'^x}^{1} \alpha(x)dx \int_0^{x-(1+\nu)y'^x} \log\left|\frac{1 - c\rho}{c - \rho}\right| d\alpha(\xi)$$

if one puts

$$y'^x = \eta_0 - y^x$$

One may make the change in variables used before which consists in replacing y,x,ξ by x,ξ,ρ; one obtains

$$\int_0^1 \alpha(x)dx \int_0^x d\alpha(\xi)(x - \xi) \int_{-1}^0 \frac{2\rho c\left(1 - c^2\right)}{(\rho - c)^2(1 - \rho c)^2} \log\left|\frac{1 - c\rho}{c - \rho}\right| d\rho$$

which is evidently zero.

This justifies a remark of Th. von Kármán (ref. 4).

For wings of high-aspect ratio, one may adopt, without large error, the formula (IV.35) for the total drag.

The calculation of the drag of an infinite sweptback wing (fig. 72), on the hypothesis that $\nu > 1$, is perfectly analogous to the one just performed. It suffices to replace, according to section 3.1.2.2, in the preceding formulas

$$\log\left|\frac{1 - c\rho}{c - \rho}\right| \qquad \text{by} \qquad \log\left|\frac{1 - c\rho}{c - \rho}\right| + \log\left|\frac{1 + c\rho}{c + \rho}\right|$$

Since

$$\log\left|\frac{1 + c\rho}{c + \rho}\right| = \log\left|\frac{1 + c^2}{2c}\right| + \frac{c^2 - 1}{2\left(1 + c^2\right)} r$$

it is sufficient to combine the expression

$$\frac{4}{\beta^2\pi\sqrt{\nu^2 - 1}} \frac{c}{1 + c^2} = \frac{2}{\pi} \frac{\cos^2\gamma}{\sin\gamma\left(1 - M^2\cos^2\gamma\right)^{1/2}}$$

with the coefficient of the double integral of the formula (IV.35).

However, one thus attains only the drag for half the wing $\left(x_2 > 0\right)$; one must therefore multiply by 2 in order to obtain the desired formula

$$C_x = \frac{4}{\pi} \frac{\cos^2\gamma}{\sin\gamma} \frac{1 + 2\sin^2\gamma - M^2\cos^2\gamma}{\left(1 - M^2\cos^2\gamma\right)^{3/2}} \int_0^1 \alpha(x)dx \int_0^1 (\xi - x)\log|x - \xi|\, d\alpha(\xi) \tag{IV.37}$$

According to the remark just made, this formula gives, for a sweptback wing of high-aspect ratio, an approximate value of the total drag[45].

We shall borrow from the memorandum Th. Von Kármán's the figure 73 which illustrates the usefulness of the formulas found above for the study of the variation of the C_x of a sweptback wing of high-aspect ratio with the Mach number (the profile is rhombic, the sweepback angle $\gamma = 45^{\circ}$, and the reduced aspect ratio $\eta_0 = 4$). We obtained in the course of this investigation the value of the $C_{x_{max}}$ (point A of the figure) by the formula (IV.29), and the portion of the curve from B (formula IV.27). The dotted part at the right of the abscissa $M = \sqrt{2}$ is calculated by that same formula. One sees that it indicates also the behavior of the exact curve. Finally, for the values of $M < \sqrt{2}$, the dotted part corresponds to the formula (IV.37). It represents a good approximation of the rigorous values, except for the immediate surroundings of $M = \sqrt{2}$.

Here we shall stop the investigation of "symmetrical" wing problems. One sees that this method leads to simple results and that the calculations are always elementary. The field of application may easily be extended to more general cases (trapezoidal wings, leading edge curvature, etc.).

4.1.2 - Lifting Problems

Study of the lifting problems is generally more delicate. In fact, the boundary conditions furnish on the wing the values of w, but outside of the wing (in the general case) w is different from zero; on the other hand, continuity of the pressure is required which leads to supposing (in pursuance of the hypothesis of linearization as noted in chapter III) that $u = 0$ in the plane Ox_1x_2 outside of the region (R) occupied by the wing. The difficulty lies in the fact that, in the general case, the boundary conditions bear up on two of the velocity components.

4.1.2.1 - Problems where the condition $u = 0$ may be replaced by $w = 0$

The rule of "forbidden signals" permits to define a general class of lifting problems where it will be possible to replace the

[45]Compare appendix No. 5.

condition $u = 0$ by the simpler condition $w = 0$. This will be the case for wings, the projection (R) on Ox_1x_2 of which will satisfy the following condition:

With C designating the contour of the plan form (R), the tangent to (C) forms, at every point of (C), with Ox_1 an angle which is larger than the Mach angle.

Naturally, such a contour (C) will present angular points. It is understood that, at these points, each of the semitangents must satisfy the condition stated. For the sake of abbreviation, we shall say that this contour is entirely supersonic.

Let us consider a point M of (R). As we remarked in section 1.1.4, the state of the fluid at M depends only on the perturbations inside of the Mach forecone of the point M; this forecone cuts off, in Ox_1x_2, a portion of (R) on which w is given, and a portion of the plane Ox_1x_2 in which the general flow is not disturbed (section 1.1.4) and on which $u = v = w = 0$. In order to calculate the pressure at the point M, one may suppose that $w = 0$ outside of (R). One may also say that, under these conditions, the upper and the lower surface of the wing are independent. The solution of the corresponding lifting problems is therefore perfectly analogous to that of the symmetrical problems visualized in the previous paragraph.

Let us assume, for instance, a flat plate of the plan form indicated by figure 74, with the contour (C) being entirely supersonic; the pressure at every point of this plate has been calculated in chapter III. We intend to calculate the total C_z. One has obviously

$$C_z = -\frac{1}{S}\int_{\omega_1}^{\omega_0} C_p r^2 d\varphi$$

if one puts

$$x = \frac{\beta x_2}{x_1} = \beta \tan \varphi \qquad r = OM$$

with S being the area of the region (R). Let us put furthermore

$$\lambda = \beta \tan \omega_1 \qquad \mu = \beta \tan \omega_0 \qquad P(x) = \frac{\beta r^2}{\beta^2 + x^2} = \frac{r^2 \cos^2\varphi}{\beta}$$

$P(x)$ depends uniquely on the trailing edge B'AB.

One then obtains the formula

$$C_z = -\frac{1}{S}\int_{\lambda}^{\mu} C_p(x)P(x)dx \tag{IV.38}$$

Let us recall that

$$C_p(x) = \begin{cases} -\dfrac{2i}{\beta}\dfrac{1}{\sin\theta_0} & \text{if} \quad 1 < x < \mu \\ -\dfrac{2i}{\beta\pi}\left(\dfrac{1}{\sin\theta_0}\,\text{Arc cos}\,\dfrac{\cos\theta_0 - x}{1 - x\cos\theta_0} + \dfrac{1}{\sin\theta_1}\,\text{Arc cos}\,\dfrac{x - \cos\theta_1}{1 - x\cos\theta_1}\right) & \text{if} \quad -1 < x < +1 \\ -\dfrac{2i}{\beta}\dfrac{1}{\sin\theta_1} & \text{if} \quad \lambda < x < -1 \end{cases}$$

with i designating the incidence counted according to the usual conventions.

In a recent memorandum, M. Snow (ref. 32) has applied this method to the calculation of the total C_z of a plate in the shape of a quadrilateral. We simply want to point out that, in a certain number of cases, it is possible to calculate the integral (IV.38) very simply. This simplification becomes apparent when $P(x)$ is analytic. It is then possible to use integrals in the complex field (variable z or Z).

Let us suppose to begin with that the contour B'AB is rectilinear and that its polar equation is written

$$r = \frac{r_0}{\sin(\varphi_0 - \varphi)}$$

$$OA = l = \frac{r_0}{\sin\varphi_0} \qquad x_0 = \beta\tan\varphi_0$$

$$S = \frac{l^2}{2}\frac{\tan^2\varphi_0(\tan\omega_0 - \tan\omega_1)}{(\tan\varphi_0 - \tan\omega_0)(\tan\varphi_0 - \tan\omega_1)} = \frac{l^2}{2\beta}\frac{x_0^{\,2}(\mu - \lambda)}{(x_0 - \mu)(x_0 - \lambda)}$$

$$P(x) = \frac{\beta r_0^2}{\cos^2\varphi_0 (x - x_0)^2}$$

and consequently

$$C_z = -\frac{\beta r_0^2}{S\cos^2\varphi_0}\int_\lambda^\mu \frac{C_p(x)}{(x - x_0)^2}\,dx = \frac{2\beta r_0^2}{S\cos^2\varphi_0}\int_\lambda^\mu \frac{u(x)dx}{(x - x_0)^2}$$

$$= \frac{2\beta r_0^2}{S\cos^2\varphi_0}\,\underline{R}\left[\int_\lambda^\mu \frac{U(z)dz}{(z - x_0)^2}\right] = -\frac{2\beta r_0^2}{S\cos^2\varphi_0}\,\underline{R}\left(i\pi R_0\right)$$

with R_0 designating the residue at the point $z = x_0$.

However,

$$R_0 = \frac{dU}{dz}_{(z=x_0)} = \frac{x_0}{\beta\sqrt{x_0^2 - 1}}\,\frac{dW}{dz}_{(z=x_0)} = -i\,\frac{w_0}{\pi\beta}\,\frac{(\mu - \lambda)x_0}{(x_0 - \lambda)(x_0 - \mu)\sqrt{x_0^2 - 1}}$$

and

$$C_z = -\frac{2r_0^2}{S\cos^2\varphi_0}\,\frac{(\mu - \lambda)x_0w_0}{(x_0 - \lambda)(x_0 - \mu)\sqrt{x_0^2 - 1}} = \frac{2l^2x_0^2}{\beta^2S}\,\frac{(\mu - \lambda)x_0w_0}{(x_0 - \lambda)(x_0 - \mu)\sqrt{x_0^2 - 1}}$$

or

$$C_z = -\frac{4w_0}{\beta}\,\frac{x_0}{\sqrt{x_0^2 - 1}} = \frac{4i}{\beta}\,\frac{\sin\varphi_0}{\sqrt{M^2\sin^2\varphi_0 - 1}} \qquad \text{(IV.39)}$$

The C_z is independent of ω_0 and of ω_1; this generalizes a result already found[46] in section 3.2.2.2.

Let us now suppose that the arc BAB' is an arc of an ellipse with the polar equation

$$r^2 = \frac{a^2b^2}{b^2\cos^2\varphi + a^2\sin^2\varphi}$$

and let us, for simplification, assume that $\omega_0 = -\omega_1$.

$$P(x) = \frac{\beta a^2b^2}{\beta^2b^2 + a^2x^2}$$

whence

$$C_z = -\frac{\beta a^2b^2}{S}\int_\lambda^\mu \frac{C_p(x)dx}{\beta^2b^2 + a^2x^2} = \frac{2\beta a^2b^2}{S}\,\underline{R}\left[\int_\lambda^\mu \frac{U(z)dz}{\beta^2b^2 + a^2z^2}\right]$$

$$= \frac{2\beta a^2b^2}{S}\,\underline{R}\left[2i\pi R_1\right]$$

R_1 being the residue at the point $z = i\,\dfrac{\beta b}{a}$.

In order to calculate this residue, one must know the value of U, for $z = i\,\dfrac{\beta b}{a}$; this value is very easily obtained from the formula (III.51). One finds

$$U\left(i\,\frac{\beta b}{a}\right) = -\frac{2w_0}{\pi\beta\sin\theta}\,\mathrm{Arc\,sin}\,\frac{a\sin\theta}{\sqrt{a^2 + b^2\beta^2\cos^2\theta}}$$

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[46]In a general manner, one can obtain the C_z of a wing, the surface of which is a portion of a cone bisecting the Mach cone, with the vertex O and a rectilinear trailing edge by measurement of the electric intensity in the tank. This result may be extended to the case where the cone is placed in any arbitrary relation to the Mach cone of O provided the trailing edge is rectilinear.

On the other hand

$$S = ab \text{ Arc tan } \frac{a}{b\beta \cos \theta}$$

Thus, if one puts $w_0 = -i$ (incidence)

$$C_z = \frac{4i}{\beta \sin \theta \text{ Arc tan } \frac{a}{b\beta \cos \theta}} \text{ Arc sin } \frac{a \sin \theta}{\sqrt{a^2 + b^2\beta^2\cos^2\theta}} \qquad \text{(IV.40)}$$

So far we have visualized only the case where the flow on the plate was conical due to the shape of the leading edge. To terminate these few remarks about the flat plate of supersonic contour, we shall now examine the case where the leading edge is curvilinear.

We shall start with the case of a polygonal leading edge (fig. 75). The investigation is based on the following remark: if one superposes at a point A_1 two elementary lifting flows, which completely bisect the Mach cone of A_1 and the first of which has as bounding generatrices $A_1 \Delta$, A_1D_1, so that $w = -w_0$ on (ΔA_1D_1), while the second has as bounding generatrices $A_1 \Delta$, A_1B_1, so that $w = w_0$ on (ΔA_1B_1), one obtains a resultant flow of such a type that, if $A_1\gamma_1$ and $A_1\gamma_1'$ are the sections of the Mach cone of A_1 in the plane Ox_1x_2, $w = 0$ outside of the angle $(B_1A_1D_1)$, whereas $w = -w_0$ on that angle; on the other hand, $u = 0$ outside of the angle $(\gamma_1'A_1D_1)$. Besides, one can easily verify that the resultant flow thus obtained is independent of the generatrix Δ (provided, however, that the latter is outside of $(\gamma'A_1\gamma_1)$), and that, if one puts as usual

$$\cos \theta_0 = \frac{1}{\beta \tan \omega_0} \qquad \cos \theta_1 = \frac{1}{\beta \tan \omega_1}$$

the pressure coefficient is equal to

$$\frac{2w_0}{\beta}\left(\frac{1}{\sin \theta_1} - \frac{1}{\sin \theta_0}\right) \qquad \text{on } \widehat{(\gamma_1A_1B_1)}$$

and to

$$\frac{2w_0}{\beta\pi}\left(\frac{1}{\sin\theta_1}\text{Arc cos}\frac{\cos\theta_1 - x}{1 - x\cos\theta_1} - \frac{1}{\sin\theta_0}\text{Arc cos}\frac{\cos\theta_0 - x}{1 - x\cos\theta_0}\right)$$

on $\widehat{(\gamma_1'A_1\gamma_1)}$

x represents as usual a semi-infinite line inside of $\widehat{(\gamma_1'A_1\gamma_1)}$.

We shall note the resultant flow

$$\overrightarrow{C}(A_1,\theta_0,\theta_0 - \theta_1)$$

The flow about the plate schematized in figure 75 is then obtained by superimposing on the conical flow of the vertex 0 and the bounding generatrices OD_1' and OD_1 the flows

$$\overrightarrow{C}(A_1,\theta_0,\theta_0 - \theta_1) \qquad \text{and} \qquad \overleftarrow{C}(A_1',\theta_0',\theta_0' - \theta_1')$$

with θ_0', θ_1', θ_0, θ_1 characterizing the directions of the straight lines OA_1', $A_1'B_1'$, OA_1, A_1B_1.

If the leading edge is curvilinear (fig. 76), let us assume $A\left[x_1(t),x_2(t)\right]$ the point moving along this leading edge, $\omega(t)$ the angle between the tangent at the moving point and Ox_1, and let us put

$$\cos\theta(t) = \frac{1}{\beta\tan\omega(t)}$$

Assuming $M(x_1,x_2)$ to be the point where one desires to calculate the pressure, one will put

$$x(t) = \beta\frac{x_2(t) - x_2}{x_1(t) - x_1}$$

The flow will be obtained by subtracting the flow symbolized by

$$\int_{(c)} C\left[A(t),\theta(t),d\theta(t)\right]$$

from the flow around a plate of infinite aspect ratio, with the leading edge Ox_2.

If MA_1 and MA_2 are, in Ox_1x_2, the two semigeneratrices of the Mach forecone at the point M, one has therefore as value of the pressure coefficient at M by putting

$$\frac{d}{d\theta}\left|\frac{1}{\sin\theta}\text{ Arc cos }\frac{\cos\theta - x}{1 - x\cos\theta}\right| = F(\theta,x)$$

$$C_p(M) = \frac{2w_0}{\beta}\left[1 - \frac{1}{\pi}\int_{t_1}^{t_2} F\left[\theta(t),x(t)\right] d\theta\right] \tag{IV.41}$$

At a point such as M' (compare fig. 76) a slight modification of the formula will be convenient; one must write

$$C_p(M') = \frac{2w_0}{\beta \sin\theta_1} - \frac{2w_0}{\beta\pi}\int_{t_1}^{t_2} F\left[\theta(t),x(t)\right] d\theta$$

One thus obtains the C_p by a simple integral.

We shall point out a very remarkable result for the total C_z of such a plate when the trailing edge is rectilinear. We shall show that the C_z of such a plate depends only on the trailing edge; this fact generalizes the result of the formula (IV.39). It suffices, of course, to demonstrate the result in the case of a polygonal leading edge; thence the general case is deduced by passing to the limit (fig. 77). According to the formula (IV.39), the resultant of the normal forces due to the flow

$$\vec{C}\left(A_1,\theta_0,\theta_0 - \theta_1\right)$$

acting on the region (R) is equal to that of the normal forces acting on the triangle $A_1B_1D_1$ in the conical flow with the vertex O and the bounding generatrices OD_1, OD_1'. The result stated above results from this remark. Thus one verifies that on this plate the total C_z is the same as if the direction of the flow had been reversed[47].

4.1.2.2 - Infinitely thin rectangular wing

We shall now investigate the case of a rectangular wing, the profile of which is an arc segment (fig. 78). In accordance with what was said before, this arc segment will be defined by the angle $j_0(x_1)$ which is formed by the tangent and the chord at the point with the abscissa x_1; if the wing has a geometric incidence defined by the angle i, we put

$$j(x_1) = j_0(x_1) - i \tag{IV.42}$$

w must be equal to $j(x_1)$ on (R), and u must be zero outside of (R).

We shall designate by $\overrightarrow{C_p}(M,\alpha)$ the lifting elementary conical flow, with the vertex M, which furnishes the value $w = \alpha$ on the two faces of the quadrant M, x_1, x_2. With the notations of figure 58, the formula (III.60) is then written

$$\left.\begin{array}{lll} C_p = \dfrac{2\alpha}{\beta\pi} \text{Arc cos } (1 - 2\beta \tan \varphi & \text{for} & 0 < \beta \tan \varphi < 1 \\ & & \\ C_p = \dfrac{2\alpha}{\beta} & \text{for} & \beta \tan \varphi > 1 \end{array}\right\} \tag{IV.43}$$

By an argument analogous to the one of section 4.1.1.1.2 we are induced to define the desired flow by the symbolic notation

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[47]One finds here anew a remark made before by M. Snow (ref. 32) in a particular case. Besides, this result may be extended without great difficulties to any arbitrary plate of supersonic contour.

$$\int_{AA'} \vec{C}_p(M,dj) + \int_{BB'} \overleftarrow{C}_p(M,dj) - E\left[j\left(x_1\right)\right] \tag{IV.44}$$

The flow thus defined does satisfy the conditions concerning w; however, one sees immediately that the flow gives a component u, zero outside of (R) only in the case where the aspect ratio $\beta\lambda$ is smaller than or equal to 1. The limiting case $\beta\lambda = 1$ corresponds to the disposition of the Mach cones given by figure 79. We shall use here[48] the hypothesis where $\beta\lambda \geqslant 1$, and shall then be able to calculate the flow by the formula (IV.44).

<u>4.1.2.2.1. Study of the flow $\int_{AA'} \vec{C}_p(M,dj)$.</u>- We shall use the same notations as in section 4.1.1.1.3. According to equation (IV.43), the pressure coefficient C_p at a point $\left(x^x,y^x\right)$ is written[49] $\left(0 < x^x < 1\right)$

$$C_p\left(x^x,y^x\right) = \frac{2}{\beta}\int_0^{x^x} dj(\xi) = \frac{2}{\beta}\, j\left(x^x\right) \qquad \text{if} \quad 0 < x^x < y^x$$

$$C_p\left(x^x,y^x\right) = \frac{2}{\beta}\int_{x^x-y^x}^{x^x} dj(\xi) + \frac{2}{\beta\pi}\int_0^{x^x-y^x} \text{Arc cos}\left(1 - \frac{2y^x}{x - \xi}\right)dj(\xi)$$

$$\text{if} \quad x^x > y^x$$

---

[48] It is not impossible to investigate the case where $\beta\lambda < 1$. One must then superimpose on the flow given by (equation (IV.44)) other conical flows, the vertices of which describe the two edges of the wing, in order to establish pressure continuity without changing the w value on the wing. This investigation is clearly more complicated than the one we shall make. We shall not enter on it in order to limit ourselves to the simplest results. Further on (section 4.1.2.3.2.) one will find an application of this method in a special case.

[49] Strictly speaking, the slope of the wing should be noted $j^x(x^x)$ when one expresses it as a function of the reduced abscissa. We shall omit the asterisk in order to simplify the notations.

These two formulas may be written

$$C_p = \frac{2}{\beta}\left[j\left(x^x\right) - R\left(x^x,y^x\right)\right]$$

with

$$R\left(x^x,y^x\right) = j\left(x^x - y^x\right) - \frac{1}{\pi}\int_0^{x^x-y^x} \text{Arc cos}\left(1 - \frac{2y^x}{x^x - \xi}\right)dj(\xi) \tag{IV.45}$$

stating that the function $j\left(x^x\right)$ is zero outside of the interval (0.1).

It is then easy to calculate the local c_z of a section y^x with this coefficient defined by

$$c_z\left(y^x\right) = -2\int_0^1 C_p\left(x^x,y^x\right)dx^x$$

Remarking that

$$j\left(x^x\right) = j_0\left(x^x\right) - i$$

and putting

$$f\left(x^x\right) = \int_0^{x^x} j_0(\xi)d\xi \qquad \left[f(1) = 0\right]$$

one has

$$c_z = \frac{4i}{\beta} + \frac{4}{\beta}\int_{y^x}^1 R\left(x^x,y^x\right)dx^x$$

Now

$$\int_{y^x}^{1} R\left(x^x, y^x\right) dx^x = \int_{y^x}^{1} j\left(x^x - y^x\right) dx^x -$$

$$\frac{1}{\pi}\int_{y^x}^{1} dx^x \int_{0}^{x^x - y^x} \text{Arc cos}\left(1 - \frac{2y^x}{x^x - y^x}\right) dj(\xi)$$

$$= f\left(1 - y^x\right) - i\left(1 - y^x\right) -$$

$$\frac{1}{\pi}\int_{0}^{1-y^x} dj(\xi) \int_{y^x+\xi}^{1} \text{Arc cos}\left(1 - \frac{2y^x}{x^x - y^x}\right) dx^x$$

However,

$$\int_{y^x}^{1-\xi} \text{Arc cos}\left(1 - \frac{2y^x}{u}\right) du = 2\left[(1 - \xi)\text{Arc sin}\sqrt{\frac{y^x}{1 - \xi}} + \sqrt{y^x\left(1 - y^x - \xi\right)}\right] - \pi y^x$$

Thus we put

$$\left.\begin{aligned} k\left(y^x, \xi\right) &= \frac{4}{\pi}\left[(1 - \xi)\text{Arc sin}\sqrt{\frac{y^x}{1 - \xi}} + \sqrt{y^x\left(1 - y^x - \xi\right)}\right] \\ &\qquad \text{if} \quad y^x < 1 - \xi \\ k\left(y^x, \xi\right) &= 2(1 - \xi) \qquad \text{if} \quad y^x > 1 - \xi \end{aligned}\right\} \quad (\text{IV.46})$$

$$\int_{y^x}^{1} R(x^x,y^x)dx^x = f(1 - y^x) - i(1 - y^x) + y^x\left[-i + j_0(1 - y^x)\right] - \frac{1}{2}\int_0^{1-y^x} k(y^x,\xi)\,dj(\xi)$$

$$= -i + \frac{i}{2}k(y^x,0) + f(1 - y^x) + y^x j_0(1 - y^x) - \frac{1}{2}\int_0^{1-y^x} k(y^x,\xi)\,dj_0(\xi)$$

$$= -i + \frac{i}{2}k(y^x,0) + p_0(y^x)$$

with

$$p_0(y^x) = f(1 - y^x) + y^x j_0(1 - y^x) - \frac{1}{2}\int_0^{1-y^x} k(y^x,\xi)\,dj_0(\xi) \tag{IV.47}$$

Consequently

$$c_z(y^x) = \frac{2k(y^x,0)}{\beta}\,i + \frac{4}{\beta}\,p_0(y^x) \tag{IV.48}$$

One will find in figure 80 the curve giving the variation of $k(y^x,0)$ and of $k(y^x,1/2)$.

We remark that

$$\int_0^1 dy^x \int_0^{1-y^x} k(y^x,\xi)dj_0(\xi) = \int_0^1 dj_0(\xi) \int_0^{1-\xi} k(y^x,\xi)\,dy^x$$

$$= \frac{3}{2}\int_0^1 (1 - \xi)^2 dj_0(\xi)$$

because

$$\int_0^{1-\xi} k\left(y^x,\xi\right) dy^x = \frac{3}{2}(1 - \xi)^2$$

However

$$\int_0^1 (1 - \xi)^2 dj_0(\xi) = 2\int_0^1 (1 - \xi)j_0(\xi) = 2\int_0^1 f(x)dx = 2\mu$$

putting

$$\int_0^1 f(x)dx = \mu$$

On the other hand

$$\int_0^1 f\left(1 - y^x\right)dy^x = \int_0^1 f(x)dx = \mu$$

and

$$\int_0^1 y^x j_0\left(1 - y^x\right)dy^x = \int_0^1 (1 - t)j_0(t)dt = -\int_0^1 tj_0(t)dt = \mu$$

Consequently

$$\int_0^1 c_z\left(y^x\right)dy^x = \frac{3i}{\beta} + \frac{2\mu}{\beta} \tag{IV.49}$$

4.1.2.2.2 - Study of the thin rectangular wing in the case where $\beta\lambda > 1$.- As we have said in section 4.1.2.2, one can apply to this case a method analogous to the one employed in section 4.1.1.1.4. The pressure coefficient at a point of the wing situated on the surface $x_3 = +0$ of reduced coordinates t, η, can immediately be written

$$C_p(t,\eta) = \frac{2}{\beta}\left[j_0(t) - i - \left[R\left(t,\eta_0 + \eta\right) + R\left(t,\eta_0 - \eta\right)\right]\right]$$

Consequently, the local c_z of the section η is obtained by the formula

$$c_z(\eta) = -2\int_0^1 C_p(t,\eta)dt$$

$$= \frac{4i}{\beta} + \frac{4}{\beta}\left[-2i + \frac{i}{2}\left[k\left(\eta_0 + \eta,0\right) + k\left(\eta_0 - \eta,0\right)\right] + p_0\left(\eta_0 + \eta\right) + p_0\left(\eta_0 - \eta\right)\right]$$

or

$$c_z(\eta) = \frac{2i}{\beta}\left[k\left(\eta_0 + \eta,0\right) + k\left(\eta_0 - \eta,0\right) - 2\right] + \frac{4}{\beta}\left[p_0\left(\eta_0 + \eta\right) + p_0\left(\eta_0 - \eta\right)\right]$$

with the functions k and p_0 being defined by the equalities (IV.46) and (IV.47). Finally, let us calculate the total C_z

$$C_z = \frac{1}{2\eta_0}\int_{-\eta_0}^{\eta_0} c_z(\eta)d\eta = -\frac{4i}{\beta} + \frac{2i}{\beta\eta_0}\int_0^{2\eta_0} k(t,0)dt + \frac{4}{\beta\eta_0}\int_0^{2\eta_0} p(t)dt$$

and since

$$2\eta_0 = 1 + \beta\lambda - 1$$

one has, applying the results established at the end of the preceding paragraph,

$$C_z = -\frac{4i}{\beta} + \frac{1}{\eta_0}\left(\frac{3i}{\beta} + \frac{2\mu}{\beta}\right) + (\beta\lambda - 1)\frac{4i}{\beta\eta_0}$$

because

$$k(t,0) = 2 \qquad \text{if} \quad t > 1$$

whence

$$C_z = \frac{4i}{\beta}\left(1 - \frac{1}{2\beta\lambda}\right) + \frac{4\mu}{\beta^2\lambda} \tag{IV.50}$$

Figure 81 gives the variation of $\left(\frac{1}{2}\frac{dC_z}{di}\right)$, as a function of M, for various values of λ[50].

