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SPIRAL MOTIONS OF VISCOUS FLUIDS

By Georg Hamel

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INTRODUCTION

The equations for the plane motion of viscous fluids of constant volume are, after elimination of the pressure and introduction of the stream function ψ by which the velocity components

$$v_x = \frac{\partial\psi}{\partial y} \qquad v_y = -\frac{\partial\psi}{\partial x}$$

are expressed, reduced to the one equation

$$\frac{\partial\,\Delta\psi}{\partial t} + \frac{\partial\,\Delta\psi}{\partial x}\frac{\partial\psi}{\partial y} - \frac{\partial\,\Delta\psi}{\partial y}\frac{\partial\psi}{\partial x} = \sigma\,\Delta\,\Delta\psi \tag{I}$$

therein σ indicates the ratio between viscosity coefficient and specific mass μ, and Δ signifies the Laplace operator.

This equation is satisfied by all potential motions

$$\Delta\psi = 0$$

however, this fact is of little significance since viscous fluids adhere to solid walls and, from well-known considerations of function theory, there cannot exist a potential motion which would do so. Otherwise, properly speaking, only Poiseuille's laminar motion is known as exact solution of equation (I) and that solution does not even show the significance of the quadratic terms because they identically disappear there.

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*"Spiralförmige Bewegungen zäher Flüssigkeiten." Jahresber. d. deutschen Math. Ver. 25, 1917, pp. 34-60.

Under these circumstances it seems perhaps useful to know a few more exact solutions of equation (I) for which the quadratic terms do not disappear; such solutions will be indicated below according to two methods.

In both cases, one deals with motions in spiral-shaped streamlines (which are observed frequently).

Third, we shall, in addition, investigate the neighborhood solutions to pure radial flow.

FIRST PART

We raise the question:

Are there solutions of equation (I) which are not potential motions for which, however, the stream paths are the same as for a potential motion whereas the velocity distribution is to be different?

We shall be able to indicate such solutions, in fact all of them: the streamlines are logarithmic spirals (including concentric circles and pure radial flow); for the velocity distribution, one arrives at an ordinary differential equation which for pure radial flow leads to elliptic functions. In the discussion, the influence of the quadratic terms becomes manifest in a considerable difference between inflow and outflow (see paragraphs 7, 8, and 9).

We require, therefore, solutions ψ of equation (I) for which

$$\psi = f(\varphi)$$

and $\Delta\varphi = 0$, but not $\Delta\psi = 0$. The latter condition excludes

$$f''(\varphi) = 0$$

We limit ourselves to steady motions $\frac{\partial\psi}{\partial t} = 0$.

1. The calculation becomes clearer if first the auxiliary problem has been solved:

Transformation of equation (I) into isometric coordinates, that is, such curvilinear coordinates φ,χ that

$$\varphi + i\chi = w(x + iy) = w(z)$$

Let us thus assume

$$\psi = \psi(\varphi,\chi) \qquad \frac{\partial\varphi}{\partial x} = \frac{\partial\chi}{\partial y} \qquad \frac{\partial\varphi}{\partial y} = -\frac{\partial\chi}{\partial x}$$

If one denotes $\frac{\partial^2\psi}{\partial\varphi^2} + \frac{\partial^2\psi}{\partial\chi^2}$ by $\Delta'\psi$, there results first, with the abbreviation

$$\left|\frac{dw}{dz}\right|^2 = Q$$

$$\Delta\psi = Q \qquad \Delta'\psi$$

With the double integral extended over an arbitrary region, one has

$$\iint\left(\frac{\partial\,\Delta}{\partial x}\frac{\partial\psi}{\partial y} - \frac{\partial\,\Delta}{\partial y}\frac{\partial\psi}{\partial x}\right)dx\,dy = \iint d\,\Delta\,d\psi$$

$$= \iint\left[\frac{\partial(Q\,\Delta')}{\partial\varphi}\frac{\partial\psi}{\partial\chi} - \frac{\partial(Q\,\Delta')}{\partial\chi}\frac{\partial\psi}{\partial\varphi}\right]d\varphi\,d\chi$$

$$= \iint\left[\frac{\partial(Q\,\Delta')}{\partial\varphi}\frac{\partial\psi}{\partial\chi} - \frac{\partial(Q\,\Delta')}{\partial\chi}\frac{\partial\psi}{\partial\varphi}\right]\left(\frac{\partial\varphi}{dx}\frac{\partial\chi}{\partial y} - \frac{\partial\varphi}{\partial y}\frac{\partial\chi}{\partial x}\right)dx\,dy$$

Since, however,

$$\frac{\partial\varphi}{\partial x}\frac{\partial\chi}{\partial y} - \frac{\partial\varphi}{\partial y}\frac{\partial\chi}{\partial x} = \left|\frac{dw}{dz}\right|^2 = Q$$

there follows

$$\frac{\partial \Delta}{\partial x} \frac{\partial \psi}{\partial y} - \frac{\partial \Delta}{\partial y} \frac{\partial \psi}{\partial x} = Q^2 \left[\left(\frac{\partial \Delta'}{\partial \varphi} \frac{\partial \psi}{\partial \chi} - \frac{\partial \Delta'}{\partial \chi} \frac{\partial \psi}{\partial \varphi} \right) + \Delta' \left(\frac{\partial \ln Q}{\partial \varphi} \frac{\partial \psi}{\partial \chi} - \frac{\partial \ln Q}{\partial \chi} \frac{\partial \psi}{\partial \varphi} \right) \right]$$

However,

$$\frac{\partial \ln Q}{\partial \varphi} = 2 \frac{\partial}{\partial \varphi} R \ln \frac{dw}{dz} = 2R \frac{d}{dw} \ln \frac{dw}{dz} = 2R \frac{\frac{d^2w}{dz^2}}{\left(\frac{dw}{dz}\right)^2}$$

and

$$\frac{\partial \ln Q}{\partial \chi} = 2 \frac{\partial}{\partial \chi} R \ln \frac{dw}{dz} = -2 \frac{\partial}{\partial \varphi} J \ln \frac{dw}{dz} = -2J \frac{d}{dw} \ln \frac{dw}{dz} = -2J \frac{\frac{d^2w}{dz^2}}{\left(\frac{dw}{dz}\right)^2}$$

are valid. If one puts the analytic function of z

$$2 \frac{\frac{d^2w}{dz^2}}{\left(\frac{dw}{dz}\right)^2} = a + bi$$

one obtains

$$\frac{\partial \Delta}{\partial x} \frac{\partial \psi}{\partial y} - \frac{\partial \Delta}{\partial y} \frac{\partial \psi}{\partial x} = \left|\frac{dw}{dz}\right|^4 \left[\left(\frac{\partial \Delta'}{\partial \varphi} \frac{\partial \psi}{\partial \chi} - \frac{\partial \Delta'}{\partial \chi} \frac{\partial \psi}{\partial \varphi} \right) + \Delta' \left(a \frac{\partial \psi}{\partial \chi} + b \frac{\partial \psi}{\partial \varphi} \right) \right]$$

Finally, there results

$$\Delta\,\Delta\psi = Q\,\Delta'(Q\,\Delta'\psi) = Q^2\Delta'\,\Delta'\psi + Q\,\Delta'Q\,\Delta'\psi + 2Q\left(\frac{\partial\,\Delta'\psi}{\partial\varphi}\frac{\partial Q}{\partial\varphi} + \frac{\partial\,\Delta'\psi}{\partial\chi}\frac{\partial Q}{\partial\chi}\right)"$$

$$= Q^2\left[\Delta'\,\Delta'\psi + \Delta'\psi\,\frac{\Delta'Q}{Q} + 2\left(\frac{\partial\,\Delta'\psi}{\partial\varphi}\,a - \frac{\partial\,\Delta'\psi}{\partial\chi}\,b\right)\right]$$

