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By H. Schlichting

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AIRFOIL THEORY AT SUPERSONIC SPEED*

By H. Schlichting

A theory is developed for the airfoil of finite span at supersonic speed analogous to the Prandtl airfoil theory of 1918-19 for incompressible flow. In addition to the profile and induced drags, account must be taken at supersonic flow of still another drag, namely, the wave drag, which is independent of the wing aspect ratio. Both wave and induced drags are proportional to the square of the lift and depend on the Mach number, that is, the ratio of the flight to sound speed. In general, in the case of supersonic flow, the drag-lift ratio is considerably less favorable than is the case for incompressible flow. Among others, the following examples are considered:

1. Lifting line with constant lift distribution (horseshoe vortex).
2. Computation of wave and induced drag and the twist of a trapezoidal wing of constant lift density.
3. Computation of the lift distribution and drag of an untwisted rectangular wing.

I. INTRODUCTION

The basic principles for the following computation of airfoil flow at supersonic speed are presented in the paper of Professor Prandtl (reference 1), and a detailed explanation of the method may therefore be dispensed with here.

The potential Φ_T of a lifting line at supersonic speed may be derived in a simple manner from the potential Φ_Q of a stationary source in the presence of a supersonic flow.

*"Tragflügeltheorie bei Überschallgeschwindigkeit." Jahrbuch 1937 der deutschen Luftfahrtforschung, pp. I 181-97.

If Φ_Q denotes the source potential of strength 4π , the potential Φ_T of the lifting line element dy with circulation Γ about the y axis is given by

$$\Phi_T = 2 \frac{\Gamma}{4\pi} \int_{x' = -\infty}^{x' = x} dx' \frac{\partial \Phi_Q}{\partial z} \quad (1)$$

The potential of a source at the point $x = y = z = 0$ in the presence of a flow with velocity $u_0 > c$ in the direction of the positive x axis is

$$\Phi_Q = \frac{1}{\sqrt{x^2 - \left[\left(\frac{u_0}{c} \right)^2 - 1 \right] (y^2 + z^2)}} \quad (2)$$

The potential (2) is real within the double cone with half cone angle α , the axis of which cone is parallel to the direction of flow ($\sin \alpha = c/u_0$). Outside of this cone the potential, according to the formula, is imaginary. Actually, Φ_Q is there to be taken identically equal to zero. The potential has physical reality only in the "after cone" of the point $x = y = z = 0$. In the "forward cone" it is similarly to be taken identically equal to zero.

The potential Φ_Q is the starting point for constructing the airfoil potential. We shall first derive from it the potential of a line source of finite length, then with the aid of the operation indicated in equation (1) we shall obtain the potential of a lifting line of finite length for various lift distributions. From the lifting line, there is finally obtained by the familiar method, the lifting surface. In this manner, a theory of the airfoil of finite span for supersonic speed is obtained that forms the counterpart of the Prandtl airfoil theory for the incompressible flow case (reference 2).

II. CONSTANT LIFT DISTRIBUTION (LIFTING LINE)

We shall now assume a line source of length b (later = span of wing) which lies in the direction of the y axis and extends from $y' = -b/2$ to $y' = +b/2$ (fig. 1). Let $g(y')$ be the initially given local source intensity (later = the lift distribution). Further, let x, y, z be the coordinates of a point in the flow and $0, y', 0$, the coordinates of a source point. Then from equation (2) the potential of the line source is

$$\Phi_Q = \int_{y' = -\frac{b}{2}}^{y' = +\frac{b}{2}} \frac{g(y') \, dy'}{\sqrt{x^2 - \left[\left(\frac{u_0}{c} \right)^2 - 1 \right] \{ (y - y')^2 + z^2 \}}} \quad (3)$$

We introduce nondimensional coordinates by dividing all lengths by the half-length $b/2$ of the line source and accordingly set

$$\frac{2x}{b} = \xi; \quad \frac{2y}{b} = \eta; \quad \frac{2z}{b} = \zeta; \quad \frac{2y'}{b} = \eta'$$

Further, we introduce the abbreviated notation

$$\left. \begin{aligned} \frac{u_0^2}{c^2} - 1 &= \kappa^2 \\ \text{or} \\ \tan \alpha &= \frac{1}{\sqrt{\frac{u_0^2}{c^2} - 1}} = \frac{1}{\kappa} \end{aligned} \right\} \quad (4)$$

where α denotes the Mach angle.

The potential of the line source then becomes

$$\Phi_Q(\xi, \eta, \zeta) = \int_{\eta' = -1}^{\eta' = +1} \frac{g(\eta') \, d\eta'}{\sqrt{\xi^2 - \kappa^2 [(\eta - \eta')^2 + \zeta^2]}} \quad (5)$$

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We shall now carry out the integration in equation (5) for the simplest case of the line source with constant source density, that is, $g(\eta') = \text{const.} = 1$. This gives

$$\Phi_Q = \int_{\eta' = -1}^{\eta' = +1} \frac{d\eta'}{\sqrt{\xi^2 - \kappa^2 [(\eta - \eta')^2 + \zeta^2]}} \quad (6)$$

Writing $\eta - \eta' = \vartheta$; $\eta' = -1$: $\vartheta = \vartheta_1 = \eta + 1$

$\eta' = +1$: $\vartheta = \vartheta_2 = \eta - 1$

and

$$(\xi/\kappa)^2 - \zeta^2 = a^2$$

equation (6) becomes

$$\Phi_Q = -\frac{1}{\kappa} \int_{\vartheta = \vartheta_1}^{\vartheta = \vartheta_2} \frac{d\vartheta}{\sqrt{a^2 - \vartheta^2}} = -\frac{1}{\kappa} \left(\text{arc sin} \frac{\vartheta_2}{a} - \text{arc sin} \frac{\vartheta_1}{a} \right) \quad (7)$$

By the operation in equation (1) there is then obtained the potential Φ_T of the lifting line with the constant lift distribution Γ_0 , setting

$$\Phi_T = \frac{\Gamma_0}{2\pi} \int_{\xi' = -\infty}^{\xi' = \xi} \frac{\partial \Phi_Q}{\partial \xi} d\xi' \quad (8)$$

The first step of the above operation, differentiation with respect to ξ , may be carried out immediately but the integration requires a somewhat longer computation. There is obtained

$$-\kappa \frac{\partial \Phi_Q}{\partial \xi} = \frac{\zeta \vartheta_2}{a^2 \sqrt{a^2 - \vartheta_2^2}} - \frac{\zeta \vartheta_1}{a^2 \sqrt{a^2 - \vartheta_1^2}} \quad (9)$$

In integrating with respect to ξ , ϑ_2 and ϑ_1 are constant. For the first term, there is obtained

$$\int \frac{\xi \vartheta_2 d\xi'}{a^2 \sqrt{a^2 - \vartheta_2^2}} = \frac{\kappa^3}{2} \int \frac{\xi \vartheta_2 d\xi^*}{\xi^* \sqrt{\xi^{*2} + \kappa^2 \xi^* (\xi^2 - \vartheta_2^2) - \kappa^4 \xi^2 \vartheta_2^2}} \quad (10)$$

where there has been set

$$\xi^* = \xi^2 - (\kappa \xi)^2$$

The evaluation of the integral gives

$$\begin{aligned} \int \frac{\xi \vartheta_2 d\xi'}{a^2 \sqrt{a^2 - \vartheta_2^2}} &= \\ &= \frac{\kappa}{2} \arctan \frac{-2\kappa^2 \vartheta_2^2 \xi^2 + (\xi^2 - \vartheta_2^2) \xi^*}{2\xi \vartheta_2 \sqrt{\xi^{*2} + \kappa^2 \xi^* (\xi^2 - \vartheta_2^2) - \kappa^4 \xi^2 \vartheta_2^2}} \\ &= \frac{\kappa}{2} \arctan \frac{\xi^2 \omega - \xi^2 (\eta - 1)^2}{2\xi (\eta - 1) \xi \sqrt{\omega}} \end{aligned} \quad (11)$$

where

$$\omega = \xi^2 - \kappa^2 [(\eta - 1)^2 + \xi^2] \quad (12)$$

Since the integral (10) outside of the Mach cone, at the end point $\eta = 1$ of the lifting line with axis parallel to the x axis,

$$\xi^2 - \kappa^2 [(\eta - 1)^2 + \xi^2] = 0$$

is imaginary, i.e., is to be taken equal to zero, the integration with respect to ξ' need not be extended from $\xi' = -\infty$ but only from the cone surface along lines parallel to the x axis. For the lower limit of integration, we have thus the constant $\arctan \infty = \pi/2$, which we may suppress. There is thus found from (8), (9), and (11) the required potential of the lifting line with constant circulation

$$\begin{aligned} \Phi_{\Gamma} &= -\frac{\Gamma_0}{2\pi} \kappa \int_{\xi'=-\infty}^{\xi'=\xi} \frac{\xi \partial_a d\xi'}{a^2 - \sqrt{a^2 - \partial_a^2}} = -\frac{\Gamma_0}{4\pi} \text{arc tan} \frac{\xi^2 \omega - \xi^2 (\eta - 1)^2}{2 \xi (\eta - 1) \xi \sqrt{\omega}} \\ &= \frac{\Gamma_0}{2\pi} \text{arc tan} \frac{\xi (\eta - 1)}{\xi \sqrt{\omega}} + \text{similar term for the cone at} \\ &\qquad \qquad \qquad \eta = -1 \qquad \qquad \qquad (13) \end{aligned}$$

This potential is different from zero only within the two Mach cones arising at the ends of the lifting line ($\omega > 0$) while in the entire remaining space it is equal to 0. For a complete circuit about each of the cone axes $\eta = \pm 1$, $\xi = 0$, the arc tan increases by 2π . The enclosed vortex filament therefore has the circulation Γ_0 . The lifting line assumed to extend from $y = -b/2$ to $y = +b/2$ with the constant circulation Γ_0 along the span continues behind as a free vortex line in the two axes of the Mach cones. Equation (13) thus gives the potential of a "horseshoe vortex" at supersonic flow. As in the case of the incompressible flow, this simple horseshoe vortex becomes the starting point for more complicated lifting systems.

In order to obtain an idea as to the appearance of the supersonic flow in the neighborhood of a horseshoe vortex, we differentiate the potential (13) to find the induced velocities

$$c_x = -\frac{\partial \Phi_{\Gamma}}{\partial x}; \quad c_y = -\frac{\partial \Phi_{\Gamma}}{\partial y}; \quad c_z = -\frac{\partial \Phi_{\Gamma}}{\partial z}$$

and obtain

$$\left. \begin{aligned} c_x &= \kappa \frac{\Gamma_0}{\pi b} \frac{(\eta - 1) \xi}{(\xi^2 - \kappa \xi^2) \sqrt{\omega}} \\ c_y &= -\frac{\Gamma_0}{\pi b} \frac{\xi \xi}{[\xi^2 + (\eta - 1)^2] \sqrt{\omega}} \\ c_z &= \frac{\Gamma_0}{\pi b} \frac{\xi (\eta - 1) (\omega - \kappa^2 \xi^2)}{[\xi^2 \omega + (\eta - 1)^2 \xi^2] \sqrt{\omega}} \end{aligned} \right\} (14a, b, c)^1$$

¹See footnote on next page.

The field of these velocities exhibits a number of singularities. On the cone surface all three velocity components become infinite. On the cone axis $c_x = 0$, but c_y and c_z become infinite as $1/r$ (where r is the distance from the axis). In the neighborhood of the cone axis, c_y and c_z thus behave exactly as in the neighborhood of a vortex filament in the incompressible flow. The field of the induced velocities gives a motion which encircles the vortex filament traveling downstream from the end of the lifting line $\eta = 1$, $\xi = \zeta = 0$, as may be seen immediately from (14).

In the plane $\eta - 1 = 0$ through the end of the lifting line $c_z = 0$ and

$$\xi > 0 : c_y < 0$$

$$\xi < 0 : c_y > 0$$

In the plane $\zeta = 0$, which contains the lifting line, $c_y = 0$ and

$$\eta - 1 > 0 : c_z > 0$$

$$\eta - 1 < 0 : c_z < 0$$

The flow picture in the cone, however, in its detail is essentially different from that in the neighborhood of a vortex filament in the incompressible case. Figure 2 shows the flow picture of the y and z velocities in a plane perpendicular to the cone axis downstream of the lifting line. The figure was obtained by computing the isocline field $c_z/c_y = \text{const.}$ On the cone surface, as has been said, c_z and c_y are infinite, although for the slope of the streamlines c_z/c_y there is here obtained the simple value

$$\frac{c_z}{c_y} = \frac{\xi}{\eta - 1}$$

¹A check for the correctness of this solution is obtained by substituting in the linearized continuity equation

$$-\kappa^2 \frac{\partial c_x}{\partial x} + \frac{\partial c_y}{\partial y} + \frac{\partial c_z}{\partial z} = 0$$

which must be identically satisfied.

The direction of the streamlines is therefore radial to the center. The flow consists partly of the closed streamlines which circulate about the vortex filament and partly of the streamlines that enter on one side of the cone and leave it again on the other side.