One may also plan the calculation of the drag of this wing. First of all the local drag

$$c_x(\eta) = ic_z(\eta) + 2\int_0^1 C_p(t,\eta)j_0(t)dt$$

or

$$c_x(\eta) = ic_z(\eta) + \frac{4}{\beta}\bar{j}_0{}^2 - \frac{4}{\beta}\left[T\left(\eta_0 + \eta\right) + T\left(\eta_0 - \eta\right)\right] \tag{IV.51}$$

putting

$$T\left(y^x\right) = \int_{y^x}^1 R\left(t,y^x\right)j_0(t)dt$$

$$= if\left(y^x\right) + \int_{y^x}^1 j_0\left(t - y^x\right)j_0(t)dt -$$

$$\frac{1}{\pi}\int_{y^x}^1 j_0(t)dt\int_0^{t-y^x} \operatorname{Arc\,cos}\left(1 - \frac{2y^x}{t - \xi}\right)dj(\xi)$$

---

[50]A. Bonney has already obtained this formula in the case where $j_0 = 0$, $\mu = 0$ (rectangular flat plate); compare reference 33.

whence

$$T(y^x) = if(y^x) + \frac{2i}{\pi}\int_{y^x}^{1} \text{Arc sin}\sqrt{\frac{y^x}{t}}\, j_0(t)dt + \int_{y^x}^{1} j_0(t - y^x)j_0(t)dt -$$

$$\frac{2}{\pi}\int_0^{1-y^x} dj_0(\xi)\int_{y^x+\xi}^{1} \text{Arc sin}\sqrt{\frac{y^x}{t - \xi}}\, j_0(t)dt$$

The total drag C_x will be obtained by taking the mean value

$$C_x = \frac{1}{2\eta_0}\int_{-\eta_0}^{\eta_0} c_x(\eta)d\eta = iC_z + \frac{4}{\beta}\,\bar{j}_0{}^2 - \frac{4}{\beta\eta_0}\int_0^{2\eta_0} T(t)dt$$

$$C_x = iC_z + \frac{4}{\beta}\,\bar{j}_0{}^2 - \frac{4}{\beta\eta_0}\int_0^{1} T(t)dt$$

It is easy to calculate the mean value of $T(t)$ in the interval (0.1), since

$$\int_0^1 dy^x \int_{y^x}^{1} j_0(t - y^x)j_0(t)dt = 0$$

and

$$\int_0^1 dy^x \int_0^{1-y^x} dj_0(\xi)\int_{y^x+\xi}^{1} \text{Arc sin}\sqrt{\frac{y^x}{t - \xi}}\, j_0(t)dt =$$

$$\int_0^1 j_0(t)dt\int_0^t dj(\xi)\int_0^{t-\xi} \text{Arc sin}\sqrt{\frac{y^x}{t - \xi}}\, dy^x = 0$$

The calculations are analogous to those carried out at the end of section 4.1.1.1.3.

However,

$$\int_0^1 dy^x \int_{y^x}^1 \text{Arc sin} \sqrt{\frac{y^x}{t}}\, j_0(t)dt =$$

$$\int_0^1 j_0(t)dt \int_0^t \text{Arc sin} \sqrt{\frac{y^x}{t}}\, dy^x = -\mu \frac{\pi}{4}$$

consequently

$$C_x = \frac{4}{\beta} \bar{j}_0{}^2 + iC_z - \frac{4\mu i}{\beta^2\lambda} = \frac{4i^2}{\beta}\left(1 - \frac{1}{2\beta\lambda}\right) + \frac{4}{\beta} \bar{j}_0{}^2 \qquad \text{(IV.53)}$$

We shall make an application to the case where the profile is defined by

$$j_0(x) = j_0 \qquad \text{if} \quad 0 < x^x < \frac{1}{2}$$

$$j_0(x) = -j_0 \qquad \text{if} \quad \frac{1}{2} < x^x < 1$$

$$f(x) = \begin{cases} j_0 x & \text{if} \quad 0 < x < \frac{1}{2} \\ j_0(1 - x) & \text{if} \quad \frac{1}{2} < x < 1 \end{cases}$$

$$\mu = \int_0^{\frac{1}{2}} j_0 x\, dx + \int_{\frac{1}{2}}^1 j_0(1 - x)dx = \frac{j_0}{4}$$

In order to determine the local forces, one must calculate the functions $p_0(y^x)$ and $T(y^x)$; now

$$p_0(y^x) = \begin{cases} j_0 k\left(y^x,\frac{1}{2}\right) - \frac{j_0}{2} k\left(y^x,0\right) & \text{if} \quad 0 \leqslant y^x \leqslant \frac{1}{2} \\ j_0 - \frac{j_0}{2} k\left(y^x,0\right) & \text{if} \quad \frac{1}{2} \leqslant y^x \leqslant 1 \end{cases} \qquad \text{(IV.54)}$$

The variation of $p_0(y^x)$ for a wing of reduced aspect ratio equal to 2 is given by the figure 82.

On the other hand, $T(y^x)$ can be expressed simply as a function of $k(y^x,\xi)$. In fact

$$\int_{y^x}^{1} j_0(t - y^x) j_0(t)dt = \begin{cases} (1 - 3y^x) j_0{}^2 & \text{if } 0 \leqslant y^x \leqslant \frac{1}{2} \\ (y^x - 1) j_0{}^2 & \text{if } \frac{1}{2} \leqslant y^x \leqslant 1 \end{cases}$$

$$\int_{y^x}^{1} \text{Arc sin} \sqrt{\frac{y^x}{t}}\, j_0(t)dt = \begin{cases} \frac{\pi j_0}{4}\left[2k\left(y^x,\frac{1}{2}\right) - k\left(y^x,0\right) - 2y^x\right] & \text{if } 0 \leqslant y^x \leqslant \frac{1}{2} \\ \frac{\pi j_0}{4}\left[2y^x - k\left(y^x,0\right)\right] & \text{if } \frac{1}{2} \leqslant y^x \leqslant 1 \end{cases}$$

These formulas one can establish immediately, remarking that

$$\int_{y^x}^{1-\xi} \text{Arc sin} \sqrt{\frac{y^x}{u}}\, du = \frac{\pi}{4}\left[k\left(y^x,\xi\right) - 2y^x\right]$$

Finally

$$\int_{0}^{1-y^x} dj_0(\xi) \int_{y^x+\xi}^{1} \text{Arc sin} \sqrt{\frac{y^x}{t - \xi}}\, j_0(t)dt = \begin{cases} j_0{}^2\, \frac{\pi}{4}\left[4k\left(y^x,\frac{1}{2}\right) - 6y^x - k\left(y^x,0\right)\right] & \text{if } 0 \leqslant y^x \leqslant 1 \\ j_0{}^2\, \frac{\pi}{4}\left[2y^x - k\left(y^x,0\right)\right] & \text{if } \frac{1}{2} \leqslant y \leqslant 1 \end{cases}$$

whence

$$T\left(y^{x}\right) = \begin{cases} \frac{i}{2}\, j_0\left[2k\left(y^{x},\frac{1}{2}\right) - k\left(y^{x},0\right)\right] + \frac{1}{2}\, j_0{}^{2}\left[2 + k\left(y^{x},0\right) - 4k\left(y^{x},\frac{1}{2}\right)\right] & \\ & \text{if} \quad 0 \leqslant y^{x} \leqslant \frac{1}{2} \\ \frac{i}{2}\, j_0\left[2 - k\left(y^{x},0\right)\right] + \frac{1}{2}\, j_0{}^{2}\left[k\left(y^{x},0\right) - 2\right] & \text{if} \quad \frac{1}{2} \leqslant y \leqslant 1 \end{cases}$$

(IV.55)

In figure 83, one will find the distribution of the drags over the span for a wing of this type of réduced aspect ratio equal to 2.

One will remark that

$$\int_0^1 p_0\left(y^{x}\right) dy^{x} = \frac{\mu}{2} = \frac{j_0}{8}$$

and

$$\int_0^1 T\left(y^{x}\right) dy^{x} = \frac{\mu i}{2} = \frac{i}{8}\, j_0$$

This results from the equality previously demonstrated

$$\int_0^{1-\xi} k\left(y^{x},\xi\right) dy^{x} = \frac{3}{2}(1 - \xi)^{2}$$

4.1.2.2.3 - Effect of flaps and ailerons.- We shall begin with the case of a flat plate; the formulas can easily be generalized in the case where the wing profile is curved. The ailerons are, for instance, disposed on the plate in the arrangement indicated by figure 84; γ_1 designates the deflection of the first aileron A'CDD', γ_2 that of the second B'EFF'.

For study of the flow one must utilize conical flows $\vec{T}(M,\alpha)$ which one can define in the following manner. In the region $x_3 > 0$ the flow $\overrightarrow{T}(M,\alpha)$ is identical with the flow $\overrightarrow{C}_s(M,-\alpha)$; in the region $x_3 < 0$ it is identical with the flow $\overrightarrow{C}_s(M,\alpha)$. One can immediately make an interpretation of the flows $\overrightarrow{T}$ which gives account of the possible utilization in the effects of flaps and ailerons;[51] the flow $\overrightarrow{T}(M,\alpha)$ is established when, after the plane Ox_1x_2 has been materialized, one makes the quadrant Mx_1x_2 pivot around Mx_2 by an angle $-\alpha$ (fig. 58). Hence the investigated flow may be obtained immediately by superposition of conical flows schematized in the following manner

$$\left.\begin{array}{lll} \overrightarrow{C}_p(A,-i) & \overleftarrow{C}_p(B,-i) & E(AB,+i) \\ \\ \overrightarrow{C}_p(C,\gamma_1) & \overleftarrow{T}(D,\gamma_1) & E(CD,-\gamma_1) \\ \\ \overleftarrow{C}_p(E,\gamma_2) & \overrightarrow{T}(F,\gamma_2) & E(EF,-\gamma_2) \end{array}\right\} \qquad (IV.56)$$

If such a scheme is to be valid without further complications, the pressure coefficient outside of (R) must, of course, be zero. This will be the case if the reduced aspect ratio of the plate and the flaps is greater than, or equal to 1.

Let us apply these principles to the calculation of the local C_z of a plate for which the Mach cones of the points A_1, B, C, D, E, F are disposed as shown in figure 84. One may then place the origin at the point A and immediately write the local C_z as a function of $y^x\left(y^x = \beta x_2\right)$; one will put $AA' = 1$, $CA' = c$, $\beta CD = l$, according to

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[51]We have indicated this method in a note to the reports on the proceedings of the Academy of Sciences in December 1947 (ref. 37). The advantage of the flows T we indicate here has also been pointed out in the article of M. Snow, published at the same time as our note (ref. 32).

the results obtained in sections 4.1.1.1.5 and 4.1.2.2.1:

$$c_z = \frac{8}{\beta\pi}\left[\gamma_1\left[c \text{ Arc sin}\sqrt{\frac{y^x}{c}} + \sqrt{y^x\left(c - y^x\right)}\right] + i\left[\text{Arc sin}\sqrt{y^x} + \sqrt{y^x\left(1 - y^x\right)}\right]\right] \qquad \text{if} \quad y^x < c$$

$$c_z = \frac{4}{\beta}\left[\gamma_1 c + \frac{2i}{\pi}\left[\text{Arc sin}\sqrt{y^x} + \sqrt{y^x\left(1 - y^x\right)}\right]\right] \qquad c < y^x < 1$$

$$c_z = \frac{4}{\beta}\left(i + \gamma_1 c\right) \qquad 1 < y^x < l - c$$

$$c_z = \frac{4}{\beta}\left[i + \frac{\gamma_1 c}{2} + \frac{\gamma_1 c}{\pi}\left[\text{Arc sin}\frac{l - y^x}{c} + \left(l - y^x\right)\text{Arg ch}\frac{c}{l - y^x}\right]\right]$$

$$l - c < y^x < c$$

$$c_z = \frac{4}{\beta}\left[i + \frac{\gamma_1 c}{2} - \frac{\gamma_1 c}{\pi}\left[\text{Arc sin}\frac{y^x - l}{c} + \left(y^x - l\right)\text{Arg ch}\frac{c}{y^x - l}\right]\right]$$

$$l < y^x < l + c$$

In figure 85, one will find the distribution of c_z over the span. Besides, it will be possible to write in a general manner the local c_z of any slender rectangular wing provided with flaps or ailerons.

In fact, if one puts

$$\left.\begin{aligned}
f(u,c) &= c\left[1 + \frac{2}{\pi}\left(\text{Arc sin}\,\frac{u}{c} + u\ \text{Arg ch}\,\frac{c}{|u|}\right)\right] && \text{if}\quad -c < u < c \\
f(u,c) &= 0 && \text{if}\quad u + c < 0 \\
f(u,c) &= 2c && \text{if}\quad u - c > 0
\end{aligned}\right\} \qquad \text{(IV.58)}$$

one has with the customary notations

$$\begin{aligned}
c_z(\eta) = {} & \frac{2i}{\beta}\left[k\left(\eta_0 + \eta;0\right) + k\left(\eta_0 - \eta,0\right) - 2\right] + \frac{4}{\beta}\left[\left(\eta_0 + \eta\right) + p_0\left(\eta_0 - \eta\right)\right] + \\
& \frac{2\gamma_1}{\beta}\left[k\left(\eta_0 + \eta,l - c\right) - f\left(\eta_0 - l + \eta,c\right)\right] + \\
& \frac{2\gamma_2}{\beta}\left[k\left(\eta_0 - \eta,l - c\right) - f\left(\eta_0 - l - \eta,c\right)\right]
\end{aligned} \qquad \text{(IV.59)}$$

The total C_z may be easily calculated. We remark for this purpose that

$$\int_0^{2\eta_0} k(u,l - c)du = \int_0^{c} k(u,l - c)du + \int_c^{2\eta_0} k(u,l - c)du$$

$$= \frac{3}{2}\,c^2 + 2c\left(2\eta_0 - c\right)$$

The mean value of f is very easily obtained whence

$$C_z = \frac{4i}{\beta}\left(1 - \frac{1}{2\beta\lambda}\right) + \frac{4\mu}{\beta^2\lambda} + 2\,\frac{\left(\gamma_1 + \gamma_2\right)}{\beta^2\lambda}\left(2cl - \frac{1}{2}\,c^2\right) \qquad \text{(IV.60)}$$

One also sees that the calculation of the moments does not present any difficulties.

4.1.2.3 - A few remarks regarding the study of the effect of sweepback

We cannot here develop a theory of the sweptback wing. We therefore shall content ourselves with a few remarks.

4.1.2.3.1 - Study of the sweptback wing with "supersonic leading edge" ($\beta \cot \gamma > 1$), compare figure 66.- This investigation does not present any difficulty in the case where the reduced aspect ratio η_0 is greater than $\frac{1}{1 - \cos \theta}$. We reassume the notations of section 4.1.1.2.1; let $j(\xi)$ be the angle defining the infinitely slender profile of the wing supposed to be constant over the span $\left[j(\xi) = j_0(\xi) - i\right]$; the flow will be obtained by superimposing as before:

(1) Conical flows bisecting the Mach cone, centered on OO'.

(2) Lifting flows centered on AA' and BB'.

(c) Finally flows about the wing of infinite span with a fin with the same profiles as the wing profile and leading edges which coincide with OA and OB.

In order to simplify the investigation, we shall assume that the Mach cones of the points O, A, B do not interfere with the wing; this will permit one to study separately the "head effect" (conical flows centered on OO') and the "end effect" (conical flows centered on AA' or BB'). The "head effect" can be investigated immediately, according to the formulas of section 4.1.1.2.1.

The pressure coefficient on the surface $x_3 = +0$ is written

$$C_p = \frac{1}{\beta \sin \theta}\left[2j(x) - 2Q\left(x, y^x, \theta\right)\right]$$

Q being defined by the formula (IV.21) in which $\alpha(\xi)$ has been replaced by $j(\xi)$.

The local-lift coefficient is written

$$c_z\left(y^x\right) = \frac{4i}{\beta \sin \theta} + \frac{8}{\beta\pi \sin \theta} \int_{y^x(1-\cos\theta)}^{1} \int_0^{x-y^x(1-\cos\theta)} \text{Arc cos } \frac{\sin \theta\left(x - \xi + y^x\cos \theta\right)}{\sqrt{(x - \xi)\left(x - \xi + 2y^x\cos \theta\right)}} \, dj(\xi)$$

The mean value of $c_z\left(y^x\right)$ in the region $0 < y^x < \frac{1}{1 - \cos \theta}$ is equal to

$$\frac{4i}{\beta \sin \theta} + \frac{8(1 - \cos \theta)}{\beta\pi \sin \theta} \int_0^1 dx \int_0^x dj(\xi) \int_0^{\frac{x-\xi}{1-\cos\theta}} \text{Arc cos } \frac{\sin \theta\left(x - \xi + y^x\cos \theta\right)}{\sqrt{(x - \xi)\left(x - \xi + 2y^x\cos \theta\right)}} \, dy^x =$$

$$\frac{4i}{\beta \sin \theta} + \frac{8I(\theta)(1 - \cos \theta)}{\beta\pi \sin \theta}\left(- \frac{i}{2} + \mu\right) \tag{IV.61}$$

with

$$I(\theta) = \int_0^1 \text{Arc cos } \frac{\sin \theta}{\sqrt{1 - t^2\cos^2\theta}} \frac{dt}{(1 - t \cos \theta)^2}$$

$$= \frac{1}{2 \sin \theta}\left[1 + \frac{1}{\cos \theta \sin \theta}\left[\left(\frac{\pi}{2} - \theta\right)\cos 2\theta + \frac{\pi}{2} \cos^2\theta\right]\right]$$

Likewise, the local drag is written

$$c_x\left(y^x\right) = \frac{4\bar{j}^2}{\beta \sin \theta} - \frac{4}{\beta \sin \theta} G\left(y^x, \theta\right)$$

under the condition that in the formula defining G, (formula (IV.25)), $\alpha(\xi)$ is replaced by $j(\xi)$. The mean value of this drag in the region $0 < y^x < \frac{1}{1 - \cos \theta}$ is written

$$\frac{4\left(i^2 + \bar{j}_0^{\,2}\right)}{\beta \sin \theta} - \frac{8(1 - \cos \theta)}{\beta\pi \sin \theta} \int_0^1 j(x)dx \int_0^x dj(\xi) \int^{\frac{x - \xi}{1-\cos\theta}} \operatorname{Arc} \sin \frac{\sin \theta\left(x - \xi + y^x \cos \theta\right)}{\sqrt{(x - \xi)\left(x - \xi + 2y^x \cos \theta\right)}} dy^x =$$

$$\frac{4\left(i^2 + \bar{j}_0^{\,2}\right)}{\beta \sin \theta} - \frac{8(1 - \cos \theta)}{\beta\pi \sin \theta} I(\theta) \int_0^1 j(x)dx \int_0^x (x - \xi)dj(\xi) =$$

$$\frac{4\left(i^2 + \bar{j}_0^{\,2}\right)}{\beta \sin \theta} - \frac{4(1 - \cos \theta)}{\beta\pi \sin \theta} I(\theta) i^2 \qquad \text{(IV.63)}$$

One may study in the same manner the "end effect" by compounding the flows investigated in section 3.2.3.2. Taking the reduced coordinates, referred to the point A, one has immediately

$$C_p = \frac{2}{\beta \sin \theta}\left[j(x) - R\left[x, y^x(1 + \cos \theta)\right]\right]$$

R being the function defined by equation (IV.54). Consequently

$$c_z(y^x) = \frac{2k\left(y^x(1 + \cos\theta), 0\right)}{\beta \sin\theta} i + \frac{4}{\beta \sin\theta} p_0\left[y^x(1 + \cos\theta)\right]$$

the mean value of c_z in the interval $0 < y^x < \frac{1}{1 + \cos\theta}$ is written

$$C_z = \frac{3i}{\beta \sin\theta} + \frac{2\mu}{\beta \sin\theta}$$

In the same manner, one obtains without any difficulty the value of the local drag

$$c_x(y^x) = ic_z(y^x) + \frac{4}{\beta \sin\theta} \overline{j_0}^2 - \frac{4}{\beta \sin\theta} T\left[y^x(1 + \cos\theta)\right]$$

and its mean value in the interval $0 < y^x < \frac{1}{1 + \cos\theta}$

$$\frac{3i^2}{\beta \sin\theta} + \frac{4}{\beta \sin\theta} \overline{j_0}^2$$

One may summarize these results in the following manner: we consider a wing of an aspect ratio equal to $2\eta_0$ (fig. 86); the total C_z of this wing is written

$$C_z = \frac{4i}{\beta \sin\theta\eta_0}\left[1 - \frac{I(\theta)}{\pi}(1 - \cos\theta)\right]\frac{1}{1 - \cos\theta} + \frac{3i}{\eta_0\beta \sin\theta}\frac{1}{1 + \cos\theta} +$$

$$\frac{4i}{\eta_0\beta \sin\theta}\frac{\eta_0 \sin^2\theta - 2}{\sin^2\theta} + \frac{2\mu}{\beta \sin\theta\eta_0}\left[\frac{1}{1 + \cos\theta} + \frac{4I(\theta)}{\pi}\right]$$

or

$$C_z = \frac{4i}{\beta \sin\theta}\left[1 - \frac{1}{4\eta_0}\left(\frac{1}{1 + \cos\theta} + \frac{4I(\theta)}{\pi}\right)\right] + \frac{2\mu}{\beta\eta_0 \sin\theta}\left[\frac{1}{1 + \cos\theta} + \frac{4I(\theta)}{\pi}\right] \tag{IV.64}$$

Likewise, for the total drag

$$C_x = \frac{4\left(i^2 + \bar{j}_0^{\,2}\right)}{\eta_0 \beta \sin\theta} \frac{1}{1 - \cos\theta} - \frac{4I(\theta)}{\beta\pi \sin\theta\eta_0} i^2 +$$

$$\left[\frac{3i^2}{\beta \sin\theta} + \frac{4}{\beta \sin\theta} \bar{j}_0^{\,2}\right] \frac{1}{(1 + \cos\theta)\eta_0} + \frac{4\left(i^2 + \bar{j}_0^{\,2}\right)}{\beta \sin\theta\eta_0} \frac{\eta_0 \sin^2\theta - 2}{\sin^2\theta}$$

or

$$C_x = \frac{4i^2}{\beta \sin\theta}\left[1 - \frac{1}{4\eta_0}\left(\frac{1}{1 + \cos\theta} + \frac{4I(\theta)}{\pi}\right)\right] + \frac{4\bar{j}_0^{\,2}}{\beta \sin\theta} \qquad \text{(IV.65)}$$

These formulas remain applicable as long as $\eta_0 > \dfrac{1}{1 - \cos\theta}$.

4.1.2.3.2 - The study of the sweptback wing with a supersonic leading edge when $\eta_0 < \dfrac{1}{1 - \cos\theta}$, or with a subsonic leading edge, presents more serious difficulties.- A complete investigation of this kind would lead us too far. We shall content ourselves with treating a simple example which will show how to proceed in order to surmount the difficulties.

We attain this aim by introducing conical flows which we shall denote

$$S\left(M, t_0, u_0\right)$$

defined in the plane Z by the following boundary conditions (fig. 87):

(1) On $\left(C_0\right)$, $u = v = w = 0$.

(2) On OA, $w = 0$.

(3) On the upper edge of OC, $u = u_0$.
On the lower edge of OC, $u = -u_0$.

u_0 is a given constant, the point C is the image of the number $Z = -a^2$; one puts as usual

$$t_0 = \frac{2a^2}{1 + a^4}$$

The methods of chapter III permit one to write very easily the function $U(Z)$ the real part of which gives the component u of such a flow; if one puts

$$Z = s^2$$

one has

$$U(Z) = -\frac{iu_0}{\pi} \log\left[\frac{s - ia}{s + ia}\,\frac{1 + ias}{1 - ias}\right] \tag{IV.66}$$

One verifies readily that this flow satisfies all boundary conditions. Besides, if one puts

$$t = \frac{2s^2}{1 + s^4}$$

one has on OA

$$u = \pm \frac{u_0}{\pi} \text{Arc cos}\left[1 - \frac{2t_0(1 - t)}{t + t_0}\right] = \pm \frac{2u_0}{\pi} \text{Arc sin}\sqrt{\frac{t_0(1 - t)}{t + t_0}} \tag{IV.67}$$

These flows will enable us to make the pressure discontinuities appearing outside of the wing disappear, without modification of the boundary conditions on the wing itself.

Let us take for instance the case of a plate of the plan form indicated in figure 88. With γ being the sweepback angle, one will put as usual

$$\beta \text{ cotan } \gamma = \frac{1}{\cos \theta}$$

One assumes that the Mach cone A does not intersect the segment OO', but that the Mach cone of O does intersect the segment AA' at the point M_0. According to what was said above, one will obtain a flow which satisfies the boundary conditions on the wing portion $y^x < 0$ by superimposing a conical flow of the vertex O and bounding generatrices OA, OB, a flow of the vertex A and bounding generatrices AA', AO, and by subtracting the flow about a plate of infinite span with AO as leading edge; however, the region M_0P_0A' then is a zone of discontinuity for the pressure. If M_0P_1 represents the other generatrix of the Mach cone of M_0 in x_1Ox_2, the pressures obtained in the region $M_0A'P_1$ will thus be erroneous.

One will obtain the desired result by superimposing on the preceding flow a flow schematized by

$$\int_{M_0A'} S(M,t_0,u_0)$$

In this formula

$$u_0 = -\frac{2i}{\beta\pi \sin\theta}\frac{\partial}{\partial\xi}\left[\text{Arc sin}\frac{\sin\theta\left(\xi + \eta_0\cos\theta\right)}{\sqrt{\xi\left(\xi + 2\eta_0\cos\theta\right)}}\right]d\xi$$

$$t_0 = \frac{\eta_0}{\xi + \eta_0\cos\theta}$$

if M is at M_0, $t_0 = 1$, $\xi = \eta_0(1 - \cos\theta)$.

The pressure coefficient[52] in the region $M_0A'P_1$ is given by the following formula (y^x is negative):

[52]One will find in appendix No. 6 the explicit calculation of this pressure coefficient and a few important brief remarks regarding certain pecularities occurring in analogous problems.

$$C_p = -\frac{2i}{\beta \sin\theta}\left[\frac{2}{\pi}\left[\text{Arc sin}\,\frac{\sin\theta\left(x - y^x\cos\theta\right)}{\sqrt{x\left(x - 2y^x\cos\theta\right)}} + \text{Arc sin}\sqrt{\frac{\left(\eta_0 + y^x\right)\left(1 + \cos\theta\right)}{x}}\right] - 1 - \frac{4}{\pi^2}\int_{\eta_0(1-\cos\theta)}^{x-(\eta_0+y^x)(1+\cos\theta)} \text{Arc sin}\sqrt{\frac{\eta_0\left[x - \xi - \left(\eta_0 + y^x\right)\left(1 + \cos\theta\right)\right]}{\eta_0 x + y^x \xi}}\,\frac{\partial}{\partial\xi}\left[\text{Arc sin}\,\frac{\sin\theta\left(\xi + \eta_0\cos\theta\right)}{\sqrt{\xi\left(\xi + 2\eta_0\cos\theta\right)}}\right]d\xi\right] \tag{IV.68}$$

One verifies in particular that C_p becomes zero for $y^x = -\eta_0$, that is, on the edge AA .

One can see that this formula is rather complicated. We shall content ourselves with examining the case where $\theta = 0$. The formula then presents an indeterminate form which may, however, easily be eliminated by developing the terms in brackets up to the first order inclusively (in θ). It is convenient to perform an integration by parts before making this development.

The result then is considerably simpler; one finds

$$C_p = -\frac{4i}{\pi\beta}\left[\frac{\left(x - y^x\right)}{\sqrt{x\left(x - 2y^x\right)}} + \frac{2}{\pi}\int_0^{x-2(\eta_0+y^x)} \frac{\xi + \eta_0}{\sqrt{\xi\left(\xi + 2\eta_0\right)}}\,\frac{\partial}{\partial\xi}\left[\text{Arc sin}\sqrt{\frac{\eta_0\left[x - \xi - 2\left(\eta_0 + y^x\right)\right]}{\eta_0 x + y^x\xi}}\right]d\xi\right] \tag{IV.69}$$

The integral of the second term represents the "end effect" of the wing AA' while the first term represents simply the pressure coefficient in the conical flow with the vertex 0[53].