$\ln Q = \ln\left|\frac{dw}{dz}\right|^2$ is a harmonic function, thus

$$\Delta'\ \ln Q = 0$$

hence,

$$\frac{\Delta'Q}{Q} = \left(\frac{\partial\ln Q}{\partial\varphi}\right)^2 + \left(\frac{\partial\ln Q}{\partial\chi}\right)^2 = a^2 + b^2$$

Thus one obtains as the result of the conversion of equation (I) to isometric coordinates φ,χ for steady motion

$$\frac{\partial\,\Delta'\psi}{\partial\varphi}\frac{\partial\varphi}{\partial\chi} - \frac{\partial\,\Delta'\psi}{\partial\chi}\frac{\partial\psi}{\partial\varphi} + \Delta'\psi\left(a\,\frac{\partial\psi}{\partial\chi} + b\,\frac{\partial\psi}{\partial\varphi}\right) = \sigma\left[\Delta'\,\Delta'\psi + \right.$$

$$\left.\Delta'\psi\left(a^2 + b^2\right) + 2\left(\frac{\partial\,\Delta'\psi}{\partial\varphi}\,a - \frac{\partial\,\Delta'\psi}{\partial\chi}\,b\right)\right] \tag{II}$$

therein, $a + bi$ is the analytic function

$$2\,\frac{\frac{d^2w}{dz^2}}{\left(\frac{dw}{dz}\right)^2} \qquad (w = \varphi + i\chi,\quad z = x + iy)$$

and Δ' denotes the operator

$$\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial\chi^2}$$

2. We return to the question on page 2: ψ must be a mere function of φ

$$\psi = f(\varphi)$$

If derivatives, with respect to φ, are denoted by primes, equation (II) becomes

$$f''f'b = \sigma\left[f^{IV} + f''\left(a^2 + b^2\right) + 2f'''a\right] \tag{III}$$

f may depend only on φ but must not depend on χ.

This is certainly possible if a and b do not depend on χ, thus, since $a + bi$ is an analytic function of $\varphi + \chi i$, do not depend on φ either, if $a + bi$ is, therefore

$$a + bi = 2\,\frac{\dfrac{d^2w}{dz^2}}{\left(\dfrac{dw}{dz}\right)^2} = C$$

that is, constant. We shall see later (paragraph 3) that this is the only possibility.

From a and b being constant, there follows

$$w = -\frac{2}{a + bi}\ln\left(z - z_0\right) + w_0$$

thus, after introduction of the polar coordinates

$$z - z_0 = re^{\vartheta i}$$

$$\varphi = -\frac{2}{a^2 + b^2}(a \ln r + b\vartheta) + \varphi_0$$

Thus, the streamlines $\varphi = \text{const}$ are identical with the logarithmic spirals

$$a \ln r + b\vartheta = \text{const}$$

$a = 0$ signifies pure radial flow, $b = 0$ flow in concentric circles. The velocity distribution, however, is given by equation (III): the radial component is

$$\frac{\partial\psi}{r\,\partial\vartheta} = f'\,\frac{\partial\varphi}{r\,\partial\vartheta} = -\frac{2b}{a^2 + b^2}\,f'\,\frac{1}{r}$$

the circular component

$$-\frac{\partial\psi}{\partial r} = -f'\,\frac{\partial\varphi}{\partial r} = \frac{2a}{a^2 + b^2}\,f'\,\frac{1}{r}$$

consequently $\dfrac{2}{\sqrt{a^2 + b^2}}\,f'\,\dfrac{1}{r}$ the magnitude of the velocity. Therefore, f' must disappear on solid walls.

Without restriction of the generality, one may presuppose left-hand spirals so that r increases with ϑ, thus a and b have different signs; since, furthermore, $-(\varphi + i\chi)$ is an analytical function just as $\varphi + i\chi$, and equation (III) is actually invariant with respect to a simultaneous sign[1] change of φ, a, b, one may presuppose

$$a \geqq 0$$

$$b \leqq 0$$

Therefore, positive velocity components $\dfrac{\partial\psi}{r\,\partial\vartheta}$ and $-\dfrac{\partial\psi}{\partial r}$ signify for

$$f' > 0 \qquad \text{outflow,}$$

in contrast for

$$f' < 0 \qquad \text{inflow.}$$

[1]Translator's note: The original says "time change," obviously a misprint.

Since φ may be replaced by $c\varphi$, one may in addition impose a condition on the constants a and b.

3. We now want to conduct the proof that on the basis of our requirements a and b must be constant, that therefore the flows in logarithmic spirals are the only ones the flow patterns of which correspond to a potential motion without themselves being a potential motion.

If a and b were not constant, the analytical function a + bi would produce a conformal transformation of the $\varphi + i\chi$-plane; by virtue of equation (III) which with the abbreviations

$$A = -\frac{f'''}{f''} \qquad B = \frac{1}{2\sigma} f' \qquad C = \frac{f^{IV}}{f''}$$

($f'' = 0$ is excluded) may also be written

$$a^2 + b^2 - 2A(\varphi)a - 2B(\varphi)b + C(\varphi) = 0 \tag{III'}$$

the circles (equation (III')) would correspond to the straight lines $\varphi = \text{const}$ in this transformation.

These circles would therefore have to form an isometric curve family.

However, if the family of curves

$$g(a,b,\varphi) = 0 \tag{III'}$$

is to be an isometric one so that $\Delta\varphi = 0$, the function g must satisfy the equation

$$\Delta g g_\varphi^{\ 2} - 2g_\varphi\left(g_{\varphi,a}g_a + g_{\varphi,b}g_b\right) + g_{\varphi,\varphi}\left(g_a^{\ 2} + g_b^{\ 2}\right) = 0 \tag{IV}$$

$$\left(\Delta = \frac{\partial^2}{\partial a^2} + \frac{\partial^2}{\partial b^2}\right)$$

and this equation must either be identically satisfied, or be a consequence of equation (III').

One has

$$g_a = 2(a - A), \quad g_b = 2(b - B), \quad \Delta g = 4, \quad g_{\varphi a} = -2A', \quad g_{\varphi b} = -2B'$$

$$g_\varphi = -2A'a - 2B'b + C', \quad g_{\varphi\varphi} = -2A''a - 2B''b + C''$$

$$g_a^{\ 2} + g_b^{\ 2} = 4(a - A)^2 + 4(b - B)^2 = 4(A^2 + B^2 - C)$$

Thus, equation (IV) is quadratic in a and b; an easy calculation shows the result that the quadratic terms are automatically eliminated. Therefore, the coefficients of the two terms must be zero whence follow three conditions

$$0 = \frac{A''}{A'} - \frac{C' - 2AA' - 2BB'}{C - A^2 - B^2} = \frac{B''}{B'} - \frac{C' - 2AA' - 2BB'}{C - A^2 - B^2}$$

$$= \frac{C''}{C'} - \frac{C' - 2AA' - 2BB'}{C - A^2 - B^2}$$

Hence, there follows further that A', B', C', must be proportional and furthermore that

$$\frac{B'}{C - A^2 - B^2}$$

must be constant.

The final condition yields

$$C = \alpha_1 B + \beta \qquad A = \gamma_1 B - \delta$$

or with

$$\frac{\alpha_1}{2\sigma} = \alpha \qquad \frac{\gamma_1}{2\sigma} = -\gamma$$

$$f^{IV} = \alpha f'f'' + \beta f'' \qquad \text{and} \qquad f''' = \gamma f'f'' + \delta f''$$

which, integrated, yields

$$f''' = \frac{1}{2}\alpha f'^2 + \beta f' + \epsilon \qquad \text{and} \qquad f'' = \frac{1}{2}\gamma f'^2 + \delta f' + \eta$$

Comparison of the two values for f''' results in

$$f'' = \frac{\frac{1}{2}\alpha f'^2 + \beta f' + \epsilon}{\gamma f' + \delta}$$

which must be identical with the preceding value of f''. The comparison requires

$$\gamma = 0 \qquad \alpha = 0 \qquad \beta = \delta^2 \qquad \epsilon = \delta\eta$$

thus $C = A^2 = \delta^2$ constant and $f'' = \delta f' + \eta$.