In addition to the two Mach cones that arise from each of its ends, the lifting line generates two plane waves, which enclose a "wedge space" and which appear in the streamline picture as the common tangents of the two cones.

For the downwash distribution in the plane $\xi = 0$ through the cone center, there is obtained from (14c) the simple formula

$$\frac{2 \pi x}{\Gamma_0} \tan \alpha c_{z_0} = \frac{\sqrt{1 - \delta^2}}{\delta} \quad (15)$$

where

$$\delta = \frac{y - \frac{b}{2}}{x \tan \alpha} \quad (15a)$$

This downwash distribution is shown in figure 3.

In order to study the processes on an airfoil of finite length at supersonic speed, particularly the induced drag, the replacement of the wing by a lifting line with constant circulation as in the case of the incompressible flow, appears inadmissible since on account of the infinite velocity at the end of the lifting line an infinite induced drag would be obtained. This difficulty in the case of the incompressible flow is avoided, as is known, by allowing the circulation to drop to zero in a suitable manner toward the wing tips. The induced drag is then computed by the formula.

$$W_i = \rho \int_{y = -b/2}^{y = +b/2} c_{z_0}(y) \Gamma(y) dy \quad (16)$$

(where $c_{z_0}(y)$ is the induced downwash velocity at the place of the lifting line, and ρ the density).

In the case of the supersonic flow, the relations are complicated by the fact that in spite of the assumption of a lift distribution decreasing to zero toward the wing tips, there are obtained singularities at the lifting line position of such a character as to make the computation of the induced drag by formula (16), which maintains its validity for supersonic flow, impossible. As closer investigation shows, this is due to the fact that the lifting line is the geometric locus of the vertices of all the Mach cones that pass down behind. This difficulty may be overcome by passing from the lifting line to the lifting surface.

III. WAVE RESISTANCE (DRAG)

Before proceeding to the corresponding computations, we shall discuss briefly the supersonic flow about an infinitely long airfoil (two-dimensional problem), a problem that had been considered by J. Ackeret in 1925 (reference 3).

The simplest and at the same time the ideal supersonic profile is that of the infinitely thin flat plate of chord t set to a small angle of attack β_0 (fig. 4). For such a plate the lift per unit span is

$$A = 2 \tan \alpha \beta_0 t \rho u_0^2 \quad (17)$$

or

$$\frac{A}{\frac{\rho}{2} u_0^2 t} = c_a = 4 \tan \alpha \beta_0 \quad (18)$$

On account of $A = \rho u_0 \Gamma_0$, the relation between the angle of attack of the wing and the circulation is

$$\beta_0 u_0 \tan \alpha = \frac{1}{2} \frac{\Gamma_0}{t} \quad (18a)$$

From the incompressible flow, the supersonic flow about the airfoil differs in that, for the latter case, even if the fluid friction is neglected, there is always associated a drag that originates from the plane waves which start out from the lifting surface and are inclined to the latter by the Mach angle and which, therefore may

be denoted as the wave drag. For the flat plate, the wave drag per unit span is

$$W_{\text{wave}} = -\beta_0 A = -2 \tan \alpha \beta_0^2 t \rho u_0^2 \quad (19)$$

or

$$\frac{W_{\text{wave}}}{\frac{\rho}{2} u_0^2 t} = c_{w\text{wave}} = 4 \tan \alpha \beta_0^2 \quad (19a)$$

The resultant of the lift and the drag is here at right angles to the plate. This comes from the fact that at supersonic flows there is no suction force at the leading edge of the plate. From equations (18) and (19a), there is obtained for the polar of the wave drag

$$c_{w\text{wave}} = \frac{c_a^2}{4 \tan \alpha}$$

which is thus a parabola as in the case of the incompressible flow.

Plane waves start out from the leading and trailing edges of the inclined flat plate (fig. 4) and in the space between them the induced downwash velocity is

$$c_{z\text{wave}} = -\beta_0 u_0 = -\frac{1}{2} \frac{\Gamma_0}{t \tan \alpha} \quad (20)$$

The wave drag, on the other hand, can also be computed from this downwash velocity induced by the plane waves, according to the formula

$$W_{\text{wave}} = \rho \Gamma_0 c_{z\text{wave}} \quad (21)$$

as may be seen by comparison with (19) and (20). In the next section it will be shown that, for a lifting surface, the velocity induced by the tip vortices like $c_{z\text{wave}}$ is proportional to $\Gamma_0/t \tan \alpha$. It then follows from equations (21) and (16) that the wave drag behaves in exactly the same way as the induced drag from the tip vortices. For practical applications it is therefore of no interest to consider the induced drag alone, but it is the sum of the induced and wave drags that must be considered.

For an airfoil of finite span and constant chord with circulation that is constant along the chord and variable along the span $\Gamma(y) = t \gamma(y)$ the total lift and wave drag are given by

$$A = \rho u_0 \int_{y=-b/2}^{y=+b/2} \Gamma dy = \frac{1}{2} \rho u_0 b t \int_{\eta=-1}^{\eta=+1} \gamma(\eta) d\eta \quad (22)$$

$$W_{\text{wave}} = \rho \int_{y=-b/2}^{y=+b/2} c_{z_{ow}} \Gamma dy = \frac{\rho}{2} b t \int_{\eta=-1}^{\eta=+1} c_{z_{ow}} \gamma d\eta \quad (23)$$

where

$$c_{z_{ow}} = - \frac{1}{2} \frac{\gamma}{\tan \alpha} \quad (24)$$

is the induced wave velocity. Accordingly

$$W_{\text{wave}} = - \frac{\rho}{4} \frac{b t}{\tan \alpha} \int_{\eta=-1}^{\eta=+1} \gamma^2 d\eta \quad (25)$$

By comparison of equations (22) and (25), there is found the relation between drag and lift

$$W_{\text{wave}} = 2 \frac{Z}{\rho t \tan \alpha} \left(\frac{A}{u_0 b} \right)^2 \quad (26)$$

In the above equation Z is a nondimensional coefficient that depends only on the lift distribution

$$Z = \frac{1}{2} \frac{\int_{\eta=-1}^{\eta=+1} \gamma^2 d\eta}{\left(\int_{\eta=-1}^{\eta=+1} \gamma d\eta \right)^2} \quad (27)$$

From equation (26), it follows that:

$$c_{w\text{wave}} = \frac{Z}{\tan \alpha} c_a^2 \quad (28)$$

The numerical values of Z are given in table I for several simple lift distributions.

TABLE I - Values of the Coefficient Z for Various Lift Distributions (Wing of Rectangular Plan Form)

| Number | Lift distribution | Z |
|--------|-------------------|--|
| 1 | rectangular | $1/4 = 0.250$ |
| 2 | ellipse | $\frac{8}{3\pi^2} = .271$ |
| 3 | trapezoidal | $\frac{2 + \frac{b'}{b}}{3 \left(1 + \frac{b'}{b}\right)^2}$ |
| | " $b' = b/2$ | $= .370$ |
| 4 | parabola | $\frac{3}{10} = .300$ |
| 5 | triangle | $\frac{2}{3} = .667$ |

IV. LIFTING SURFACE WITH CONSTANT LIFT DISTRIBUTION

For the successful computation of the induced drag for supersonic flow, according to section II, the simultaneous assumptions must be made of a suitable drop in lift toward the edges and a surface distribution of the bound vortices. This twofold extension means naturally a considerable swelling of the computation of the field of induced velocities as compared with the incompressible flow where the computation involves mostly a lifting line. In order to be able to recognize more clearly the effect of each of these two extensions, we proceed in two steps.

We first maintain the lift distribution constant along the span and consider only the transition from lifting line to lifting surface. The field of induced velocities thus obtained for a wing with constant spanwise circulation distribution and constant chord, while it does not enable as yet the computation of the induced drag nevertheless furnishes many useful results so that we proceed first to compute this field.

We assume therefore the circulation Γ_0 constant along the span b as uniformly distributed over a rectangular lifting surface of chord t and extending from $x = 0$ to $x = t$ (fig. 5). The circulation for a strip of the lifting surface of unit width is therefore $\gamma_0 = \Gamma_0/t$. It would be most convenient to make the transition from the lifting line to the lifting surface directly on the potential (13). On account of the integration difficulties that arise, however, the transition will be made on the velocity components (equation (14)), first for the z component since the latter is the most important for the computation of the induced drag.

A strip of the lifting surface of width dx' at a distance x' from the leading edge contributes to the induced z component c_z at the point x, y, z , if the point lies within the Mach cone arising from the end of the strip the amount

$$d \bar{c}_z = \frac{\gamma_0}{\pi b} dx' f(\xi - \xi', \eta, \zeta) = \frac{\gamma_0}{2\pi b} d\xi' f(\xi - \xi', \eta, \zeta)$$

where, according to equation (14)

$$f(\xi, \eta, \zeta) = \frac{\xi(\eta - 1)(\omega - \kappa \zeta^2)}{[\zeta^2 \omega + (\eta - 1)^2 \xi^2] \sqrt{\omega}} \quad (29)$$

If the point x, y, z lies outside the cone, the amount contributed is zero. The contributions from the plane waves starting out from the lifting surface will be separately considered. Integration over the wing chord therefore gives for the downwash velocity induced by the lifting surface

$$2\pi \bar{c}_z = \gamma_0 \int_{\xi'=0}^{\xi'=\xi_1} f(\xi - \xi', \eta, \zeta) d\xi'$$

or written out in full

$$2\pi\bar{c}_2 = \gamma_0(\eta-1) \int_{\xi'=0}^{\xi'=\xi_1} \frac{(\xi-\xi') \left\{ (\xi-\xi')^2 - \kappa^2 [(\eta-1)^2 + \xi^2] \right\} d\xi'}{\sqrt{(\xi-\xi')^2 - \kappa^2 [(\eta-1)^2 + \xi^2]} \left\{ (\xi-\xi')^2 - \kappa^2 [(\eta-1)^2 + \xi^2] \right\}} \quad (30)$$

The upper integration limit $\xi' = \xi_1$ is different according to whether the point lies within the Mach cone II arising from the end point of the trailing edge (fig. 5) or between the latter and cone I arising from the end point of the leading edge. The corresponding limits will be

$$\left. \begin{aligned} \xi_1 &= 2t/b = \epsilon \quad (\text{within cone II}) \\ \xi_1 &= \xi - \kappa \sqrt{(\eta-1)^2 + \xi^2} \quad (\text{between cones I and II}) \end{aligned} \right\} (31)$$

as may be easily seen after some consideration. Introducing the new variables of integration

$$(\xi - \xi')^2 - \kappa \xi^2 = \tau$$

and writing for briefness

$$a_1^2 = (\eta - 1)^2 + \xi^2$$

we have

$$\begin{aligned} 2\pi \frac{\bar{c}_2}{\gamma_0} &= - \frac{\eta - 1}{a_1^2} \int_{\tau=\tau_1}^{\tau=\tau_2} \frac{(\tau - \kappa^2 a_1^2) d\tau}{\tau \sqrt{\tau - \kappa^2 (\eta - 1)^2}} \\ &= - \frac{\eta - 1}{a_1^2} \int_{\tau=\tau_1}^{\tau=\tau_2} \frac{d\tau}{\sqrt{\tau - \kappa^2 (\eta - 1)^2}} \\ &\quad + \frac{\kappa^2}{2} (\eta - 1) \int_{\tau=\tau_1}^{\tau=\tau_2} \frac{d\tau}{\tau \sqrt{\tau - \kappa^2 (\eta - 1)^2}} \end{aligned}$$

The evaluation of the integral gives

$$2 \pi \frac{\bar{c}_z}{\gamma_0} = - \frac{\eta - 1}{(\eta - 1)^2 + \zeta^2} \left[\sqrt{(\xi - \xi')^2 - \kappa^2 \{(\eta - 1)^2 + \zeta^2\}} \right]_{\xi' = \xi_1}^{\xi' = 0} - \kappa \left[\text{arc tan} \frac{\sqrt{(\xi - \xi')^2 - \kappa^2 \{(\eta - 1)^2 + \zeta^2\}}}{\kappa(\eta - 1)} \right]_{\xi' = 0}^{\xi' = \xi_1}$$

Taking account of the different upper integration limits according to whether the point considered is within cone II or between cones I and II, equation (31), and setting for briefness

$$\omega \epsilon = (\xi - \epsilon)^2 - \kappa^2 [(\eta - 1)^2 + \zeta^2] \quad (32)$$

there is obtained as the final expression for \bar{c}_z :

For cone II

$$2 \pi \frac{\bar{c}_z}{\gamma_0} = \frac{\eta - 1}{(\eta - 1)^2 + \zeta^2} \{ \sqrt{\omega} - \sqrt{\omega \epsilon} \} - \kappa \left\{ \text{arc tan} \frac{\sqrt{\omega}}{\kappa(\eta - 1)} - \text{arc tan} \frac{\sqrt{\omega \epsilon}}{\kappa(\eta - 1)} \right\} \quad (33a)$$

Between cones I and II

$$2 \pi \frac{\bar{c}_z}{\gamma_0} = \frac{\eta - 1}{(\eta - 1)^2 + \zeta^2} \sqrt{\omega} - \kappa \text{ arc tan} \frac{\sqrt{\omega}}{\kappa(\eta - 1)} \quad (33b)$$

where

$$- 1 < \frac{\kappa(\eta - 1)}{\xi} < + 1 : - \frac{\pi}{2} < \text{arc tan} < \frac{\pi}{2}$$

From these formulas it follows that \bar{c}_z on the surface of cone I is equal to zero and on account of $\omega = 0$ is con-

tinuous in passing through cone II. On the common axis of cones I and II, there still occurs the same singularity as in the case of the lifting line, \bar{c}_z , there becoming infinite as $1/r$.