As an application, we shall calculate the total C_z of this wing

$$C_z = \frac{8i}{\beta\pi\eta_0}\left[\int_0^{\eta_0} dy^x \int_0^1 \frac{\left(x + y^x\right)dx}{\sqrt{x\left(x + 2y^x\right)}} + \frac{1}{\pi}\int_0^1 dz \int_z^1 dx \int_0^{x-z} F(z,x,\xi)d\xi\right]$$

---

[53]Had one wanted to study directly the case where $\theta = 0$ by application of the preceding method, one would have been led to write

$$C_p = -\frac{4i}{\beta\pi}\left[\frac{x - y^x}{\sqrt{x\left(x - 2y^x\right)}} - \right.$$

$$\left.\frac{2}{\pi}\int_0^{x-2\left(\eta_0+y^x\right)} \text{Arc sin}\sqrt{\frac{\eta_0\left[x - \xi - 2\left(\eta_0 + y^x\right)\right]}{\eta_0 x + y^x\xi}}\ \frac{\partial}{\partial\xi}\left[\frac{\xi + \eta_0}{\sqrt{\xi\left(\xi + 2\eta_0\right)}}\right]d\xi\right]$$

However, the integral of the second term has no meaning since the differential element is in $\xi^{-3/2}$. In order to eliminate this difficulty, one must utilize the conception: "finite part" of an integral introduced into the analysis by M. Hadamard (compare ref. 7). One has in fact

$$\int_0^{x-2\left(\eta_0+y^x\right)} \text{Arc sin}\sqrt{\frac{\eta_0\left[x - \xi - 2\left(\eta_0 + y^x\right)\right]}{\eta_0 x + y^x\xi}}\ \frac{\partial}{\partial\xi}\ \frac{\xi + \eta_0}{\sqrt{\xi\left(\xi + 2\eta_0\right)}}\ d\xi =$$

$$-\int_0^{x-2\left(\eta_0+y^x\right)} \frac{\xi + \eta_0}{\sqrt{\xi\left(\xi + 2\eta_0\right)}}\ \frac{\partial}{\partial\xi}\left[\text{Arc sin}\sqrt{\frac{\eta_0\left[x - \xi - 2\left(\eta_0 + y^x\right)\right]}{\eta_0 x + y^x\xi}}\right]d\xi$$

This justifies once more the interest in the motion of the "finite" part of an integral which permits a very easy performance of limiting process which may be delicate.

if one puts

$$2\left(\eta_0 + y^x\right) = z$$

with $F(z,x,\xi)$ designating the quantity under the sign $\int$ in the formula (IV.69). The double integral may be calculated immediately (compare the end of section 4.1.1.2.1)

$$\int_0^{\eta_0} dy^x \int_0^1 \frac{x + y^x}{\sqrt{x\left(x + 2y^x\right)}} dx = \int_0^1 dx \int_0^{\eta_0} \frac{x + y^x}{\sqrt{x\left(x + 2y^x\right)}} dy^x =$$

$$\frac{1}{3}\int_0^1 \sqrt{\frac{x + 2\eta_0}{x}}\left[2x + \eta_0\right] dx - \frac{1}{3} = \frac{1}{3}\left[\left(1 + 2\eta_0\right)^{\frac{3}{2}} - 1\right]$$

As to the triple integral, one may write it changing the order of integrations

$$\frac{1}{\sqrt{2}}\int_0^1 \frac{\left(\xi + \eta_0\right)\sqrt{\eta_0}\, d\xi}{\sqrt{\xi\left(\xi + 2\eta_0\right)}}\int_\xi^1 dx \int_0^{x-\xi} \frac{z - x - 2\eta_0}{z\xi + 2\eta_0(x - \xi)}\sqrt{\frac{z}{x - \xi - z}}\, dz$$

In order to calculate the last integral, one puts

$$\frac{z}{x - \xi - z} = t^2$$

It is then written

$$\int_0^\infty 2\,\frac{t^2(x - \xi) - \left(x + 2\eta_0\right)\left(1 + t^2\right)}{2\eta_0\left(1 + t^2\right) + t^2\xi}\,\frac{t^2 dt}{\left(1 + t^2\right)^2}$$

and may be calculated rapidly by residues. It has the value

$$\frac{\pi x}{\xi^2}\sqrt{2\eta_0(\xi + 2\eta_0)} - \frac{\pi x}{2\xi^2}(\xi + 4\eta_0) - \frac{\pi}{2}$$

Therefore, it suffices for calculating the triple integral to utilize the following results

$$\int_0^1 \frac{(\xi + \eta_0)(1 - \xi)}{\sqrt{\xi(\xi + 2\eta_0)}}\,d\xi = 2\left(\frac{2}{3} + \eta_0\right) - \frac{2\eta_0(1 + 2\eta_0)}{\sqrt{2\eta_0}}\operatorname{Arc\,tan}\frac{1}{\sqrt{2\eta_0}}$$

$$\int \frac{(\xi + \eta_0)(\xi + 4\eta_0)(1 - \xi^2)}{\xi^{3/2}(\xi + 2\eta_0)}\,d\xi = -\left[\frac{4\eta_0}{3\xi^{3/2}} + \frac{3}{\sqrt{\xi}} + 6\eta_0\sqrt{\xi} + \frac{2}{3}\xi^{3/2} + \frac{1 - 4\eta_0^2}{\sqrt{2\eta_0}}\operatorname{Arc\,tan}\sqrt{\frac{\xi}{2\eta_0}}\right]$$

$$\int \frac{(\xi + \eta_0)(1 - \xi^2)\,d\xi}{\sqrt{\xi(\xi + 2\eta_0)}\,\xi^2} = -\left[\sqrt{\xi(\xi + 2\eta_0)} + \sqrt{\frac{\xi + 2\eta_0}{\xi}}\,\frac{2\xi + \eta_0}{3\eta_0\xi}\right]$$

The triple integral thus has the value

$$\frac{\pi}{8}\sqrt{2\eta_0}\left(1 + \frac{10\eta_0}{3}\right) + \frac{(1 + 2\eta_0)^2}{8}\pi\operatorname{Arc\,tan}\frac{1}{\sqrt{2\eta_0}} - \pi\frac{(1 + 2\eta_0)^{\frac{3}{2}}}{3}$$

which leads to the following value for the desired C_z

$$C_z = \frac{i}{\beta\pi\eta_0}\left[\frac{(3 + 10\eta_0)}{3}\sqrt{2\eta_0} + (1 + 2\eta_0)^2\operatorname{Arc\,tan}\frac{1}{\sqrt{2\eta_0}} - \frac{8}{3}\right] \qquad \text{(IV.70)}$$

One will find in figure 89 the variation of C_z as a function of η_0.

As an application, we have traced in figure 90 the variation of $\frac{1}{2}\frac{dc_z}{di}$ as a function of the Mach number for plates of the plan form defined by figure 86. The angle of sweepback is 45°; the geometric aspect ratios are, respectively, equal to 1, 2, and 8. The points situated on the abscissa $M = \sqrt{2}$ are obtained exactly [formula (IV.70)]. The parts traced in solid lines are given by the formula (IV.64). The dotted parts are obtained by interpolation. In order to obtain them in full rigor, one would have to calculate the C_z from the formula (IV.68).

4.1.2.4 - The Uniformly Lifting Segments

The role played by the "horseshoe vortex" or "uniformly lifting segment" in the subsonic wing theory is well-known; the linear theory of Prandtl is based on this conception. We shall show how easy it is to obtain the corresponding supersonic flow, and shall indicate a few possible applications.

According to section 3.2.3.1, the conical flow for which

$$U(Z) = u_0 + i\,\frac{u_0}{\pi}\log\frac{Z}{1 + Z^2} \tag{IV.71}$$

represents a flow for which u has the value zero in the plane $x_3 = 0$ except on the quadrant Ox_1, Ox_2 where u assumes the values $\pm u_0$. Let us then apply the results of section 1.3. The homogeneous flow of zero order, defined by the complex potential

$$\Phi(Z) = -i\,\frac{p_0}{2\pi}\log\frac{Z}{1 + Z^2} \tag{IV.72}$$

may be considered as a derivative of the flow in the direction Ox_1 of the conical flow defined by equation (IV.71), and consequently defines the flow corresponding to the uniformly lifting semi-infinite line Ox_2, with the uniform lift being equal to p_0. The velocity field inside of the Mach cone Γ of O is obtained by application of the formulas (I.29)

$$u = \frac{p_0}{2\pi x_1} \frac{\rho^2 + 1}{\rho^2 - 1} \underline{R}\left[i \frac{Z^2 - 1}{Z^2 + 1}\right]$$

$$v = \frac{p_0}{2\pi x_1} \frac{\rho^2 + 1}{\rho^2 - 1} \underline{R}\left[-i \frac{\beta}{2} \frac{Z^2 - 1}{Z}\right]$$

$$w = \frac{p_0}{2\pi x_1} \frac{\rho^2 + 1}{\rho^2 - 1} \underline{R}\left[- \frac{\beta}{2} \frac{\left(Z^2 - 1\right)^2}{Z\left(Z^2 + 1\right)}\right]$$

Outside of (Γ) the velocities are zero.

If one calculates the velocity field in a plane $x_1 = x_1{}^0$ $\left(x_1{}^0\right.$ being very large$\left.\right)$, one has therefore

$$\rho \sim \frac{\beta r}{2x_1}$$

Consequently

$$u \sim 0 \qquad v = - \frac{p_0}{2\pi} \frac{\sin \theta}{r} \qquad w \frac{p_0}{2\pi} \frac{\cos \theta}{r}$$

that is, the classical vortex field.

In order to obtain the flow corresponding to the uniformly lifting segment, one visualizes the superposition of two homogeneous flows of this type. Let, for instance, A_1 and A_2 be two points of Ox_2, and Z_1 and Z_2 the values of the variable Z if one takes, respectively, A_1 and A_2 as origin. The desired flow is determined by the complex potential[54]

$$\Phi(Z) = \frac{ip_0}{2\pi}\left[\log \frac{Z_2}{1 + Z_2{}^2} - \log \frac{Z_1}{1 + Z_1{}^2}\right]$$

---

[54]The formulas here obtained have been obtained by another method by Schlichting (ref. 34).

This could form the basis of a theory of straight wings (without sweepback) analogous to the Prandtl theory for subsonic flows. However, one has not succeeded in linking the local lift with the general inclination of the profile.

On the other hand, one can apply these formulas for the study, at least in an approximate manner, of the velocity field behind a straight wing when the distribution of the circulations is known. This seems to us to be a method which should permit a first investigation of the interaction of a wing and the controls[55].

Likewise, it is very easy to define, following the same principle, the flows corresponding to two uniformly lifting semi-infinite lines $0\,\Delta_1$, $0\,\Delta_2$ (compare fig. 91). If $\beta \cot \gamma = \frac{1}{\cos \theta}$, we are dealing with a homogeneous flow of zero order, defined by the complex potential

$$\Phi(Z) = -\frac{ip_0}{2\pi} \log \frac{1 + Z^2 - 2Z \cos \theta}{1 + Z^2 + 2Z \cos \theta} \qquad \text{(IV.73)}$$

This results immediately from the formula (III.47). Likewise does the semi-infinite line $0\,\Delta_1$ when uniformly loaded, give rise to the flow defined by

$$\Phi(Z) = -\frac{ip_0}{2\pi} \log \frac{Z}{1 + Z^2 - 2Z \cos \theta}$$

In each of these cases, one can immediately write the velocity field, applying the formulas (I.29).

This permits one to define the flow about a lifting line such as A_1OA_2 which is uniformly loaded. As in the case of a straight wing, one may utilize these flows for the study of the velocity field behind a sweptback wing.

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[55]The investigation made in section 4.1.2.2 for the rectangular wing permits in fact calculation of the forces acting on the wing but does not in any case permit the study of the field behind the wing.

4.2 - Study of Fuselages

4.2.1 - Generalities Concerning the Flows Past Bodies of Revolution of Fuselage Shape

By composition of conical flows, we shall obtain a new method for the investigation of flows past bodies of revolution. The results relating to these flows have formed the subject of numerous reports (refs. 35, 36, 5); however, the methods we shall describe seem to us to permit certain generalizations. The given parameter in this problem is the value of the radial velocity v_r along a meridian line. This velocity is equal to $\frac{dr}{dx_1}(x_1)$; $r(x_1)$ is the function defining the meridian line in a plane r, x_1. However, we shall see that $v_r(x_1,r)$ is a function which is, when x_1 is fixed, of the order $\frac{1}{r}$, for a small r. The boundary condition may also be written

$$rv_r = r\frac{dr}{dx_1} = \frac{1}{2\pi}\frac{dS}{dx_1}$$

with $S(x_1) = \pi r^2$ designating the area of the fuselage section of the abscissa x_1. If one makes r tend toward zero, rv_r will maintain a finite value. In a precise manner, we shall state that the investigated flow will have to verify the following boundary condition

$$\lim_{r \to 0} rv_r = \frac{1}{2\pi}\frac{dS}{dx_1} \tag{IV.74}$$

4.2.2 - Investigation of a Particular Case

Let us consider the flow around a cone of revolution; the formulas $V(Z)$, $W(Z)$, $U(Z)$ are functions of Z which admit inside of $|Z| = 1$ only the point $Z = 0$ as a singularity. Thus they may be continued analytically to the interior of the circle (C), image of the conical obstacle in the plane Z, under the condition of excluding the origin 0 from this circle.

After this statement we shall determine the flow around a body of revolution the meridian line of which has the simple form given by figure 92. θ_0 naturally is an infinitely small angle. A first idea for obtaining such a flow consists in subtracting from a flow around a cone of revolution of the vertex O and the angle θ_0 a similar flow of vertex A. Let us put

$$\frac{x_1}{\beta r} = \frac{1 + \rho^2}{2\rho} \qquad \frac{x_1 - a}{\beta r} = \frac{1 + \rho_1^2}{2\rho_1}$$

The radial velocity of the resultant flow is

$$v_r = \frac{\beta\theta_0^2}{2}\left[\frac{1}{\rho} - \rho - \left(\frac{1}{\rho_1} - \rho_1\right)\right]$$

Let us assume that ρ and ρ_1 are infinitely small which is the case for points M which are sufficiently distant from A

$$v_r \sim \frac{\beta\theta_0^2}{2}\left[\frac{1}{\rho} - \frac{1}{\rho_1}\right] \sim \frac{\theta_0^2}{r}\left[x_1 - \left(x_1 - a\right)\right] = \frac{a\theta_0^2}{r}$$

In order to obtain the desired flow, it will therefore be necessary (which is, besides, in accordance with the theorem of section 1.1.3) to add a homogeneous flow of zero order with the vertex A defined by the complex potential

$$\Phi\left(Z_1\right) = -a\theta_0^2 \log Z_1$$

with Z_1 designating the complex variable Z for a flow with the vertex A $\left(\text{in particular } \left|Z_1\right| = \rho_1\right)$.

The resultant flow has for $x_1 > a$ the radial velocity (compare formula (I.29))

$$v_r = \frac{\beta\theta_0^2}{2}\left[\left(\frac{1}{\rho} - \rho\right) - \left(\frac{1}{\rho_1} - \rho_1\right)\right] - \frac{a\theta_0^2}{x_1 - a}\,\frac{1 + \rho_1^2}{1 - \rho_1^2}\left(\rho_1 + \frac{1}{\rho_1}\right)$$

or

$$v_r = \frac{\beta\theta_0^2}{2}\left[2\rho_1 - 2\rho - a\,\frac{\left(1 + \rho_1^2\right)^2}{1 - \rho_1^2}\,\frac{\beta r}{\left(x_1 - a\right)^2}\right]$$

The obtained radial velocity is therefore not identically zero along the conical obstacle, but it is very small when x_1 is not too close to a since ρ, ρ_1 and r are infinitely small quantities. For the rest, the equality (IV.74) is satisfied for any value $x_1 > a$. In first approximation, we regard the flow obtained as satisfying the conditions posed, although of course the value of v_r is not negligible if x_1 is close to a.

Let us now suppose that we would want to study the flow around a body of revolution which has a meridian line schematized by figure 93. One is led to visualize the flow as a resultant of the previously defined flow and a conical flow of revolution of vertex A relative to the angle θ_1. At a point M situated on the meridian line (when the abscissa of M is distinctly larger than a), one has as the radial velocity

$$v_r \sim \theta_0^2\left(\frac{x_1}{r} - \frac{x_1 - a}{r}\right) - \frac{\theta_0^2 a}{r} + \frac{\theta_1^2(x_1 - a)}{r}$$

where

$$r = (x_1 - a)\theta_1 + a\theta_0 = r(a) + (x_1 - a)\theta_1$$

If one puts

$$r(a) = a\theta_0$$

$r(a)$ designates the radius of the abscissa section $x_1 = a$.

Hence

$$v_r \sim \frac{\theta_1(r - a\theta_0)}{r} \sim \theta_1 - \frac{r(a)\theta_1}{r}$$

Since one must have $v_r = \theta_1$, one sees that one must, moreover, add the homogeneous flow of vertex A of complex potential

$$\Phi(Z) = r(a)\theta_1 \log Z$$

Finally, the case investigated is obtained:

(1) By adding a conical flow of the vertex 0 relative to the angle θ_0.

(2) By adding a conical flow of the vertex A relative to the angle θ_1.

(3) By subtracting a conical flow of the vertex A relative to the angle θ_0.

(4) By adding a homogeneous flow of zero order of complex potential

$$r(a)\Delta\theta(a)\log Z$$

where $r(a)$ is the value of the radius for $x_1 = a$ and $\Delta\theta(a)$ is the discontinuity of the angle θ for $x_1 = a$.

4.2.3 - Approximate Study of a Body of Revolution of Fuselage Shape

The application of the above said permits to obtain, in an approximate manner, the flow about a fuselage-shaped body the meridian line of which is polygonal and, by limiting process, the flow about a body of revolution the meridian line of which possesses a continuous tangent.

If one assumes first a_1, a_2, . . . a_n . . ., as the abscissas of the vertices of the polygonal line which constitutes the meridian, the desired flow will arise from the superposition:

(1) Of a succession of conical flows which cause an axial velocity of the form (formula (II.23))

$$\theta_n^2 \log \frac{\rho_n}{\rho_n^2 + 1}$$

where

$$\frac{1 + \rho_n^2}{2\rho_n} = \frac{x_1 - a_n}{\beta r}$$

with θ_n being the value of θ for $a_n < x_1 < a_{n+1}$;

(2) Of a succession of homogeneous flows which cause an axial velocity of the form (formula (I.29))

$$-\frac{r_n\,\Delta\theta_n}{x_1 - a_n}\,\frac{1 + \rho_n^{\,2}}{1 - \rho_n^{\,2}}$$

where

$$r_n = r\left(a_n\right) \qquad \Delta\theta_n = \Delta\left(\theta_n\right) = \theta_n - \theta_{n-1}$$

However, ρ_n will be very small except in the immediate neighborhood of a_n, consequently one may expect the reduced axial velocity to be written

$$u\left(x_1,r\right) = \sum \theta_n^{\,2}\log\frac{x_1 - a_{n+1}}{x_1 - a_n} - \sum\frac{r_n\,\Delta\theta_n}{x_1 - a_n}$$

with the sums $\sum$ extending to all points A_n the abscissa a_n of which is smaller than $x_1 - \beta r$. The case of a meridian with a continuous tangent is obtained by performing the limiting process in the preceding expression which leads to

$$u = -\int_0^{x_1-\beta r}\frac{\theta^2(\xi)d\xi}{x_1 - \xi} - \int_0^{x_1-\beta r}\frac{r(\xi)\theta'(\xi)}{x_1 - \xi}\,d\xi$$

However,

$$\theta(\xi) = r'(\xi) \qquad r'^2(\xi) + rr''(\xi) = \frac{1}{2\pi}\,\frac{d^2}{d\xi^2}\,S(\xi)$$

if $S(\xi) = \pi r^2(\xi)$.

One obtains

$$u = -\frac{1}{2\pi}\int_0^{x_1-\beta r}\frac{\dfrac{d^2S}{d\xi^2}}{x_1 - \xi}\,d\xi \tag{IV.75}$$

This expression is exactly the one given by Laitone (ref. 5); it is, besides, equivalent to those suggested by the other authors named before.

However, the argument just produced is somewhat summary due to the difficulties arising in the neighborhood of the points a_1, a_2, . . . a . . . In the following paragraph, we shall justify the aforesaid, in particular the important formula (IV.75).

4.2.4 - Justification of the Method

The question is to calculate the radial and axial velocities according to the rigorous formulas, and to take the possible simplifications into account only in the final result. The radial velocity comprises two terms, the first of which results from the composition of the homogeneous flows of zero order; the differential element of the corresponding integral is

$$\frac{\beta}{2}\frac{1}{x_1 - \xi}\left(\rho + \frac{1}{\rho}\right)\frac{1 + \rho^2}{1 - \rho^2}\, r(\xi)\Delta\theta(\xi) = \frac{1}{r}\,\frac{1 + \rho^2(\xi)}{1 - \rho^2(\xi)}\, r(\xi)\Delta\theta(\xi)$$

or

$$\frac{1}{r}\left[r(\xi)\theta'(\xi)d\xi + \beta r^2(\xi)\theta'(\xi)d\rho\right]$$

if one assumes $\theta(\xi)$ differentiable since

$$\frac{2\rho^2 d\xi}{1 - \rho^2} = \beta r\ d\rho$$

hence the contribution due to these flows to the radial velocity

$$\frac{1}{2}\int_0^{x_1-\beta} r(\xi)\theta'(\xi)d\xi + \int_{\rho_0}^{1} r(\xi)\theta'(\xi)d\rho$$

ρ_0 being the value of $\rho(\xi)$ for $\xi = 0$.

Likewise, the composition of the conical flow causes an integral the differential element of which is written

$$\sum \frac{\beta}{2}\,\theta^2(\xi)\frac{d}{d\xi}\left[\frac{1}{\rho(\xi)} - \rho(\xi)\right]d\xi = \sum \frac{\beta}{2}\,\theta^2(\xi)\frac{d}{d\xi}\,\frac{2(x_1 - \xi)}{\beta r} + \sum \beta\theta^2(\xi)d\rho$$

Hence the desired integral

$$\frac{1}{2}\int_0^{x_1-\beta r} \theta^2(\xi)d\xi + \beta\int_{\rho_0}^{1} \theta^2(\xi)d\rho$$

Thus the velocity is written

$$v_r = \frac{1}{2\pi r}\int_0^{x_1-\beta r} \frac{d^2S(\xi)}{d\xi^2}\,d\xi + \frac{\beta}{2\pi}\int_{\rho_0}^{1} \frac{d^2S(\xi)}{d\xi^2}\,d\rho$$

The last integral is bounded by the upper boundary of $\dfrac{d^2S(\xi)}{d\xi^2}$ and consequently the condition (IV.74) is thus verified. The calculation of u is made by a quite analogous method and leads to the formula

$$u = \frac{1}{2\pi}\int_0^{x_1-\beta r} \frac{1}{x_1 - \xi}\,\frac{d^2S}{d\xi^2}\,d\xi - \int_0^{x_1-\beta r} \frac{r(\xi)\theta^2(\xi)}{x_1 - \xi}\,\frac{2\rho^2(\xi)}{1 - \rho^2(\xi)}\,d\xi$$

$$= -\frac{1}{2\pi}\int_0^{x_1-\beta r} \frac{\dfrac{d^2S}{d\xi^2}}{x_1 - \xi}\,d\xi - \frac{1}{2\pi}\int_{\rho_0}^{1} \frac{2\rho(\xi)}{1 + \rho^2(\xi)}\left(\frac{d^2S}{d\xi^2}\right)d\rho$$

Now it is quite obvious that this last integral is negligible compared to the first. Thus the formula (IV.75) is established. It furnishes the following approximation for the pressure coefficient

$$C_p = \frac{1}{\pi}\int_0^{x_1-\beta r} \frac{1}{x_1 - \xi}\,\frac{d^2S}{d\xi^2}\,d\xi \tag{IV.76}$$

Remark.

In chapter II, we had utilized the formula (I.10) for writing the pressure coefficient. This formula would lead to write here

$$C_p = \frac{1}{\pi} \int_0^{x_1 - \beta r} \frac{d^2 S}{d\xi^2} \frac{d\xi}{x_1 - \xi} - r'^2(x)$$

One will compare this formula with the one given in reference 36. Nevertheless, the analysis just made does not guarantee that the term r'^2 represents all terms of the second order; therefore, besides, in accordance with Laitone, we shall content ourselves with the formula (IV.76).

4.2.5 - Generalizations

The method indicated above has the advantage not only of giving a new demonstration of the formulas relative to flows of revolution, but also of furnishing a more general method which lends itself to application to numerous fuselage problems.

Let us take, for instance, the case of fuselages of revolution the axis of which is slightly inclined toward the wind direction. One may reassume the preceding method, starting out from the flow about a cone of revolution inclined toward the wind (formulas (II.24) and (II.25)). The desired flow is obtained by suitable superposition of those conical flows and of homogeneous flows of zero order which one deduces from them by differentiating these flows in the direction of the axis of the fuselage (compare section 1.3).

It is permissible to assume that this method will also permit the study of fuselages which are not bodies of revolution but the cross section of which remains, for instance, homothetic. Certain difficulties make their appearance, but do not seem insurmountable. In entering on the investigation of fuselages by the method of conical flows, we aimed only at indicating the principle of a new method. We reserve the development for a later report[56].

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[56]Compare in appendix No. 7 the development of this idea.

4.3 - First Investigation Regarding the Conical Flows Past a Flat Dihedral. Applications to the Fins and Control Surfaces.

We have already indicated in the course of this chapter that there exist other conical flows than the flows with infinitesimal cone angles or the flows flattened in one direction. In this last paragraph, we shall give a few examples of flows past a flat dihedral. These flows may be utilized either for the study of the effect of dihedral on a lifting wing or for the study of the fins and control surfaces. We can here not consider developing the complete theory of these flows. We shall content ourselves with indicating a few examples.

4.3.1 - Effect of Dihedral on a Wing Completely Bisecting the Mach Cone

Let us consider a Δ wing having dihedral; this wing is infinitely flattened into two planes which intersect in Ox_1. For simplification, we shall assume that the plane Ox_1x_3 is a plane of symmetry, the wing completely bisecting the Mach cone; upper and lower sides are therefore "independents." This signifies that in the plane Z the region inside of (C_0) is divided into two domains (fig. 94). The wing portion inside of the Mach cone (Γ) is represented by two radii OD, OD' which form with OX the angles θ_0 and $\pi - \theta_0$. The bounding generatrices of the Δ have as images the points E and E' of the argument θ_1 and $\pi - \theta_1$ on the circle. One will assume, in order to better establish the ideas, that $0 < \theta_0 < \theta_1 < \frac{\pi}{2}$.

The boundary conditions which permit determination of the unknown functions U(Z), V(Z), W(Z) in the region ODEE'D'O are:

(1) On the arc EE' $\qquad u = v = w = 0$

(2) On the arc ED and on the segment OD $\qquad w \cos \theta_0 - v \sin \theta_0 = \alpha$

(3) On the arc E'D' and on the segment OD' $\qquad w \cos \theta_0 + v \sin \theta_0 = \alpha$

We shall treat here the elementary case; consequently, α will be considered constant. The condition

$$w \cos \theta_0 - v \sin \theta_0 = \alpha$$

entails that on OD

$$\underline{R}\left[Z \frac{dW}{dZ} \cos \theta_0 - Z \frac{dV}{dZ} \sin \theta_0\right] = 0$$

or also

$$\underline{R}\left[\left[\left(Z + \frac{1}{Z}\right)\sin \theta_0 + i\left(Z - \frac{1}{Z}\right)\cos \theta_0\right] Z \frac{dU}{dZ}\right] = 0$$

whence

$$\underline{T}\left[Z \frac{dU}{dZ}\right] = 0$$

The normal derivative of u is zero along OD.

One would have an analogous result on the segment OD'.

On the other hand, on ED

$$\underline{T}\left[Z \frac{dW}{dZ} \cos \theta_0 - Z \frac{dV}{dZ} \sin \theta_0\right] = 0$$

which entails also

$$\underline{T}\left[Z \frac{dU}{dZ}\right] = 0$$

Consequently, u maintains a constant value on ED and E'D'.

Besides, it is easy to calculate this value owing to the formulas (III.46); one finds

$$u_0 = \frac{\alpha}{\beta \sin\left(\theta_0 - \theta_1\right)}$$

In order to achieve the calculation of U(Z) it is then necessary to carry out the conformal transformation

$$\tau = i^{1-m}Z^m$$

where

$$m = \frac{\pi}{\pi - 2\theta_0}$$

The domain investigated is represented on a semicircle of the plane τ (compare fig. 95). The homologous point of E has as argument

$$\varphi_1 = \frac{\pi(\theta_1 - \theta_0)}{\pi - 2\theta_0}$$

Now the function $U(\tau)$ can be written immediately on the strength of the results of section 3.2.2.1

$$U(\tau) = \frac{-i\alpha}{\pi\beta \sin(\theta_0 - \theta_1)} \log \frac{\left(\tau + e^{-i\varphi_1}\right)\left(1 - \tau e^{i\varphi_1}\right)}{\left(1 + \tau e^{-i\varphi_1}\right)\left(e^{i\varphi_1} - \tau\right)} \tag{IV.77}$$

and according to formula (III.53) one may write the value of the pressure coefficient on the wing portion inside of (Γ)

$$C_p = \frac{4\alpha}{\pi\beta \sin(\theta_1 - \theta_0)} \text{Arc sin} \frac{\sin \varphi_1}{\sqrt{1 - x^2\cos^2\varphi_1}}$$

putting

$$x = \beta \tan \omega$$

In order to link θ_1 to the angle ω_0 defining the bounding generatrices of the cone, one will remark that

$$\theta_1 = \eta_0 + \theta_0 \qquad \text{with} \qquad \cos \eta_0 = \frac{1}{\beta \tan \omega_0}$$

It is easy to obtain the component of the normal forces on the upper surface of each half wing; one will express this component by the dimensionless coefficient

$$C_N = -\frac{1}{\beta \tan \omega_0}\int_0^{x_0} C_p \, dx \qquad x_0 = \beta \tan \omega_0$$

In order to calculate C_N one will use the plane τ

$$C_N = -\frac{1}{2\beta \tan \omega_0}\left[\int_{\underline{L}} C_p \frac{2\left(1 - \tau^2\right)}{\left(1 + \tau^2\right)^2} d\tau\right]$$

$$= \frac{2}{\beta \tan \omega_0}\int_{\underline{L}} \underline{R}\left[U(\tau)\frac{1 - \tau^2}{\left(1 + \tau^2\right)^2} d\tau\right]$$

$$= \frac{2}{\beta \tan \omega_0} \underline{R}\left[\int_{\underline{L}} U(\tau)\frac{1 - \tau^2}{\left(1 + \tau^2\right)^2} d\tau\right]$$

with $\underline{L}$ denoting the contour e'd'de in the plane τ (fig. 95).