The second condition

$$\frac{B}{A^2 + B^2 - C} = \frac{B'}{B^2} = \text{const}$$

however would yield $\frac{f''}{f'^2} = \text{const}$ and this together with $f'' = \delta f' + \eta$ would result in the contradiction

$$f' = \text{const}$$

therewith, the proof has been produced.

4. We now turn to the determination of the velocity according to differential equation (III) which may be integrated once and assumes, after introduction of the quantity proportional to the velocity at unit distance

$$u = f'(\varphi)$$

the form

$$u'' + 2au' + u\left(a^2 + b^2\right) - \frac{b}{2\sigma} u^2 + C = 0$$

This equation is identical with a damped oscillation which takes place under the influence of the potential

$$-\frac{b}{6\sigma} u^3 + \frac{1}{2}\left(a^2 + b^2\right)u^2 + Cu$$

We start with the limiting cases:

1. <u>The streamlines are concentric circles: b = 0.</u>

Then

$$u = -\frac{C}{a^2} + e^{-a\varphi}(A + B\varphi)$$

and, because of

$$\varphi = -\frac{2}{a} \ln r$$

$$u = \text{const} + r^2\left(A + B_1 \ln r\right)$$

whereby, the velocity distribution

$$v = \frac{2}{a}\frac{u}{r}$$

is given. The exact solution of Conette's case is also contained therein: the three constants here are determined from the two limiting values of the velocity and from the fact that in case of a full turn around the circular annulus, the pressure must revert to its initial value. An easy calculation yields $B_1 = 0$ and thus

$$v = \frac{c}{r}\left(r^2 - r_1^2\right)$$

(More details on the determination of the pressure are seen in paragraph 10.)

2. The flow is purely radial: $a = 0$.

The differential equation reads

$$u'' = b^2u - \frac{b}{2\sigma}u^2 + C = 0$$

and leads to elliptic functions

$$u' = \sqrt{-\frac{b}{3\sigma}}\sqrt{-u^3 + 3\sigma bu^2 + \text{const } u + \text{const}}$$

$$= \sqrt{-\frac{b}{3\sigma}}\sqrt{\left(e_1 - u\right)\left(e_2 - u\right)\left(e_3 - u\right)}$$

where the three e's are only subject to the one condition

$$e_1 + e_2 + e_3 = 3\sigma b$$

but otherwise are still at disposal.

Since, according to the remark on page 8 one relation between a,b is still unused, it will be expedient to put

$$b = -2$$

so that one obtains, according to page 7

$$\varphi = \vartheta$$

Then the conditional equation for the e reads

$$e_1 + e_2 + e_2 = -6\sigma$$

and one has

$$u' = \sqrt{\frac{2}{3\sigma}}\sqrt{\left(e_1 - u\right)\left(e_2 - u\right)\left(e_3 - u\right)}$$

thus

$$u = -2\sigma + \wp\left[\frac{i}{\sqrt{6\sigma}}\left(\vartheta - \vartheta_0\right);\ g_2,g_3\right]$$

where ϑ_0, g_2, g_3 are the three integration constants. For the pressure (see paragraph 10) there results the equation

$$\frac{\partial}{\partial\varphi}\left(\frac{p}{u} + \frac{1}{2}v^2\right) = \frac{1}{r^2}f'f'' + \frac{2\sigma}{r^2}f'' = \frac{\partial}{\partial\varphi}\frac{1}{r^2}\left(\frac{1}{2}f'^2 + 2\sigma f'\right)$$

its uniqueness is a priori ensured, thus does not determine here any of the constants.

Discussion of the Radial Flow

5. The condition

$$e_1 + e_2 + e_3 = -6\sigma \tag{1}$$

requires at least one e to have a negatively real part, for instance

$$R\left(e_1\right) \geqq R\left(e_2\right) \geqq R\left(e_3\right)$$

then

$$R\left(e_3\right) \leqq -2\sigma$$

with the equality sign being valid only when all three e's have the same real part.

Furthermore, since this part is real, there must apply

(a) for three real e's either

$$-\infty < u \leqq e_3 \leqq -2\sigma$$

or

$$e_2 \leqq u \leqq e_1$$

(b) for one real e

$$-\infty \leqq u \leqq e$$

where, however, this e may be positive.

Furthermore, two possible types of flow must be distinguished:

1. Either there are no solid walls, thus a source or sink in an unlimited fluid. Then u must be a periodic function of φ, with a period which is an integral part of 2π. $u = -\infty$ is excluded, $u = 0$ need not occur. Therefore, this case can occur only for three real e's, and

$$e_2 \leqq u \leqq e_1$$

must be valid.

2. Or there are two solid walls, for instance for $\vartheta = 0$ and for $\vartheta = \vartheta_1$ (which may also be equal 2π); then at these walls u must be $u = 0$.

(a) In case of three real e's there must be, additionally

$$e_2 \leqq 0 \qquad e_1 \geqq 0$$

and either

(α) $$e_2 \leqq u \leqq 0$$

or

(β) $$0 \leqq u \leqq e_1$$

(b) In the case of one real e, this e must be positive and

$$0 \leqq u \leqq e$$

One remembers, furthermore, that according to page 7, paragraph 2, $u > 0$ signifies outflow, $u < 0$ signifies inflow; so that one has inflow in the case of 2(a)(α), and outflow in the case of 2(a)(β), and 2(b) above. For the case 1, both cases may occur.

First Case: Free Flow

6. One must assume $\vartheta = 0$ for $u = e_2$ and has therefore

$$\vartheta = \sqrt{\frac{3\sigma}{2}} \int_{e_2} \frac{du}{\sqrt{(e_1 - u)(u - e_2)(u - e_3)}}$$

Hence, there must

$$\int_{e_2}^{e_1} \frac{du}{\sqrt{(e_1 - u)(u - e_2)(u - e_3)}} = \sqrt{\frac{2}{3\sigma}} \frac{\pi}{n} \tag{2}$$

with n being an integral.

By the known substitution

$$u = e_2 + (e_1 - e_2) \sin^2\psi \qquad \varkappa^2 = \frac{e_1 - e_2}{e_2 - e_3}$$

equation (2) becomes

$$\frac{2}{\sqrt{e_2 - e_3}} \int_0^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{1 + \varkappa^2 \sin^2\psi}} = \sqrt{\frac{2}{3\sigma}} \frac{\pi}{n}$$

If one now introduces the mean velocity[2]

$$u_m = \frac{1}{2}(e_1 + e_2)$$

and the velocity fluctuation[2]

$$\delta = e_1 - e_2$$

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[2]At the distance $r = 1$.

there becomes because of equation (1)

$$e_2 - e_3 = 6\sigma + 3u_m - \frac{1}{2}\delta > 0 \tag{3}$$

thus

$$\varkappa^2 = \frac{\delta}{6\sigma + 3u_m - \frac{1}{2}\delta}$$

and

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{1 + \varkappa^2\sin^2\psi}} = \frac{2}{n}\,\frac{\sqrt{6\sigma + 2u_m - \frac{1}{2}\delta}}{\sqrt{6\sigma}} \tag{2'}$$

From this, one may draw several interesting conclusions.