In order to obtain the total downwash velocity, there is still to be added to equations (33a) and (33b) the portion contributed by the plane wave. This contribution is different from zero only between the plane waves starting from the leading and trailing edges (fig. 4). According to equation (20)

$$c_{z\text{wave}} = -\frac{\kappa}{2} \gamma_0; \quad c_{y\text{wave}} = c_{x\text{wave}} = 0 \quad (34)$$

The expressions for the two remaining components of the induced velocity \bar{c}_y and \bar{c}_x are found by similar integrations. We shall only indicate the results:

Cone II:

$$2 \pi \frac{\bar{c}_y}{\gamma_0} = - \frac{\xi}{(\eta - 1)^2 + \xi^2} (\sqrt{\omega} - \sqrt{\omega_\epsilon})$$

$$2 \pi \frac{\bar{c}_x}{\gamma_0} = \text{arc tan} \frac{(\xi - \epsilon)(\eta - 1)}{\xi \sqrt{\omega_\epsilon}} - \text{arc tan} \frac{\xi(\eta - 1)}{\xi \sqrt{\omega}}$$

Between cones I and II:

$$2 \pi \frac{\bar{c}_y}{\gamma_0} = - \frac{\xi}{(\eta - 1)^2 + \xi^2} \sqrt{\omega}$$

$$2 \pi \frac{\bar{c}_x}{\gamma_0} = - \text{arc tan} \frac{\xi(\eta - 1)}{\xi \sqrt{\omega}}$$

(35)

In the above equations, the arc tan is to be taken $-\pi/2$ and $+\pi/2$. As may be seen from equations (33) and (35) by comparison with equation (14) in passing from the lifting line to the lifting surface, the difficulty of the infinite velocity at the cone surface has been set aside. The singularity of \bar{c}_y and \bar{c}_z on the cone axis (infinite as $1/r$) still remains, however, and prevents the computation of the induced drag for this lift distribution.

For the downwash distribution in the plane $z = 0$, at 1) the location of the wing $x < t$, and at 2) behind the wing $x > t$, taking account of the plane wave, there is obtained the following:

1) For $x < t$:

$$\left. \begin{aligned} -1 < \vartheta < 0: \frac{2\bar{c}_{z_0}}{\kappa\gamma_0} &= -1 + \frac{1}{\pi} \left\{ \frac{\sqrt{1 - \vartheta^2}}{\vartheta} - \arctan \frac{\sqrt{1 - \vartheta^2}}{\vartheta} \right\} \\ 0 < \vartheta < +1: \frac{2\bar{c}_{z_0}}{\kappa\gamma_0} &= \frac{1}{\pi} \left\{ \frac{\sqrt{1 - \vartheta^2}}{\vartheta} - \arctan \frac{\sqrt{1 - \vartheta^2}}{\vartheta} \right\} \end{aligned} \right\} \quad (36a)$$

where for the arc tan the same values are to be taken as in (33) and ϑ is given by equation (15a).

2) For $x > t$: The plane waves do not contribute anything but the formulas obtained differ according as the region considered is within cone II or between cones I and II (fig. 5).

$$\begin{aligned} - \left(1 - \frac{t}{x} \right) < \vartheta < 1 - \frac{t}{x}: \frac{2\bar{c}_{z_0}}{\kappa\gamma_0} &= \\ &= \frac{1}{\pi \vartheta} \left(\sqrt{1 - \vartheta^2} - \sqrt{\left(1 - \frac{t}{x} \right)^2 - \vartheta^2} \right) \\ &- \frac{1}{\pi} \left(\arctan \frac{\sqrt{1 - \vartheta^2}}{\vartheta} - \arctan \frac{\sqrt{\left(1 - \frac{t}{x} \right)^2 - \vartheta^2}}{\vartheta} \right) \\ -1 < \vartheta < - \left(1 - \frac{t}{x} \right) & \\ 1 - \frac{t}{x} < \vartheta < +1 & \end{aligned} \left. \vphantom{\frac{2\bar{c}_{z_0}}{\kappa\gamma_0}} \right\} :$$

$$\frac{2\bar{c}_{z_0}}{\kappa\gamma_0} = \frac{1}{\pi} \left(\frac{\sqrt{1 - \vartheta^2}}{\vartheta} - \arctan \frac{\sqrt{1 - \vartheta^2}}{\vartheta} \right) \quad (36b)$$

The downwash distribution for $x < t$ and for $x = 2t$ computed by the above equations is shown in figure 6.

Further, we have in the same manner as for the lifting line determined for the lifting surface the streamline field of the y and z velocities in a plane at right angles to the cone axis. At the location of the wing ($x < t$, fig. 7) there is obtained outside of the cones springing from the wing tips a constant downwash, due to the plane waves, along the span. The streamline picture within the Mach cone in the outer half is similar to that of the lifting line (fig. 2); the inner half however is entirely changed by the additional downwash velocity from the plane wave.

The streamline picture behind the wing ($x = 2t$, fig. 8) has, outside the Mach cones springing from the wing tips, a constant downwash velocity due to the plane waves in two strips symmetrical to the plane $z = 0$. These two strips are limited by the plane waves starting out from the forward and the trailing edges of the lifting surface. Within the Mach cone the streamline picture in the outer ring is the same as for $x < t$ and is changed only in the inner region.

We shall yet consider briefly the question, what the form of the wing surface must be that corresponds to the assumed lift distribution. The wing plan form we have assumed as rectangular. Angle of attack and twist are obtained from the consideration that at the wing, i.e., in the plane $z = 0$ in each section parallel to the flow direction, the direction of the streamlines must be parallel to the wing tangent. Let $z = z(x, y)$ be the equation of the wing surface and $z(0, y) \equiv 0$, i.e., straight leading edge. Then we have

$$\frac{dz}{dx} = \frac{c_{z_0}(x, y)}{u_0}$$

where c_{z_0} includes the induced velocities from both the plane waves and the edge cones. There is thus obtained for the wing surface

$$z(x, y) = \frac{1}{u_0} \int_{x'=0}^{x'=x} c_{z_0}(x', y) dx' \quad (37)$$

so that a further quadrature is required to compute the form of surface wing.

For the case considered of constant lift distribution there is obtained for the region outside of the two Mach cones at the wing tips, from equations (37) and (20):

$$z(x, y) = -\beta_0 x$$

that is, a flat surface with angle of attack β_0 . Within the Mach cone the surface bends downward more and more strongly as the edge is approached. The edge itself ($y = \pm b$) is bent infinitely downward, i.e., actually the rectangular surface with constant spanwise and chordwise lift distribution is not possible. For this reason we may dispense with the further computation of the wing-surface shape.

V. TRAPEZOIDAL WING WITH CONSTANT LIFT DISTRIBUTION

We consider now a trapezoidal wing with constant surface density of the lift γ_0 (fig. 9). If the wing is cut away behind (taper angle τ , fig. 9) in such a manner that the Mach cone at the tip of the leading edge does not overlap the wing ($\tau > \alpha$), the induced drag is obviously equal to zero and only the wave drag exists (reference 4).

$$\tau > \alpha: W_i = 0$$

The trapezoidal wing with constant surface density of the lift γ_0 is plane outside the Mach cone and has the angle of attack β_0 where

$$\gamma_0 = 2 \beta_0 u_0 \tan \alpha$$

The trapezoidal flat surface with constant lift distribution whose cut-away angle τ is greater than the Mach cone angle may be looked upon as the "ideal supersonic wing with finite span" since for it the ratio of drag to lift is no greater than for the wing of infinite span.

The computation of the induced drag for $\tau < \alpha$ is possible in a simple manner from the above results. By a lifting element we shall mean a strip of the lifting surface of chord $d x$ and therefore with circulation $\gamma_0 d x$,

Such a lifting element at $x = 0$ generates at a lifting element of chord $d x'$ at $x = x'$

$$y = y_1(x)$$

$$d^2 W_{i0x'} = \rho \gamma_0 d x' \int_{y=y_0(x)}^{y=y_1(x)} d c_{z_0}^{(0x')} d y \quad (38)$$

where $d c_{z_0}^{(0x')}$ denotes the downwash velocity induced by the lifting element $x = 0$ at the position $x = x'$. The integration limits are the surface of the Mach cone arising from the tip of the wing leading edge and the side edge of the plate. For the downwash velocity $c_{z_0}^{(0x')}$ in the plane $\xi = 0$, we have according to equation (14c)

$$d c_{z_0}^{(0x')} = \frac{\gamma_0}{\pi b} \frac{\sqrt{\xi^2 - \kappa^2(\eta - 1)^2}}{\xi(\eta - 1)} d x \quad (39)$$

with the aid of which equation (38) becomes

$$d^2 W_{i0x'} = \frac{\rho \gamma_0^2}{2 \pi} d x d x' \int_{\eta=\eta_0}^{\eta=\eta_1} \frac{\sqrt{\xi^2 - \kappa^2(\eta - 1)^2}}{\xi(\eta - 1)} d \eta$$

or with $\vartheta = \kappa \frac{\eta - 1}{\xi}$, according to equation (15a) and

$$\theta = \frac{\tan \tau}{\tan \alpha} \quad (40)$$

as the reduced angle of taper

$$d^2 W_{i0x'} = - \frac{\rho}{2 \pi} \gamma_0^2 d x d x' \int_{\vartheta=0}^{\vartheta=1} \frac{\sqrt{1 - \vartheta^2}}{\vartheta} d \vartheta$$

$$= - \frac{\rho}{2 \pi} \gamma_0^2 d x d x' g(\theta) \quad (41)$$

The evaluation of the definite integral gives

$$g(\theta) = -\sqrt{1-\theta^2} - \log \frac{1-\sqrt{1-\theta^2}}{\theta} \quad (42)$$

According to equation (41) the induced drag from the lifting element $x=0$ at the position $x=x'$ is independent of the distance between the two elements. All elements lying between $x=0$ and $x=x'$ accordingly produce the same drag, so that the total drag induced at $x=x'$ amounts to

$$dW_{ix'} = -\frac{\rho}{2\pi} \gamma_0^2 x' dx' g(\theta)$$

The drag for the entire wing is obtained from the above by integrating over x' between the limits $x'=0$ and $x'=t$ and multiplying by two (both ends)

$$\begin{aligned} W_i &= -\frac{\rho}{\pi} \gamma_0^2 g(\theta) \int_{x'=0}^{x'=t} x' dx' \\ &= -\frac{\rho}{2\pi} \gamma_0^2 t^2 g(\theta) = -\frac{\rho}{2\pi} \Gamma_0^2 g(\theta) \quad (43) \end{aligned}$$

The minus sign is explained by the fact that with our choice of coordinate system the drag component of a force is in the direction of the negative z axis. Formula (43) for the induced drag of a trapezoidal wing with constant surface density of the lift is of the same structural form that is found for the incompressible flow. For triangular lift distribution (lifting line) in the case of incompressible flow, we have, for example,

$$W_i = -\frac{\log 2}{\pi} \rho \Gamma_0^2$$

where Γ_0 is the circulation at the wing center. For equal total circulation Γ_0 , W_i according to equation (43) is independent of θ , i.e., the ratio of the tangent

of the angle τ to the tangent of the Mach angle (equation (40)). In passing to the rectangular wing, $\theta \rightarrow 0$, the induced drag according to equations (42) and (43) becomes logarithmically infinite; in agreement with our results of the previous section.