The calculation of this integral has already been performed in section 3.2.2.2. Hence

$$C_N = -\frac{2\alpha}{\beta \tan \omega_0}\frac{1}{2\beta}\frac{2}{\cos \varphi_1}\frac{\sin \varphi_1}{\sin\left(\theta_1 - \theta_0\right)}$$

$$= -\frac{2\alpha}{\beta}\frac{\cos\left(\eta_0 + \theta_0\right)}{\cos \varphi_1}\frac{\sin \varphi_1}{\sin\left(\theta_1 - \theta_0\right)} \qquad \text{(IV.78)}$$

Remarks.

(1) It is obvious that the general case where α would be variable over the span can be investigated without difficulty with the aid of electric analogies.

(2) The treatment of the case where the cone representing a dihedral is entirely inside the Mach cone is more difficult. The domain where the functions $U(Z)$, $V(Z)$, $W(Z)$ must be studied is annular, and in contrast to what occurred in section 3.1, the conformal representation of such a domain on a circular annulus does not seem to follow immediately.

(3) It is possible to study the effect of dihedral on a rectangular or on a sweptback wing by "composition" using the methods developed in section 4.1.

4.3.2 - Fin at the Wing Tip

Let us consider, for instance, the edge AA' of a rectangular wing of large aspect ratio; we shall assume the fin to be formed by a triangular plate ABB' (fig. 96) which we shall suppose, to start with, as lined up with the wind. We aim to calculate the effect of this fin on the flow.

4.3.2.1 - It is almost evident that if the semi-infinite lines AB, AB' are outside of the Mach cone of A, the fin suppresses the end effect of AA'

Let us consider, for instance, the case where the wing is reduced to a lifting plate in the plane Z; the boundary conditions for the quadrant OAB read, in fact, as follows:

$$w = w_0, \text{ on } OA \text{ and } AB$$

$$v = 0, \text{ on } OB$$

They are the same that would be valid for a flow around a plate of infinite span placed at a certain incidence with respect to the wind.

In contrast, the perturbation flow in the quadrant OA'B is identically zero. This result applies, by the way, likewise to the "thickness effect." We deal, therefore, not with a new mathematical problem, but simply with a remark which can be utilized in certain technical problems.

If now the fin is itself a lifting surface, that is, if v assumes on the fin a constant value different from zero, the case is particularly simple and one may conclude immediately that it is the one where the bounding generatrices of the fin are symmetrical with respect to the plane x_1Ox_2. In fact, if the fin were by itself, it would give rise to a flow of such a type that the component w would be zero in the plane Ox_1x_2. Thus it suffices to add this flow to the one found in the case where the fin is lined up with the wind[57].

4.3.2.2 - The case where the bounding generatrices of the fin are inside of the Mach cone gives rise to a new problem

If C and C' are the images of these generatrices in the plane Z (fig. 97), we shall suppose, for instance, that C and C' are symmetrical with respect to O, and shall study the effect of the fin on an elementary symmetrical problem. The boundary conditions are:

$w = w_0$ on the upper edge of OA and on the arc $\widehat{AB}$

$w = -w_0$ on the lower edge of OA and on the arc $\widehat{AB'}$

$v = 0$ on the two edges of OC and of OC'

For reasons of symmetry one also has $w = 0$ on OA'.

We shall discuss the function $Z\frac{dU}{dZ}$ $\left(\text{the function } F(Z) \text{ introduced in section 3.1.1 is proportional to } Z\frac{dU}{dZ}\right)$. The boundary conditions inform us that $Z\frac{dU}{dZ}$ is real on the contour ABA'OCOA. On the other hand, according to the results obtained in chapter 3, B is a simple pole for this function while C is a critical point of the order $p + 1/2$, p being an integer. Reassuming the arguments raised in section 3.1, one sees that the simplest (in the sense of the principle of minimum singularities) of the functions which satisfy these conditions is written

$$Z\frac{dU^{(1)}}{dZ} = -\frac{k}{\beta}\frac{Z^2}{\left(Z^2 + 1\right)\left[\left(Z^2 + c^2\right)\left(1 + c^2Z^2\right)\right]^{1/2}} \qquad \text{(IV.79)}$$

We denote by the index (1) the corresponding solution.

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[57]For reasons of simplification, we have visualized the case where the bounding generatrices of the cone were normal to the wind; it is easy to treat in the same manner the case where this condition is not satisfied.

Besides, u is zero on the arc A'B, and $u = -\frac{w_0}{\beta}$ on the arc AB. Consequently, one has, if one takes as the initial determination for the radical the positive one on the upper edge of OA, according to equation (III.46)

$$k = \frac{2w_0}{\pi}\left(1 - c^2\right)$$

The integration of equation (IV.79) does not present any difficulty; naturally, the integration constant must be chosen in such a manner that $u = 0$ for $Z = -1$. One finds

$$U^{(1)}(Z) = \frac{w_0 i}{\beta\pi} \log\left[\frac{i\left(1 - Z^2\right)\left(1 - c^2\right) - 2\sqrt{\left(Z^2 + c^2\right)\left(1 + c^2Z^2\right)}}{\left(Z^2 + 1\right)\left(1 + c^2\right)}\right] \tag{IV.80}$$

with the logarithm having the value $i\pi$ for $Z = 1$.

The explicit calculation of $W^{(1)}(Z)$ and $V^{(1)}(Z)$ may be made by the elliptic functions. One must, in fact, examine whether all boundary conditions are satisfactorily verified. Now

$$\frac{dV^{(1)}}{dZ} = + \frac{w_0\left(1 - c^2\right)}{\beta\pi} \frac{1}{\left[\left(Z^2 + c^2\right)\left(1 + c^2Z^2\right)\right]^{1/2}}$$

Consequently, if one puts

$$Z = ic \operatorname{sn}\left(\tau, c^2\right)$$

the investigated region of the plane Z has as image in the plane τ a rectangle (compare section 3.1.1.8 and fig. 34) and one obtains

$$\frac{dV^{(1)}}{d\tau} = \frac{dV^{(1)}}{dZ} \frac{dZ}{d\tau} = \frac{iw_0}{\pi\beta}\left(1 - c^2\right)$$

$$V^{(1)} = \frac{iw_0}{\beta\pi}\left(1 - c^2\right)\left(\tau - i \frac{K'}{2}\right)$$

The integration constant is chosen in such a manner that $v = 0$ on the circle (C_0). The solution $U^{(1)}(Z)$, $V^{(1)}(Z)$, $W^{(1)}(Z)$ thus does not satisfy the boundary conditions posed; it corresponds to the case where the fin itself is inclined toward the wind direction with the value of v on the fin being equal to

$$v^{(1)} = \frac{w_0}{\beta\pi} \frac{\left(1 - c^2\right)K'}{2} \tag{IV.81}$$

On the other hand, one finds for $W^{(1)}(Z)$

$$\frac{dW^{(1)}}{d\tau} = \frac{dW^{(1)}}{dZ} \frac{dZ}{d\tau} = - \frac{w_0\left(1 - c^2\right)}{\beta\pi} \frac{1 + c^2 sn^2\tau}{1 - c^2 sn^2\tau}$$

$W^{(1)}$ is, therefore, expressed as a function of τ by an elliptic integral of the third kind.

After having thus defined the solution $U^{(1)}(Z)$, $V^{(1)}(Z)$, $W^{(1)}(Z)$ it is easy to obtain the one which is relative to the posed boundary problem; it suffices to add a solution $U^{(2)}(Z)$, $V^{(2)}(Z)$, $W^{(2)}(Z)$ so that

(1) $u^{(2)} = v^{(2)} = w^{(2)} = 0$, on (C_0)

(2) $w^{(2)} = 0$, on OA and OA'

(3) $v^{(2)} = -v^{(1)}$, on the two edges of the cut CC'

This flow is, except for the notations, the one which has been studied in section 3.1.1.7. In particular, the value of the function $U^{(2)}(Z)$ is written

$$U^{(2)}(Z) = \frac{2}{\beta} \frac{c^2}{c^2 + 1} \frac{v^{(1)}}{E\left(\frac{1 - c^2}{1 + c^2}\right)} \frac{1 - Z^2}{\left[\left(c^2 + Z^2\right)\left(1 + Z^2c^2\right)\right]^{1/2}} \tag{IV.82}$$

One obtains thus the following general result: if one must on the fin have $v = v_0$, the value of the function $U(Z)$ is given by the formula

$$U(Z) = \frac{iw_0}{\beta\pi} \log\left[\frac{i(1 - Z^2)(1 - c^2) - 2\sqrt{(Z^2 + c^2)(1 + c^2Z^2)}}{(1 + Z^2)(1 + c^2)}\right] +$$

$$\frac{2c^2}{\beta(1 + c^2)E\left(\frac{1 - c^2}{1 + c^2}\right)}\left[-v_0 + \frac{w_0}{2\beta\pi}(1 - c^2)K' \frac{(1 - Z^2)}{\sqrt{(Z^2 + c^2)(1 + c^2Z^2)}}\right] \tag{IV.83}$$

One will see that in the case where $v_0 \rightarrow 0$ and $c^2 \rightarrow 1$, one finds, at the limit, the result foreseen in the case where the fin bisects the Mach cone (4.3.2.1); and that, if $c \rightarrow 0$, one falls back on the solution of section 3.2.2.1 (equation (III.57)). One may then calculate the pressure coefficient on the wing (Z real and positive), and finds

$$C_p = \frac{2w_0}{\beta\pi}\left[\frac{\pi}{2} + \text{Arc cos } \sqrt{(1 - x^2)(1 - \gamma^2)}\right] +$$

$$\frac{\gamma^2}{\beta E\left(\sqrt{1 - \gamma^2}\right)}\left[2v_0 - \frac{w_0}{\beta\pi}(1 - c^2)K'\right]\sqrt{\frac{1 - x^2}{x^2(1 - \gamma^2) + \gamma^2}}$$

putting

$$x = \frac{2\rho}{1 + \rho^2} \qquad \gamma = \frac{2c}{1 + c^2}$$

4.3.3 - Crossed Wings

To terminate these few remarks regarding the calculation of the effects of dihedral, we shall give a few indications regarding the case of crossed wings.

Let us consider a cone flattened in two directions of the planes Ox_1x_2, Ox_1x_3. The function w on the two faces of the triangle OAA' and the function v on the two faces of the triangle OBB' are known.

Let us suppose that OB and OB' are symmetrical with respect to Ox_1x_2, and that OA and OA' are symmetrical with respect to Ox_1x_3; under these conditions the flow around the crossed wing is obtained in a particularly simple manner. It suffices to superimpose the flow which is infinitely flattened into the plane Ox_1x_3 and realizes the desired values for v, and the flow which is infinitely flattened into the plane Ox_1x_2 and realizes the desired values for w. In fact, due to the symmetry, the first flow gives a value of zero for w in the plane Ox_1x_2, and the second a value of zero for v in the plane Ox_1x_3.

The case where the crossed wing does not admit two planes of symmetry cannot be treated as simply in the general case. Particularly, the case where the bounding generatrices are all entirely inside the Mach cone leads doubtlessly to analytical solutions which can be explicitly expressed only with difficulty, even in the elementary case. However, as in all these problems concerning the effect of dihedral, the solution is facilitated by the utilization of conformal representations. Although they are hard to obtain in explicit analytical form, they may be determined accurately by judicious utilization of the general method of electric analogies.

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Note: The principal questions treated in this memorandum have been summarized in a certain number of notes to the Comptes rendus de l'Académie des Sciences:

Volume 224 - 1947 p. 183
Volume 225 - 1947 p. 487
Volume 226 - 1948 p. 311
Volume 226 - 1948 p. 1126

APPENDIX

No. 1 - Theorem of Existence and Singularities of the Solution for a Flow Infinitely Flattened in One Direction

1. Generalities.- The source method which should be called more exactly the "method of the fundamental solution of Hadamard" permits the general investigation of the flows about obstacles which are infinitely flattened in one direction. Several authors (compare refs. 1, 2, 3, and 4 of the references for the appendix) have independently investigated this problem. We ourselves have studied this question in collaboration with M. R. Bader. Since the corresponding report (ref. 5) has not been officially published, we shall give here the results which seem to us original with regard to the investigations quoted. With the same notations as in the text, the problem may be formulated in the following manner (see fig. 1)*:

Find a solution $\varphi(x_1,x_2,x_3)$ satisfying the equation

$$L(\varphi) = \beta^2 \frac{\partial^2\varphi}{\partial x_1{}^2} - \frac{\partial^2\varphi}{\partial x_2{}^2} - \frac{\partial^2\varphi}{\partial x_3{}^2} = 0$$

and the boundary conditions:

(1) at infinity upstream: $\varphi = 0$, $\overrightarrow{\text{grad}}\,\varphi = 0$;

(2) on (S), projection on Ox_1x_2 of the obstacle:

$$\frac{\partial\varphi}{\partial x_3} = k^+(x_1,x_2) \qquad \text{for} \qquad x_3 = +0$$

$$\frac{\partial\varphi}{\partial x_3} = k^-(x_1,x_2) \qquad \text{for} \qquad x_3 = -0$$

k^+ and k^- are known functions which satisfy the conditions of regularity (II) relative to $\partial\varphi/\partial x_3$ which will be specified below.

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*Figures for this appendix are found on pp. 332-333.

In order to pose the problem correctly, one must furthermore state exactly the hypothesis of regularity which one imposes on the solution; we shall denote by (R) the portion of Ox_1x_2 which corresponds to the wake of the flattened body on (S).

(I) φ is continuous, except for, eventually, across the plane $x_1 = 0$ on (S) and (R).

(II) The first and second derivatives of φ exist and are generally continuous outside of (S); a possible exception may occur across certain characteristic surfaces where the derivatives may have either discontinuities of the first kind at a regular point or infinities at an exceptional point. Nevertheless, they may have infinities on (S) in order to satisfy the hypothesis of linearization; $\partial\varphi/\partial x_3$ can become infinite only on parts of the boundary of (S) and only when one approaches it by remaining outside of (S).

Furthermore, we shall assume $\partial\varphi/\partial x_3$ and $\partial\varphi/\partial x_1$ to be continuous if one traverses Ox_1x_2 at a point outside of (S). This hypothesis has an immediate physical significance for $\partial\varphi/\partial x_3$; the same holds true for $\partial\varphi/\partial x_1$ if one recalls that this quantity is proportional to the pressure. In other words, only $\partial\varphi/\partial x_2$ can have a discontinuity of the first kind across Ox_1x_2.

Finally, φ can be divided (as in chapter III) into its odd and even parts with respect to x_3. If φ is odd in x_3 (symmetrical problem), $\partial\varphi/\partial x_3 = 0$ outside of (S). If φ is even in x_3 (lifting problem), $\partial\varphi/\partial x_1 = 0$ in Ox_1x_2 outside of (S) as it results from the hypothesis (II).

2. Fundamental formula.- We shall utilize the generalized formula of Green

$$\iiint_V \left[uL(v) - vL(u)\right]d\tau = - \iint_{\Sigma}\left[u\frac{dv}{d\nu} - v\frac{du}{d\nu}\right]d\sigma$$

Σ is the surface having an element $d\sigma$ which bounds the volume V having an element $d\tau$; the derivatives $d/d\nu$ are the derivatives in the transverse direction. Thus one has, if Σ is defined by $F(x_1,x_2,x_3) = 0$ with $F(x_1,x_2,x_3) > 0$ outside of V

$$- \frac{d}{d\nu} = \beta^2 \frac{\partial F}{\partial x_1} \frac{\partial}{\partial x_1} - \frac{\partial F}{\partial x_2} \frac{\partial}{\partial x_2} - \frac{\partial F}{\partial x_3} \frac{\partial}{\partial x_3}$$

Finally, utilizing the conception of the "finite part" of an integral originated by Hadamard, one may apply Green's formula to functions u and v which cause the employed integrals to become infinite. One then writes

$$\overline{\iiint_V} \left[uL(v) - vL(u)\right] d\tau = - \overline{\iint_{\Sigma}} \left[u \frac{dv}{d\nu} - v \frac{du}{d\nu}\right] d\sigma$$

Let us consider at a point $P(\xi_1,\xi_2,\xi_3)$ $(\xi_3 > 0$ for instance), the Mach forecone Γ and let us intersect it by the plane $x_1 = -A$ where A is positive and very large, and by the plane $x_3 = 0$. We determine thus a volume V in the region $x_3 > 0$, bounded by a surface Σ. Admitting the existence of φ, we apply Green's formula to the pair

$$u = \varphi(x_1,x_2,x_3)$$

$$v = H = \frac{1}{\sqrt{(\xi_1 - x_1)^2 - \beta^2 \left[(\xi_2 - x_2)^2 + (\xi_3 - x_3)^2\right]}}$$

H is the fundamental solution, in the sense of Hadamard, for the wave equation.

We cannot discuss here all the details and all justifications but we shall note the principal stages of the demonstration.

(a) It is shown that the generalized formula of Green can be applied effectively to the pair φ, H, even if the derivatives of φ present discontinuities of the first kind, owing to (II) which informs us that these discontinuities occur on characteristic surfaces.

(b). For the part of Σ situated on $x_1 = -A$, the double integral becomes zero due to the boundary conditions.

(c) On the cone Γ the double integral must be taken at its finite part. Let us introduce the cone Γ_ϵ with the equation

$$\beta^2\left[(x_2 - \xi_2)^2 + (x_3 - \xi_3)^2\right] = (1 - \epsilon)^2(x_1 - \xi_1)^2$$

and the plane P_δ

$$x_1 = \xi_1 - \delta \qquad (\delta > 0)$$

Since ϵ and δ are small, one will calculate the double integral on the surface adjoining $\sum$, formed on one hand by Γ_ϵ and on the other by the circle $C_{\epsilon\delta}$, with the section of Γ_ϵ made by the plane P_δ. One can easily show that the contribution due to Γ_ϵ has a finite part of zero, and that the one due to $C_{\epsilon\delta}$ is $-2\pi\varphi(P)$. Consequently, one obtains, denoting by h the section of (Γ) by $x_3 = 0$, the relation

$$\varphi(P) = \frac{1}{2\pi}\overline{\iint_h \varphi \frac{\partial H}{\partial x_3}\, d\sigma} - \frac{1}{2\pi}\iint_h H \frac{\partial \varphi}{\partial x_3}\, d\sigma$$

(d) In order to eliminate φ in the second term, one may apply the image method utilized by V. Volterra in an analogous problem. Let P' be the symmetric point of P with regard to Ox_1x_2; let us apply Green's formula to the volume V situated in $x_3 > 0$, bounded by the planes $x_1 = -A$, $x_3 = 0$, and the Mach forecone of P' by putting

$$u = \varphi(x_1, x_2, x_3) \qquad v = \overline{H} = H(P')$$

One thus obtains

$$0 = \frac{1}{2\pi}\overline{\iint_h \varphi \frac{\partial \overline{H}}{\partial x_3}\, d\sigma} - \frac{1}{2\pi}\iint_h \overline{H} \frac{\partial \varphi}{\partial x_3}\, d\sigma$$

and since for $x_3 = 0$

$$\overline{H} = H \qquad \frac{\partial \overline{H}}{\partial x_3} = -\frac{\partial H}{\partial x_3}$$

one has

$$\overline{\iint_h} \varphi \frac{\partial H}{\partial x_3} d\sigma = - \iint_h H \frac{\partial \varphi}{\partial x_3} d\sigma$$

Combining this result with the preceding one, one obtains the desired fundamental formula

$$\varphi(P) = - \frac{1}{\pi} \iint_h H \frac{\partial \varphi}{\partial x_3} d\sigma = \frac{1}{\pi} \overline{\iint_h} \varphi \frac{\partial H}{\partial x_3} d\sigma$$

3. The theorem of existence for the symmetrical problem.- In a symmetrical problem $\partial\varphi/\partial x_3$ is known on every face of $x_3 = 0$; consequently φ may be calculated in the entire space. The existence of the solution will be established if one verifies that this function φ satisfies $L(\varphi) = 0$, the boundary conditions, and the conditions of regularity.

(a) $L(\varphi) = 0$, for the functions $k(x_1,x_2)$ satisfying the hypothesis of regularity; one may calculate the derivatives of φ by deriving under the sum sign with respect to the coordinates of P. Since only H depends on these coordinates and H satisfies $L(H) = 0$, the result follows from it as Hadamard has shown in a very general manner.

(b) In order to verify the boundary conditions, one must show that

$$\lim_{\xi_3 \to 0} \frac{\partial \varphi}{\partial \xi_3}(\xi_1,\xi_2,\xi_3) = \lim_{\xi_3 \to 0} - \frac{1}{\pi} \overline{\iint_h} \frac{\partial H}{\partial \xi_3} \frac{\partial \varphi}{\partial x_3} d\sigma = k^+(\xi_1,\xi_2) \qquad (\xi_3 > 0)$$

This verification is easy if one puts

$$x_1 = \xi_1 + \frac{\beta \xi_3}{(1 - \mu)\sin \theta} \qquad x_2 = \xi_2 - \xi_3 \cot \theta$$

in the integral and then going to the indicated limit.

(c) Verification of the conditions of regularity leads to a careful study of the behavior of φ and its derivatives. We can give here only the conclusions of this study.

A. In the plane Ox_1x_2, let P^+ and P^- be two points lined up with P so that

$$P^+P = P^-P = \epsilon$$

(1) If there are only isolated points of discontinuity of $\partial\varphi/\partial x_3$ on the Mach lines ahead of P, and if $\partial\varphi/\partial x_3$ is continuous at P

$$\varphi(P^+) = \varphi(P^-) + O(\epsilon)$$

that is, φ is continuous at P, of the order ϵ. An analogous result is valid for the first derivatives $\partial\varphi/\partial x_1$, $\partial\varphi/\partial x_2$.

(2) If there is only a finite number of points of discontinuity on the Mach lines ahead of P and if P is a point of a supersonic line (compare chapter IV) of discontinuity for $\partial\varphi/\partial x_3$, φ is continuous of the order ϵ, but $\partial\varphi/\partial x_1$ and $\partial\varphi/\partial x_2$ have discontinuities of the first kind. In particular, if the tangent to the line of discontinuity at P forms with Ox_1 the angle ω, the discontinuities of $\partial\varphi/\partial x_1$ and of $\partial\varphi/\partial x_3$ are connected by the well-known relation

$$\Delta\left(\frac{\partial\varphi}{\partial x_1}\right) = -\frac{\tan\omega}{\sqrt{\beta^2\tan^2\omega - 1}}\,\Delta\left(\frac{\partial\varphi}{\partial x_3}\right)$$

(3) If there is only a finite number of points of discontinuity on the Mach lines ahead of P, and if P is a point of a subsonic line of discontinuity, the first derivatives of φ become infinite as $\log\epsilon$ when one tends toward P.

(4) If there is a discontinuity of $\partial\varphi/\partial x_3$ on an entire segment of one of the Mach lines ahead of P, the first derivatives of φ become there infinite as $\epsilon^{-1/2}$.

B. Outside of the plane Ox_1x_2 one has the following results:

(1) If the boundary of h is not at any point tangent to a line of discontinuity of $\partial\varphi/\partial x_3$, and does not contain any finite part of such a line, the first derivatives of φ are continuous and of the order ϵ.

(2) If the boundary of h is at certain points tangent to a line of discontinuity of $\partial\varphi/\partial x_3$ without containing any finite part of such a line, P is situated on the characteristic surface which has this line of discontinuity as directrix, and the first derivatives of φ admit discontinuities of the first kind at P when traversing this surface.

(3) If in exceptional cases the boundary of h contains a part of a line of discontinuity of $\partial\varphi/\partial x_3$, the first derivatives of φ become infinite as $\epsilon^{-1/2}$; besides, such a point is necessarily isolated.

All these results taken together show that the conditions of regularity are satisfied which proves the existence of the solution found in this manner.

4. The theorem of existence for the lifting problem.- We shall insist less on the calculation of the solution, which one can find in the published memoranda quoted before, particularly in reference 4, than on the study of its singularities. However, in order to make this investigation, we must indicate briefly the procedure of the calculation; we shall do so for the simplest case, the one where the edges of the wing are independent. (Compare fig. 2.)

The fundamental formula permits the calculation of the potential when one knows $\partial\varphi/\partial x_3$ on the entire plane Ox_1x_2.

It is clear that this quantity is zero upstream from the line AMM_1, with MM_1 being the characteristic tangent to the leading edge of the wing.

In order to calculate this quantity in the regions where it remains provisionally unknown, it is advisable to make the change of variable

$$\left.\begin{aligned} x_1 - \beta x_2 &= \lambda \\ x_1 + \beta x_2 &= \mu \end{aligned}\right\} \qquad k\left(x_1,x_2\right) = K(\lambda,\mu)$$

If $\mu = \mu_1(\lambda)$ and $\mu = \mu_2(\lambda)$ are the equations of the arcs AM and MNQ, one has at a point λ_0, μ_0 of the region M_1MNN_1 (since in this region φ is zero) the equation

$$I(\lambda_0,\mu_0) = \int_{\lambda_M}^{\lambda_0} \frac{d\lambda}{\sqrt{\lambda_0 - \lambda}} \int_{\mu_1(\lambda)}^{\mu_2(\lambda)} \frac{K(\lambda,\mu)\,d\mu}{\sqrt{\mu_0 - \mu}} +$$

$$\int_{\lambda_M}^{\lambda_0} \frac{d\lambda}{\sqrt{\lambda_0 - \lambda}} \int_{\mu_2(\lambda)}^{\mu_0} \frac{\partial\varphi}{\partial x_3} \frac{d\mu}{\sqrt{\mu_0 - \mu}} = 0$$

this equation entails the equality

$$\int_{\mu_2(\lambda)}^{\mu_0} \frac{\partial\varphi}{\partial x_3} \frac{d\mu}{\sqrt{\mu_0 - \mu}} + \int_{\mu_1(\lambda)}^{\mu_2(\lambda)} \frac{K(\lambda,\mu)}{\sqrt{\mu_0 - \mu}}\,d\mu = 0$$

which determines $\partial\varphi/\partial x_3(\lambda_0,\mu_0)$ by the inversion of an equation of Abel. One finds (ref. 4)

$$\frac{\partial\varphi}{\partial x_3} = -\frac{1}{\pi\sqrt{\mu_0 - \mu_2(\lambda_0)}} \int_{\mu_1(\lambda_0)}^{\mu_2(\lambda_0)} \frac{K(\lambda_0,\mu')\sqrt{\mu_2(\lambda_0) - \mu'}}{\mu - \mu'}\,d\mu'$$

thus one knows $\partial\varphi/\partial x_3$ in the region M_1MNN_1.

At a point where $\varphi(\lambda_0,\mu_0)$ is not zero (for instance on the wake), one has

$$-2\pi\beta\varphi(\lambda_0,\mu_0) = I(\lambda_0,\mu_0)$$

which gives, after a double Abel inversion

$$\frac{\partial\varphi}{\partial x_3}(\lambda_2,\mu_2) = -\frac{1}{\pi}\frac{\partial}{\partial\mu_2}\int_{\mu_2(\lambda_2)}^{\mu_2}\frac{d\mu_0}{\sqrt{\mu_2-\mu_0}}\int_{\mu_1(\lambda_2)}^{\mu_2(\lambda_2)}\frac{K(\lambda_2,\mu)\,d\mu}{\sqrt{\mu_0-\mu}} -$$

$$\frac{2\beta}{\pi}\frac{\partial}{\partial\mu_2}\int_{\mu_2(\lambda_2)}^{\mu_2}\frac{d\mu_0}{\sqrt{\mu_2-\mu_0}}\frac{\partial}{\partial\lambda_2}\int_{\lambda_N}^{\lambda_2}\frac{\varphi\,d\lambda_0}{\sqrt{\lambda_2-\lambda_0}}.$$

This equation contains two unknown functions, and in general it will be impossible to determine them both without introducing a supplementary hypothesis. But if one supposes that:

<u>$\frac{\partial\varphi}{\partial x_3}$ is continuous in Ox_1x_2 when traversing the subsonic trailing edge,</u>

it will be seen that it is easy to calculate first φ on the wake, and then $\partial\varphi/\partial x_3$ in Ox_1x_2. The preceding equation is written

$$\frac{\partial\varphi}{\partial x_3} = -\frac{1}{\pi}\frac{1}{\sqrt{\mu_2-\mu_2(\lambda_2)}}\int_{\mu_1(\lambda_2)}^{\mu_2(\lambda_2)}\frac{K(\lambda_2,\mu)\,d\mu}{\sqrt{\mu_2(\lambda_2)-\mu}} +$$

$$\frac{1}{2\pi}\int_{\mu_2(\lambda_2)}^{\mu_2}\frac{d\mu_0}{\sqrt{\mu_2-\mu_0}}\int_{\mu_2(\lambda_2)-\epsilon}^{\mu_2(\lambda_2)}\frac{K(\lambda_2,\mu)\,d\mu}{(\mu_0-\mu)^{3/2}} +$$

$$\frac{1}{2\pi}\int_{\mu_2(\lambda_2)}^{\mu_2}\frac{d\mu_0}{\sqrt{\mu_2-\mu_0}}\int_{\mu_1(\lambda_2)}^{\mu_2(\lambda_2)-\epsilon}\frac{K(\lambda_2,\mu)\,d\mu}{(\mu_0-\mu)^{3/2}} -$$

$$\frac{2\beta}{\pi}\frac{1}{\sqrt{\mu_2-\mu_2(\lambda_2)}}\int_{\lambda_N}^{\lambda_2}\frac{\varphi_{\lambda_0}'\left[\lambda_0,\mu_2(\lambda_2)\right]}{\sqrt{\lambda_2-\lambda_0}}\,d\lambda_0 -$$

$$\frac{2\beta}{\pi}\int_{\mu_2(\lambda_2)}^{\mu_2}\frac{d\mu_0}{\sqrt{\mu_2-\mu_0}}\frac{\partial}{\partial\mu_0}\int_{\lambda_N}^{\lambda_2}\frac{\varphi_{\lambda_0}'(\lambda_0,\mu_0)}{\sqrt{\lambda_2-\lambda_0}}\,d\lambda_0$$

If one makes μ_2 tend toward $\mu_2(\lambda_2)$ with ϵ being a small quantity, one sees that, according to the previous hypothesis, the second term of the second member tends toward $\partial\varphi/\partial x_3(\lambda_2,\mu_2)$ whereas the third tends toward zero. Let us moreover make the provisional hypothesis that the last term tends toward zero (this hypothesis will have to be verified later on), and we obtain

$$\int_{\mu_1(\lambda_2)}^{\mu_2(\lambda_2)} \frac{K(\lambda_2,\mu)\,d\mu}{\sqrt{\mu_2(\lambda_2) - \mu}} + 2\beta \int_{\lambda_N}^{\lambda_2} \frac{\varphi'\left[\lambda_0,\mu_2(\lambda_2)\right]}{\sqrt{\lambda_2 - \lambda_0}}\,d\lambda_0 = 0$$

However, since φ maintains in the wake a constant value on the lines parallel to Ox_1, it suffices to know, for instance, the values of the potential on the straight line QI (fig. 2) in order to know them everywhere. In accordance with this remark

$$\int_{\lambda_I}^{\lambda} \frac{\varphi'(\lambda_0,\mu_Q)}{\sqrt{\lambda - \lambda_0}}\,d\lambda_0 = -\frac{1}{2\beta}\int_{\mu_1(\lambda_{P'})}^{\mu_2(\lambda_{P'})} \frac{K(\lambda_{P'},\mu)}{\sqrt{\mu_2(\lambda_{P'}) - \mu}}\,d\mu$$

or

$$\varphi(\lambda_0,\mu_Q) = -\frac{1}{2\pi\beta}\int_{\lambda_I}^{\lambda_0} \frac{d\lambda}{\sqrt{\lambda_0 - \lambda}}\int_{\mu_1(\lambda_{P'})}^{\mu_2(\lambda_{P'})} \frac{K(\lambda_{P'},\mu)}{\sqrt{\mu_2(\lambda_{P'}) - \mu}}\,d\mu$$

if one defines $\lambda_{P'}$ by

$$\lambda - \lambda_{P'} = \mu_Q - \mu_2(\lambda_{P'})$$

We note that the circulation along the subsonic trailing edge is thus calculated.