One has

$$\int_0^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{1 + \varkappa^2\sin^2\psi}} = \int_0^{\frac{\pi}{4}} \frac{d\psi}{\sqrt{1 + \varkappa^2\sin^2\psi}} + \int_0^{\frac{\pi}{4}} \frac{d\psi}{\sqrt{1 + \varkappa^2\cos^2\psi}}$$

$$= \int_0^{\frac{\pi}{4}} d\psi\left[\frac{1}{\sqrt{1 + \frac{1}{2}\varkappa^2(1 - \cos 2\psi)}} + \frac{1}{\sqrt{1 + \frac{1}{2}\varkappa^2(1 + \cos 2\psi)}}\right] > 2\int_0^{\frac{\pi}{4}} \frac{d\psi}{\sqrt{1 + \frac{1}{2}\varkappa^2}}$$

thus

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{1 + \varkappa^2 \sin^2\psi}} = \frac{1}{\sqrt{1 + \frac{1}{2}\,\epsilon\varkappa^2}}$$

where

$$0 \leqq \epsilon \leqq 1$$

Thus the relation (2') between u_m,δ,n,σ reads, due to the significance of $\varkappa^2$,

$$\frac{1}{\sqrt{6\sigma + 3u_m - \frac{1}{2}(1 - \epsilon)\delta}} = \frac{2}{n}\,\frac{1}{6\sigma}$$

or

$$6\sigma + 3u_m - \frac{1}{2}\,\eta^2\delta = 6\sigma\,\frac{n^2}{4} \tag{2''}$$

with η being a proper fraction.

Since, furthermore

$$\int_0^{\frac{\pi}{2}} \frac{\varkappa\, d\psi}{\sqrt{1 + \varkappa^2\sin^2\psi}} > \int_0^{\frac{\pi}{2}} \frac{\varkappa\, d\psi}{\sqrt{1 + \varkappa^2\psi^2}} = \sinh^{-1}\varkappa\,\frac{\pi}{2}$$

thus becomes arbitrarily large with increasing $\varkappa$, one has $\lim\limits_{\varkappa=\infty} \epsilon = 0$, thus,

$$\lim_{\varkappa=\infty} \eta = 1$$

From equation (2'') there follows

$$u_m > -2\sigma\left(1 - \frac{n^2}{4}\right)$$

which, with $u = 1$, gives as the minimum value

$$u_m > -\frac{3}{2}\sigma$$

The mean inflow velocity is therefore considerably limited upward, the more so, the easier movable the fluid.

However, this is the only restriction: If u_m and the integer n are selected so that

$$6\sigma + 3u_m > 6\sigma\frac{n^2}{4}$$

there exists, certainly, a pertaining δ.

For if $\frac{1}{2}\delta$ increases from zero to the value $6\sigma + 3u_m$, $\frac{1}{2}\eta^2\delta$ lies between zero and $6\sigma + 3u_m$ (because for the second value $\varkappa^2$ becomes infinite and, hence, $\eta^2 = 1$) so that certainly sometime $\frac{1}{2}\eta^2\delta$ becomes equal to $6\sigma + 3u_m - 6\sigma\frac{n^2}{4}$ which is presupposed to be positive.

One sees, furthermore, that for a prescribed period number n the fluctuation δ and for a prescribed fluctuation δ the period number n must increase to infinity with the mean velocity u_m.

Second Case: Outflow Between Solid Walls

7. The cases $2(\alpha)(\beta)$, and $2(b)$ may be summarized thus

$$\vartheta = \sqrt{\frac{3\sigma}{2}}\int_0^u \frac{du}{\sqrt{(e - u)\left(u^2 + 2\alpha u + \beta\right)}}$$

$e > 0$ is the maximum velocity (at the distance $r = 1$); because of

$$2\alpha = -e_2 - e_3 = 6\sigma + e$$

and $\beta = e_2 e_3 > 0$, otherwise, however, arbitrary

$$u^2 + 2\alpha u + \beta$$

may for prescribed e assume all values from $u^2 + 2\alpha u$ to ∞, so that

$$\vartheta_1 = 2\sqrt{\frac{3\sigma}{2}}\int_0^e \frac{du}{\sqrt{(e - u)\left(u^2 + 2\alpha u + \beta\right)}}$$

appears not at all restricted downward, but upward restricted by

$$\vartheta_{1,\max} = 2\sqrt{\frac{3\sigma}{2}}\int_0^e \frac{du}{\sqrt{(e - u)u(u + e + 6\sigma)}}$$

since

$$\int_0^e \frac{dn}{\sqrt{(e - u)u}} = \pi$$

one has

$$\vartheta_{1,\max} = 2\pi\sqrt{\frac{3\sigma}{2e(1 + \epsilon) + 12\sigma}} \tag{4}$$

where ϵ signifies a positive proper fraction.

For the outflow, the width of the wall opening appears therefore restricted, according to the preceding equation, by the maximum value e of the velocity. For small velocity and large viscosity, the maximum lies near π, otherwise, however, lower; with increasing e it drops below all limits.

If, therefore, an angle opening smaller than π is prescribed, it permits an outflow only up to a certain maximum value. If a greater outflow quantity is prescribed, the jet will, therefore, actually probably separate from the walls.

Also, there is, of course, for any prescribed angle ϑ_1 a flow possible where partly inflow partly outflow occurs.

Third Case: Inflow Between Solid Walls

8. There remains the case 2(a)(α)

$$e_2 \leqq u \leqq 0$$

all three roots e real, e_3, e_2 negative, e_1 positive.

Here

$$\vartheta_1 = 2\sqrt{\frac{3\sigma}{2}}\int_{e_2}^{0}\frac{du}{\sqrt{(u - e_2)(u - e_3)(e_1 - u)}}$$

$$= \sqrt{6\sigma}\int_{e_2}^{0}\frac{du}{\sqrt{(u - e_2)(-u^2 - 2\alpha u - \beta)}}$$

where

$$2\alpha = -(e_1 + e_3) = 6\sigma + e_2$$

$$\beta = e_1 e_3 < 0$$

otherwise, however, arbitrary. Thus, the angle ϑ_1 may be made arbitrarily small for prescribed e_2. On the other hand, however, it may also be made arbitrarily large: one takes, for prescribed e_2, the negative value e_3 sufficiently close to e_2, as far as this is not made impossible by $e_1 < 0$. The sole relation between the e

$$e_1 + e_2 + e_3 = -6\sigma$$

however, results with $e_1 > 0$ in

$$-e_2 - e_3 > 6\sigma$$

If $-e_2 \geqq 3\sigma$, e_3 may actually be assumed arbitrarily close to e_2.

If the maximum inflow velocity is larger than 3σ, any angle ϑ_1 is possible between the solid walls.

If, however, $-e_2 < 3\sigma$, say $\epsilon 3\sigma$, where ϵ is a positive proper fraction, only

$$-e_3 = e_1 + (6 - 3\epsilon)\sigma = -e_2 + e_1 + 6(1 - \epsilon)\sigma$$

is possible and

$$\vartheta_1 = \sqrt{6\sigma} \int_{e_2}^{0} \frac{du}{\sqrt{(u - e_2)\left[u + (6 - 3\epsilon)\sigma + e_1\right](e_1 - u)}}$$

attains its highest value for $e_1 = 0$

$$\vartheta_{1,max} = \sqrt{6\sigma} \int_{-3\epsilon\sigma}^{0} \frac{du}{\sqrt{(u + 3\epsilon\sigma)\left[u + (6 - 3\epsilon)\sigma\right](-u)}}$$

$$= \frac{\pi\sqrt{6\sigma}}{\sqrt{\left[6 - 3\epsilon(2 + \eta)\right]\sigma}} = \frac{\pi}{\sqrt{1 - \frac{1}{2}\epsilon(1 + \eta)}} = \frac{\pi}{\delta}$$

where η and δ are positive proper fractions. Thus, the maximum of ϑ_1 is larger than π.

When the maximum inflow velocity is smaller than 3σ, the angle openings of the solid walls also may attain any magnitude up to π.

Flow in Spirals

9. Because of the damping $2au'$ (see paragraph 4, page 11), a periodic solution, aside from $u = \text{const}$, is not possible.

A free motion in logarithmic spirals is always a potential motion. In contrast, there exist other flows on logarithmic spirals between solid walls.

