ON A CLASS OF EXACT SOLUTIONS OF THE EQUATIONS OF MOTION OF A VISCOUS FLUID

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The general solution is obtained herein of the equations of motion of a viscous fluid in which the velocity field is inversely proportional to the distance from a certain point. Some particular cases of such motion are investigated.

1. The motion of a viscous fluid with velocity field and pressure in spherical coordinates can be given by the following expressions:

\[ v_r = \frac{F(\theta)}{r}, \quad v_\theta = \frac{f(\theta)}{r}, \quad v_\varphi = 0, \quad \frac{p}{\rho} = \frac{g(\theta)}{r^2} \quad (1) \]

A particular solution of the equations of Navier-Stokes for this case was obtained by Landau (reference 1). In the present paper a general solution is given of the equations of Navier-Stokes for the motion of the class under consideration.

Substituting expressions (1) in the equations of Navier-Stokes and in the equation of continuity yields the following system:

\[ F^2 + f^2 - fF' + 2g - \nu \left[ F'' + F' \cot \theta - 2f' - 2F + 2f \cot \theta \right] = 0 \quad (2) \]

\[ ff' + g' - \nu \left[ f'' + f' \cot \theta + 2F' - f (1 + \cot^2 \theta) \right] = 0 \quad (3) \]

\[ F + f' + f \cot \theta = 0 \quad (4) \]

Determining \( F \) from equation (4) and substituting in equations (2) and (3) give

Differentiating expression (6)

\[ f'^2 + ff'' + g' + \nu \left[ f'' + f' \cot \theta - f' (1 + \cot^2 \theta) \right] = 0 \] (6)

Eliminating the nonlinear terms \( f'^2, ff'', \) and \( ff' \) from equation (5) with the aid of equations (6) and (7) yields a linear equation in the function \( g + 2\nu f' \):

\[ (g + 2\nu f')'' + 3 \cot \theta (g + 2\nu f')' - 2 (g + 2\nu f') = 0 \] (8)

the general solution of which is in the form

\[ g + 2\nu f' = 2\nu^2 \frac{b \cos \theta - a}{\sin^2 \theta} \] (9)

where \( 2\nu^2a \) and \( 2\nu^2b \) are constants of integration.

Integrating equation (6)

\[ f^2 + 2g + 2\nu (f' + f \cot \theta) = -2\nu^2c \] (10)

where \( 2\nu^2c \) is the constant of integration.

The function \( g(\theta) \) is eliminated from equations (9) and (10) to give an equation of the Riccati type for the function \( f' \):

\[ f' = \frac{1}{2\nu} f^2 + f \cot \theta + 2\nu \left( \frac{b \cos \theta - a}{\sin^2 \theta} + \frac{c}{2} \right) \] (11)

\(^1\)After sending the manuscript to press the author obtained from L. D. Landau a communication on the work of N. Slezkin (reference 2) in which he arrived at the same equation by a different method.
The substitution

\[ f = - \frac{2\nu x'(\theta) / x(\theta)}{\nu} \]  

(12)

reduces equation (11) to the linear equation:

\[ x'' - x' \cot \theta + \left( \frac{b \cos \theta - a}{\sin^2 \theta} + \frac{c}{2} \right) x = 0 \]  

(13)

which by the substitution

\[ z = \cos^2 (\theta/2) \]  

(14)

is transformed into an equation of the Fuchsian type:

\[ \frac{d^2 x}{dz^2} - \frac{a + b - 2 (b + c) z + 2cz^2}{4z^2 (z - 1)} x = 0 \]  

(15)

The usual computations (reference 3), which are omitted herein, give the general solution of equation (15) as:

\[ x(\theta) = \left( \cos \frac{\theta}{2} \right)^y \left( \sin \frac{\theta}{2} \right)^{1+\alpha+\beta-y} \left\{ \begin{aligned} & c_1 F \left( \alpha, \beta, \gamma, \cos^2 \frac{\theta}{2} \right) + \\
& c_2 F \left( \alpha + 1 - \gamma, \beta + 1 - \gamma, 2 - \gamma, \cos^2 \frac{\theta}{2} \right) \end{aligned} \right\} \]  

(16)

where the parameters of the hypergeometric function \( \alpha, \beta, \gamma \) (which can also have complex values) are connected with the constants of integration \( a, b, c \) by the formulas:

\[ \begin{aligned} a &= r^2 - (1 + \alpha + \beta) \gamma + \frac{(\alpha + \beta)^2}{2} - \frac{1}{2} \\
b &= (\alpha + \beta - 1) \gamma - \frac{(\alpha + \beta)}{2} + \frac{1}{2} \\
c &= \frac{(\alpha - \beta)^2}{2} - 1 \end{aligned} \]  

(17)
Formulas (4), (9), (16), and (17) give the general solution, depending on the four constants $a$, $b$, $c$, and $A = c_2/c_1$, of the Navier-Stokes equations for the class of motion of a viscous fluid under consideration. The constants of integration $a$, $b$, and $c$ are expressed in terms of the corresponding tensor components of the density of the momentum transfer:

$$
\Pi_{ik} = p\delta_{ik} + \rho \nu_i \nu_k - \rho \nu \left( \frac{\partial \nu_i}{\partial x^k} + \frac{\partial \nu_k}{\partial x^i} \right)
$$