It remains to be verified that the provisional hypothesis adopted in the course of the calculation is well founded which can be accomplished without difficulties. One sees thus how the solution of the lifting problem can be determined.

In order to establish that the calculated solution completely fulfills the problem that is the theorem of existence, one proceeds as in the symmetrical case; thus the whole matter finally amounts to an investigation of the singularities of this solution; this investigation permits a verification a posteriori of the conditions of regularity. In order to make this investigation, it is necessary to study first of all the behavior of $\partial\varphi/\partial x_3$ in the plane Ox_1x_2. As before, we shall indicate the results without demonstration.

(a) Study of $\partial\varphi/\partial x_3$ in Ox_1x_2.- First, one sees immediately that $\partial\varphi/\partial x_3$ increases indefinitely as $\epsilon^{-1/2}$ when one tends toward the subsonic leading edge MN, remaining outside of (S). On the other hand, according to hypothesis, this quantity is continuous on the subsonic trailing edge NQ. We shall now specify its behavior along the characteristic NN_1; a rather simple calculation which we cannot reproduce here, in order to avoid postponement of publication, permits to show that:

Along the line $\lambda = \lambda_N$, $\mu > \mu_N$, $\partial\varphi/\partial x_3$ undergoes a discontinuity of the first kind equal to

$$- \frac{1}{\pi} \frac{1}{\sqrt{\mu_2 - \mu_2(\lambda_N)}} \int_{\mu_1(\lambda_N)}^{\mu_2(\lambda_N)} \frac{K(\lambda_N,\mu)}{\sqrt{\mu_2(\lambda_N) - \mu}} \, d\mu$$

The manner in which $\partial\varphi/\partial x_3$ is calculated shows then readily that $\partial\varphi/\partial x_3$ has no other discontinuities in the plane Ox_1x_2, outside of (S), of course.

(b) Study of the solution in Ox_1x_2.- What has been said for the symmetrical problem remains valid by means of the following modification: First of all, $\partial\varphi/\partial x_1$ and $\partial\varphi/\partial x_2$ become infinite like $\epsilon^{-1/2}$ along the subsonic leading edge. On the other hand, a very important fact, the derivatives $\partial\varphi/\partial x_1$ and $\partial\varphi/\partial x_2$ undergo discontinuities of the first kind along the characteristics issuing from the boundary points between subsonic leading edge and subsonic trailing edge. [For $\partial\varphi/\partial x_1$, however, such a discontinuity can occur only on (S).]

(c) Study in space.- The only really new fact to be pointed out is that across the Mach cones behind the boundary points between subsonic leading and trailing edges, the first derivatives undergo a discontinuity of the first kind.

5. Final remarks.-

(a) We have adhered to demonstrating the existence of the solution, but the employed procedure of demonstration shows at the same time that the solution is unique. Consequently, every solution which corresponds to the hypothesis found by other methods (particularly by the method of conical and homogeneous flows) represents the unique solution to the problem posed.

(b) One will also note that the supplementary hypothesis introduced along the subsonic trailing edge in the case of a lifting problem may also be expressed by saying that the pressure remains continuous along this line. This is an immediate consequence of the investigation of the behavior of the solution.

(c) We have not attempted to investigate here the most general type of surface (S). In general, the method can be applied by means of a few precautions (compare ref. 4 or ref. 5). Nevertheless, there exist cases where the application of this method actually fails, for instance, the case where the wing does not possess a supersonic leading edge, or also for certain dispositions of the trailing edge. Figure 3 shows such examples; if one traces a few Mach lines, one will understand immediately the reason for this failure.

(d) One of the advantages of the method just described is the fact that it may be effectively applied to very general problems. Nevertheless, it does, in our opinion, not minimize the advantages of the method of conical flows, since in many particular problems arising in aeronautics, the method of conical flows (and the method of homogeneous flows) lead in a simpler manner to the desired result.

(e) The method of the fundamental solution has the great merit of permitting the study of the general conditions of the flow, particularly the study of certain pressure discontinuities which one encounters on the surface of the wing in certain lifting problems.

No. 2 - On Homogeneous Flows

We developed the theory of homogeneous flows[58] and gave a few applications in a recent article (ref. 7). We shall give here a few supplements to the general study made in section 1.3. If one puts

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[58]Simultaneously, this problem has formed the subject of an article by M. Poritzky (ref. 6). However, this author does not seem to us to have gone as far as we have in the investigation of the homogeneous flows.

$$\varphi^{(n)}_{(p,q,r)} = \frac{\partial^n \varphi}{\partial x_1{}^p \partial x_2{}^q \partial x_3{}^r} \qquad (p + q + r = n)$$

the $\varphi^{(n)}_{(p,q,r)}$ depend in a homogeneous flow of the order n only on X and θ. Inside of the Mach cone (Γ) these quantities may be considered as the real parts of analytic functions of the variable Z defined except for an additive purely imaginary constant which we shall denote

$$\Phi^{(n)}_{(p,q,r)}(Z)$$

A problem of homogeneous flows is treated for the nth derivatives. These nth derivatives are connected by the relations of compatibility which may be expressed in the following manner:

All the expressions

$$\left(-\frac{1}{\beta}\right)^{p+q}\left(\frac{2Z}{Z^2 + 1}\right)^p\left(\frac{2iZ}{Z^2 - 1}\right)^q \frac{d\Phi^{(n)}_{(n-p-q,p,q)}}{dZ}$$

are identical whatever the integers p and q may be which satisfy the inequalities

$$0 \leqslant p + q \leqslant n$$

In order to express the boundary conditions with the nth derivatives, and to enter the nth derivatives into the calculation of the potential or of the pressure $\left(C_p = -u\right)$, one will utilize a generalization of Euler's identity

$$\varphi = \frac{1}{n!}\left[x_1\varphi^{(1)}_{(1,0,0)} + x_2\varphi^{(1)}_{(0,1,0)} + x_3\varphi^{(1)}_{(0,0,1)}\right]^n$$

a formula in which one must use the following convention concerning the $\varphi^{(k)}$

$$\left[\varphi^{(1)}_{(1,0,0)}\right]^p\left[\varphi^{(1)}_{(0,1,0)}\right]^q\left[\varphi^{(1)}_{(0,0,1)}\right]^r = \varphi^{(p+q+r)}_{(p,q,r)}$$

One will find in the quoted article an application of these general principles to the case of the flows flattened in one direction. The methods used in chapter III can be generalized without any difficulties; also, one may utilize in this investigation the analogy of the electrolytic tank. A superposition of homogeneous flows permits, in a very simple manner, the investigation[59] of a rather large group of Δ wings: "the Δ wings with affine sections."

No. 3 - On the Methods Utilized in Chapter III

The exposition of certain problems of chapter III could be somewhat simplified not only by omitting certain intermediary calculations of wholly elementary character which we have mentioned to facilitate the reading, but also by employing slightly different methods[60]. First of all, as we have remarked in the text, certain simplifications appear if one places oneself in the plane z. Thus the symmetrical problem may be solved by the same formulas whatever the position of the obstacle may be with respect to the Mach cone. Nevertheless one has to be very careful regarding the determinations of the solution when one passes from one case to another since the solution should be characterized by continuity. We have elected to utilize here the plane Z because the relations of compatability in Z do not cause the appearance of multiform functions and the theoretical difficulties are, consequently, of distinctly lesser importance even though the calculations may sometimes be a little lengthier. Particularly, the demonstration of the theorems of sections 3.1.1.3 and 3.1.1.4 is markedly simpler if one utilizes the plane Z. Summarizing one may say that the plane Z is simpler theoretically while the plane z is simpler for the calculations[61].

Mr. Ward has stated the solution of certain elementary problems relative to obstacles flattened in one direction using a very elegant method (ref. 8). His study is based on a solution of the equation of cylindrical waves given by Whittaker. With our notations

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[59]One will also refer to the article of Mr. Fenain which will appear shortly in "La Recherche Aéronautique"; in it one will find a complete study of a certain number of these particulars.

[60]In conferences at the 'Centre d'Etudes supérienres de mécainque (1949) we have made an exposition regarding conical flows flattened in one direction which is very different in form from the one given in this report.

[61]The same may hold true for the electric analogies (compare on this subject the article of Mr. Fenain quoted before).

$$\varphi = \int_C \left(x_1 - \beta x_2 \text{ ch } u + i\beta x_3 \text{ sh } u\right) f(u) du$$

is the potential of a conical flow provided that the contour C joins two points u_1 and u_2 so that u_1 and u_2 are roots of the equation

$$x_1 - \beta x_2 \text{ ch } u + i\beta x_3 \text{ sh } u = 0$$

In contrast, the function $f(u)$ is arbitrary.

This very refined expression for φ furnishes the relations of compatibility and permits solution of the particular problems. The homogeneous flows are given by the solutions of the wave equation of the form

$$\int_C \left(x_1 - \beta x_2 \text{ ch } u + i\beta x_3 \text{ sh } u\right)^n f(u) du$$

In the case of homogeneous problems of the order n, it seems nevertheless difficult to state the boundary problem clearly and to solve it by this method without falling back on methods strictly equivalent to those reemployed.

No. 4 - On the Complementary Hypothesis at the Subsonic Trailing Edge

The question posed in section 3.3, which we left pending, seems to admit a practically definitive answer; one must maintain the flows of the type II which give rise to a discontinuity of the potential along the wake of the wing. But as we have said before, this results from a hypothesis clearly formulated in the appendix No. 1 which may be stated as follows:

The gradient of the potential is continuous across a subsonic trailing edge. All the remarks made in section 3.3 concerning the consequences of this hypothesis remain valid.

The most decisive argument in favor of this hypothesis is that it appears to be the simplest of all one may set up that insures the continuity of φ.

In the case of conical flows infinitely flattened in one direction, we have seen that it entails a line of singularities following Ox_1 along which w is infinite when the body has a trailing edge. Such an occasion does not arise in the general case (compare appendix No. 1). All methods of chapter III can be applied to the calculation of the conical flows for which this complementary hypothesis must be taken into account. In particular, we have indicated elsewhere[62] how one must operate in this case for the analogical calculation of the solution.

No. 5 - Remark on Sweptback Wings

With Subsonic Leading Edge[63]

The formula (IV.37) may be written also

$$C_x = -\frac{4}{\pi}\frac{\cos^2\gamma\left(1 + 2\sin^2\gamma - M^2\cos^2\gamma\right)}{\sin\gamma\left(1 - M^2\cos^2\gamma\right)^{3/2}}\int_0^1 \alpha(x)dx\int_0^1 \alpha(\xi)\log\left|x - \xi\right|d\xi$$

This formula lends itself well to an investigation of the optimum. We shall search, in fact, for the profile which, in delimiting a given area, provides a minimum drag; putting

$$e(x) = \int_0^x \alpha(t)dt$$

one is led to seek the minimum absolute value of the integral

$$\int_0^1 de(x)\int_0^1 de(\xi)\log\left|x - \xi\right|$$

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[62]Communication to the 7th Congrès International de Mécanique appliquée (1948).

[63]This remark has been made by the author in the course of his communication to the 7th Congrès International de Mécanique appliquée (1948), quoted above.

It is easily seen, and the fact is well-known to aerodynamists, that the solution of the function $e(x)$ of this problem has the form

$$e(x) = k\sqrt{1 - x^2}$$

that is, that the desired profile is an ellipse.

The train of thought which leads to (IV.37) cannot be applied to the case where the profile has a tangent normal to the symmetry axis; but according to a remark already made more than once, one may nevertheless assume that the obtained result does not lack connection with reality.

This leads to the idea that, for a wing with subsonic leading edge, it may be practical to utilize profiles with rounded leading edges.

One will note that this is not the case in supersonic regime. If one takes up this problem for a wing of infinite span normal to the wind, one finds readily that the optimum profile is formed by two symmetrical parabolic arcs.

No. 6 - Remarks on Lifting Sweptback Wings With Sonic and Subsonic Leading Edges

(Compare Section 4.1.2.3.2)

The formula (IV.69) may also be written by putting

$$\frac{\eta_0\left[x - \xi - 2\left(\eta_0 + y^x\right)\right]}{\eta_0 x + y^x \xi} = \frac{t^2}{1 + t^2} \qquad \frac{2\left(\eta_0 + y^x\right)}{x} = \frac{1}{1 + t_0^2}$$

in the form

$$C_p = -\frac{4i}{\pi\beta\sqrt{x\left(x - 2y^x\right)}}\left\{x - y^x - \frac{2}{\pi}\sqrt{\frac{x}{2\left(\eta_0 + y^x\right)}}\int_0^{t_0} \frac{x - y^x - \left(\eta_0 + y^x\right)\left(1 + t^2\right)}{1 + t^2} \frac{dt}{\sqrt{t_0^2 - t^2}}\right\}$$

One could make the calculation of the C_z of the plate studied in section 4.1.2.3.2 in a different manner by obtaining first the preceding integral, and integrating the pressures along the plate. Thus one finds that the C_p has in the region AA'A" of figure 88 the simple value

$$C_p = -\frac{2i}{\pi\beta}\sqrt{\frac{2\left(\eta_0 + y^x\right)}{x - 2y^x}}$$

Along AA" there exists therefore a pressure discontinuity equal to

$$\Delta C_p = \frac{2i}{\pi\beta}\frac{x - 2y^x}{\sqrt{x\left(x - 2y^x\right)}}$$

Besides, this discontinuity may be calculated immediately from the formula giving the C_p in making t_0 tend toward zero since it is clear that the integral tends toward a finite value when t_0 tends toward zero.

If the leading edge is subsonic, the same theory is applicable. In this case, the C_p cannot be expressed with the aid of elementary functions[64]. However, the pressure discontinuity along the Mach line issuing from A may be calculated directly. One will compare this important phenomenon with the general investigation made at the end of the appendix 1 which anticipates the existence of such discontinuities on the Mach cones which have as vertices the ends of the subsonic leading edges.

No. 7 - Calculation of Fuselage Shaped Bodies With Infinitesimal Opening Angle

At the end of section 4.2.4 we indicated that by composition of conical flows one could give a complete study of any arbitrary spindle-shaped bodies with infinitesimal cone angle. In a communication to the 7th Congrès International de Mécanique appliquée (September 1948), Mr. Ward described an elegant method based on a solution of the wave equation with the aid of symbolic calculation; this report has been published (ref. 9). We shall show here the accuracy of our anticipation by

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[64]Compare an investigation of this problem with numerical applications in an article to appear shortly in "La Recherche Aéronautique."

establishing through the method of composition of conical flows the fundamental formulas given by Mr. Ward.

The notations which are not defined here are the same as those of chapter II. In this chapter we have shown that in the neighborhood of the obstacle, the complex velocity $U(Z)$ had the form

$$U(Z) = A_0 \log Z + \sum_1^\infty \frac{A_n}{Z^n}$$

with the A_n being numerical coefficients depending on the shape of the cone. Let us put

$$z = re^{i\theta}$$

as in the neighborhood of the obstacle

$$\rho \cong \frac{\beta r}{2x_1} \qquad Z \cong \frac{\beta z}{2x_1}$$

and

$$U(z) = A_0\left(\log z + \log \frac{\beta}{2x_1}\right) + \sum_1^\infty \frac{A_n' x_1^{\,n}}{z^n}$$

with the A_n' being new coefficients. Hence one deduces that the potential of perturbation has the form

$$\varphi = \underline{R}_0\left[K_0(z)\right]$$

with

$$K_0(z) = A_0\left[x_1 \log z + \int_0^{x_1} \log \frac{\beta}{2t}\, dt\right] + \sum_1^\infty \frac{A_n'' x_1^{\,n+1}}{z^n}$$

with the A_n'' denoting new numerical coefficients.

More generally, the potential of the conical flow with infinitesimal cone angle, the vertex of which is situated in $x_1 = \sigma$, $r = 0$, can be expressed (in the neighborhood of the obstacle)

$$\varphi = \underline{R}_0\left[K_\sigma(z)\right]$$

with

$$K_\sigma(z) = A_0(\sigma)\left[\left(x_1 - \sigma\right)\log z + \int_0^{x_1-\sigma} \log \frac{\beta}{2t}\, dt\right] + \sum_1^\infty \frac{A_n''(\sigma)\left(x_1 - \sigma\right)^{n+1}}{z^n}$$

A superposition of conical flows the vertices of which are situated on Ox_1 causes a flow which in the neighborhood of the obstacle depends on the potential

$$\varphi = \underline{R}_0\left[f(z)\right] \tag{1}$$

where $f(z)$ has the form

$$f(z) = a_0 \log z + b_0 + \sum_1^\infty \frac{a_n}{z^n} \tag{2}$$

the coefficients a_0, b_0, a_n being defined by the integrals

$$a_0 = \int_0^{x_1} \left(x_1 - \sigma\right) dA_0(\sigma)$$

$$b_0 = \int_0^{x_1} dA_0(\sigma) \int_0^{x_1-\sigma} \log \frac{\beta}{2t}\, dt$$

$$a_n = \int_0^{x_1} \left(x_1 - \sigma\right)^{n+1} dA_n''(\sigma)$$

One will remark immediately that

$$b_0 = a_0 \log \frac{\beta}{2} - \int_0^{x_1} dA_0(\sigma) \int_0^{x_1-\sigma} \log t \, dt$$

$$= a \log \frac{\beta}{2} - \int_0^{x_1} \log\left(x_1 - \sigma\right) d\sigma \int_0^{\sigma} dA_0(t)$$

or

$$b_0 = a_0 \log \frac{\beta}{2} - \int_0^{x_1} \frac{da_0}{d\sigma} \log\left(x_1 - \sigma\right) d\sigma \tag{3}$$

Reciprocally, it is clear that under very broad conditions a function $f(z)$ like (2) (in which the coefficients a_0, b_0, a_n are functions of x_1, a_0, and b_0 connected by (3)) determines by (1) the potential of a flow with infinitesimal cone angle in the neighborhood of the axis Ox_1. This constitutes the fundamental result of Mr. Ward.

Thus we are in a position to construct such flows. The only theoretical question to be examined is the following: Can one determine the coefficients $a_n\left(x_1\right)$ so that φ represents the potential of a flow around a given obstacle. We shall see that, visualizing the boundary conditions, we may answer this question in the affirmative.

Let us designate by

$$r = F\left(\theta, x_1\right)$$

the equation defining the obstacle by C_{x_1} the section of the abscissa x_1 and by φ_{x_1} the function of the two variables r and θ obtained by considering x_1 in φ as parameter.

The normal derivative of φ_{x_1} along C_{x_1} is given by

$$\frac{d\varphi_{x_1}}{dn} = \frac{\frac{\partial\varphi}{\partial r} - \frac{1}{F}\frac{\partial F}{\partial\theta}\frac{\partial\varphi}{\partial\theta}}{\sqrt{F^2 + (\partial F/\partial\theta)^2}}$$

Now the boundary conditions along C_{x_1} are written, taking into account the usual approximations,

$$F \frac{\partial\varphi}{\partial r} - \frac{1}{F} \frac{\partial F}{\partial\theta} \frac{\partial\varphi}{\partial\theta} = F \frac{\partial F}{\partial x_1}$$

hence the relation

$$\frac{d\varphi_{x_1}}{dn} = \frac{F \dfrac{\partial F}{\partial x_1}}{\sqrt{F^2 + (\partial F/\partial\theta)^2}}$$

Thus one has, denoting by s and ψ_{x_1}, respectively, the arc of C_{x_1} and the conjugate function of φ_{x_1}

$$\frac{d\psi_{x_1}}{ds} = F \frac{\partial F}{\partial x_1} \frac{d\theta}{ds}$$

The coefficient a_0 is given by

$$a_0 = \frac{1}{2\pi} \int_{C_{x_1}} \frac{d\psi_{x_1}}{ds} ds = \frac{1}{2\pi} \int_0^{2\pi} F \frac{\partial F}{\partial x_1} d\theta = \frac{1}{2\pi} \frac{dS}{dx_1}$$

$S(x_1)$ denotes the area delimited by C_{x_1}; the coefficients a_n are then obtained by solving an exterior Dirichlet problem for the contour C_{x_1}. Thus the flow around any obstacle with infinitesimal opening angle can be completely determined.

Mr. Ward (ref. 9) has given in his memorandum splendid applications of these results. In particular, he has shown, taking for expressing the pressure the formula (I.11), that the total lift is uniquely expressed as a function of the coefficient a_1 of the terminal section of the obstacle and that the drag depended only on the coefficients a_n of this section.

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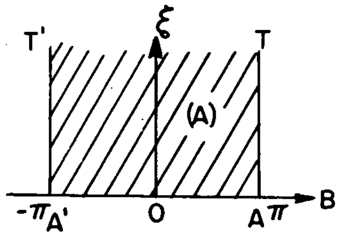


Figure 1

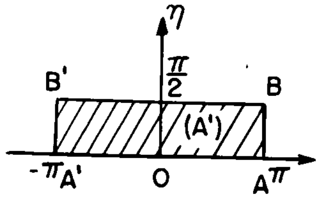
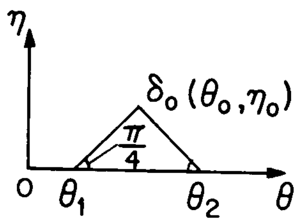


Figure 2



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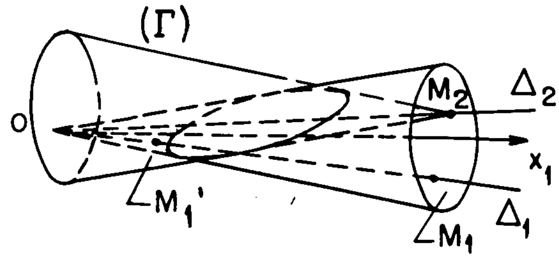

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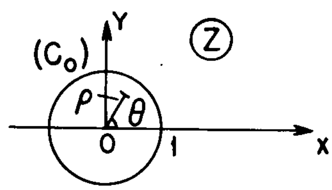


Figure 5

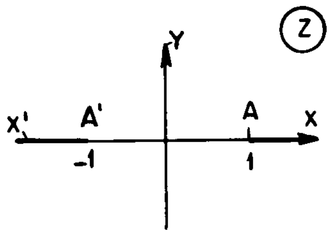
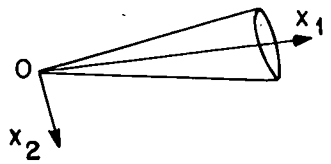


Figure 6



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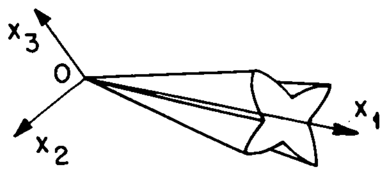

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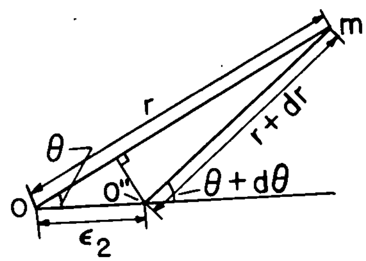

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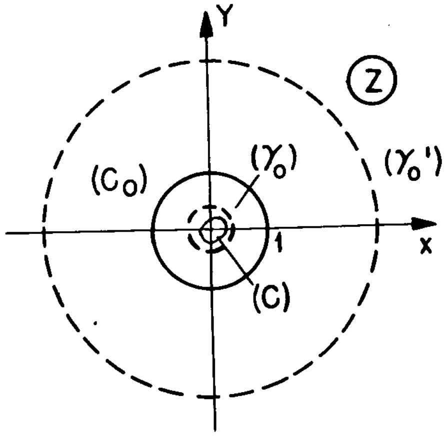


Figure 10

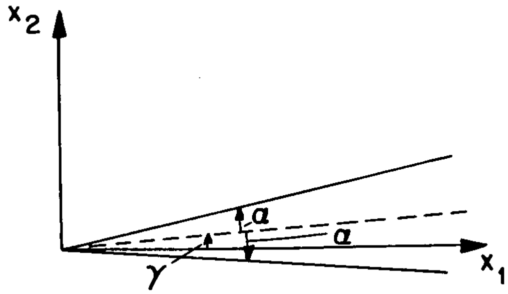
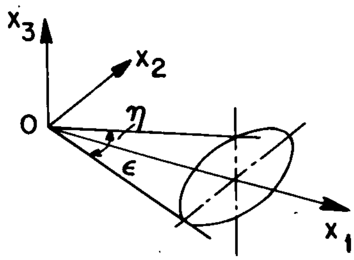


Figure 11



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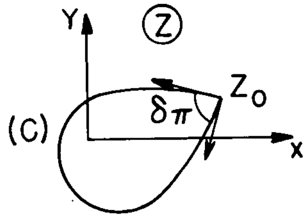

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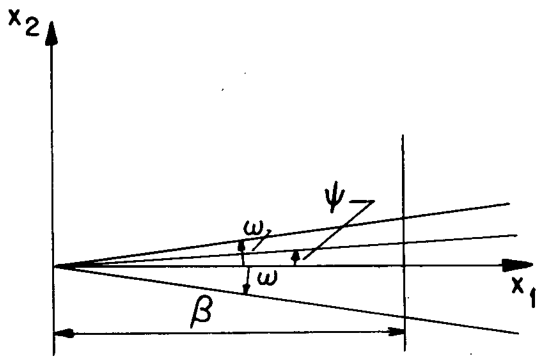


Figure 14

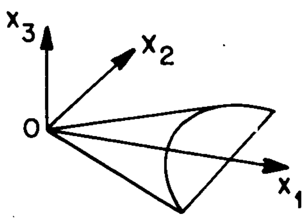


Figure 15

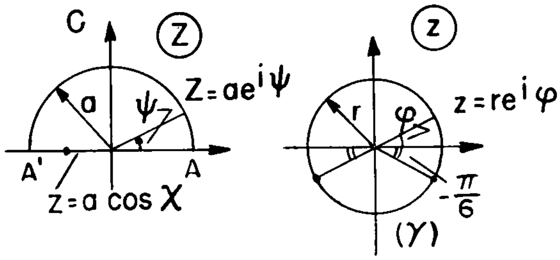
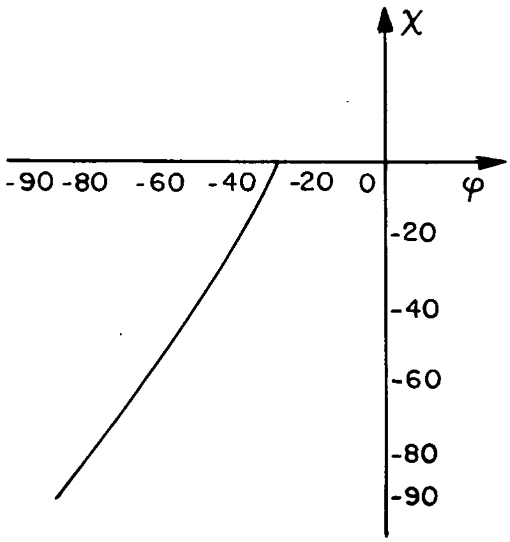