In order to have, for $r = \text{const}$, the variable φ agree with the angle ϑ, one may furthermore prescribe for the constants a,b, in such a manner that one obtains

$$-\frac{2b}{a^2 + b^2} = 1$$

thus

$$b = -1 \pm \sqrt{1 - a^2}$$

a must be a proper fraction, otherwise it remains arbitrary.

Equation III, once integrated, (see page 10) then reads

$$u'' + 2au' + \beta^2 u + \frac{\beta^2}{4\sigma} u^2 + C = 0$$

where $\beta^2 = -2b = 2 \mp 2\sqrt{1 - a^2} < 4$, but $> a^2$.

The velocity at unit distance is

$$\frac{2}{\sqrt{a^2 + b^2}} u = \sqrt{\frac{2}{1 \mp \sqrt{1 - a^2}}} u$$

u is, therefore, the velocity at the distance

$$r = \sqrt{\frac{2}{1 \mp \sqrt{1 - a^2}}} = \frac{2}{\beta}$$

If one first omits the damping, one has exactly the same case one had before except that

$$\sqrt{\frac{\beta^2}{6\sigma}} \text{ instead of } \sqrt{\frac{2}{3\sigma}}$$

is in front of the square root (see page 12). The relation for the e remains the former one. Since $\beta^2 < 4$, the angle opening is increased by this influence ϑ_1.

The damping, however, takes effect in the same sense. Nevertheless, the main result remains correct.

For outflow the admissible angle opening ϑ_1 is restricted by the maximum flow velocity in such a manner that it tends toward zero when this velocity increases beyond all limits.

If one puts

$$u = ve^{-a\varphi}$$

the above differential equation becomes

$$v'' + \left(\beta^2 - a^2\right)v + \frac{\beta^2}{4\sigma} v^2 e^{-a\varphi} + Ce^{a\varphi} = 0$$

If $\varphi = 0$ is assumed to be the location of the maximum v_0 for v, multiplication by $2v'$ and integration yields

$$v'^2 + \left(\beta^2 - a^2\right)\left(v^2 - v_0^2\right) + \frac{\beta^2}{2\sigma}\int_{v_0}^{v} v^2 e^{-a\varphi} dv + 2C\int_{v_0}^{v} e^{a\varphi} dv = 0$$

From the corresponding equation for u

$$u'^2 + 4a\int_{u_0}^{u} u'\ du + \beta^2\left(u^2 - u_0^2\right) + \frac{\beta^2}{6\sigma}\left(u^3 - u_0^3\right) + 2C\left(u - u_0\right) = 0$$

one can see that for equal u_0 the u-curve will be the steeper, ϑ_1 therefore the smaller, the larger C. For value close to u_0 this is immediately clear from the differential equation, for in case of $u' = 0$, u'' will be the smaller, the larger C, thus $|u'|$ the larger; from the preceding equation one may see, however, that for larger C

$$u'^2 + 4a\int_{u_0}^{u} u'\ du = u'^2 - 4a\int_{u}^{u_0} u'\ du$$

has the higher value. Hence, follows directly for the ascending branch $(u' > 0)$ that always $|u'|$ has the higher value when C is the larger value. For if u' would once reach for the initially flatter curve (C smaller) the value of the steeper curve, $\int_{u_0}^{u} u'\ du$ would have to have for the former the smaller, thus $\int_{u}^{u_0} u'\ du$ the higher value which for $u' > 0$ immediately leads to contradiction since, up to then, that is between u and u_0, u' had been the smaller value. If, however, $u' < 0$, one has in case of a variation of the C by ΔC

$$\Delta \frac{u'^2}{u_0 - u} + 4a \frac{1}{u_0 - u}\int_{u}^{u_0} \Delta|u'|\ du = 2\ \Delta C$$

or, since

$$\Delta u'^2 = \Delta|u'|(2|u'| + \Delta|u'|)$$

$$\frac{\Delta|u'|(2|u'| + \Delta|u'|)}{u_0 - u} + \frac{4a}{u_0 - u}\int_u^{u_0} \Delta|u'| \, dn + 2\,\Delta C$$

If there were at one point $\Delta|u'| = 0$, then at this point the first term would, for fixed ΔC, decrease with decreasing u, that is, go over from positive to negative values; the second term also would decrease since the part of the integral $\int_u^{u_0} \Delta|u'| \, du$ supervening with decreasing u would be negative. This is impossible, however, since the sum of both terms is supposed to be constant $2\,\Delta C$.

Therewith, it has been generally proved that the angle opening ϑ_1 decreases with increasing C (for fixed u_0); since the maximum possible ϑ_1 is desired, the minimum admissible value for C may be assumed.

This value is determined from $v'^2 > 0$

$$2C\int_v^{v_0} e^{a\varphi}dv \geqq -\left(\beta^2 - a^2\right)\left(v_0^{\,2} - v^2\right) - \frac{\beta^2}{2\sigma}\int_v^{v_0} v^2 e^{-a\varphi}dv$$

whence, one may see that the minimum admissible value of C is zero or negative[3].

The above inequality must be valid for all v's between v_0 and zero and for the positive and negative φ attained. One may write them

$$2C \geqq -\left(\beta^2 - a^2\right)\left(v_0 + v\right)e^{-a\varphi_0} - \frac{\beta^2}{6\sigma}\left(v_0^{\,2} + v_0 v + v^2\right)e^{-a\left(\varphi_2 + \varphi_0\right)}$$

---

[3]According to page 22, $\beta^2 - a^2 > 0$.

where φ_0,φ_2 are certain mean values. One must further note that, for $v = v_0$, $|\varphi_0|$ and $|\varphi_2|$ must be zero whereas they have maximum values for $v = 0$. The severest restriction is due to the absolutely smaller value of the right side, thus the minimum admissible C is given by

$$2C = -\left(\beta^2 - a^2\right)v_0 e^{-a\varphi_0'} - \frac{\beta^2}{6\sigma} v_0^2 e^{-a\left(\varphi_2'+\varphi_0'\right)}$$

φ_0',φ_2' are positive and the maximum of φ_0 and φ_2 which occur for $v = 0$.

Consequently, one has for the maximum possible ϑ_1

$$v'^2 = \left(\beta^2 - a^2\right)\left[\left(v_0^2 - v^2\right) - v_0 e^{-a\varphi_0'} \int_v^{v_0} e^{a\varphi} dv\right] +$$

$$\frac{\beta^2}{2\sigma}\left[\int_v^{v_0} v^2 e^{-a\varphi} dv - \frac{1}{3} v_0^2 e^{-a\left(\varphi_2'+\varphi_0'\right)} \int_v^{v_0} e^{a\varphi} dv\right]$$

$$= \left(\beta^2 - a^2\right)\left[\left(v_0^2 - v^2\right) - v_0\left(v_0 - v\right)e^{-a\left(\varphi_0'-\varphi_0\right)}\right] +$$

$$\frac{\beta^2}{2\sigma}\left[\frac{1}{3}\left(v_0^3 - v^3\right)e^{-a\varphi_2} - \frac{1}{3} v_0^2\left(v_0 - v\right)e^{-a\varphi_2'-a\left(\varphi_0'-\varphi_0\right)}\right]$$

Since $\varphi_0' > \varphi_0$ and $\varphi_2' > \varphi_2$

$$v'^2 > \left(v_0 - v\right)v\left[\left(\beta^2 - a^2\right) + \frac{\beta^2}{6\sigma}\left(v + v_0\right)e^{-a\varphi_2}\right]$$

and since $\varphi_2' < \vartheta_1$

$$v'^2 > \left(v_0 - v\right)v\left[\left(\beta^2 - a^2\right) + \frac{\beta^2}{6\sigma}\left(v + v_0\right)e^{-a\vartheta_1}\right]$$

whence follows

$$\vartheta_1 < \frac{2\pi}{\sqrt{\left(\beta^2 - a^2\right) + \frac{\beta^2}{6\sigma} v_0\left(1 + \epsilon\right)e^{-a\vartheta_1}}}$$

(Compare formula (4), page 19).