Actually, we are not interested so much in the value of the induced drag alone as in the sum of the induced and wave drags. For the wave drag, according to equation (21) we have

$$W_{\text{wave}} = \rho F \gamma_0 c_{z_{\text{wave}}}$$

where $F = b t \left(1 - \frac{t \tan \tau}{b}\right)$, the area of the wing

$$c_{z_{\text{wave}}} = -\frac{1}{2} \frac{\gamma_0}{\tan \alpha}$$

hence

$$W_{\text{wave}} = -\frac{\rho}{2} F \frac{\gamma_0^2}{\tan \alpha}$$

} (44)

For the lift we have, on account of $\gamma_0 = 2 \beta_0 u_0 \tan \alpha$:

$$A = \rho \gamma_0 u_0 F = 2 \rho u_0^2 F \beta_0 \tan \alpha \quad (45)$$

or

$$\frac{A}{\frac{\rho}{2} u_0^2 F} = c_a = 4 \beta_0 \tan \alpha \quad (46)$$

For the wave drag we obtain from (44)

$$W_{\text{wave}} = 2 \rho F u_0^2 \beta_0^2 \tan \alpha$$

$$\frac{W_{\text{wave}}}{\frac{\rho}{2} u_0^2 F} = c_{w_{\text{wave}}} = 4 \beta_0^2 \tan \alpha = \beta_0 c_a \quad (47)$$

and for the induced drag from equation (43)

$$W_i = \frac{\rho}{2\pi} t^2 4 \beta_0^2 u_0^2 \tan^2 \alpha g(\theta)$$

$$\frac{W_i}{\frac{\rho}{2} u_0^2 F} = c_{wi} = \frac{1}{\pi} \frac{4 t \beta_0^2 \tan^2 \alpha g(\theta)}{b \left(1 - \frac{t \tan \alpha}{b}\right)}$$

$$c_{wi} = 4 \beta_0^2 \tan \alpha \frac{\lambda}{\pi} \frac{g(\theta)}{1 - \theta \lambda} \quad (48)$$

where $\lambda = \frac{t \tan \alpha}{b}$ is the "reduced aspect ratio" of the

wing. For the total drag there is thus obtained from (47) and (48)

$$(c_w)_{\text{wave} + \text{ind}} = 4 \beta_0^2 \tan \alpha \left\{ 1 + \frac{\lambda}{\pi} \frac{g(\theta)}{1 - \theta \lambda} \right\}$$

$$(c_w)_{\text{wave} + \text{ind}} = \frac{c_a^2}{4 \tan \alpha} \left\{ 1 + \frac{\lambda}{\pi} \frac{g(\theta)}{1 - \theta \lambda} \right\} \quad (49)$$

It follows therefore from the above that for supersonic speed the wave plus induced drag, like the induced drag in the incompressible flow case is proportional to the square of the lift. Equation (49) is analogous to the

well-known formula $c_{wi} = \frac{c_a^2}{\pi} \frac{F}{b^2}$ of the elliptic lift

distribution for the incompressible flow. The essential difference lies in the fact that for the supersonic flow the drag parabola for small aspect ratios t/b is to a first approximation independent of the aspect ratio. The manner in which the drag increases with increasing reduced

aspect ratio $\lambda = \frac{t \tan \alpha}{b}$ and decreasing θ is shown in

figure 10 where $c_w / \frac{c_a^2}{4 \tan \alpha}$ is plotted against λ for

various values of θ . Our formulas are valid only for

$\lambda \leq \frac{1}{2}$, i.e., for the case in which the Mach cones do not overlap on the wing.

In order to be able to predict what the wing shape must be so that our assumed lift distribution may be possible, we must first compute the field of the induced velocities. For this purpose equation (39) is to be integrated over the trapezoidal area. The value

$d c_{z_0}^{(o\xi)}$ according to (39) gives the downwash velocity induced at the position ξ by a lifting element $\gamma_0 dx$ starting at $\xi = 0$ and ending at $\eta = 1$. A lifting element which starts at $\xi = \xi'$ and ends at $\eta = \eta'$ thus produces at the position $\xi, \eta, \zeta = 0$ the downwash velocity

$$d c_{z_0}^{(\xi'\xi)} = - \frac{\gamma_0}{2\pi} d\xi' \frac{\sqrt{(\xi - \xi')^2 - \kappa^2 (\eta - \eta')^2}}{(\xi - \xi')(\eta' - \eta)}$$

For the velocity induced by the entire surface there is thus obtained

$$c_{z_0}^{(\xi)} = - \frac{\gamma_0}{2\pi} \int_{\xi'=0}^{\xi'=\xi_1} \frac{\sqrt{(\xi - \xi')^2 - \kappa^2 (\eta - \eta')^2}}{(\xi - \xi')(\eta' - \eta)} d\xi' \quad (50)$$

In order to evaluate this integral we introduce the new integration variables

$$\vartheta = \kappa \frac{\eta' - \eta}{\xi - \xi'} = \kappa \tan \varphi'' \quad (51)$$

(See fig. 9.) Since the end points of the lifting elements lie on the wing contour there exists the relation

$$1 - \eta' = \xi' \tan \tau \quad (52)$$

The upper integration limit $\xi' = \xi_1$, in equation (50) is obtained from the condition. (See fig. 9.)

$$\tan \varphi' = \tan \alpha : \xi' = \xi_1 : \delta' = 1$$

The lifting elements whose ξ' is greater than the ξ_1 thus determined give no contribution at the points $\xi, \eta, \zeta = 0$. For the lower integration limit

$$\xi' = 0 : \eta' = 1 : \delta' = \kappa \frac{1 - \eta}{\xi} = \delta$$

From equations (51) and (52) there is obtained

$$\frac{d \xi'}{\xi - \xi'} = \frac{d \delta'}{\delta' - \theta}$$

where θ is the abbreviation introduced in equation (40). There is then obtained from equation (50)

$$c_{z0}(\delta) = - \frac{\gamma_0}{2\pi} \kappa \int_{\delta'}^{\delta'=1} \frac{\sqrt{1 - \delta'^2}}{\delta'(\delta' - \theta)} d \delta' = - \frac{\gamma_0}{2\pi} \kappa F(\delta, \theta) \quad (53)$$

In evaluating the above integral the following three cases are to be distinguished:

1. $0 < \theta < \delta$; 2. $0 < \delta < \theta$; 3. $\delta < 0 < \theta$

In case 1 the point $P(\delta)$ lies within, in cases 2 and 3 without the trapezoidal wing. In case 1 the integrand is regular over the entire range of integration; in case 2 it possesses a singularity at $\delta' = \theta$; and in case 3, two singularities at $\delta' = 0$ and $\delta' = \xi$. In cases 2 and 3 the principal values are to be taken, namely,

$$0 < \delta < \theta: \quad F(\delta, \theta) = \lim_{\epsilon \rightarrow 0} \left\{ \int_{\delta'}^{\delta'=\theta-\epsilon} \frac{\sqrt{1 - \delta'^2}}{\delta'(\delta' - \theta)} d \delta' + \int_{\delta'=\theta+\epsilon}^{\delta'=1} \frac{\dots \dots \dots}{\dots \dots \dots} d \delta' \right\} \quad (54a)$$

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and

$\delta < 0 < \theta$:

$$F(\delta, \theta) = \lim_{\epsilon \rightarrow 0} \left\{ \int_{\delta'=\delta}^{\delta'=-\epsilon} \frac{\sqrt{1-\delta'^2}}{\delta'(\delta'-\theta)} d\delta' + \int_{\delta'=+\epsilon}^{\delta'=\theta-\epsilon} \frac{\dots\dots\dots}{\dots\dots\dots} d\delta' = \int_{\delta'=\theta+\epsilon}^{\delta'=1} \frac{\dots\dots\dots}{\dots\dots\dots} \right\} \quad (54b)$$

The integral

$$F(\delta, \theta) = \int_{\delta'=\delta}^{\delta'=1} \frac{\sqrt{1-\delta'^2}}{\delta'(\delta'-\theta)} d\delta' \quad (55)$$

may be obtained by elementary methods. We set

$$\sqrt{1-\delta'^2} = t\delta' - 1$$

where t is the new integration variable so that (55) becomes

$$F(\delta, \theta) = \int_{t=1}^{t=t_1} \frac{(1-t^2)^2 dt}{t(1+t^2) \{2t - \theta(1+t^2)\}}$$

where

$$t_1 = \psi = \frac{1 + \sqrt{1-\delta^2}}{\delta} \quad (56)$$

By breaking up into partial fractions there is obtained

$$\frac{(1-t^2)^2}{t(1+t^2) \{2t - \theta(1+t^2)\}} = -\frac{2}{1+t^2} - \frac{1}{\theta t} + \frac{\sqrt{1-\theta^2}}{\theta} \left(\frac{1}{t-t_2} - \frac{1}{t-t_1} \right)$$

where

$$t_{1,2} = \frac{1 \pm \sqrt{1-\theta^2}}{\theta}$$

Performing the integration, there is obtained

$$F(\theta, \theta) = \left[-2 \arctan t - \frac{\log t}{\theta} + \frac{\sqrt{1-\theta^2}}{\theta} \left\{ \log(t-t_2) - \log(t-t_1) \right\} \right]_{t=1}^{t=\psi}$$

For $0 < \theta < 1$ there is therefore obtained directly

$$F(\theta, \theta) = \frac{\pi}{2} - 2 \arctan \psi - \frac{\log \psi}{\theta} + \frac{\sqrt{1-\theta^2}}{\theta} \log \left(\frac{1 + \sqrt{1-\theta^2} \psi \theta - 1 + \sqrt{1-\theta^2}}{1 - \psi \theta + \sqrt{1-\theta^2}} \right) \quad (57)$$

while the formation of the principal value according to equation (54a,b) gives

$0 < \delta < \theta$:

$$F(\delta, \theta) = \frac{\pi}{2} - 2 \arctan \psi - \frac{\log \psi}{\epsilon} - \frac{\sqrt{1 - \epsilon^2}}{\theta} \log \left(\frac{1 + \sqrt{1 - \theta^2}}{\epsilon} \frac{\psi \theta - 1 + \sqrt{1 - \epsilon^2}}{\psi \theta - 1 - \sqrt{1 - \theta^2}} \right) \quad (57)$$

$\delta < 0 < \theta$:

$$F(\delta, \theta) = -\frac{\pi}{2} - 2 \arctan \psi - \frac{\log(-\psi)}{\epsilon} + \frac{\sqrt{1 - \epsilon^2}}{\theta} \log \left(\frac{1 + \sqrt{1 - \theta^2}}{\epsilon} \frac{\psi \theta - 1 + \sqrt{1 - \epsilon^2}}{\psi \theta - 1 - \sqrt{1 - \theta^2}} \right)$$

where

$$-\frac{\pi}{2} < \arctan \psi < \frac{\pi}{2}$$

There is thus found the downwash distribution in the entire Mach cone springing from $y = b/2$, $x = 0$. For $F(\delta, \theta)$ we have

$$\delta = \pm 1: F(\pm 1; \theta) = 0$$

$$\delta = \theta, \delta = 0: F(\theta, \theta) = F(0, \theta) = \infty \text{ as } \log \delta \text{ at } \delta = 0 \quad (58)$$

On the two rims of the cone ($\delta = \pm 1$) the induced velocity is thus zero and on the edge of the trapezoidal wing ($\delta = \theta$) and on the cone axis ($\delta = 0$) it is infinite. In figures 11 and 12 for the particular case $\epsilon =$

$$\frac{1}{3} \left(\tan \alpha = \sqrt{3}, \tan \tau = \frac{1}{\sqrt{3}} \right) \text{ there is shown the induced}$$

downwash velocity in a section parallel and perpendicular, respectively, to the principal stream direction.

To the above velocity field of the tip vortices there is still to be added the velocity field due to the plane wave. The latter in the plane $z = 0$ within the wing area is

$$c_{z_0, \text{wave}} = - \frac{1}{2} \frac{\gamma_0}{\tan \alpha} = - \beta_0 u_0$$

and outside the wing area

$$c_{z_0, \text{wave}} \equiv 0$$

From the velocity field it is now possible to compute the form of the trapezoidal wing surface that has constant lift distribution. Outside of the Mach cone we have, according to equation (37)

$$z(x, y) = \frac{1}{u_0} \int_{x'=0}^{x'=x} c_{z_0, \text{wave}}(x', y) dx' = - \beta_0 x$$

that is, a flat surface with the angle of attack β_0 given by equation (18). The twist of this flat surface within the edge region of the trapezoidal area that is overlapped by the Mach cone is given by

$$z(x, y) = \frac{1}{u_0} \int_{x'=(b/2-y)/\tan \alpha}^{x'=x} c_{z_0 i} dx'$$

and according to equation (53)

$$z(x, y) = - \frac{1}{2\pi} \frac{\gamma_0}{u_0} \kappa \int_{x'=(b/2-y)/\tan \alpha}^{x'=x} F(\vartheta', \theta) dx' = - \frac{\beta_0}{\pi} \int_{x'=(b/2-y)/\tan \alpha}^{x'=x} F(\vartheta', \theta) dx'$$

On account of $\vartheta = \frac{\kappa \left(\frac{b}{2} - y \right)}{x}$ there is obtained

$$z(x, y) = \frac{\beta_0}{\pi} \kappa \left(\frac{b}{2} - y \right) \int_{\vartheta'=1}^{\vartheta'=\vartheta} \frac{F(\vartheta', \theta)}{\vartheta'^2} d\vartheta' \quad (59)$$

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Since the function $F(\vartheta, \theta)$ is known from equation (57), it is possible from the equation above to compute for a given θ the profile sections of the surface at various distances $(b/2 - y)$ from the edge. The ordinate of the obliquely cut-away edge of the trapezoidal area for $b/2 - y < t \tan \tau$:

$$z(x_R, y) = \frac{\beta_0}{\pi} \kappa \left(\frac{b}{2} - y \right) \int_{\vartheta'=1}^{\vartheta'=\theta} \frac{F(\vartheta', \theta)}{\vartheta'^2} d\vartheta' \quad (60)$$

The integrand becomes infinite for $\vartheta' = \theta$ (equation (58)). The integral exists, however, and may be evaluated by special computation. There is obtained

$$\int_{\vartheta'=1}^{\vartheta'=\theta} \frac{F(\vartheta', \theta)}{\vartheta'^2} d\vartheta' = \frac{\pi}{2\theta} - \frac{\arcsin \theta}{\theta} - \frac{1}{\theta^2} + \frac{1}{1 + \sqrt{1 - \theta^2}} \quad (60a)$$

(The evaluation of the integral was performed by Dr. F. Riegels.)