Figure 16



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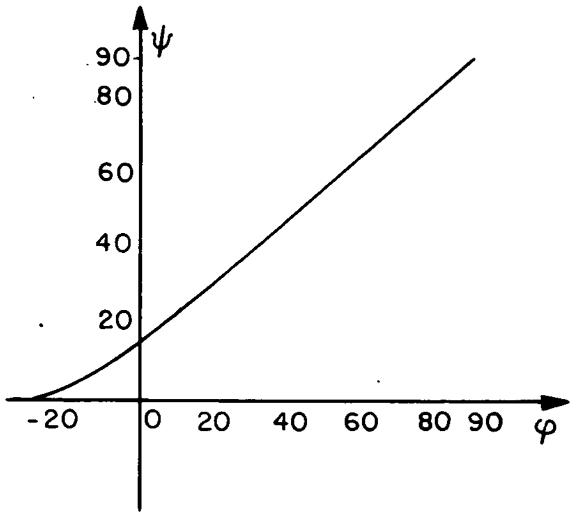
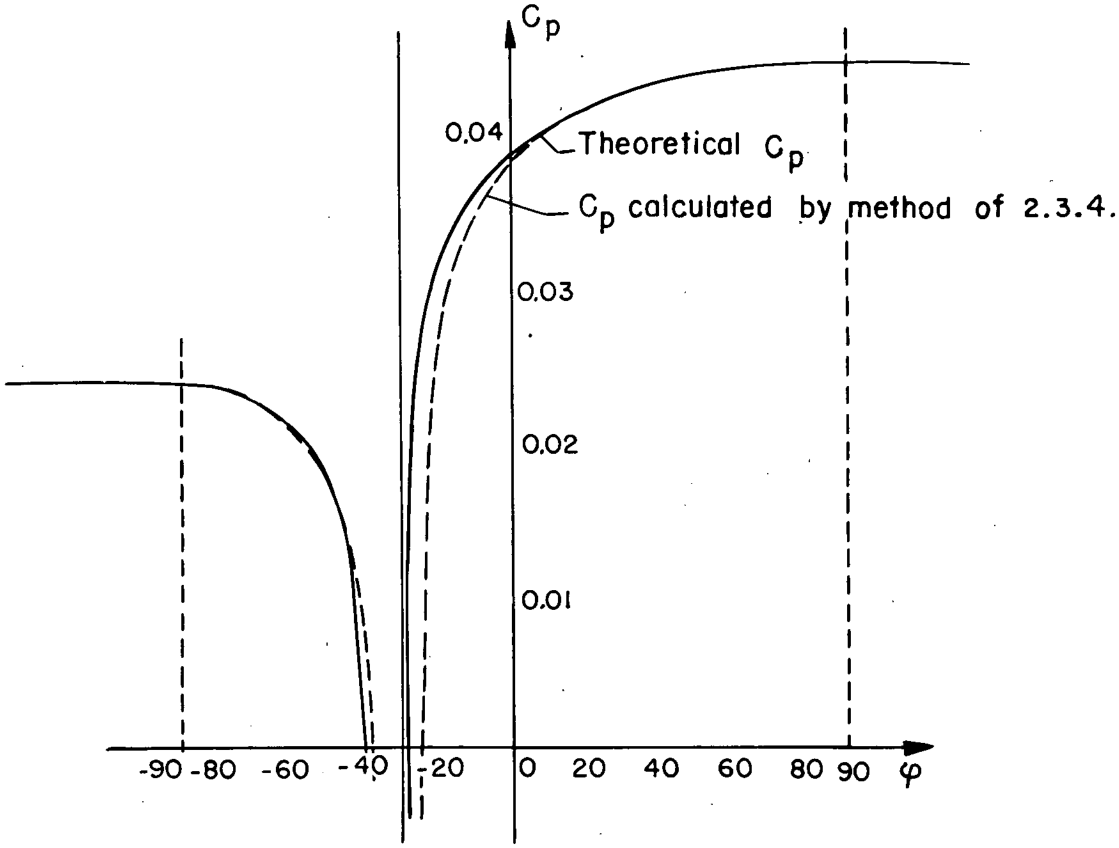


Figure 18



Cp
0.04
Theoretical Cp
Cp calculated by method of 2.3.4.
0.03
0.02
0.01
-90 -80 -60 -40 -20 0 20 40 60 80 90 φ

Figure 19

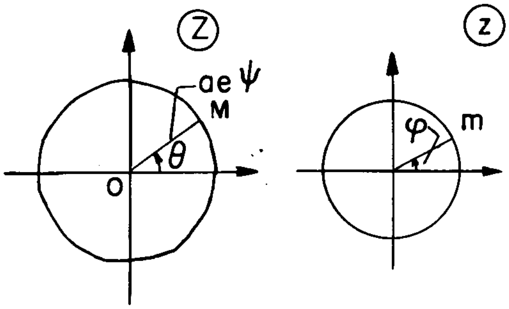


Figure 20

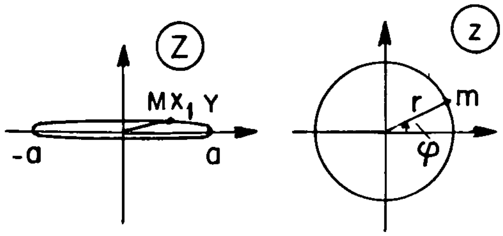
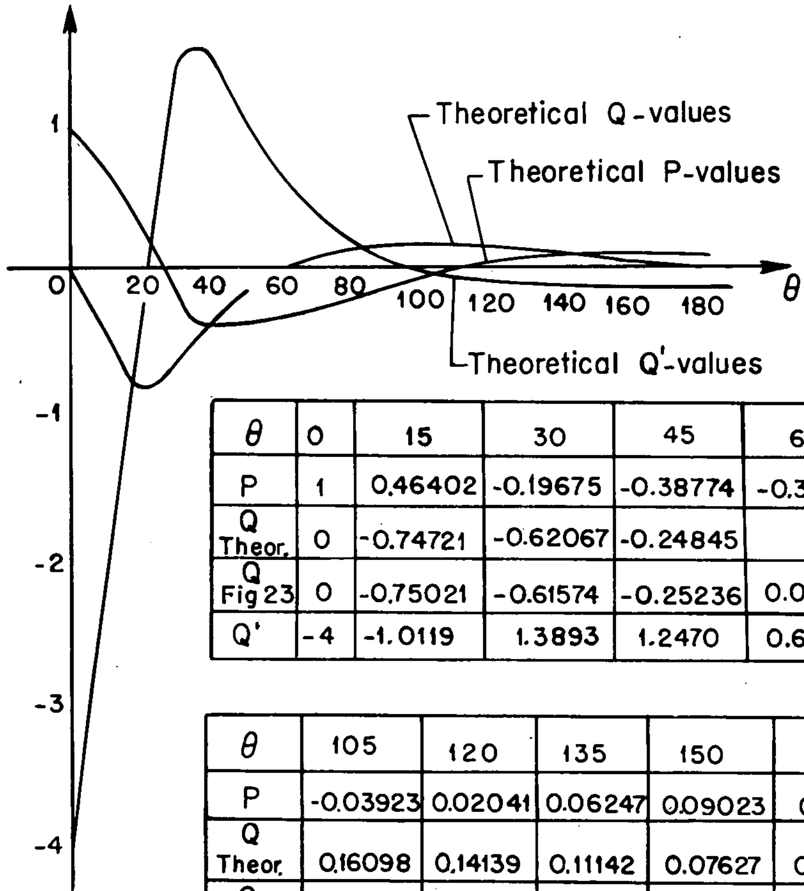
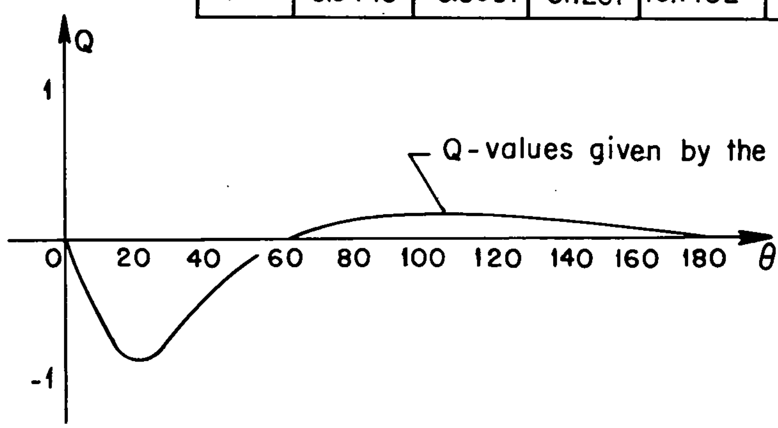


Figure 21

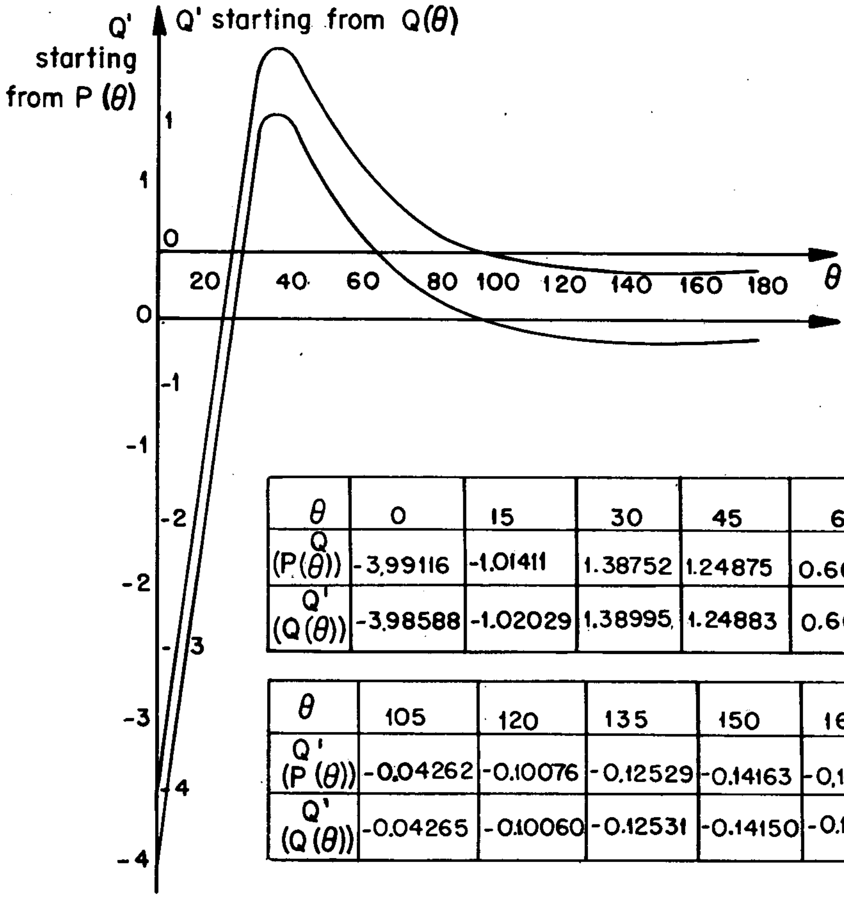


θ	0	15	30	45	60	75	90
P	1	0.46402	-0.19675	-0.38774	-0.33333	-0.22262	-0.12
Q Theor.	0	-0.74721	-0.62067	-0.24845	0	0.11856	0.16
Q Fig 23	0	-0.75021	-0.61574	-0.25236	0.00370	0.11576	0.16244
Q'	-4	-1.0119	1.3893	1.2470	0.6667	0.2755	0.0640

θ	105	120	135	150	165	180	
P	-0.03923	0.02041	0.06247	0.09023	0.106	0.11111	
Q Theor.	0.16098	0.14139	0.11142	0.07627	0.03864	0	
Q Fig 23	0.15919	0.14287	0.11041	0.07697	0.03830	0	
Q'	-0.0443	-0.0991	-0.1267	-0.1402	-0.1463	-0.1482	



Figures 22 and 23



θ	0	15	30	45	60	75	90
Q (P(θ))	-3,99116	-1.01411	1.38752	1.24875	0.66398	0.27769	0.06192
Q' (Q(θ))	-3,98588	-1.02029	1.38995	1.24883	0.66429	0.27767	0,06207

θ	105	120	135	150	165	180	
Q' (P (θ))	-0,04262	-0.10076	-0.12529	-0.14163	-0,14515	-0.14948	
Q' (Q (θ))	-0.04265	-0.10060	-0.12531	-0.14150	-0.14517	-0.14932	

Figure 24

Poisson's Integral

θ	P		0	15	30	45	60	75	90	105	120	135	150	165	180	195	210	225	240	255	270	285	300	315	330	345	
			-K1		-K3		-K5		-K7		-K9		-K11		+K11		+K9		K7		K5		K3		K1		
0	1	K_1		-0.63298		-0.201185		-0.108602		-0.063944		-0.034518		-0.010971		+0.010971		+0.034518		+0.06394		+0.108602		+0.201185		+0.632980	K_1
15	0.46402		+0.29372		-0.29372		-0.09335		-0.05039		-0.02967		-0.01601		-0.00509		+0.00509		+0.01601		+0.02967		+0.05039		+0.09335		
30	-0.19675	K_3		-0.12454		+0.12454		+0.03958		+0.02137		+0.01258		+0.00679		+0.00216		-0.00216		-0.00679		-0.01258		-0.02137		-0.03958	K_3
45	-0.38774		-0.07801		-0.24543		+0.24543		+0.07801		+0.042109		+0.02479		+0.01338		+0.00425		-0.00425		-0.01338		-0.02479		-0.042109		
60	-0.33333	K_5		-0.06706		-0.21099		+0.21099		+0.06706		+0.03620		+0.02131		+0.01151		+0.00366		-0.00366		-0.01151		*-0.02131		-0.03620	K_5
75	-0.22262		-0.02418		-0.04479		-0.14091		+0.14091		+0.04479		+0.02418		+0.01423		+0.00768		+0.00244		-0.00244		-0.00768		-0.01423		
90	-0.12	K_7		-0.01303		-0.02414		-0.07596		+0.07596		+0.02414		+0.01303		+0.00767		+0.00414		+0.00132		-0.00132		-0.00414		-0.00767	K_7
105	-0.03923		-0.00251		-0.00426		-0.00789		-0.02483		+0.02483		+0.00789		+0.00426		+0.00251		+0.00135		+0.00043		-0.00043		-0.00135		
120	+0.02041	K_9		+0.00131		+0.00222		+0.00411		+0.01292		-0.01292		-0.00411		-0.00222		-0.00131		-0.00070		-0.00022		+0.00022		+0.00070	K_9
135	0.06247		+0.00216		+0.00399		+0.00678		+0.01257		+0.03954		-0.03954		-0.01257		-0.00678		-0.00399		-0.00216		-0.00069		+0.00069		
150	0.09023	$+K_{11}$		+0.00311		+0.00577		+0.00980		+0.01815		+0.05711		-0.05711		-0.01815		-0.00980		-0.00577		-0.00311		-0.00099		+0.00099	K_{11}
165	0.106		+0.00116		+0.00366		+0.00678		+0.01151		+0.02132		+0.06709		-0.06709		-0.02132		-0.01151		-0.00678		-0.00366		-0.00116		
180	0.11111	$-K_{11}$		+0.00122		+0.00384		+0.00710		+0.01207		+0.02235		+0.07033		-0.07033		-0.02235		-0.01207		-0.00710		-0.00384		-0.00122	$-K_{11}$
195	0.106		-0.00116		+0.00116		+0.00366		+0.00678		+0.01151		+0.02132		+0.06709		-0.06709		-0.02132		-0.01151		-0.00678		-0.00366		
210	0.09023	$-K_9$		-0.00099		+0.00099		+0.00311		+0.00577		+0.00980		+0.01815		+0.05711		-0.05711		-0.01815		-0.00980		-0.00577		-0.00311	$-K_9$
225	0.06247		-0.00216		-0.00069		+0.00069		+0.00216		+0.00399		+0.00678		+0.01257		+0.03954		-0.03954		-0.01257		-0.00678		-0.00399		
240	0.02041	$-K_7$		-0.00070		-0.00022		+0.00022		+0.00070		+0.00131		+0.00222		+0.00411		+0.01292		-0.01292		-0.00411		-0.00222		-0.00131	$-K_7$
255	-0.03923		+0.00251		+0.00135		+0.00043		-0.00043		-0.00135		-0.00251		-0.00426		-0.00789		-0.02483		+0.02483		+0.00789		+0.00426		
270	-0.12	$-K_5$		+0.00767		+0.00414		+0.00132		-0.00132		-0.00414		-0.00767		-0.01303		-0.02414		-0.07596		+0.07596		+0.02414		+0.01303	$-K_5$
285	-0.22262		+0.02418		+0.01423		+0.00768		+0.00244		-0.00244		-0.00768		-0.01423		-0.02418		-0.04479		-0.14091		+0.14091		+0.04479		
300	-0.33333	$-K_3$		+0.03620		+0.02131		+0.01151		+0.00366		-0.00366		-0.01151		-0.02131		-0.03620		-0.06706		-0.21099		+0.21099		+0.06706	$-K_3$
315	-0.38774		*+0.07801		+0.04211		+0.02479		+0.01338		+0.00425		-0.00425		-0.01338		-0.02479		-0.04211		-0.07801		-0.24543		+0.24543		
330	-0.19675	$-K_1$		+0.03958		+0.02137		+0.01258		+0.00679		+0.00216		-0.00216		-0.00679		-0.01258		-0.02137		-0.03958		-0.12454		+0.12454	$-K_1$
345	+0.46402		-0.29372		-0.09335		-0.05039		*-0.02967		-0.01601		-0.00509		+0.00509		+0.01601		+0.02967		+0.05039		+0.09335		+0.29372		
	Q		-K1		-K3		-K5		-K7		-K9		-K11		+K11		K9		K7		K5		K3		K1		
			0	-0.75021	-0.61574	-0.25236	+0.00370	+0.11576	+0.16244	+0.15919	+0.14287	+0.11041	+0.07697	+0.03830	0	-0.03830	-0.07697	-0.11041	-0.14287	-0.15919	-0.16244	-0.11576	-0.00370	+0.25236	+0.61574	+0.75021	

Coefficients

K_1 = 0.632980
K_3 = 0.201185
K_5 = 0.108602
K_7 = 0.063944
K_9 = 0.034518
K_{11} = 0.010971

*NACA reviewer's note: Minor errors appearing in the original French have been corrected by the reviewer.

Figure 25

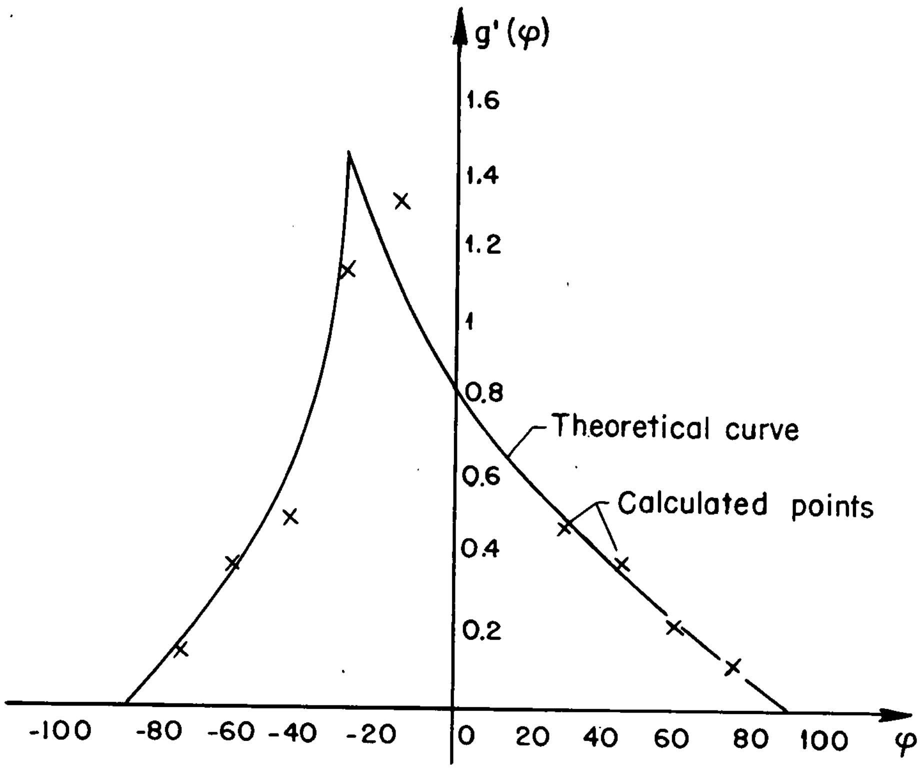

g' (φ)
1.6
1.4
1.2
1
0.8
Theoretical curve
0.6
Calculated points
0.4
0.2
-100
-80
-60
-40
-20
0
20
40
60
80
100
φ

Figure 26

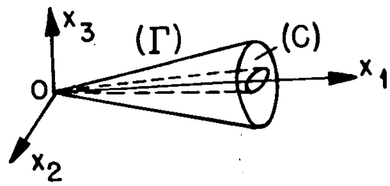


Figure 27

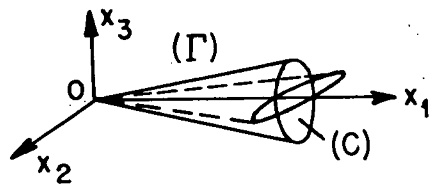


Figure 28

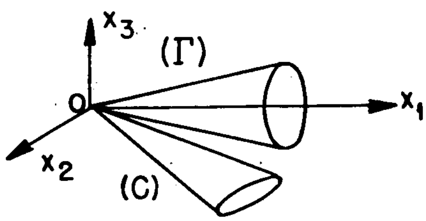


Figure 29

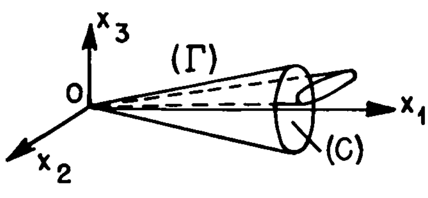


Figure 30

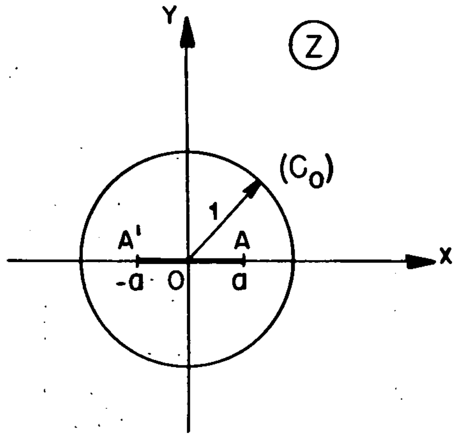


Figure 31

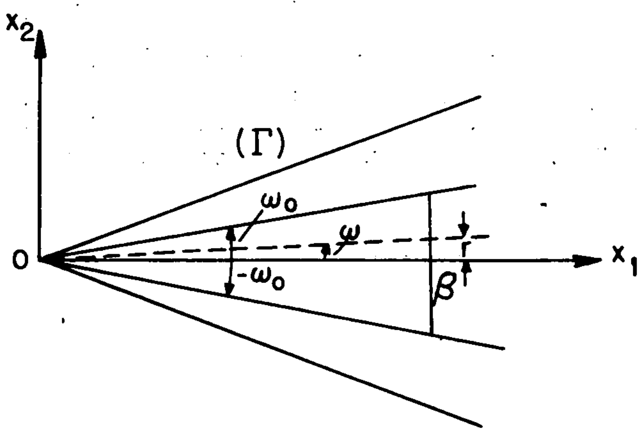
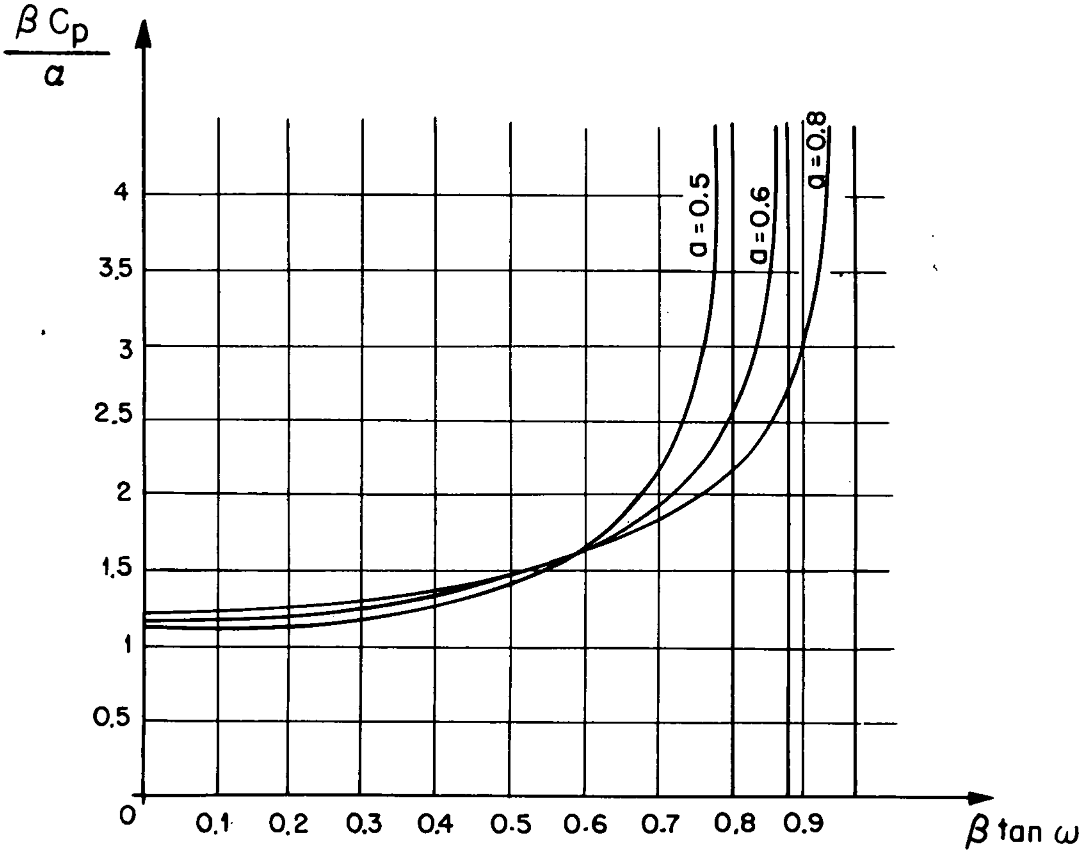


Figure 32



$\frac{\beta C_p}{a}$
4
3.5
3
2.5
2
1.5
1
0.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
$\beta \tan \omega$
a = 0.5
a = 0.6
a = 0.8

Figure 33

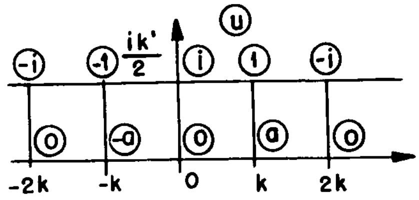


Figure 34

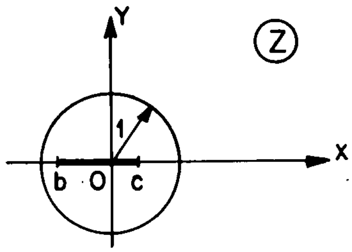


Figure 35

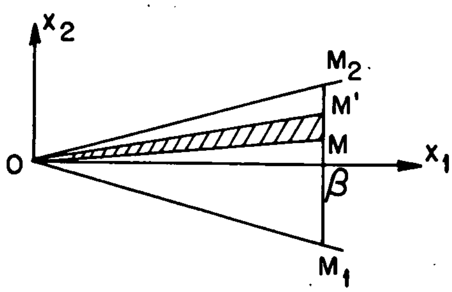


Figure 36

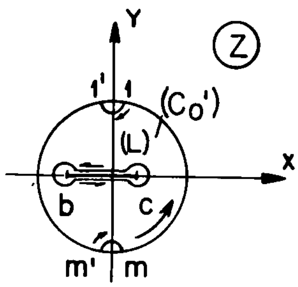
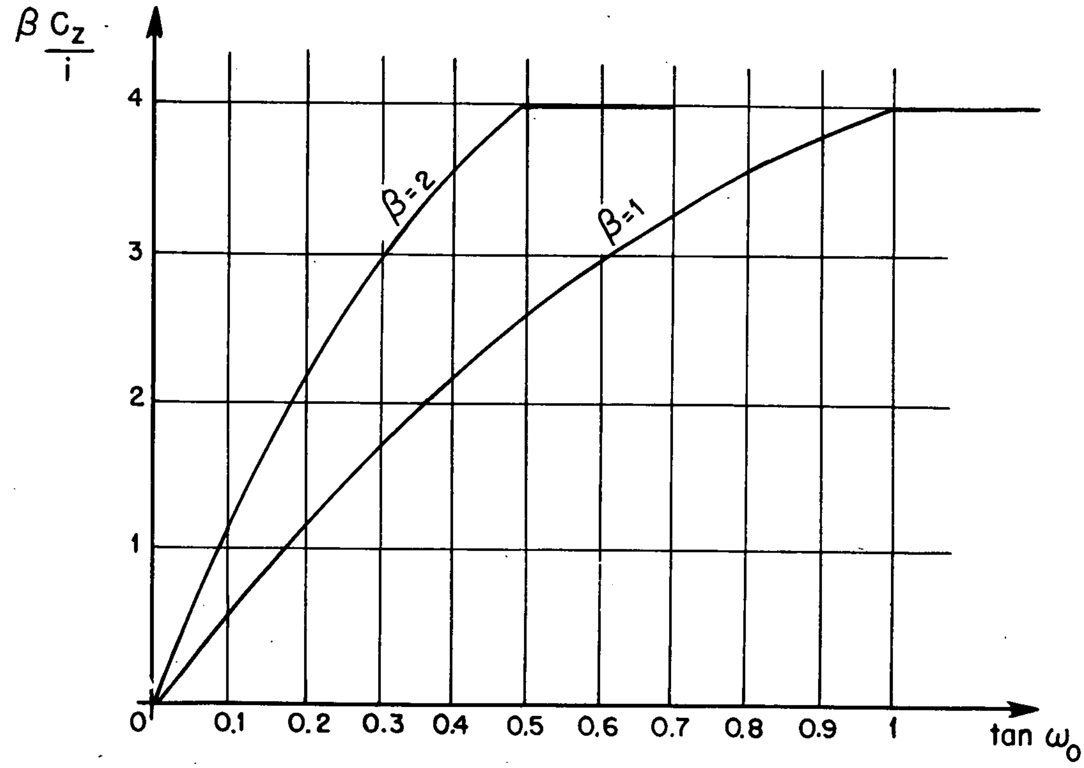