Hence follows that with increasing v_0, thus also with increasing u_0, ϑ_1 must drop below all limits: a certain width of the spiral permits only a limited outflow velocity.

SECOND PART

10. Since the only spiral motion, possible without walls, of the type used so far, lead to be a potential motion, exact steady and non-steady two-dimensional motions in free spirals will be investigated according to another method.

In polar coordinates the differential equation (I) reads

$$\frac{\partial\, \Delta\psi}{\partial t} + \frac{1}{r}\left(\frac{\partial\, \Delta\psi}{\partial r}\,\frac{\partial\psi}{\partial\varphi} - \frac{\partial\, \Delta\psi}{\partial\varphi}\,\frac{\partial\varphi}{\partial r}\right) = \sigma\, \Delta\, \Delta\psi$$

where

$$\Delta = \frac{1}{r}\,\frac{\partial}{\partial r}\, r\, \frac{\partial}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2}{\partial\varphi^2}$$

Obviously this equation permits solutions which are linear in φ

$$\psi = u + \varphi \varkappa$$

in order to make the velocity which has the components

$$v_r = \frac{\varkappa}{r} \qquad \text{and} \qquad v_\varphi = -\frac{\partial u}{\partial r} - \varphi \frac{\partial \varkappa}{\partial r}$$

unique and thus enable a free motion, $\varkappa$ must be constant. By this statement, the differential equation becomes

$$\frac{\partial \Delta u}{\partial t} + \frac{\varkappa}{r} \frac{\partial \Delta u}{\partial r} = \sigma \Delta \Delta u$$

with

$$\Delta u = \frac{1}{r} \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\right)$$

Here it is necessary also to investigate the pressure lest perhaps in a motion about the singular point $r = 0$ a multivaluedness of the pressure becomes evident.

Now one may write the equations of motion without elimination of the pressure in the following form

$$d\left(\frac{p}{\mu} + \frac{1}{2} v^2\right) = \Delta\psi \, d\psi + \left[\frac{\partial\left(\sigma \Delta - \frac{\partial}{\partial t}\right)\psi}{\partial y} dx - \frac{\partial\left(\sigma \Delta - \frac{\partial}{\partial t}\right)\psi}{\partial x} dy\right]$$

or because of the invariance of the last term

$$= \Delta\psi \, d\psi + \frac{\partial\left(\sigma \Delta - \frac{\partial}{\partial t}\right)\psi}{r \, \partial\varphi} dr - \frac{\partial\left(\sigma \Delta - \frac{\partial}{\partial t}\right)\psi}{\partial r} r \, d\varphi$$

Hence, follows

$$\frac{\partial}{\partial\varphi}\left(\frac{p}{\mu} + \frac{1}{2} v^2\right) = \Delta\psi \frac{\partial\psi}{\partial\varphi} - r \frac{\partial}{\partial r}\left(\sigma \Delta - \frac{\partial}{\partial t}\right)\psi = \varkappa \Delta u - \sigma r \frac{\partial \Delta u}{\partial r} + r \frac{\partial^2 u}{\partial t \, \partial r}$$

By virtue of the differential equation for u the right side is constant; thus it must be zero to make the pressure in case of a revolution about $r = 0$ revert to its former value, so that one obtains

$$r \frac{\partial \Delta u}{\partial r} - \frac{r}{\sigma} \frac{\partial^2 u}{\partial r \, \partial t} - \frac{\varkappa}{\sigma} \Delta u = 0$$

which by introduction of $r \frac{\partial u}{\partial r} = v$ assumes the form

$$\frac{\partial^2 v}{\partial r^2} - \left(1 + \frac{\varkappa}{\sigma}\right)\frac{1}{r} \frac{\partial v}{\partial r} = \frac{\partial v}{\partial t} \qquad \text{(V)}$$

Steady Motions

11. The solution independent of t is

$$u = c_1 r^{\frac{\varkappa}{\sigma}+2} + c_2 \ln r + c_3$$

when $\frac{\varkappa}{\sigma} \gtrless -2$, otherwise, when $\frac{\varkappa}{\sigma} = -2$

$$u = c_1 (\ln r)^2 + c_2 \ln r + c_3$$

If one disregards the trivial case of potential motion, a spiral motion the velocity of which disappears at infinity exists when

$$\frac{\varkappa}{\sigma} + 1 < 0$$

that is, $\varkappa < -\sigma$, thus a sufficiently strong inflow takes place.

The spirals then have the form

$$\varphi = -\frac{1}{\varkappa} u = C_1 r^{\frac{\varkappa}{\sigma}+2} + C_2 \ln r + C_3$$

If $\frac{\varkappa}{\sigma} + 2 < 0$, they approach at infinity the logarithmic spirals; near the sink, in contrast, they converge considerably less pronouncedly toward the sink point, and the vortex velocity is considerably higher than in case of potential flow in logarithmic spirals.

Unsteady Motions

12. If one uses the formulation

$$v = e^{nt}\chi_n(r)$$

one obtains from equation (V)

$$\chi_n'' - \frac{1}{r}\left(1 + \frac{\varkappa}{\sigma}\right)\chi_n' - \frac{n}{\sigma}\chi_n = 0$$

thus, with the abbreviation $\lambda = 1 + \frac{\varkappa}{2\sigma}$

$$\chi_n = r^{\lambda} J_{\pm\lambda}\left(\sqrt{-\frac{n}{\sigma}}\, r\right)$$

where the J are the Bessel functions

$$J_\lambda\left(\sqrt{-\frac{n}{\sigma}}\,r\right) = \text{const } r^\lambda\left[1 + \frac{n}{\sigma}\frac{r^2}{4}\frac{1}{1(1 + \lambda)} + \frac{n^2}{\sigma^2}\left(\frac{r}{2}\right)^4 \frac{1}{2!(1 + \lambda)(2 + \lambda)} + \ldots\right]$$

If λ does not happen to be an integer, $r^\lambda J_\lambda$ and $r^\lambda J_{-\lambda}$ may be regarded as independent solutions.

13. Similarly to the case of the heat conduction equation there exist also of equation (V) integrals which show for $r = 0$ and $t = 0$ an indeterminate point.

Since the differential equation (V) remains unchanged if v is multiplied by an arbitrary factor, r by a similar factor, t by its square, there must exist solutions of the form

$$v = r^\alpha t^\beta w\left(\frac{r^2}{4\sigma t}\right) = r^\alpha t^\beta w(z)$$

After substitution, one obtains for w the differential equation

$$w'' + w'\left(\frac{\alpha + 1 - \lambda}{z}\right) + w\left(\frac{\alpha^2 - 2\lambda\alpha}{4z^2} + \frac{\beta}{z}\right) = 0 \tag{VI}$$

When does this equation permit a solution of the form

$$w = e^{\mu z}$$

A simple calculation yields

$$\mu = -1$$

and then either

$$\alpha = 0 \qquad \beta = \lambda - 1$$

or

$$\alpha = 2\lambda \qquad \beta = -\lambda - 1$$

One thus has two simple integrals of the required type

$$v = t^{\lambda-1} e^{-\frac{1}{4\sigma}\frac{r^2}{t}}$$

and

$$v = r^{2\lambda} t^{-1-\lambda} e^{-\frac{1}{4\sigma}\frac{r^2}{t}}$$

for $\lambda = 0$, both are transformed into the known integral of the heat conduction equation.

Let us continue the discussion of the differential equation (VI).