For $\theta = 1/3$, we thus have

$$\theta = 1/3: \int_{\vartheta'=1}^{\vartheta'=\theta} \frac{F(\vartheta', \theta)}{\vartheta'^2} d\vartheta' = -4.92$$

The ordinate of the rear edge point $x = t$, $\frac{b}{2} - y = t \tan \tau$ for $\theta = 1/3$ is thus $z = -1.522 \beta_0 t$. (Flat surface $z = -\beta_0 t$, twist $z = -0.522 \beta_0 t$.)

For the special case $\theta = 1/3$ ($\tan \alpha = \sqrt{3}$; $\tan \tau = \frac{1}{\sqrt{3}}$) the profile sections have been computed and are given in figure 13. If the trapezoidal wing were flat there would be a drop of the lift toward the edge down to zero. In order that full lift be maintained up to the edge, the wing must be bent downward. The twist of the

wing directly at the edge is very strong as may be seen from the "elevation contour lines" (fig. 14).

VI. COMPUTATION OF THE LIFT DISTRIBUTION FOR THE UNTWISTED RECTANGULAR WING

The examples thus far considered are all in connection with the so-called first principal problem of the airfoil theory where the lift distribution is given and it is required to find the drag and the wing shape. Of greater practical importance is the second principal problem where the wing shape being given it is required to find the lift distribution and the drag. As in the case of the incompressible flow, so also in the case of the compressible flow the first problem, which leads only to quadratures, is considerably more simple than the second, which requires the solution of an integral equation.

In what follows there will now be given an example of the second principal problem, namely, the computation of the lift distribution for a plane rectangular wing (span = b , chord = t), that is to say, the same problem that was first considered by A. Betz (reference 5) for the case of incompressible flow. In the treatment of this problem we can utilize to a large extent the results we had obtained in the previous section for the trapezoidal wing with constant surface density of the lift. We consider a rectangular flat plate which extends from $x = 0$ to $x = t$ and from $y = -b/2$ to $y = +b/2$ and is set at the small angle of attack β_0 to the undisturbed velocity u_0 (fig. 5). Within the region bounded by the plane waves starting out from the leading and trailing edges and the two Mach cones there is the constant downwash velocity due to the plane waves

$$c_{z0\text{ wave}} = -\beta_0 u_0 = -\frac{1}{2} \frac{\gamma_0}{\tan \alpha} \quad (61)$$

Outside the region of the flat plate overlapped by the Mach cones at the tips there thus exists the constant lift distribution γ_0 . At the tips $y = \pm b/2$ the lift must vanish, that is, $\gamma = 0$ at $y = \pm b/2$. There is required the lift distribution $\gamma = \gamma(x, y)$ within the region overlapped by the Mach cones. The problem is considerably

simplified by the circumstance that, as will immediately become apparent, γ does not depend on the two independent variables x, y , but only on one of the variables

$$\delta = \frac{\frac{b}{2} - y}{x \tan \alpha} \quad (51)$$

(fig. 11). For the required lift distribution

$$\gamma(\delta) = \gamma_0 f(\delta) \quad (62)$$

of the rectangular wing there then exist the boundary conditions

$$\left. \begin{array}{l} \delta = 0 : f(\delta) = 0 \\ \delta = 1 : f(\delta) = 1 \end{array} \right\} \quad (63)$$

In order to be able to set up the integral equation for $\gamma(\delta)$ we must first compute the field of the downwash velocities $w(\delta)$ induced by a rectangular wing with the circulation distribution $\gamma(\delta)$ in the plane $z = 0$. The integral equation for $\gamma(\delta)$ is then obtained in the known manner from the consideration that for each position of the wing the sum of the effective angle of attack

$$\beta(\delta) = \frac{1}{2} \frac{\gamma(\delta)}{u_0 \tan \alpha} \quad (64)$$

and the induced angle of attack $-\frac{w(\delta)}{u_0}$ must be equal to the geometrical angle of attack β_0

$$\beta(\delta) - \frac{w(\delta)}{u_0} = \beta_0 \quad (65)$$

The velocity field $w(\delta)$ induced by the edge vortices is obtained by considering the rectangular wing with the variable lift distribution $\gamma(\delta) = \gamma_0 f(\delta)$ as built up by the superposition of trapezoidal wings with various taper angles each of which wings possesses a constant lift distribution. Again, let $\theta = \frac{\tan \tau}{\tan \alpha}$ be the "reduced taper angle" (equation 40), then the lift distribution $\gamma = \gamma_0 f(\delta)$ may

be obtained by the superposition of trapezoids with angles θ and lift densities $\gamma_0 f'(\theta) d\theta$. Each of these trapezoids produces, according to equation (53) the velocity field

$$dw(\vartheta) = - \frac{\gamma_0 f'(\theta)}{2\pi \tan \alpha} F(\vartheta, \theta) d\theta$$

and integration over θ from $\theta = 0$ to $\theta = 1$ then gives the induced velocity field over the rectangular wing

$$w(\vartheta) = - \frac{\gamma_0}{2\pi \tan \alpha} \int_{\theta=0}^{\theta=1} f'(\theta) F(\vartheta, \theta) d\theta \quad (65)$$

By substituting the above expression for $w(\vartheta)$ in equation (65), there is finally obtained, taking account of (61) and (64) the required integral equation for $f(\vartheta)$:

$$f(\vartheta) + \frac{1}{\pi} \int_{\theta=0}^{\theta=1} f'(\theta) F(\vartheta, \theta) d\theta = 1 \quad (67)$$

to which are added the boundary conditions (63). This integral equation for the lift distribution has the same structural form as that for the incompressible flow. It differs from the latter, however, by the different core $F(\vartheta, \theta)$, which is given by equation (57), and for the supersonic flow is of a much more complicated form than for the incompressible flow. Equation (67) also exhibits the notable property that neither the aspect ratio of the wing nor the Mach number appears explicitly, whereas in the incompressible case the characteristic value of the integral equation depends on the aspect ratio. The dependence of the lift distribution on the Mach number appears in the introduction instead of the geometric angle φ (fig. 9)

the reduced angle $\vartheta = \frac{\tan \varphi}{\tan \alpha}$ as the variable. It is neces-

sary to solve the integral equation (67) only once to obtain the lift distribution of the rectangular wing for all aspect ratios and all Mach numbers.

The solution of the integral equation (67) appears at first sight quite difficult, particularly on account of the complicated structure of the core $F(\delta, \theta)$. (See equations 56 and 57.) By a simple transformation of equation (67) it is possible, however, to simplify the problem considerably.* The equation is a nonhomogeneous integrodifferential equation for $f(\delta)$. Instead of it we shall consider the equivalent equation for $f'(\delta)$. Taking account of the singularity of the core, equation (67) may be written

$$f(\delta) + \frac{1}{\pi} \left\{ \int_{\theta=0}^{\theta=\delta} f'(\theta) F(\delta, \theta) d\theta + \int_{\theta=\delta}^{\theta=1} f'(\theta) F(\delta, \theta) d\theta \right\} = 1$$

Differentiation with respect to δ gives

$$f'(\delta) + \frac{1}{\pi} \left\{ f'(\delta) F(\delta, \delta) + \int_{\theta=0}^{\theta=\delta} f'(\theta) \frac{dF}{d\delta} d\theta - f'(\delta) F(\delta, \delta) + \int_{\theta=\delta}^{\theta=1} f'(\theta) \frac{dF}{d\delta} d\theta \right\} = 0$$

and because

$$\frac{dF}{d\delta} = - \frac{\sqrt{1-\delta^2}}{\delta(\delta-\theta)}$$

according to equation (53):

$$f'(\delta) - \frac{1}{\pi} \frac{\sqrt{1-\delta^2}}{\delta} \int_{\theta=0}^{\theta=1} \frac{f'(\theta)}{\delta-\theta} d\theta = 0 \quad (68)$$

*For this suggestion I am indebted to Doctor Lotz and for carrying out the numerical solution of the integral equation to Mr. Pretsch.

The above is the equivalent integral equation for $f'(\delta)$ which, however, is now homogeneous. The solution of this integral equation for $f'(\delta)$ is possible by building up $f'(\delta)$ in n steps and solving the corresponding system of linear equations

$$f'(\delta_{2\nu+1}) - \frac{1}{\pi} \frac{\sqrt{1-\delta^2}}{\delta_{2\nu+1}} \sum_{\lambda=0}^n f'(\delta_{2\lambda+1}) \int_{\theta=\delta_{2\lambda}}^{\theta=\delta_{2\lambda+2}} \frac{d\theta}{\delta_{2\nu+1} - \theta} = 0 \quad (\nu = 0, 1, \dots, n-1) \tag{69}$$

This is a system of n homogeneous equations for the n unknowns $f'(\delta_{2\nu+1})$ ($\nu = 0, 1, \dots, n-1$). Since, as closer investigation shows, $f'(0) = \infty$, $f'(\delta_1)$ is suitably chosen not constant but equal to

$$f'(\delta_1) = a - b \delta$$

There is then obtained in place of equation (69) a nonhomogeneous system of equations of the n th order for the n unknowns $\frac{b}{a}, \frac{1}{a} f'(\delta_{2\nu+1})$ ($\nu = 1 \dots, n-1$). The further unknown a is obtained in the numerical integration for $f(\delta)$ from the condition

$$f(1) = a \sum_{\nu=0}^{n-1} \frac{f'(\delta_{2\nu+1})}{a} \Delta \delta = 1 \tag{70}$$

In carrying out the numerical process there were first taken five steps ($\delta_{2\nu+1} = 0.1; 0.3; 0.5; 0.7; 0.9$), then ten steps ($\delta_{2\nu+1} = 0.05; 0.15; \dots; 0.95$). It was found that the ten-step approximation gives an improvement over the five-step process only in the interval $0 < \delta < 0.2$. In the third approximation therefore only the interval $0 < \delta < 0.2$ was again subdivided ($\delta_{2\nu+1} = 0.025; 0.075; 0.125; 0.175$). The values obtained in this manner for $f'(\delta)$ and $f(\delta)$ are given in table II and the function $f(\delta)$ plotted in figure 15. At $\delta = 0$ the function

$f(\vartheta)$ possesses a singularity since $f'(\vartheta)$ there becomes infinite. The mathematical nature of this singularity could not as yet be determined.

We shall now compute the lift, wave drag, and induced drag as well as the moment about the transverse axis of the rectangular flat surface.

The lift A_I of that portion of the surface which lies outside the two Mach cones is

$$A_I = \rho u_0 \gamma_0 b t \left(1 - \frac{t \tan \alpha}{b} \right)$$

while the lift of the two triangular portions overlapped by the Mach cones is

$$A_{II} = \rho u_0 t^2 \tan \alpha \int_{\vartheta=0}^{\vartheta=1} \gamma d\vartheta = \rho u_0 \gamma_0 t^2 \tan \alpha K$$

where

$$K = \int_{\vartheta=0}^{\vartheta=1} f(\vartheta) d\vartheta = 0.684 \quad (71)$$

The total lift of the rectangular plate is therefore

$$A = \rho u_0 \gamma_0 b t \left\{ 1 - (1 - K)\lambda \right\}$$

or, according to equation (20)

$$A = 2 \rho u_0^2 \beta_0 F \tan \alpha \left\{ 1 - (1 - K)\lambda \right\} \quad (72)$$

For the lift coefficient there is thus obtained

$$c_a = 4 \beta_0 \tan \alpha \left\{ 1 - (1 - K)\lambda \right\} \quad (73)$$

For the wave drag outside of the Mach cones there is obtained simply

$$W_{\text{wave I}} = \beta_0 A_I = \frac{\rho}{2} \gamma_0^2 \frac{b t}{\tan \alpha} \left\{ 1 - \frac{t \tan \alpha}{b} \right\} \quad (74)$$

The wave drag of the two triangular portions overlapped by the Mach cones is

$$W_{\text{wave II}} = 2 \rho \int \gamma c_{\text{wave}} d f$$

where

$$c_{z\text{wave}} = \frac{\gamma}{2 \tan \alpha} = \frac{\gamma_0}{2 \tan \alpha} f(\delta)$$

and

$$d f = \frac{1}{2} t^2 d (\tan \varphi)$$

Table II

Lift Distribution of the Untwisted Rectangular Wing

$f(\delta)$ and $f'(\delta)$

| δ | $f'(\delta)$ | δ | $f(\delta)$ |
|----------|--------------|----------|-------------|
| 0 | ∞ | 0 | 0 |
| 0.025 | 4.49 | 0.05 | 0.219 |
| .075 | 1.86 | .1 | .312 |
| .125 | 1.39 | .15 | .381 |
| .175 | 1.24 | .2 | .444 |
| .25 | 1.10 | .3 | .554 |
| .35 | .958 | .4 | .649 |
| .45 | .850 | .5 | .734 |
| .55 | .753 | .6 | .810 |
| .65 | .655 | .7 | .875 |
| .75 | .546 | .8 | .930 |
| .85 | .417 | .9 | .971 |
| .95 | .225 | 1.0 | 1 |