(18)

Carrying out the computations

$$
\begin{align*}
\Pi_{\varphi\varphi} &= \frac{2\nu^2 \rho}{r^2} \left( \frac{b \cos \theta - a}{\sin^2 \theta} \right) \\
\Pi_{\theta\theta} &= \frac{2\nu^2 \rho}{r^2} \left( \frac{a - b \cos \theta - c}{\sin^2 \theta} \right) \\
\Pi_{r\theta} &= \frac{2\nu^2 \rho}{r^2} \left( \frac{c \cos \theta - b}{\sin^2 \theta} \right)
\end{align*}
$$

(19)

The streamlines are determined by the equation:

$$
\frac{dr}{v_r} = \frac{rd\theta}{v_\theta}
$$

(20)

the integration of which gives

$$
\text{const}/r = f \sin \theta
$$

(21)

2. Attention is now given to two particular examples for which the equation of Fuchs degenerates.

(a) Equation (15) has only one regular singular point, $z = \infty$. In this case

$$
a = b = c = 0
$$

(22)

and therefore by equations (19)

$$
\Pi_{\varphi\varphi} = \Pi_{\theta\theta} = \Pi_{r\theta} = 0
$$

(23)
The particular solution of equation (15)

\[ X(\theta) = 2z - 1 - A \]  \hspace{1cm} (24)

leads by formulas (4), (9), and (12) to the solution found by Landau:

\[
F(\theta) = 2\nu \left[ \frac{A^2 - 1}{(A - \cos \theta)^2} - 1 \right] \\
f(\theta) = \frac{2\nu \sin \theta}{\cos \theta - A} \\
g(\theta) = 4\nu^2 \frac{1 - A \cos \theta}{(\cos \theta - A)^2} \hspace{1cm} (25)
\]

This solution is analogous to the problem of a stream flowing out of the end of a thin pipe into a region filled with the same fluid. It is the only regular solution for all values of the angle \( \theta \).

(b) Equation (15) has only two regular points \( z = 0 \) and \( z = \infty \). In this case it follows from equation (15) that

\[ a = b = c \neq 0 \]  \hspace{1cm} (26)

and equation (11) becomes Euler's equation

\[ 2z^2 \left( \frac{d^2 \chi}{dz^2} \right) - a\chi = 0 \]  \hspace{1cm} (27)

the general solutions of which are

\[
\begin{align*}
X(\theta) &= e^{x/2} \cosh (nx + A) \text{ for } a > - 1/2 \\
X(\theta) &= e^{x/2} \cos (nx + A) \text{ for } a < - 1/2 \\
X(\theta) &= e^{x/2} (1 + Ax) \text{ for } a = - 1/2
\end{align*} \hspace{1cm} (28)
\]

Correspondingly, the following equations are obtained for the function \( f(\theta) \):

\[
f = 2\nu \frac{\sin \theta}{1 + \cos \theta} \left\{ \begin{array}{l}
n \tanh (nx + A) + 1/2 \\
1/2 - n \tan (nx + A)
\end{array} \right\} \text{ for } a > - 1/2 \] \hspace{1cm} (29)

\[
f = 2\nu \frac{\sin \theta}{1 + \cos \theta} \left\{ \begin{array}{l}
1/2 - n \tan (nx + A)
\end{array} \right\} \text{ for } a < - 1/2 \] \hspace{1cm} (30)

\[
f = 2\nu \frac{\sin \theta}{1 + \cos \theta} \left\{ \begin{array}{l}
A \\
1 + Ax + 1/2
\end{array} \right\} \text{ for } a = - 1/2 \] \hspace{1cm} (31)
where \( x = \ln (1 + \cos \theta), n = \frac{1}{2} |\sqrt{1 + 2a}|. \) (For \( a = 0, n = 1/2 \) in equation (29) the solution of Landau is again obtained.)

For the solution of equation (29) by formula (4)

\[
F(\theta) = -2\nu \left\{ n \tanh (nx + A) + 1/2 \right\} + 2\nu \frac{1 - \cos \theta}{1 + \cos \theta} \frac{n^2}{\cosh^2 (nx + A)}
\]

(32)

while \( g(\theta) \) is determined by formula (9).

The equation of the streamlines is in the form

\[
\text{const}/r = (1 - \cos \theta) \left\{ n \tanh (nx + A) + 1/2 \right\}
\]

(33)

where the values of the constants \( n \) and \( A \) are determined from the conditions

\[
\begin{align*}
f \left( \frac{\pi}{2} \right) &= 2\nu \left( n \tanh A + 1/2 \right) \\
f \left( \frac{\pi}{2} \right) + F \left( \frac{\pi}{2} \right) &= 2\nu \frac{n^2}{\cosh^2 A}
\end{align*}
\]

(34)

The obtained solution corresponds to the problem of the stream flowing from the half line \( \theta = \pi \) into a region filled with the same fluid.

For solution (30), the parametric equation for the streamlines is in the form

\[
\text{const}/r = (2 - e^x) \left[ \frac{1}{2} - n \tan (nx + A) \right]
\]

\[
\theta = \arccos \left( e^x - 1 \right)
\]

(35)

The function (35) for \( \theta \to \pi \) is a strongly oscillating one. It can therefore be concluded that solution (30) has no physical sense.

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REFERENCES