The singular point $z = 0$ is a determinate point. The determining equation reads

$$\rho^2 + \rho(\alpha - \lambda) + \frac{\alpha^2 - 2\lambda\alpha}{4} = 0$$

and has the roots

$$\rho_1 = \lambda - \frac{\alpha}{2} \qquad \rho_2 = -\frac{\alpha}{2}$$

so that generally there exist developments of the form

$$w_1 = z^{\lambda - \frac{\alpha}{2}}\left(1 + c_1 z + c_2 z^2 + \ldots\right)$$

and

$$w_2 = z^{-\frac{\alpha}{2}}\left(1 + c_1' z + c_2' z^2 + \ldots\right)$$

thus

$$v_1 = r^{2\lambda} t^{\beta - \lambda + \frac{\alpha}{2}}\left[1 + c_1 \frac{r^2}{4\sigma t} + c_2 \frac{r^4}{(4\sigma t)^2} + \ldots\right]$$

and

$$v_2 = t^{\beta + \frac{\alpha}{2}}\left[1 + c_1' \frac{r^2}{4\sigma t} + c_2' \frac{r^4}{(4\sigma t)^2} + \ldots\right]$$

with the power series continuously converging since $z = 0$ is the only singular point of the differential equation.

If one assumes $\beta = 0$, that is, if one desires solution of equation (V) of the form

$$r^{\alpha} w\left(\frac{r^2}{4\sigma t}\right)$$

an integration by definite integrals is possible.

The differential equation (VI) reads after introduction of the roots ρ_1, ρ_2

$$z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz}\left(1 - \rho_1 - \rho_2 + z\right) + w\rho_1\rho_2 = 0 \qquad \text{(VI')}$$

The connection with Gauss' equation for the hypergeometric function can be easily recognized. If one makes the Euler transformation

$$w = \int e^{-3}\left(1 - \frac{s}{z}\right)^{n} y(s)\ ds$$

with the integral extended over a suitable closed path, one finds for y a differential equation which may be satisfied for

$$n = -\rho_1 \qquad \text{by} \qquad y = s^{-1+\rho_2}$$

and for

$$n = -\rho_2 \qquad \text{by} \qquad y = s^{-1+\rho_1}$$

Therefore

$$w = \int e^{-3}\left(1 - \frac{s}{z}\right)^{-\rho_1} s^{-1+\rho_2} ds$$

and

$$w = \int e^{-3}\left(1 - \frac{s}{z}\right)^{-\rho_2} s^{-1+\rho_1} ds$$

are integrals of equation (VI'). The integrals are extended best over a path which leads from $R(s) = +\infty$ around the points $s = 0$ and $s = z$ back to $R(s) = +\infty$.

Since

$$\int e^{-3}(z - s)^{-\rho_1} s^{-1+\rho_2} ds$$

is analytically regular in the neighborhood of $z = 0$, there is

$$w_1 = C_1 \int e^{-3}\left(1 - \frac{s}{z}\right)^{-\rho_1} s^{-1+\rho_2} ds$$

and

$$w_2 = C_2 \int e^{-3}\left(1 - \frac{s}{z}\right)^{-\rho_2} s^{-1+\rho_1} ds$$

One can show that

$$v = \sum_{\alpha=-\infty}^{\infty} r^{\alpha}\left[C_1 w_1\left(\frac{r^2}{4\sigma t}\right) + C_2 w_2\left(\frac{r^2}{4\sigma t}\right)\right]$$

are the general solutions of equation (V) and likewise are represented by definite integrals in closed form. I shall perhaps refer back to this and to the connection with the representation and the development in terms of Bessel functions elsewhere.

THIRD PART

Neighborhood Solutions to Radial Flow

14. We shall first look for steady neighborhood solutions to the radial flow (pages 12 and 13) by putting

$$\psi = f(\varphi) + \rho(\varphi,r)$$

where ρ is assumed to be a small quantity, the square of which is neglected.

We then obtain for f the former equation

$$f^{(IV)} + 4f'' + \frac{2}{\sigma} f'f'' = 0$$

with $f' = u$ and integrating once

$$u'' + 4u + \frac{1}{\sigma} u^2 + C = 0.$$

For ρ we obtain

$$\frac{\partial \Delta\rho}{\partial t} + \frac{u}{r}\frac{\partial \Delta\rho}{\partial r} - \frac{2u'}{r^4}\frac{\partial\rho}{\partial\varphi} - \frac{u''}{r^3}\frac{\partial\rho}{\partial r} = \sigma \Delta \Delta\rho$$

where

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}$$

Since the differential equation in the steady case remains unchanged when r and ρ each are multiplied by an arbitrary factor, there must exist solutions of the form

$$\rho = r^{\lambda} w(\varphi)$$

One obtains for w the differential equation

$$w^{IV} + w''\left(2\lambda^2 - 4\lambda + 4 + \frac{2}{\sigma} u - \frac{\lambda}{\sigma} u\right) + \frac{2}{\sigma} u'w' +$$

$$w\left(\lambda^4 - 4\lambda^3 + 2\lambda^2 + \frac{2\lambda^2 - \lambda^3}{\sigma} u + \frac{\lambda}{\sigma} u''\right) = 0 \qquad \text{(VII)}$$

We are particularly interested in free flows and thus in periodic solutions in φ.

As concerns the uniqueness of the pressure (see paragraph 10, page 28), one obtains by a simple calculation, the condition

$$\lambda^2 \int_0^{2\pi} w\left[u - (\lambda - 2)\sigma\right] d\varphi = 0$$

On the other hand, there follows from the above differential equation itself, by integration over the interval from 0 to 2π with assumption of periodicity

$$\lambda^2(\lambda - 2)\int_0^{2\pi} w\left[u - (\lambda - 2)\sigma\right]d\varphi = 0$$

so that in general the uniqueness of the pressure follows from the periodicity except for the case when $\lambda = 2$.

For free flow, u itself is a periodic function of φ; the period is an integral w part of 2π. However, it is not necessary that w have the same period as u; but this period must likewise be an integral part of 2π.

Since

$$u'' = -C - 4u - \frac{1}{\sigma}u^2$$

as well as

$$u'^2 = -2Cu - 4u^2 - \frac{2}{3\sigma}u^3 + D = \frac{2}{3\sigma}\left(e_1 - u\right)\left(u - e_2\right)\left(u - e_3\right)$$

may be rationally expressed by u, it will be useful to introduce u instead of φ as independent variable in equation (VII). Because of

$$w' = \frac{dw}{du}u'$$

$$w'' = \frac{d^2w}{du^2}u'^2 + \frac{dw}{du}u''$$

$$w''' = \frac{d^3w}{du^3}u'^3 + 3\frac{d^2w}{du^2}u'u'' + \frac{dw}{du}u'''$$

$$w^{IV} = \frac{d^4w}{du^4}u'^4 + 6\frac{d^3w}{du^3}u'^2u'' + 3\frac{d^2w}{du^2}u''^2 + 4\frac{d^2w}{du^2}u'u''' + \frac{dw}{du}u^{IV}$$

and because of

$$u^{IV} = \left(-4 - \frac{2}{\sigma} u\right)u'' - \frac{2}{\sigma} u'^2 \qquad u'u''' = \left(-4 - \frac{2}{\sigma} u\right)u'^2$$

all coefficients of the new equation are integral and rational in u; indicating the degree, one writes them

$$R_6 \frac{d^4w}{du^4} + R_5 \frac{d^3w}{du^3} + R_4 \frac{d^2w}{du^2} + R_3 \frac{dw}{du} + R_2 w = 0 \qquad \text{(VII')}$$

with

$$R_6 = u'^4 = \frac{4}{9\sigma^2}\left(e_1 - u\right)^2\left(u - e_2\right)^2\left(u - e_3\right)^2$$

From the form (VII) one can see that w possesses singularities only where they occur for u, thus certainly not in the real part of φ (which is of interest); equation (VII') shows that, as a function of u, w becomes singular only at the branch points e_1, e_2, e_3.

Since $R_5 = 6u'^2u''$ is divisible by $(e_1 - u)(u - e_2)(u - e_3)$, the points e_1, e_2, e_3 are determinate points, and since the degree of the coefficients decreases steadily by 1 with the order of the derivatives, $u = \infty$ also is a determinate point; the differential equation (VII') belongs to the Fuchs class.