We then have

$$W_{\text{wave II}} = \frac{\rho}{2} \frac{t^2}{\tan \alpha} \int \gamma^2 d (\tan \varphi) = \frac{\rho}{2} t^2 \gamma_0^2 \int_{\delta=0}^{\delta=1} f^2 d \delta$$

$$W_{\text{wave II}} = \frac{\rho}{2} \gamma_0^2 t^2 K_1 \tag{75}$$

where

$$K_1 = \int_{\vartheta=0}^{\vartheta=1} f^2(\vartheta) d\vartheta$$

Similarly there is obtained for the induced drag in the two triangular regions overlapped by the Mach cones

$$W_i = 2 \rho \int \gamma c_{zi} d f$$

where from equations (62) and (65)

$$c_{zi} = \frac{1}{2} \frac{\gamma_0}{\tan \alpha} (1 - f(\vartheta))$$

We then have

$$W_i = \frac{\rho}{2} t^2 \gamma_0^2 \int_{\vartheta=0}^{\vartheta=1} f(\vartheta) [1 - f(\vartheta)] d\vartheta$$

$$W_i = \frac{\rho}{2} t^2 \gamma_0^2 (K - K_1) \quad (76)$$

For the total drag

$$W = W_{\text{wave I}} + W_{\text{wave II}} + W_i$$

there is thus obtained from equations (74), (75), and (76)

$$W = \frac{\rho}{2} \gamma_0^2 \frac{b t}{\tan \alpha} \{1 - (1 - K)\lambda\}$$

or from equation (20)

$$W = 2 \rho u_0^2 \beta_0^2 F \tan \alpha \{1 - (1 - K)\lambda\} \quad (77)$$

and for the drag coefficient

$$c_w = 4 \beta_0^2 \tan \alpha \{1 - (1 - K)\lambda\} \quad (78)$$

From equations (72) and (77) there is obtained between the lift and the drag the simple relation

$$W = \beta_0 A \quad (79)$$

There is thus obtained for the plane surface of finite span the same simple result as for the infinitely long flat plate, namely, that the ratio of the total drag for a frictionless flow to the lift is $\beta_0 : 1$. This may also be explained by the fact that in contrast to the incompressible flow no suction force arises at the leading edge in the supersonic case and the resultant air force is therefore at right angles to the plate.

For the relation between the drag and lift coefficients, there is obtained finally from equations (73) and (78)

$$c_w = \frac{c_a^2}{4 \tan \alpha} \frac{1}{1 - (1 - K) \lambda} = \frac{c_a^2}{4 \tan \alpha} \frac{1}{1 - 0.316 \lambda} \quad (80)$$

The above formula has the same structural form as formula (49) for the trapezoidal wing with constant lift distribution. In figure 10 $c_w / \frac{c_a^2}{4 \tan \alpha}$ has been plotted against the reduced aspect ratio λ (dotted curve). It may be seen that the rectangular plane wing for the same lift has the same drag as the trapezoidal wing with constant lift distribution with the reduced taper angle $\theta = \frac{\tan \tau}{\tan \alpha} = 0.27$. For the reduced aspect ratio $\lambda = 0.3$ the rectangular plane wing has, for the same lift, about 10 percent and for $\lambda = 0.5$, 19 percent more drag than the ideal trapezoidal wing whose taper angle is greater than the Mach angle.

With the above results the theoretical polar and moment curves for the plane rectangular wing may be given for various aspect ratios and Mach numbers. For the moment M_H about the transverse axis in the wing leading edge, there is obtained

$$M_H = 2 \rho u_0^2 \beta_0 \tan \alpha b t^2 \left\{ \frac{1}{2} - \frac{2}{3} (1 - K) \lambda \right\}$$

* It is interesting to note that the constant $1 - K = 1 - \int_0^1 f(\theta) d\theta$ is equal to $1/\pi$ within the computational accuracy. That this is exactly so has as yet not been shown. For this it would be necessary to know the exact solution of the integral equation (67).

and for the moment coefficient $c_{mH} = \frac{M_H}{\frac{\rho}{2} u_0^2 b t^2}$

$$c_{mH} = 4 \beta_0 \tan \alpha \left\{ \frac{1}{2} - \frac{2}{3} (1 - K) \lambda \right\}$$

$$c_{mH} = c_a \frac{\frac{1}{2} - 0.211 \lambda}{1 - 0.316 \lambda}$$
(81)

Through equations (73), (80), and (81), the polar and moment curves not considering the frictional drag, are completely determined. In figure 16, the polars are given

for the aspect ratios $\frac{t}{b} = 0, \frac{1}{5},$ and $\frac{1}{2}$ and for the Mach numbers $\frac{u_0}{c} = 1.2, 1.5, 2.0,$ and 3.0 . The drag differences

between wings with various aspect ratios are considerably smaller in the case of the supersonic flow than for the incompressible flow since in the first case the greatest part of the drag is contributed by the wave resistance, which is independent of the aspect ratio.

The plane rectangular wing at supersonic flow is one with constant center of pressure position, if the frictional drag is disregarded. The position of the center of pressure depends only to a slight extent on the re-

duced aspect ratio $\lambda = \frac{t \tan \alpha}{b}$. For the infinitely long

wing, the center of pressure lies at the midchord position and with decreasing aspect ratio it moves forward somewhat (table III).

Table III

| $\frac{t \tan \alpha}{b}$ | 0 | 1/5 | 1/2 |
|---------------------------|---------------|-------|-------|
| $\frac{c_{mH}}{c_a}$ | $\frac{1}{2}$ | 0.489 | 0.469 |

Formula (80) for the rectangular flat plate is the analogy to the familiar $c_{w1} = c_y^2 F/\pi b^2$ of the incompressible flow. Like the latter it enables the recomputation of the drag from one reduced aspect ratio

$\lambda_1 = \frac{t_1 \tan \alpha_1}{b_1}$ to another $\lambda_2 = \frac{t_2 \tan \alpha_2}{b_2}$. From equations (73) and (80) there is obtained for the new angle of attack and the drag

$$\left. \begin{aligned} \beta_2 &= \beta_1 + \frac{c_a}{4} \left\{ \frac{1}{\tan \alpha_2 (1-0.316 \lambda_2)} - \frac{1}{\tan \alpha_1 (1-0.316 \lambda_1)} \right\} \\ c_{w2} &= c_{w1} + \frac{c_a^2}{4} \left\{ \frac{1}{\tan \alpha_2 (1-0.316 \lambda_2)} - \frac{1}{\tan \alpha_1 (1-0.316 \lambda_1)} \right\} \end{aligned} \right\} \quad (81a)$$

VII. TRAPEZOIDAL LIFT DISTRIBUTION

a) Lifting Line

As a further example we now compute the induced drag and the velocity field for trapezoidal lift distribution for both the lifting line and the lifting surface (fig. 17). Let the lift distribution therefore be given by

$$\left. \begin{aligned} \frac{\Gamma(\eta')}{\Gamma_0} &= \frac{1 - \eta'}{1 - \eta_1} \quad \text{for } \eta_1 \leq \eta' \leq 1 \\ &\quad - 1 \leq \eta' \leq -\eta_1 \\ \Gamma(\eta') &= \Gamma_0 \quad \text{for } -\eta_1 \leq \eta \leq +\eta_1 \end{aligned} \right\} \quad (82)$$

where $\eta_1 = b'/b$, according to figure 17. The field of the induced velocities and induced drag for variable lift distribution may be obtained in the familiar manner from the lift distribution by superposition. On account of integration difficulties, however, this computation can not directly be made on the potential but must be carried out separately for the three velocity components. From equation (14c) we have for the induced downwash velocity of a lifting line ending at $\eta = \eta'$ with circulation Γ_0 :

$$c_z(\eta') = \frac{\Gamma_0}{\pi b} \frac{\xi(\eta - \eta') \left\{ \xi^2 - \kappa^2 [(\eta - \eta')^2 + 2\xi^2] \right\}}{(\xi^2 - \kappa^2 \xi^2) [(\eta - \eta')^2 + \xi^2] \sqrt{\xi^2 - \kappa^2 [(\eta - \eta')^2 + \xi^2]}} \quad (83)$$

From the above there is obtained by superposition the downwash velocity c_z for variable circulation $\Gamma(\eta')$:

$$c_z = -\frac{1}{\Gamma_0} \int_{\eta'=-1}^{\eta'=+1} c_z(\eta') \frac{d\Gamma}{d\eta'} d\eta' \quad (84)$$

For the trapezoidal lift distribution according to equation (82) we have therefore if, on account of symmetry, we restrict ourselves to the half-wing $y > 0$

$$c_z = \frac{1}{1 - \eta_1} \int_{\eta'=\eta_1}^{\eta'=1} c_z(\eta') d\eta'$$

or, according to equation (83)

$$\pi \frac{b c_z}{\Gamma_0} = \frac{1}{1 - \eta_1} \int_{\eta'=\eta_1}^{\eta'=1} \frac{\xi(\eta - \eta') \left\{ \xi^2 - \kappa^2 [(\eta' - \eta)^2 + 2\xi^2] \right\} d\eta'}{(\xi^2 - \kappa^2 \xi^2) [(\eta' - \eta)^2 + \xi^2] \sqrt{\xi^2 - \kappa^2 [(\eta' - \eta)^2 + \xi^2]}}$$

or with

$$\kappa^2 (\eta' - \eta)^2 + \xi^2 = \tau$$

$$\pi \frac{b c_z}{\Gamma_0} = -\frac{1/2}{1 - \eta_1} \int_{\tau_1}^{\tau_2} \frac{d\tau}{\tau \sqrt{\xi^2 - \tau}} + \frac{1/2}{1 - \eta_1} \frac{\xi}{\xi^2 - \kappa \xi^2} \int_{\tau_1}^{\tau_2} \frac{d\tau}{\sqrt{\xi^2 - \tau}}$$

Performing the integration there is obtained for points ξ, η, ξ within the Mach cone at the wing tip $\eta = 1$

$$\pi \frac{b c_z}{\Gamma_0} = \frac{1}{1 - \eta_1} \left\{ -\frac{\xi \sqrt{w}}{\xi^2 - \kappa^2 \xi^2} + \frac{1}{2} \log \frac{\xi + \sqrt{w}}{\xi - \sqrt{w}} \right\} \quad (85)$$

and a corresponding expression with reversed sign for the Mach cone at $\eta = \eta_1$. The value of w is here given by equation (12). In the cone $\eta = 1$, $c_z > 0$ so that there is upwash velocity. In the cone $\eta = \eta_1$ there is a downwash velocity of the same absolute magnitude ($c_z < 0$) and outside of the two cones $c_z = 0$, a result which is also to be expected from reasons of symmetry since, on account of $\frac{d\Gamma}{d\eta} = \text{const.}$, all separating vortices are of the same strength. With

$$\vartheta = \frac{\kappa(\eta - 1)}{\xi} \quad \text{and} \quad = \frac{\kappa(\eta - \eta_1)}{\xi}$$

there is obtained for the downwash distribution in cone I and III respectively in the plane $z = 0$.

$$\pi \frac{b c_z}{\Gamma_0} = \pm \frac{1}{1 - \eta_1} \left\{ -\sqrt{1 - \vartheta^2} + \frac{1}{2} \log \frac{1 + \sqrt{1 - \vartheta^2}}{1 - \sqrt{1 - \vartheta^2}} \right\} \quad (86)$$

On the cone surface according to equations (85) and (86) $c_z = 0$ and is therefore continuous in passing through the cone. On the cone axis c_z now becomes logarithmically infinite, whereas with the rectangular lift distribution (horseshoe vortex) c_z becomes infinite on the axis as r^{-1} . The logarithmic singularity of c_z is no longer a disturbing factor for the computation of the induced drag.