Figure 37



$\beta \frac{C_z}{i}$
4
3
2
1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
tan ω_0
$\beta=2$
$\beta=1$

Figure 38

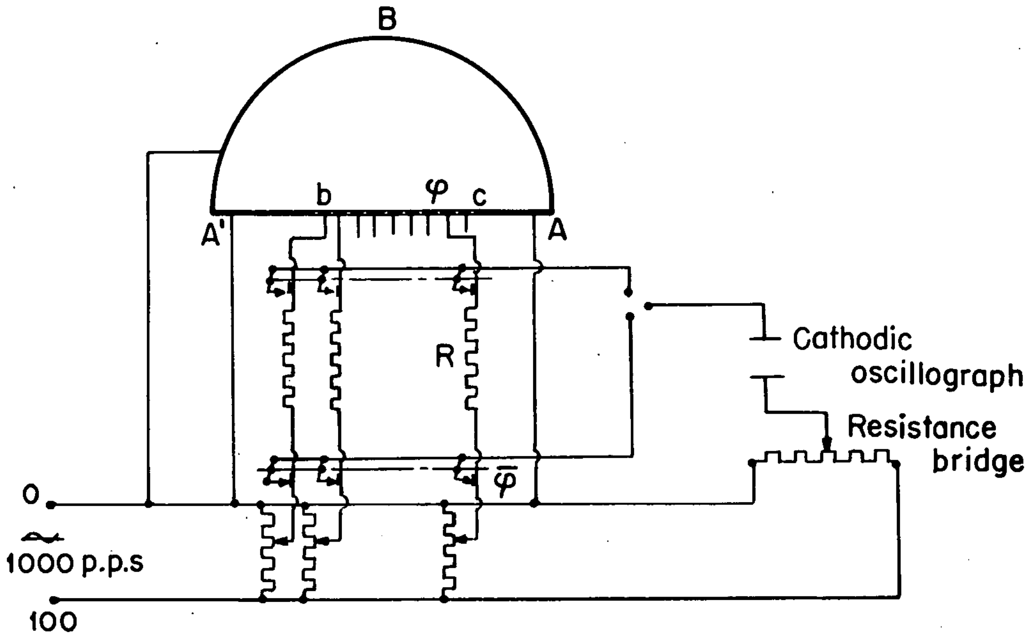

B
b
φ
c
A'
A
R
$\bar{\varphi}$
Cathodic oscillograph
Resistance bridge
0
1000 p.p.s
100

Figure 39

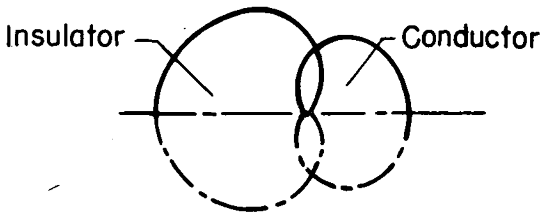


Figure 40

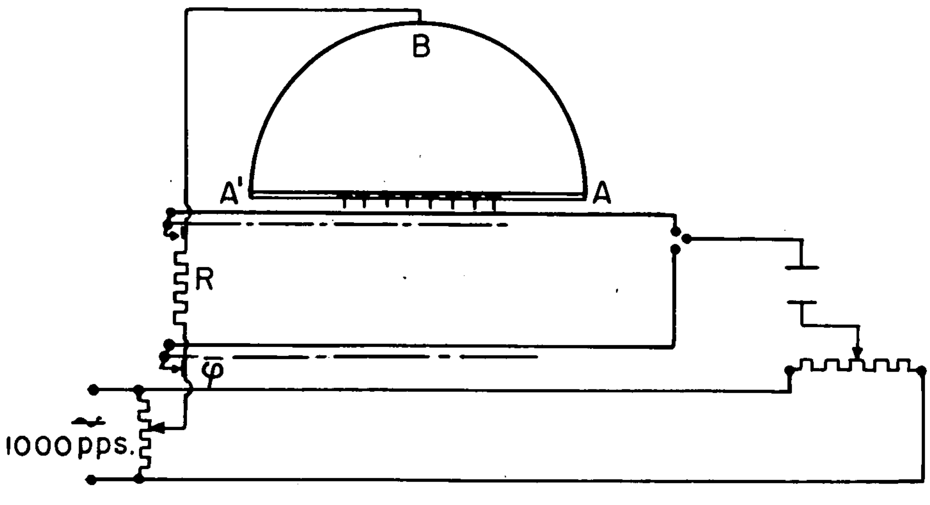


Figure 41

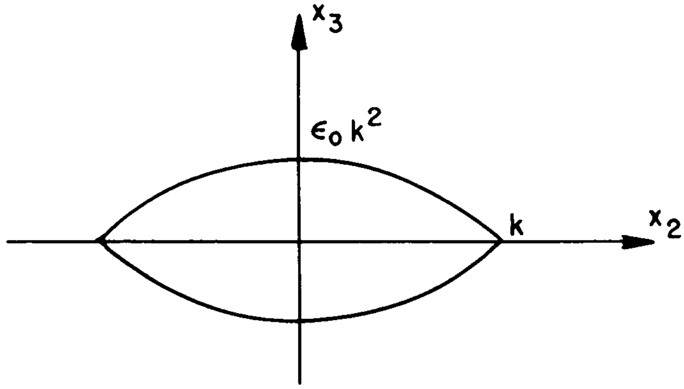


Figure 42

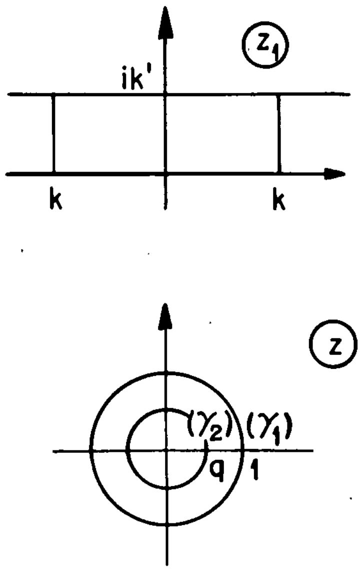


Figure 43

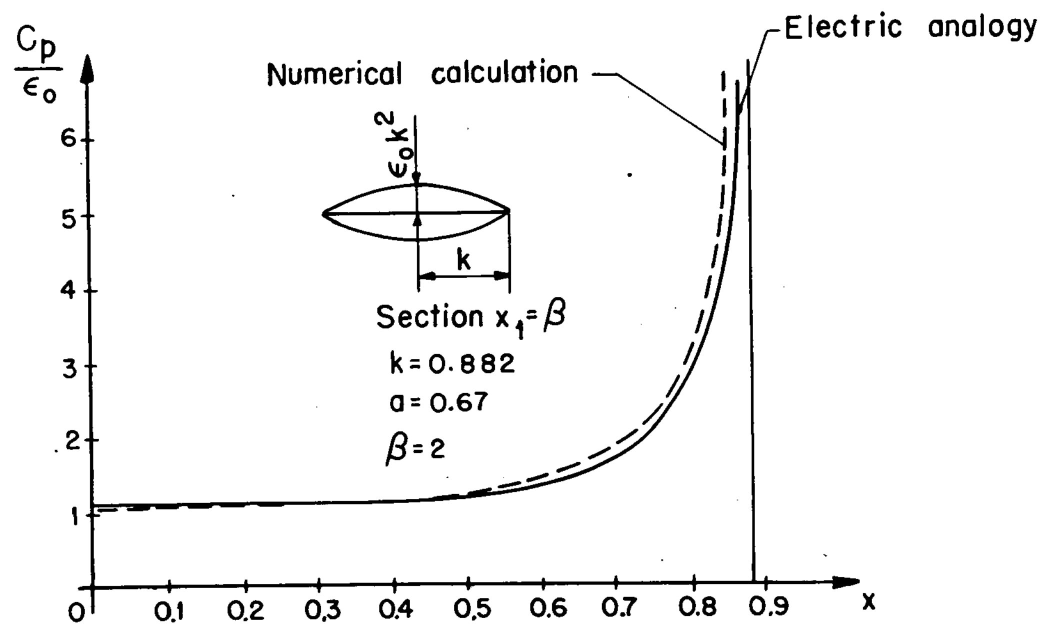

Electric analogy
Numerical calculation
$\frac{C_p}{\epsilon_o}$
$\epsilon_o k^2$
k
Section $x_1 = \beta$
k = 0.882
a = 0.67
β = 2
6
5
4
3
2
1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
x

Figure 44

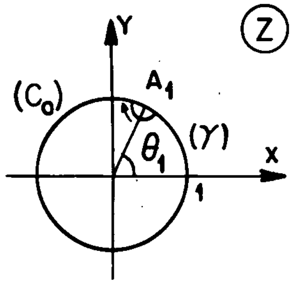


Figure 45

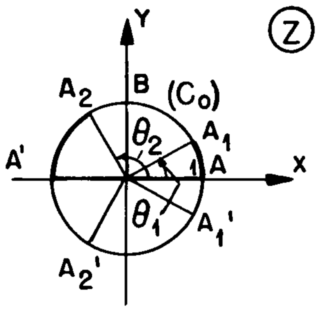


Figure 46

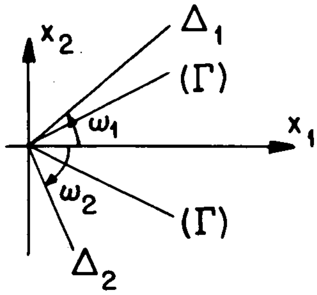
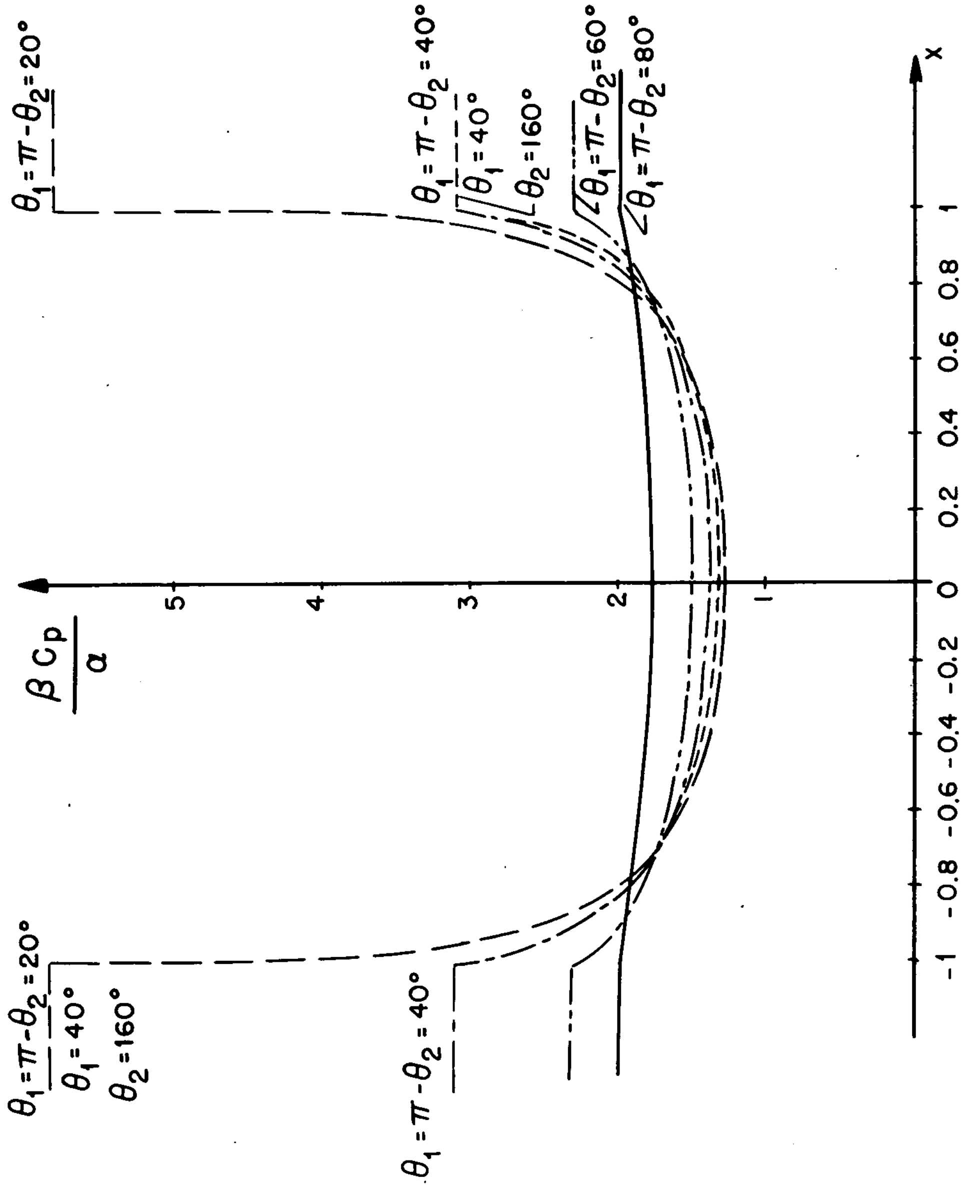


Figure 47



$\frac{\beta C_p}{\alpha}$
x
$\theta_1 = \pi - \theta_2 = 20°$
$\theta_1 = \pi - \theta_2 = 40°$
$\theta_1 = 40°$
$\theta_2 = 160°$
$\theta_1 = \pi - \theta_2 = 60°$
$\theta_1 = \pi - \theta_2 = 80°$

Figure 48

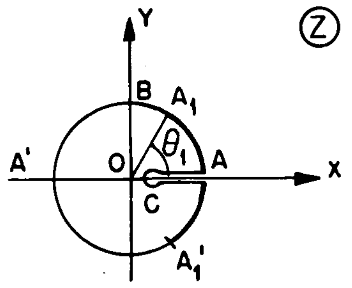


Figure 49

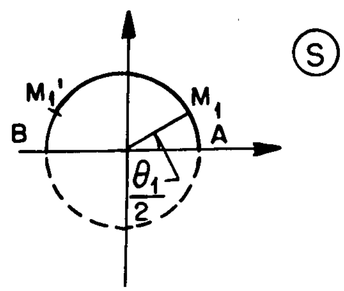


Figure 50

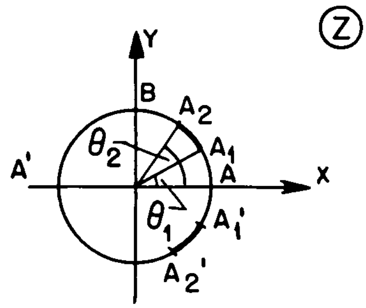


Figure 51

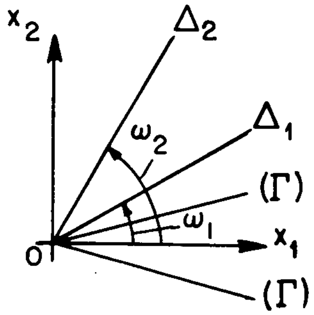
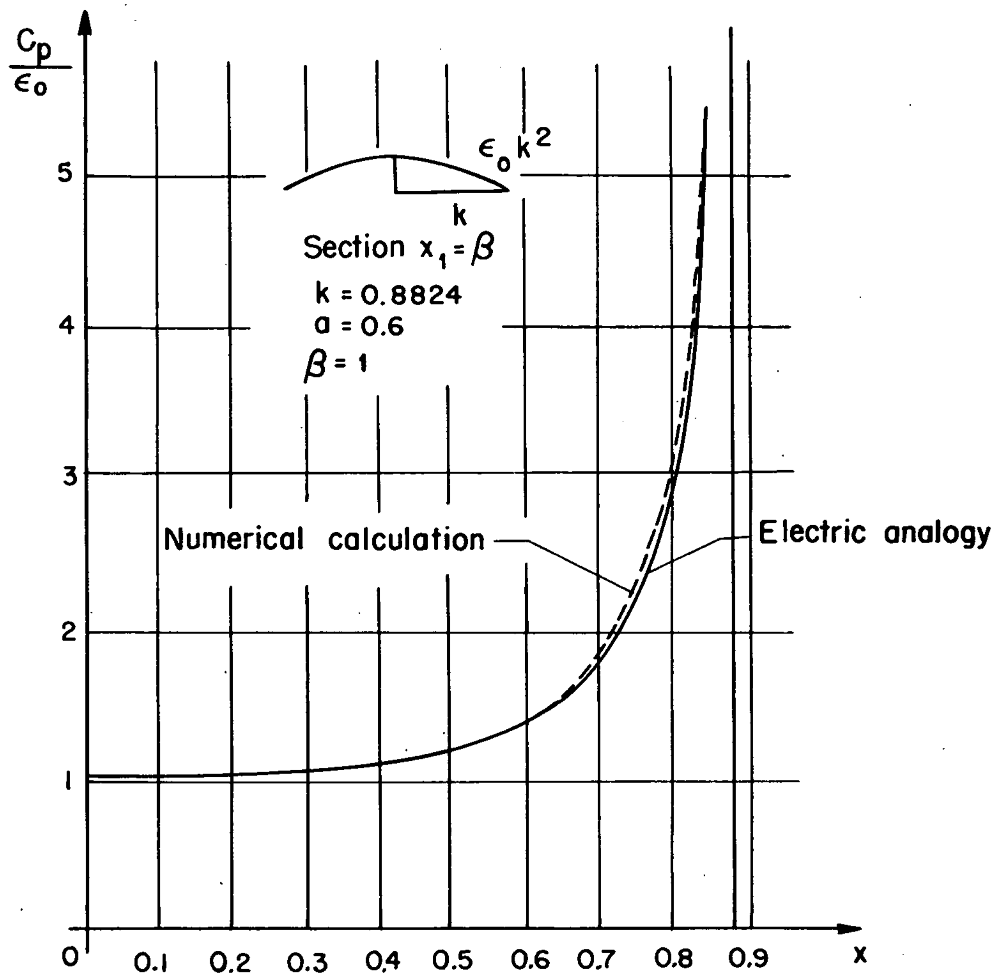


Figure 52



$\frac{C_p}{\epsilon_o}$
$\epsilon_o k^2$
k
Section $x_1 = \beta$
k = 0.8824
a = 0.6
β = 1
Numerical calculation
Electric analogy
0
1
2
3
4
5
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
x

Figure 53

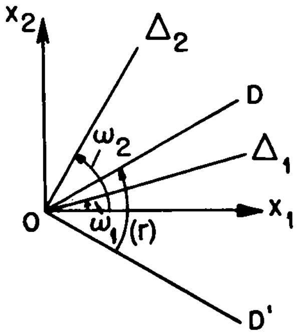


Figure 54

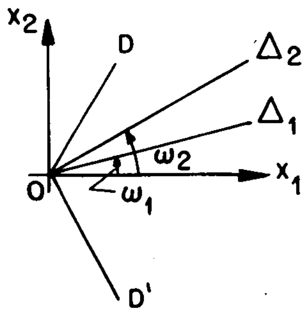


Figure 55

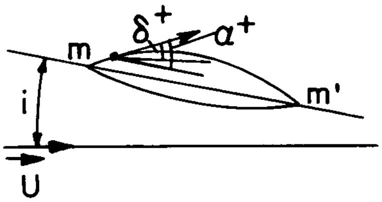


Figure 56

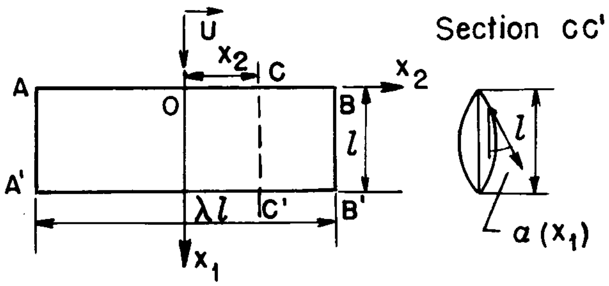


Figure 57

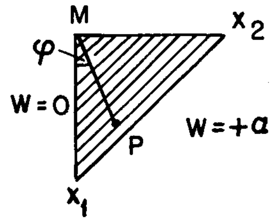


Figure 58

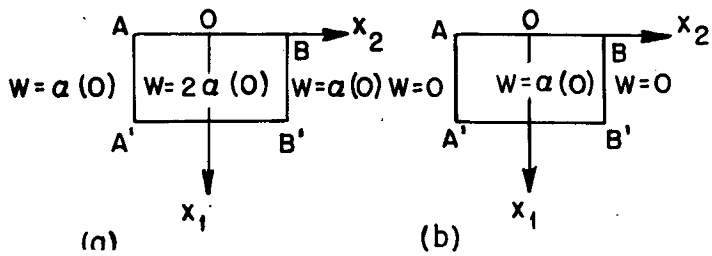


Figure 59

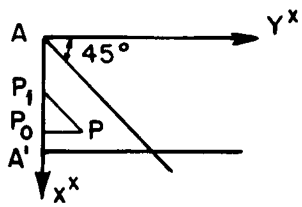
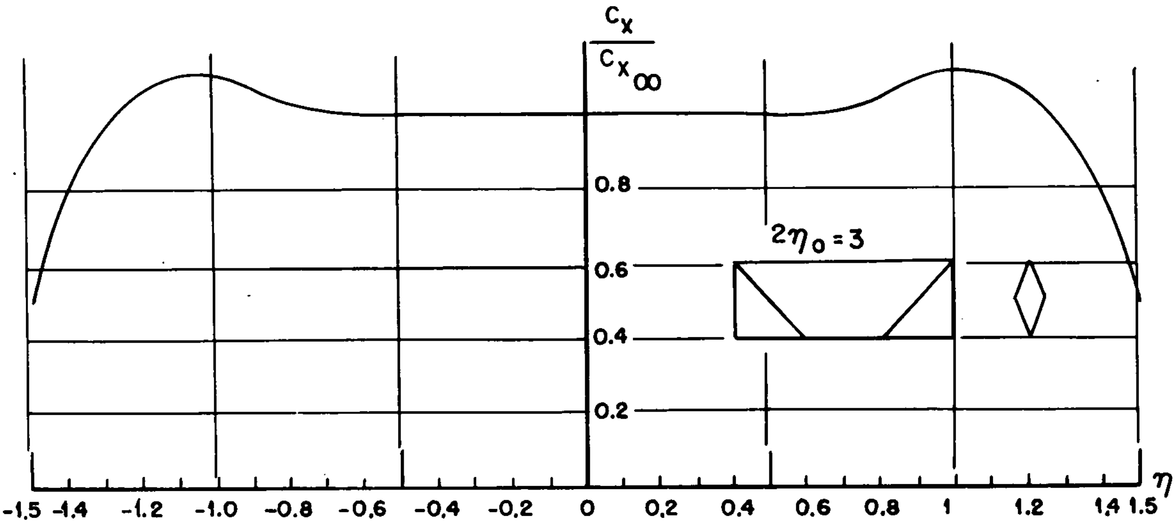
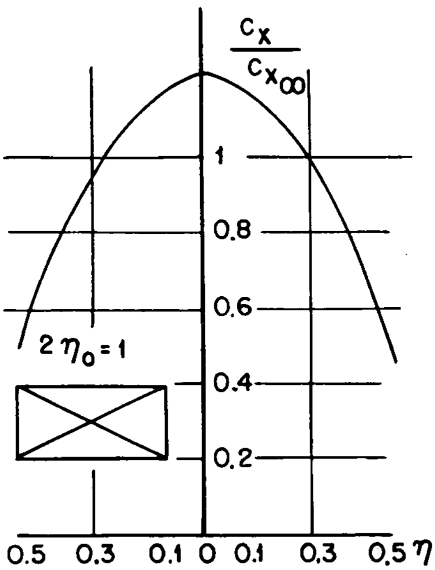


Figure 60


$\frac{c_x}{c_{x_{\infty}}}$
0.8
0.6
0.4
0.2
$2\eta_0 = 3$
-1.5 -1.4 -1.2 -1.0 -0.8 -0.6 -0.4 -0.2 0 0.2 0.4 0.6 0.8 1 1.2 1.4 1.5
η

(a)


$\frac{c_x}{c_{x_{\infty}}}$
1
0.8
0.6
0.4
0.2
$2\eta_0 = 1$
0.5 0.3 0.1 0 0.1 0.3 0.5
η

(b)