A well-known calculation yields as the four roots of the determining equation for the points e the values

$$\rho_1 = 0 \qquad \rho_2 = 1 \qquad \rho_3 = \frac{1}{2} \qquad \rho_4 = \frac{3}{2}$$

Although, therefore, two root differences here are integral, no logarithmics appear in the developments: For from the form (VII) there follows that at the points φ for which u becomes $= e$, where, therefore, u and u' are regular functions of φ, w also must be such a function, whereas $\ln(u - e)$ does not possess this regular character.

Therefore, the solutions of equation (VII') have at every point e the form

$$u = \underline{P}_1(u - e) + \sqrt{u - e}\,\underline{P}_2(u - e)$$

other singularities do not exist in a finite domain.

For $u = \infty$ there results the determining equation

$$\left(2\mu^2 + \mu - 3\right)\left(\mu^2 + \frac{5}{2}\mu + \frac{3}{2}\lambda\right) = 0$$

which has the roots

$$\mu_1 = 1 \qquad \mu_2 = -\frac{3}{2}$$

which are independent of λ, and the roots

$$\left.\begin{matrix}\mu_3\\ \mu_4\end{matrix}\right\} = -\frac{5}{4} \pm \frac{1}{4}\sqrt{25 - 24\lambda}$$

which are dependent on λ.

Continuation

15. Solutions with the real period 2π (this period must be present at least in case of free flow) will exist only for certain λ. In analogy with Hermite's method for Laine's differential equation, one can proceed as follows:

If w_1, w_2, w_3, w_4 are a fundamental system of equation (VII), the $w(\varphi + 2\pi)$ are expressed homogeneously linearly by the w

$$w_\nu(\varphi + 2\pi) = \sum_{\mu=1}^{4} a_{\nu,\mu} w_\mu(\varphi) \qquad (\nu = 1,2,3,4)$$

There certainly exist periodic functions of the second kind, that is, there exist solutions w for which

$$w(\varphi + 2\pi) = \alpha w(\varphi)$$

This α is a root of the equation of the fourth degree

$$D(\alpha;\ \lambda) \equiv \begin{vmatrix} a_{11} - \alpha & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} - \alpha & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} - \alpha & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} - \alpha \end{vmatrix} = 0$$

If a periodic solution is to exist, $\alpha = 1$ must be a root, and one obtains for λ the equation

$$D(1,\lambda) = 0$$

The characteristic exponents which were calculated suggest the attempts of putting

$$w = u + \text{const}, \quad w = \sqrt{u - e} \quad \text{and} \quad w = \sqrt{(e_\alpha - u)(u - e_\beta)}$$

Elementary calculation yields the following particular solutions:

1. The trivial possibility $w = u$ for $\lambda = 0$

2. $w = u$ for $\lambda = 2$, that is

$$\rho = \alpha r^2 u$$

$$\psi = f(\varphi) + \alpha r^2 f'(\varphi)$$

where α must be small and therefore with the same approximation

$$\psi = f\left(\varphi + \alpha r^2\right)$$

so that the streamlines are approximately the spirals

$$\varphi = \varphi_0 - \alpha r^2$$

f' remains the same elliptic function discussed before in the case of radial flow. It is true that this flow now cannot exist as free flow, since this is precisely the exceptional case $\lambda = 2$ (see page 37); and the condition for the uniqueness of the pressure can certainly not be satisfied for $w = u$.

3. $w = u + 3\sigma$, when $\lambda = 1$ and $C = 3\sigma$ whence for $e_1 - e_2 < \sigma\sqrt{3}$ no contradiction results.

4. $w = \sqrt{u - e_2}$ for $\lambda = -1$ and $e_2 = 0$. This solution has a period twice that of u; likewise, $w = \sqrt{e_1 - u}$ for $\lambda = -1$ and $e_1 = 0$.

5. $w = \sqrt{(e_1 - u)(u - e_3)}$ or $w = \sqrt{(u - e_2)(u - e_3)}$ when $\lambda = 1$ and $e_2 = 0$ or $e_1 = 0$. This solution too has a period twice that of u.

The large λ may be easily calculated approximately from equation (VII). For such large λ there is in first approximation

$$w^{IV} + 2\lambda^2 w'' + \lambda^4 w = 0$$

that is, $w = e^{\pm\lambda i\varphi}$ (we restrict ourselves to the periodic solutions), so that $\frac{2\pi}{\lambda}$ is the period. The large set-apart λ-values are therefore approximately integral.

Finally, one case may be calculated quite elementarily: the case when u is constant, the basic flow therefore an all around uniformly distributed flow.

This case is also of significance for the more general one since according to a well-known theorem by Cauchy and Boltzmann[4] the period of w in first approximation is obtained if the constant mean value is inserted for the periodic u, under the presupposition that the larger fluctuation $e_1 - e_2$ be sufficiently small.

For constant u there follows from equation (VII), page 36

$$w^{IV} + w''\left(2\lambda^2 - 4\lambda + 4 + \frac{2 - \lambda}{\sigma} u\right) + w\left(\lambda^4 - 4\lambda^3 + 4\lambda^2 + \lambda^2 \frac{2 - \lambda}{\sigma} u\right) = 0$$

thus with the formulation

$$w = e^{ui\varphi}$$

$$\mu^4 - \mu^2\left(2\lambda^2 - 2\lambda + 4 + \frac{2 - \lambda}{\sigma} u\right) + \lambda^4 - 4\lambda^3 + 4\lambda^2 + \lambda^2 \frac{2 - \lambda}{\sigma} u = 0$$

This equation has four roots

$$\mu = \pm\lambda \qquad \mu^2 = (\lambda - 2)^2 - \frac{2 - \lambda}{\sigma} u$$

so that all integral positive and negative λ are possible (potential motions) as well as all λ which are calculated from

$$\lambda = 2 + \frac{u}{4\sigma} \pm \sqrt{\left(\frac{u}{4\sigma}\right)^2 + \mu^2}$$

with integral μ.

For the case $u = \text{const}$, that is:

16. For the radial flow which is uniform all around, the unsteady neighborhood solutions also can be given.

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[4]Boltzmann, Ges. Abh., Bd. 1, S. 43.

The differential equation now reads (see page 36)

$$\frac{\partial\,\Delta\rho}{\partial t} + \frac{u}{r}\frac{\partial\,\Delta\rho}{\partial r} = \sigma\,\Delta\,\Delta\rho$$

One may integrate it either by means of the formulation

$$\Delta\rho = e^{\lambda t+ni\varphi}w(r)$$

(n integral) and thus arrives at the differential equation

$$\frac{d^2w}{dr^2} + \frac{1 - \frac{u}{\sigma}}{r}\frac{dw}{dr} - \left(\frac{n^2}{r^2} + \frac{\lambda}{\sigma}\right)w = 0$$

which may be solved by Bessel functions, or by means of the formulation (compare page 32)

$$\Delta\rho = e^{ni\varphi}r^m w\left(\frac{r^2}{4\sigma t}\right) = e^{ni\varphi}r^m w(z)$$

whereby one obtains for $w(z)$ the differential equation

$$z^2w'' + zw'\left(m + 1 - \frac{u}{2\sigma} + z\right) + w\left(\frac{m^2 - n^2}{4} - \frac{mu}{4\sigma}\right) = 0$$

For $z = 0$ this equation has a determinate point, the determining equation has the real roots

$$\rho = \frac{m}{2} - \frac{1}{4}\frac{u}{\sigma} \pm \frac{1}{2}\sqrt{n^2 + \frac{1}{4}\frac{u^2}{\sigma^2}}$$

By introduction of the roots ρ_1 and ρ_2 the differential equation assumes the form

$$z^2w'' + zw'\left(1 - \rho_1 - \rho_2 + z\right) + \rho_1\rho_2 w = 0$$

This is, however, exactly the differential equation (VI') of page 33 so that everything said about it there is also valid here.

Translated by Mary L. Mahler
National Advisory Committee
for Aeronautics