For the sake of completeness there will also be given the remaining two components of the induced velocity. There is found for the cone at $\eta = 1$:

$$\left. \begin{aligned} \pi \frac{b c_x}{\Gamma_0} &= \frac{1}{1 - \eta_1} \frac{\xi \sqrt{w}}{\xi^2 - \kappa \xi^2} \\ \pi \frac{b c_y}{\Gamma_0} &= \frac{-1}{1 - \eta_1} \arctan \frac{\xi \sqrt{w}}{\xi(\eta - 1)} \end{aligned} \right\} \quad (87)$$

and corresponding expressions with reversed signs for the cone at $\eta = \eta_1$. For the arc tan there is to be taken the principal value $0 \leq \text{arc tan} \leq \pi$. For the outer cone ($\eta = 1$) the arc tan is zero in the upper half plane on the outer quadrants of the cone surface and equal to $+\pi$ on the inner quadrants. In the wedge-shaped space between the two cones c_y is constant, being equal to

$$\xi \geq 0: c_y = \mp \frac{\Gamma_0}{b - b'}; \quad c_x = 0 \quad (88)$$

In passing through the plane $\xi = 0$, therefore there is a discontinuous increment in c_y by

$2 \frac{\Gamma_0}{b - b'}$. The region of the ξ plane limited by the cone axes $\eta = \eta_1$ and $\eta = 1$ (distance $\frac{b - b'}{2}$) is thus a vor-

tex surface with constant circulation density the total circulation of which is equal to the circulation Γ_0 of the bound vortex in the region of the constant lift.

A streamline picture of the y and z velocity components for a plane $x = \text{constant}$ that intersects both cones is drawn in figure 18. Like the streamline picture for the constant lift distribution (fig. 2) it was obtained by computing the field of isoclines. On the outer halves of the cone surfaces c_y and c_z are equal to zero but the directions of the streamlines c_z/c_y have a value different from zero. In this case, too, not all streamlines are closed, part of the streamlines entering from the undisturbed region into the one cone and coming out from the other again into the undisturbed region.

b) Lifting Surface

In order to compute the induced drag for the trapezoid-shaped lift distribution, we must, as in section IV, make the transition from the lifting line to the lifting surface. A rectangular lifting surface will therefore now be assumed of span b and extending from $x = 0$ to $x = t$. The chordwise circulation distribution is assumed to be constant of density Γ/t , while along the span the distribution is that given by equation (82). For the computation we may here restrict ourselves to the region be-

tween the cones springing from the leading and trailing edges of the lifting surface, since only this region enters into the question of the computation of the induced drag. We likewise need carry out the computation only for the cone at $\eta = 1$; for the downwash in cone $\eta = \eta_1$ there is obtained the corresponding expression with reversed sign.

For the induced z component c_z of the lifting surface, there is found, according to equation (85), with $\gamma_0 = \Gamma_0/t$:

$$2\pi \frac{\bar{c}_z}{\gamma_0} = - \frac{1}{1 - \eta_1} \int_{\xi'=0}^{\xi'=\xi_1} \frac{(\xi - \xi') \sqrt{(\xi - \xi')^2 - \kappa [(\eta - 1)^2 + \zeta^2]}}{(\xi - \xi')^2 - \kappa \zeta^2} d\xi'$$

$$+ \frac{1/2}{1 - \eta_1} \int_{\xi'=0}^{\xi'=\xi_1} \log \frac{\xi - \xi' + \sqrt{(\xi - \xi')^2 - \kappa^2 [\dots]}}{\xi - \xi' - \sqrt{(\xi - \xi')^2 - \kappa^2 [\dots]}} d\xi'$$

where

$$\xi_1' = \xi - \kappa \sqrt{(\eta - 1)^2 + \zeta^2}$$

according to equation (31). With the new integration variables $\xi - \xi' = \xi^*$, the above equation becomes

$$2\pi \frac{\bar{c}_z}{\gamma_0} = \frac{1}{1 - \eta_1} \int_{\xi^*=\xi}^{\xi_1^*} \frac{\xi^* \sqrt{\xi^{*2} - \kappa^2 [(\eta - 1)^2 + \zeta^2]}}{\xi^{*2} - \kappa^2 \zeta^2} d\xi^*$$

$$- \frac{1/2}{1 - \eta_1} \int_{\xi^*=\xi}^{\xi_1^*} \log (\xi^* + \sqrt{\xi^{*2} - \kappa^2 [\dots]}) d\xi^*$$

$$+ \frac{1/2}{1 - \eta_1} \int_{\xi^*=\xi}^{\xi_1^*} \log (\xi^* - \sqrt{\xi^{*2} - \kappa^2 [\dots]}) d\xi^*$$

where

C

$$\xi_1^* = \kappa \sqrt{(\eta - 1)^2 + \zeta^2}$$

The three integrals are evaluated as follows:

Setting $\xi^{*2} - \kappa^2 \zeta^2 = \tau$, we have for J_1

$$J_1 = \frac{1/2}{1 - \eta_1} \int_{\tau_1}^{\tau_2} \frac{\sqrt{\tau - \kappa^2(\eta - 1)^2}}{\tau} d\tau$$

$$J_1 = \frac{-1}{1 - \eta_1} \left\{ \sqrt{w} - \kappa(\eta - 1) \arctan \frac{\sqrt{w}}{\kappa(\eta - 1)} \right\}$$

With

$$(\eta - 1)^2 + \zeta^2 = a_1^2$$

and

$$\xi^* + \sqrt{\xi^{*2} - \kappa^2 a_1^2} = \tau$$

there is obtained for J_2

$$J_2 = - \frac{1}{4} \frac{1}{1 - \eta_1} \int_{\tau_1}^{\tau_2} \frac{\tau^2 - \kappa^2 a_1^2}{\tau^2} \log \tau d\tau$$

After a brief intermediate computation we have

$$J_2 = - \frac{1/2}{1 - \eta_1} \left\{ \kappa[(1 - \eta)^2 + \zeta^2] \log \left(\kappa \sqrt{(1 - \eta)^2 + \zeta^2} \right) \right. \\ \left. - \xi \log (\xi + \sqrt{w}) + \sqrt{w} \right\}$$

and similarly

$$J_3 = + \frac{1/2}{1 - \eta_1} \left\{ \kappa[(1 - \eta)^2 + \zeta^2] \log \left(\kappa \sqrt{(1 - \eta)^2 + \zeta^2} \right) \right. \\ \left. - \xi \log (\xi - \sqrt{w}) + \sqrt{w} \right\}$$

By adding we obtain

$$2 \pi \frac{\bar{c}_z}{\gamma_0} = \frac{1}{1 - \eta_1} \left\{ -\sqrt{\omega} + \kappa(\eta - 1) \arctan \frac{\sqrt{\omega}}{\kappa(\eta - 1)} + \frac{1}{2} \xi \log \frac{\xi + \sqrt{\omega}}{\xi - \sqrt{\omega}} \right\} \quad (89)$$

A corresponding expression with opposite sign is obtained for the cone $\eta = \eta_1$. The arc tan in equation (89) lies within the range $-\frac{\pi}{2} \leq \arctan \leq +\frac{\pi}{2}$ as follows from the fact that \bar{c}_z must be symmetrical in $(\eta - 1)$ since the same holds for c_z according to equation (85).

The induced z component thus found for the rectangular lifting surface with trapezoidal lift distribution has the same singularities as the corresponding formula (85) for the lifting line. On the cone surface $\bar{c}_z \equiv 0$ and on the cone axis logarithmically infinite. For the downwash distribution at the location of the wing in the plane $\xi = 0$, there is obtained

$$2 \pi \frac{\bar{c}_{z0}}{\gamma^2} = \pm \frac{\xi}{1 - \eta_1} \left\{ -\sqrt{1 - \vartheta^2} + \vartheta \arctan \frac{\sqrt{1 - \vartheta^2}}{\vartheta} + \frac{1}{2} \log \frac{1 + \sqrt{1 - \vartheta^2}}{1 - \sqrt{1 - \vartheta^2}} \right\} = \frac{\pm \xi}{1 - \eta_1} g(\vartheta) \quad (90)$$

where $\vartheta = \frac{\kappa(\eta - 1)}{\xi}$ for cone I and $\vartheta = \frac{\kappa(\eta - \eta_1)}{\xi}$ for

cone III, the upper sign holding for cone I and the lower for cone III. Equations (89) and (90) include only the downwash velocity induced by the edge vortices. In order to obtain the field of the total downwash motion, there is still to be added the induced downwash velocity due to the plane wave. In the wedge-shaped space between the leading and forward edges of the wing (fig. 4), this induced velocity component is

$$c_{z0\text{wave}} = -\frac{\kappa}{2} \frac{\Gamma}{t} = -\frac{\kappa}{2} \gamma_0 \frac{1-\eta}{1-\eta_1} = \gamma_0 \frac{1/2}{1-\eta_1} \xi \vartheta \quad (91)$$

For the total downwash velocity in the plane $z = 0$, there is thus obtained from equations (90) and (91)

For cone I:

$$-1 < \vartheta < 0: \frac{2(1-\eta_1)}{\gamma_0} \bar{c}_{z0} = \xi \left\{ \vartheta + \frac{\xi(\vartheta)}{\pi} \right\}$$

$$0 < \vartheta < +1: = \xi \frac{\xi(\vartheta)}{\pi}$$

For cone III:

$$-1 < \vartheta < 0: \frac{2(1-\eta_1)}{\gamma_0} \bar{c}_{z0} = \xi \left\{ -1 - \frac{\xi(\vartheta)}{\pi} \right\}$$

$$0 < \vartheta < +1: = \xi \left\{ \vartheta - \frac{\xi(\vartheta)}{\pi} \right\}$$

(92)

The downwash distribution thus computed is plotted in figure 19.

We are now in a position to compute, for the wing with trapezoidal lift distribution, the induced drag. In order to avoid special complications, we shall assume that the Mach cone springing from the leading edge at $\eta = \eta_1$ does not extend beyond the wing tip and does not overlap the region of dropping circulation of the other half-wing. The first is identical with the condition that the cone springing from $\eta = 1$ does not extend into the region of the wing where the circulation is constant. This gives for the Mach angle the two conditions

$$\tan \alpha \leq \frac{b-b'}{2t} \quad \text{and} \quad \tan \alpha \leq \frac{b'}{t}$$

The induced drag of one half-wing $\frac{1}{2} W_i$ is composed additively of the drag of half of cone I, W_{iI_1} and the

drags of the two half-cones of cone III, $W_{i_{III_1}}$ and $W_{i_{III_2}}$ (fig. 17).

$$\frac{1}{2} W_i = W_{i_{I_1}} + W_{i_{III_1}} + W_{i_{III_2}} \quad (93)$$

Since in cone I, in the plane $\zeta = 0$, there is upwash velocity, $W_{i_{I_1}}$ gives a forward thrust which in absolute value, however, is smaller than the back thrust in cone III, since the circulation is greater here. We have

$$W_{i_{I_1}} = \rho \int_{x=0}^{x=t} dx \int_{y=y_1}^{y=b/2} \frac{\Gamma}{t} \bar{c}_{z0} dy, \quad \left(y_1 = \frac{b}{2} - x \tan \alpha \right)$$

where \bar{c}_{z0} is known from equation (90). We thus have

$$W_{i_{I_1}} = \frac{1}{1 - \eta_1} \frac{\rho}{8\pi} \frac{\Gamma_0 b^2}{\rho t^2 \kappa} \int_{\xi=0}^{\xi=2t/b} \xi^2 d\xi \int_{\vartheta=-1}^{\vartheta=0} \Gamma(\vartheta) g(\vartheta) d\vartheta \quad (94)$$

In cone I for $-1 < \vartheta < 0$:

$$\Gamma = - \Gamma_0 \frac{\xi \vartheta}{1 - \eta_1} \kappa^{-1}$$

and therefore

$$W_{i_{I_1}} = - \frac{1}{(1 - \eta_1)^2} \frac{\rho \Gamma_0^2}{8\pi} \left(\frac{b}{\kappa t} \right)^2 \int_{\xi=0}^{\xi=2t/b} \xi^3 d\xi \int_{\vartheta=-1}^{\vartheta=0} \vartheta g(\vartheta) d\vartheta$$

For brevity we set

$$\int_{\vartheta=-1}^{\vartheta=0} g(\vartheta) d\vartheta = \int_{\vartheta=0}^{\vartheta=1} g(\vartheta) d\vartheta = K_3 \quad (95a)$$

$$-\int_{\vartheta=-1}^0 \vartheta g(\vartheta) d\vartheta = \int_{\vartheta=0}^{+1} \vartheta g(\vartheta) d\vartheta = K_2 \quad (95b)$$

These integrals may be exactly computed. There is obtained

$$K_3 = \frac{3\pi}{8} \quad K_2 = \frac{7}{18} \quad (96)$$

so that finally

$$W_{iI_1} = \frac{K_2}{(1-\eta_1)^2} \frac{\rho \Gamma_0^2}{2\pi} \left(\frac{t}{\kappa b} \right)^2 \quad (97)$$

The portion W_{iIII_1} is obtained from equation (94) by substituting $-g(\vartheta)$ for $g(\vartheta)$ and taking $\Gamma = \Gamma_0$ so that

$$W_{iIII_1} = -\frac{1}{3} \frac{K_3}{1-\eta_1} \frac{\rho \Gamma_0^2}{\pi} \frac{t}{\kappa b} \quad (98)$$

Finally, W_{iIII_2} is obtained by putting in equation (94)

$$\Gamma = \Gamma_0 \left(1 - \frac{\kappa^{-1} \xi}{1-\eta_1} \vartheta \right)$$

and substituting $-g(\vartheta)$ for $g(\vartheta)$. By comparison with equations (97) and (98) this gives

$$W_{iIII_2} = W_{iI_1} + W_{iIII_1}$$

and therefore

$$\frac{1}{2} W_i = 2 W_{iIII_1} + 2 W_{iI_1} = 2 W_{iIII_1} \left\{ 1 + \frac{W_{iI_1}}{W_{iIII_1}} \right\}$$