Figure 61

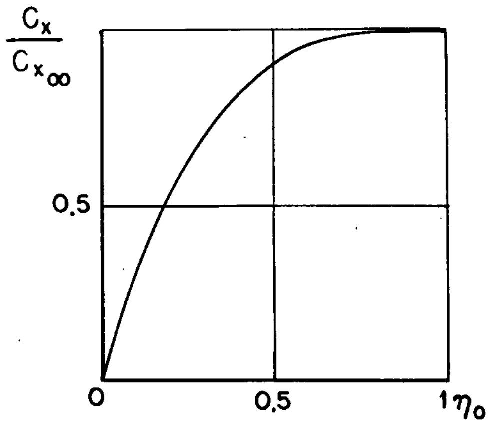


Figure 62

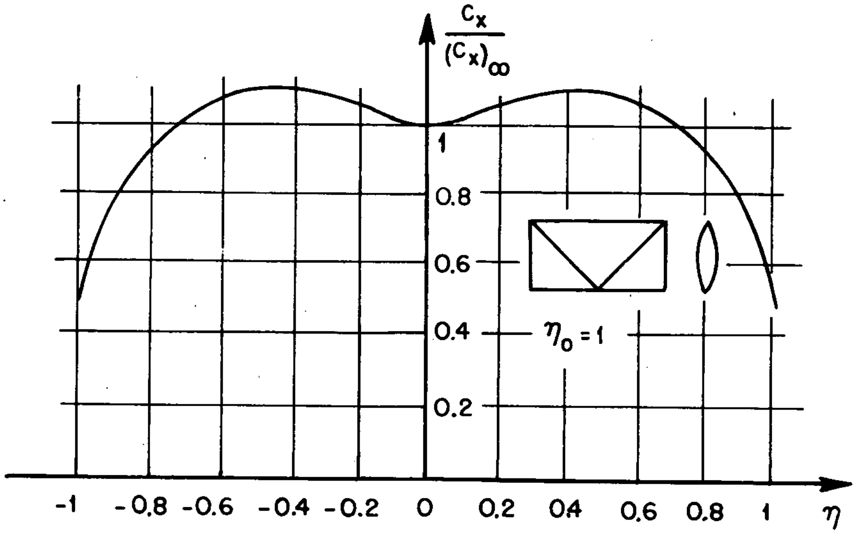


Figure 63

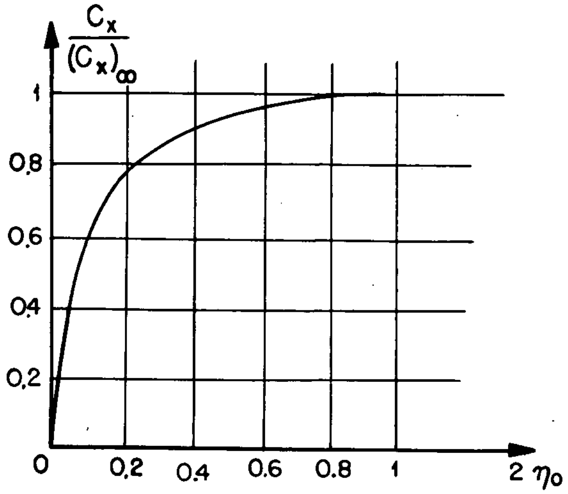


Figure 64

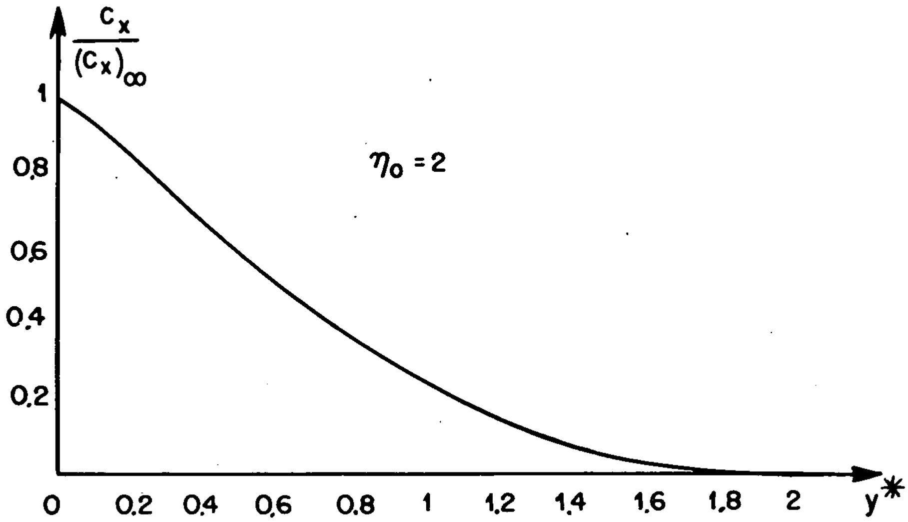


Figure 65

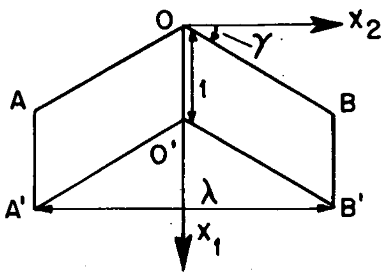


Figure 66

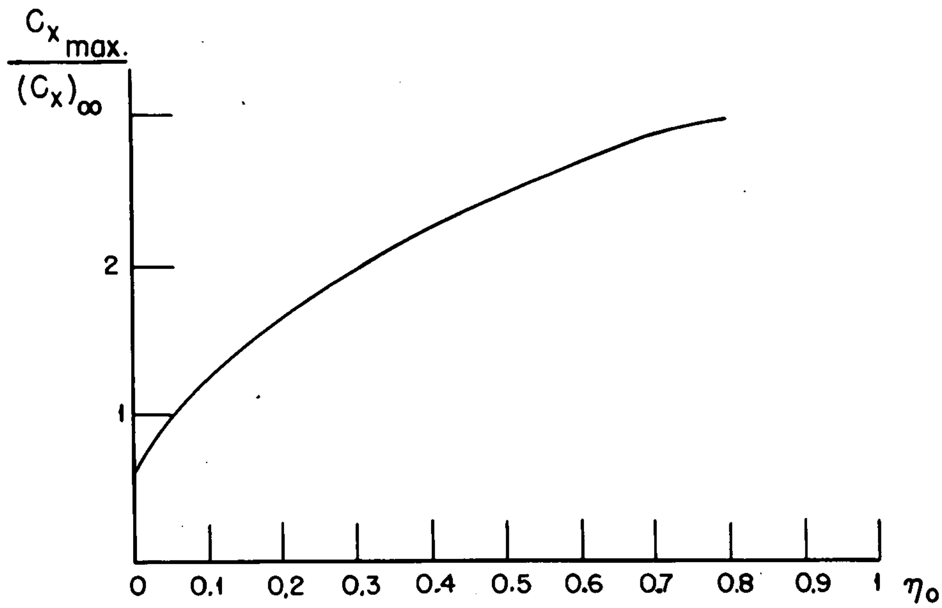


Figure 67

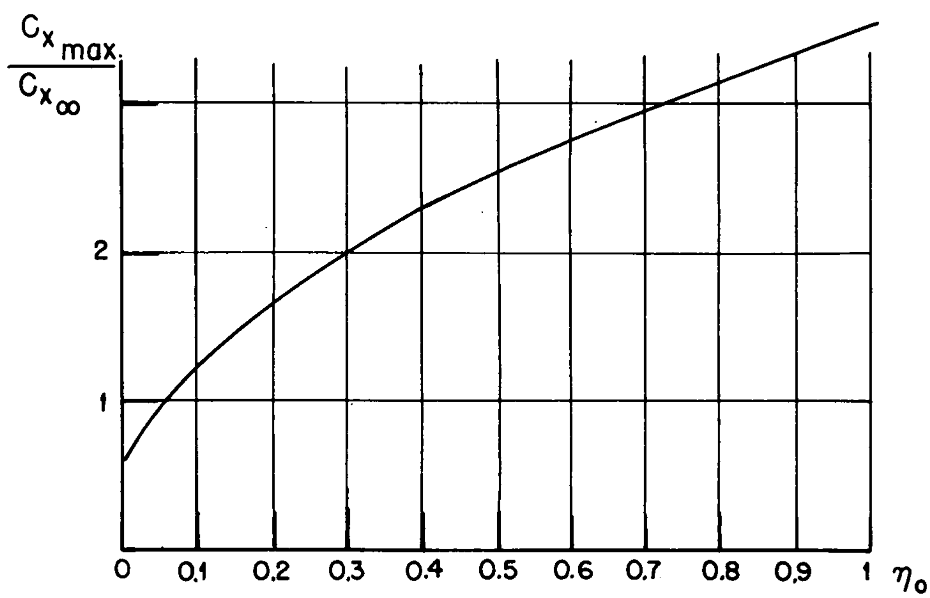


Figure 68

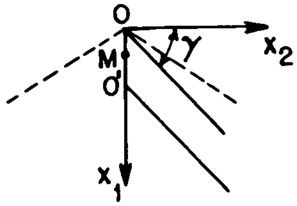


Figure 69

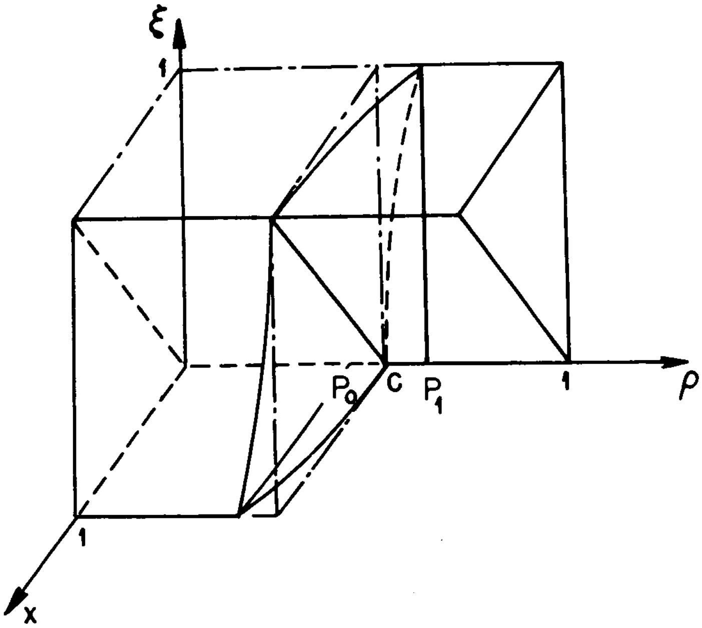
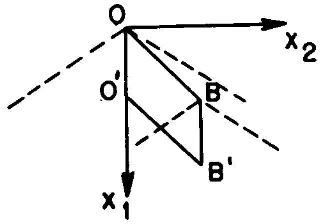


Figure 70



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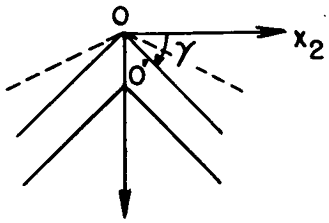
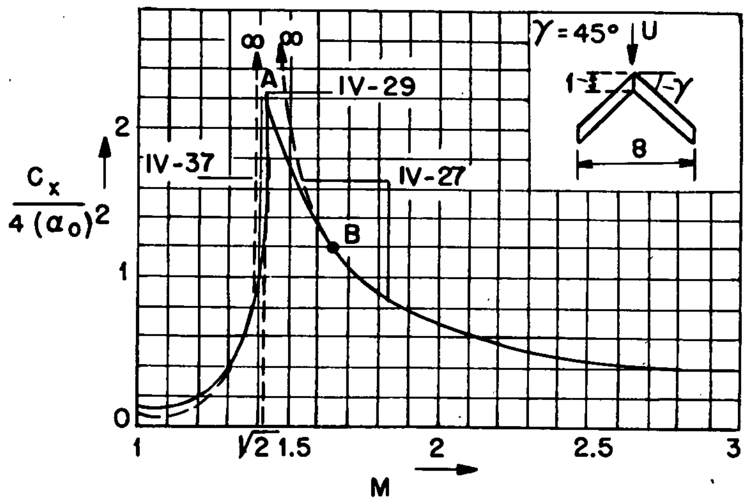


Figure 72



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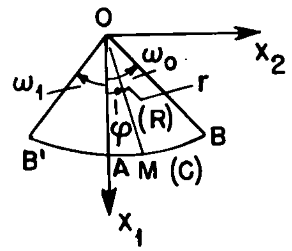
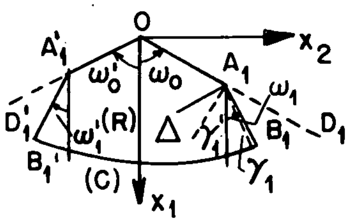


Figure 74



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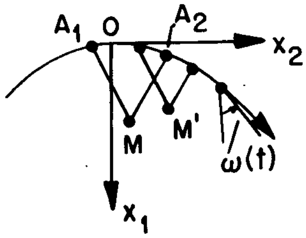


Figure 76

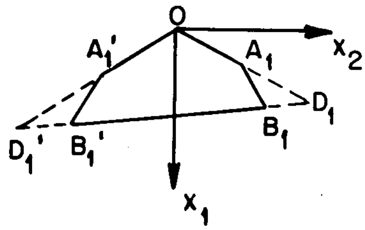


Figure 77

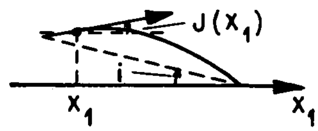
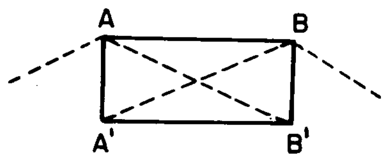


Figure 78



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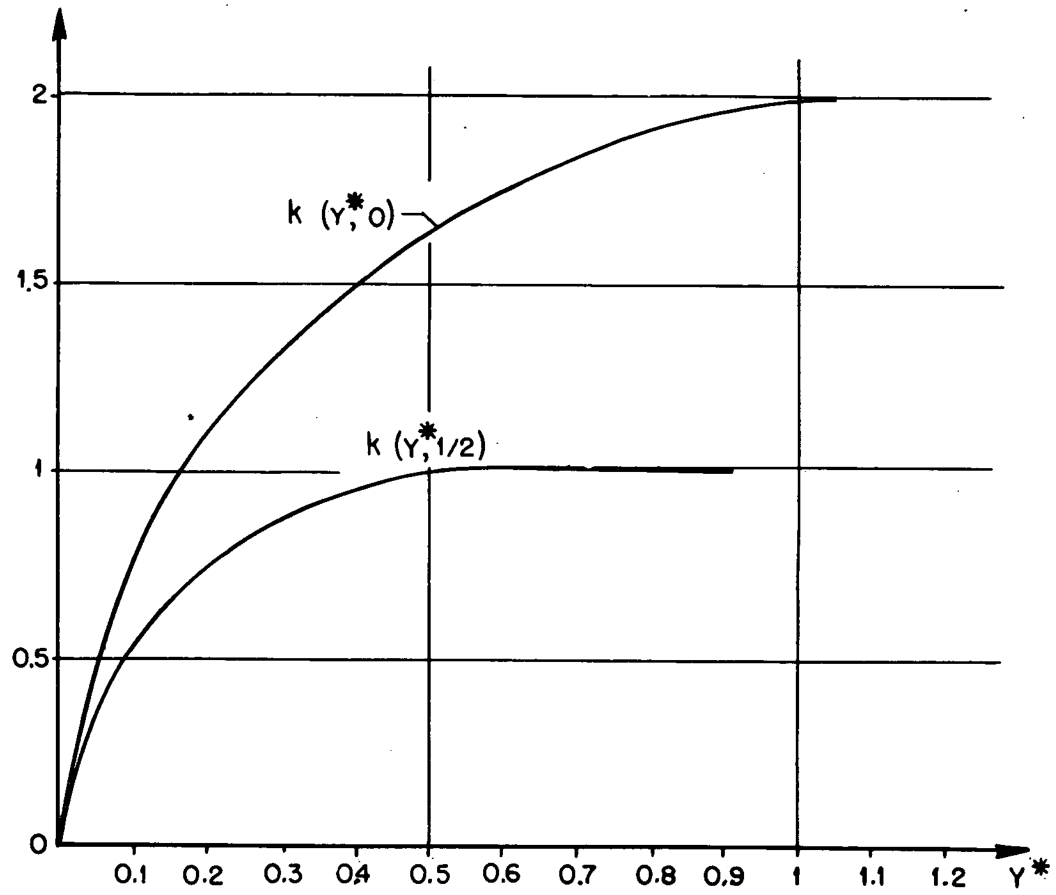


Figure 80

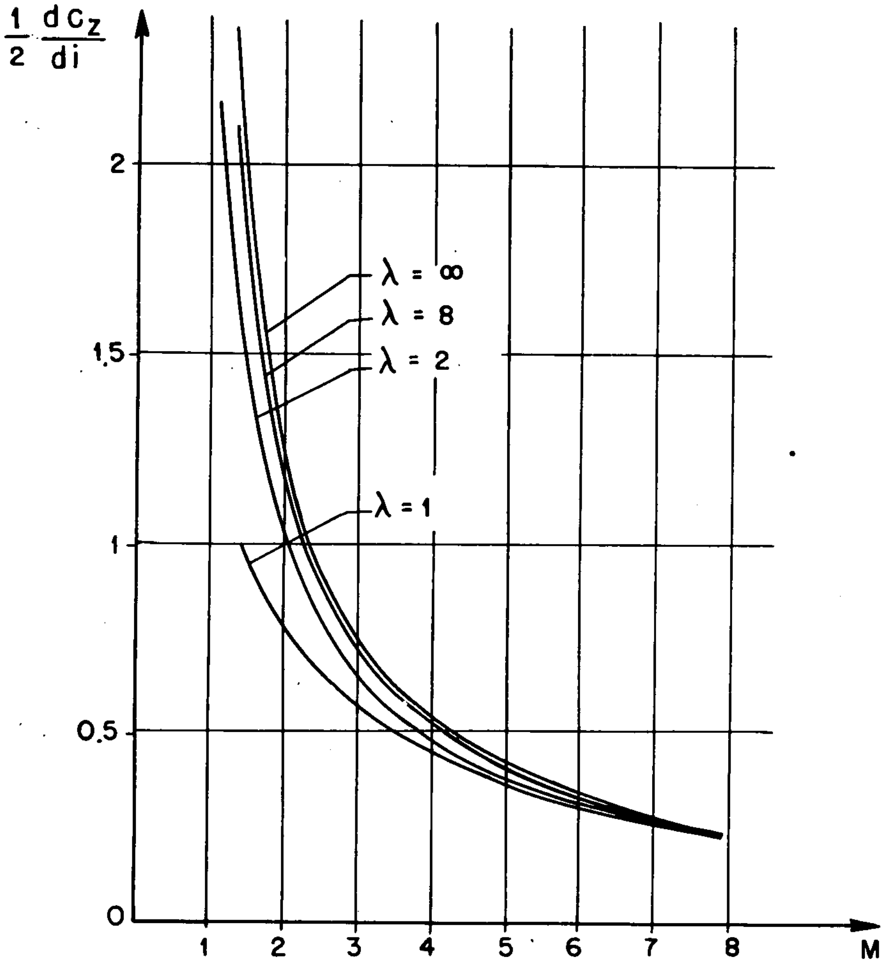

$\frac{1}{2}\frac{dc_z}{di}$
2
1.5
1
0.5
0
1
2
3
4
5
6
7
8
M
$\lambda = \infty$
$\lambda = 8$
$\lambda = 2$
$\lambda = 1$

Figure 81

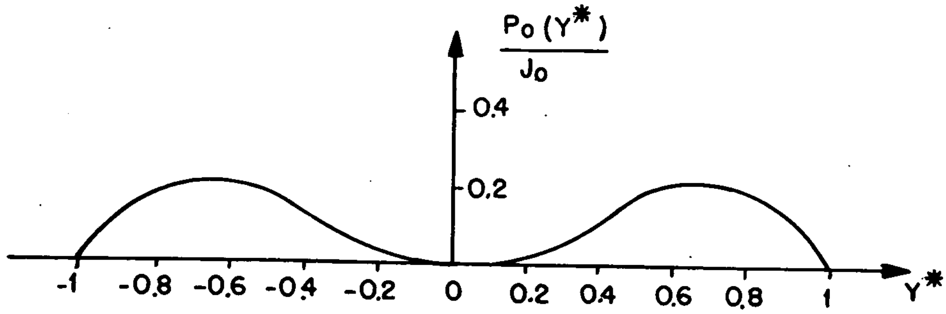


Figure 82

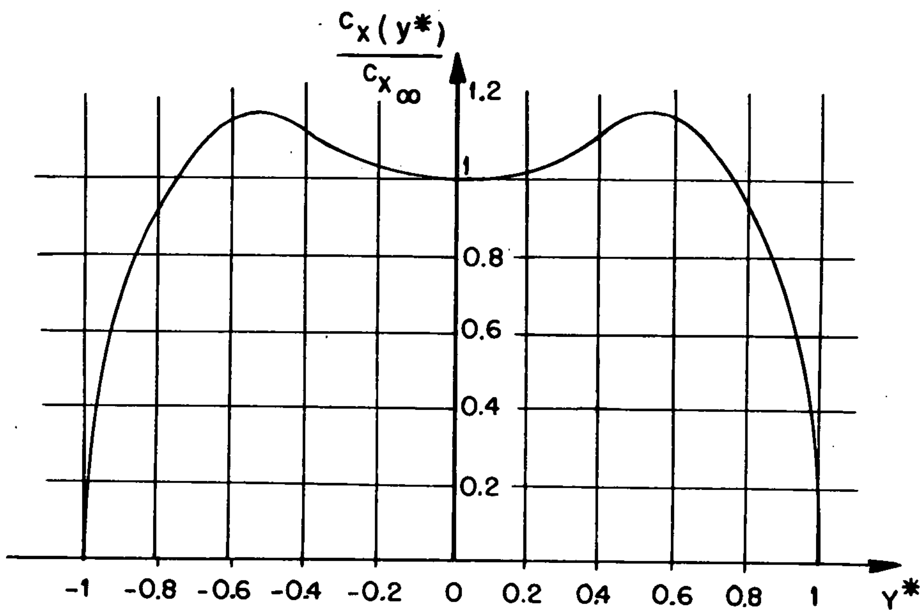


Figure 83

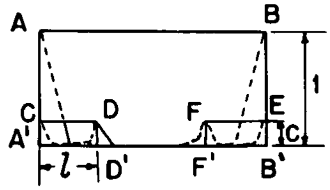


Figure 84

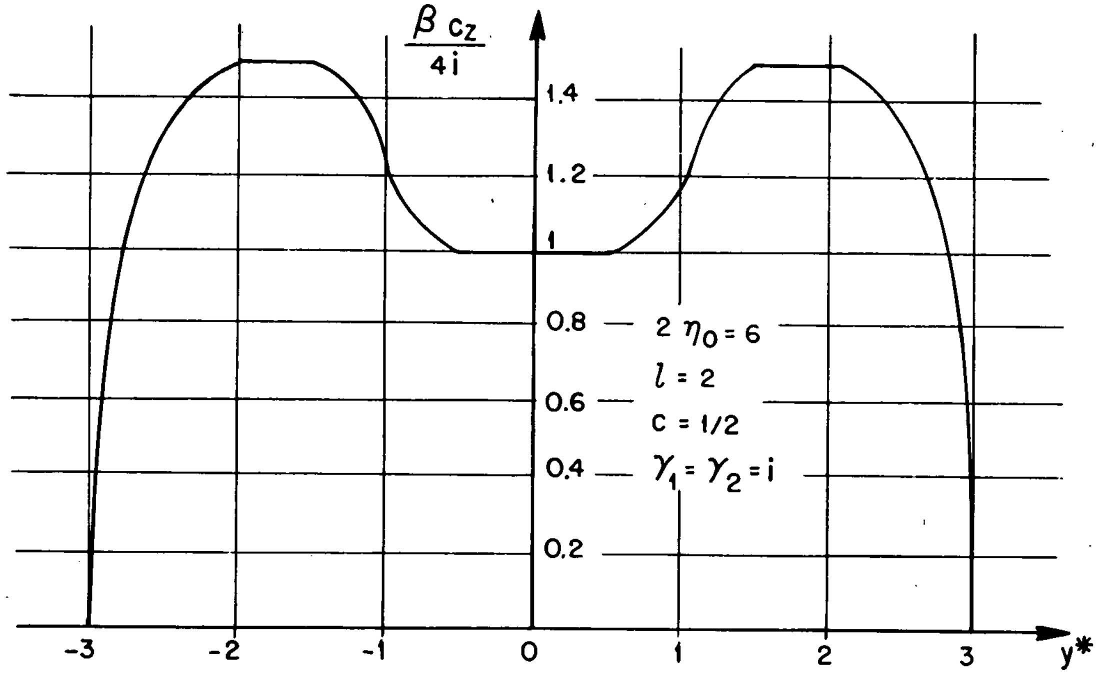


Figure 85

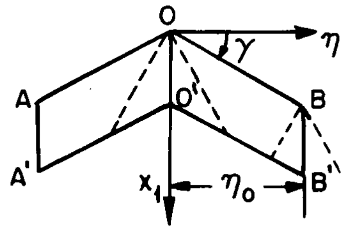


Figure 86

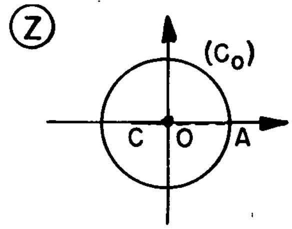


Figure 87

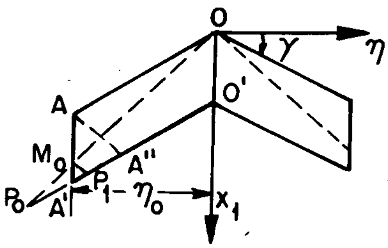


Figure 88

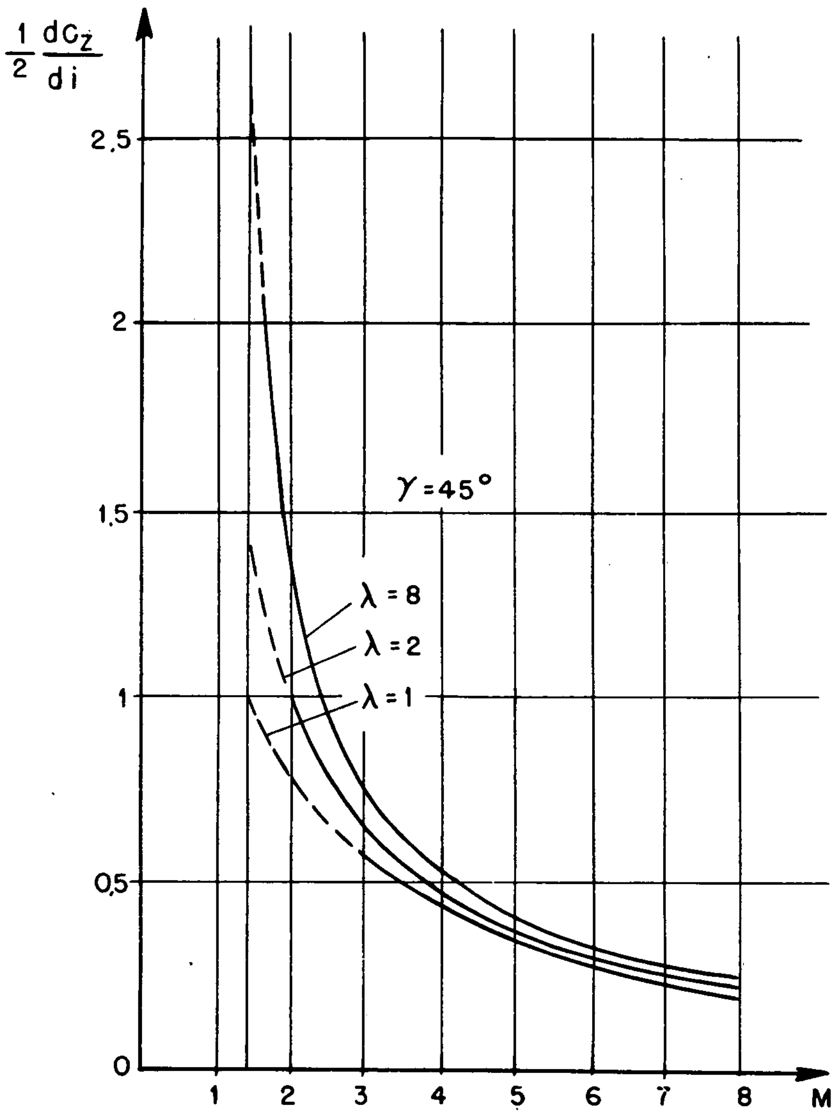
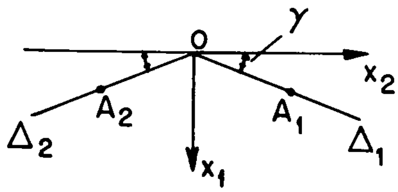

$\frac{1}{2}\frac{dc_z}{di}$
2,5
2
1,5
1
0,5
0
1
2
3
4
5
6
7
8
M
$\gamma = 45°$
$\lambda = 8$
$\lambda = 2$
$\lambda = 1$

Figure 90



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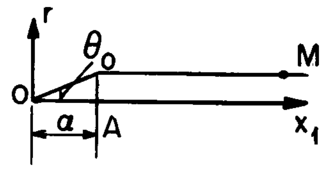

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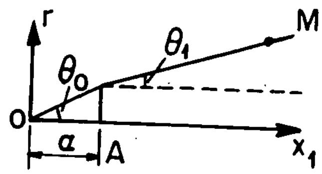

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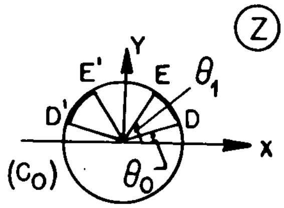


Figure 94

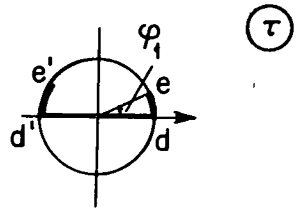
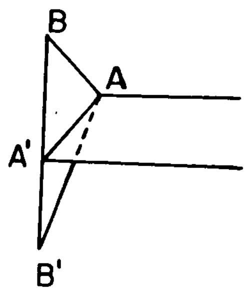


Figure 95



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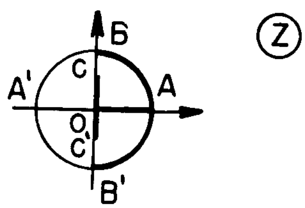

Figure 97

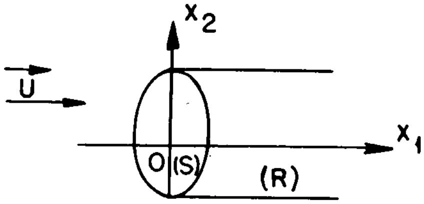

Figure 1

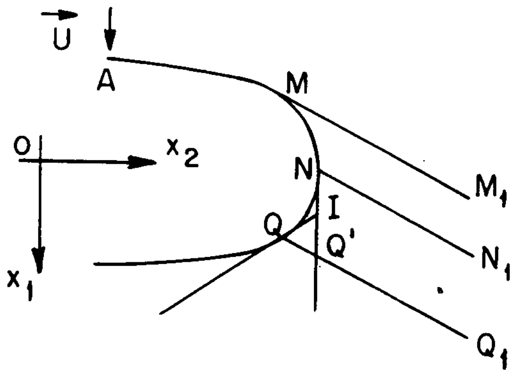

Figure 2

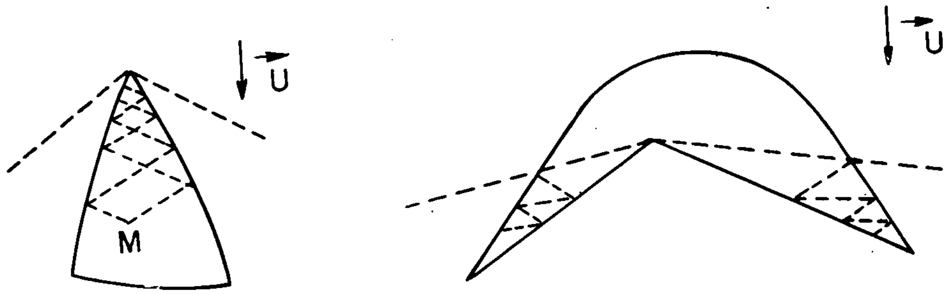

Figure 3