Substituting the values from (97) and (98) the induced drag of the entire wing is found to be

$$\bar{W}_i = -\frac{4}{3} \frac{K_3}{1 - \eta_1} \frac{\rho \Gamma_0^2}{\pi} \frac{t \tan \alpha}{b} \left\{ 1 - \frac{1}{1 - \eta_1} \frac{3 K_2}{2 K_3} \frac{t \tan \alpha}{b} \right\} \quad (99)$$

If, in place of Γ_0 , there is now substituted the lift A of the entire wing

$$A = \rho b \Gamma_0 u_0 \frac{1 + \eta_1}{2}$$

we have

$$\bar{W}_i = -\frac{16 K_3}{3 \pi} \frac{1}{(1 - \eta_1)(1 + \eta_1)^2} \frac{1}{\rho} \left(\frac{A}{b u_0} \right)^2 \frac{t \tan \alpha}{b}$$

$$\left\{ 1 - \frac{1}{1 - \eta_1} \frac{3 K_2}{2 K_3} \frac{t \tan \alpha}{b} \right\}$$

$$\bar{W}_i = -\frac{-2}{(1 - \eta_1)(1 + \eta_1)^2} \frac{1}{\rho} \left(\frac{A}{u_0 b} \right)^2 \frac{t \tan \alpha}{b}$$

$$\left\{ 1 - \frac{1}{1 - \eta_2} \frac{14}{9 \pi} \frac{t \tan \alpha}{b} \right\} \quad (100)$$

Thus the formula has been found for the induced drag with trapezoidal lift distribution. To this must be added the wave drag. The latter according to equation (26) and table I is

$$\bar{W}_{\text{wave}} = \frac{2(2 + \eta_1)}{3(1 + \eta_1)^2} \frac{b}{t \tan \alpha} \frac{1}{\rho} \left(\frac{A}{u_0 b} \right)^2 \quad (101)$$

If c_w denotes the coefficient of the wave plus induced drag then from equations (100) and (101)

$$c_w / \frac{c_a^2}{4 \tan \alpha} = \frac{4(2 + \eta_1)}{3(1 + \eta_1)^2} + \frac{4 \lambda^2}{(1 - \eta_1)(1 + \eta_1)^2} \left(1 - \frac{1}{1 - \eta_1} \frac{14}{9 \pi} \lambda \right) \quad (102)$$

The above formula differs from the corresponding formulas for the rectangular flat plate (equation (80)) and the trapezoidal wing with constant lift distribution (equation (49)) in that for small λ the induced portion of the drag is proportional to λ^2 whereas for the other two cases it is proportional to λ . In figure 20 the coefficient

$c_w / \frac{c_a^2}{4 \tan \alpha}$ is plotted against the reduced aspect ratio

$\frac{t \tan \alpha}{b} = \lambda$ for various trapezoid shapes b'/b . It may

be seen that by far the greatest portion of the drag is contributed by the wave resistance. The portion contributed by the induced drag, within the range of validity of our formulas, amounts to a maximum of 11 percent of the wave resistance for $\lambda = 0.5$ and $b'/b = 1/2$. It is therefore smaller than for the rectangular flat plate where for the same aspect ratio it amounts to 19 percent (fig. 10).

VIII. SUMMARY

With the aid of the expressions given by L. Prandtl (reference 2) a theory is developed of the airfoil of finite span at supersonic speed. As in the case of the Prandtl airfoil theory for the incompressible flow, it is a first order approximation theory. The airfoil is first replaced by a "horseshoe vortex" and the induced velocity field of the latter computed. This field is considerably different from that of the incompressible flow. From the horseshoe vortex there are obtained in the familiar manner by superposition more complicated lifting systems. The computation of the induced drag, in contrast to the incompressible case, is for the compressible flow possible only if there is first assumed a surface vortex distribution and secondly a suitable dropping off of the lift toward the wing tips.

As an example of the "first principal problem" there are computed the induced drag and the wing surface shape for a wing of trapezoidal plan form with constant surface density of the lift. The induced drag, as in the case of the incompressible flow, is found to be proportional to the square of the lift and depends on the Mach number as well as on the aspect ratio. In addition to the frictional and induced drag there is present in the supersonic case also the wave drag, produced by the sound waves, which

varies as the induced drag. It is therefore only the sum of the wave and induced drags that is of practical interest.

As an example of the "second principal problem" there is computed the lift distribution and induced drag for the rectangular flat plate (untwisted rectangular wing). Outside the two Mach cones springing from the leading edges of the wing tips the lift density is constant; within these cones the lift drops from the full value at the cone rim to the value zero at the lateral wing edge. The integral equation that arises is independent of the aspect ratio and of the Mach number and may be solved numerically by approximate methods. In general for airfoils of normal aspect ratios at supersonic flows the greatest portion of the total drag is contributed by the wave resistance while the induced drag contributes only a small proportional part.

Finally, there is considered the lifting line with trapezoidal lift distribution and the lifting surface of rectangular plan form whose lift is constant along the chord and trapezoidal along the span. For these cases the downwash distribution and induced drag are computed.

Translation by S. Reiss,
National Advisory Committee
for Aeronautics.

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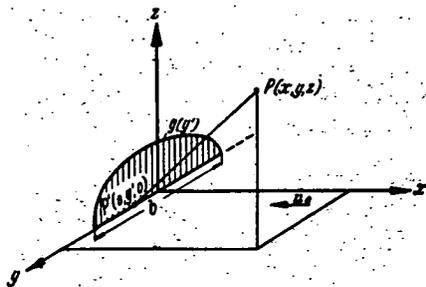


Figure 1.- Potential of the lifting line.

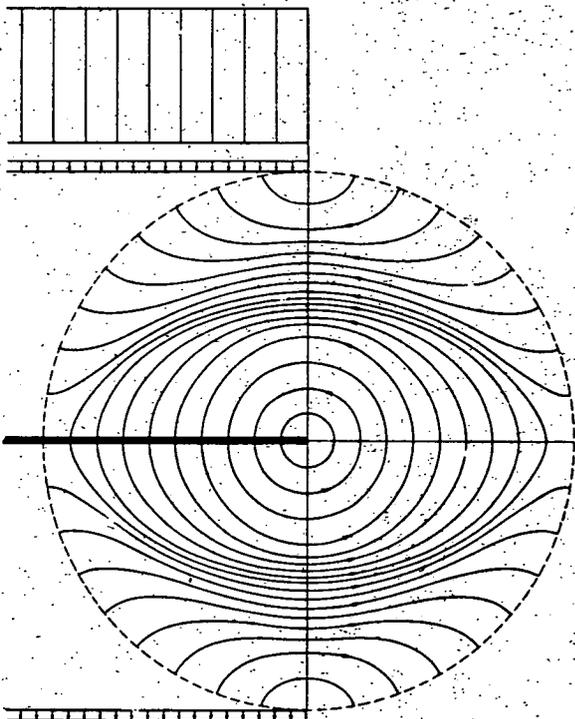


Figure 2.- Lifting line with constant lift distribution (horseshoe vortex). Streamline picture of the y- and z- velocities in a plane at right angles to the axis of the Mach cone.

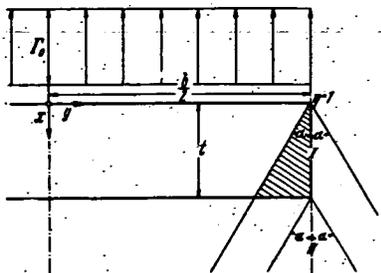


Figure 5.- Rectangular wing as lifting surface with constant lift distribution.

Figure 3.- Lifting line with constant lift distribution. Downwash distribution in Mach cone.

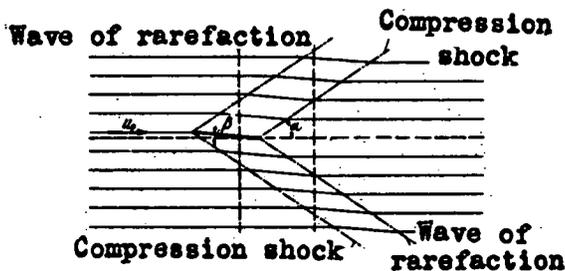
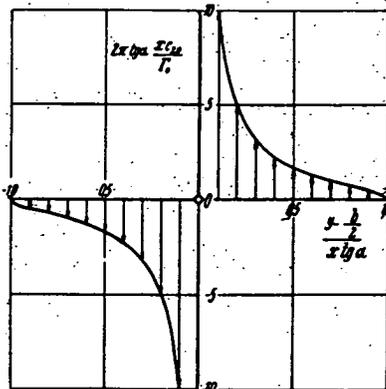


Figure 4.- Plane sound waves at a flat plate.

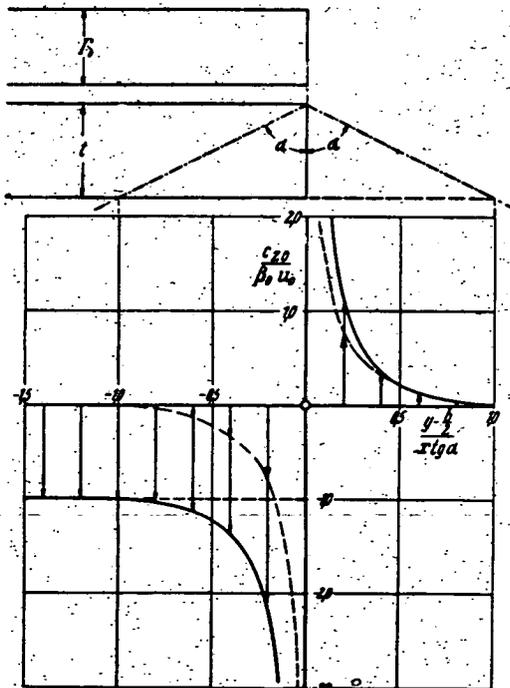


Figure 6.- Rectangular wing as lifting surface with constant lift distribution. Downwash distribution in the wing plane. Continuous curves for $x < t$ (at location of wing) dotted curves for $x = 2t$ (behind the wing).

C

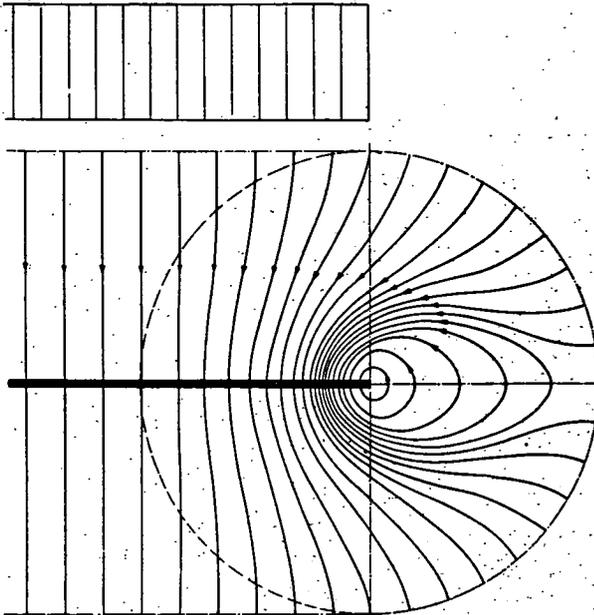


Figure 7.- Rectangular wing as lifting surface with constant lift distribution. Streamline picture of the y- and z-velocities in a plane $x < t$ at right angles to the axis of the Mach cone.

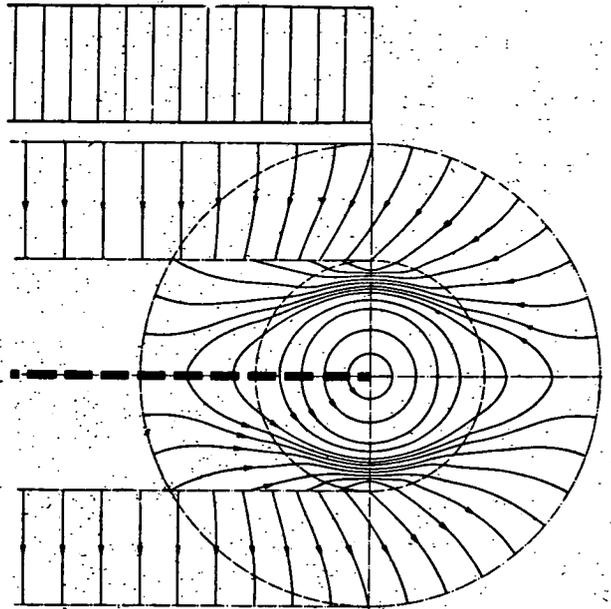


Figure 8.- Rectangular wing as lifting surface with constant lift distribution. Streamline picture of the y- and z-velocity components in the plane $x = 2t$ at right angles to the axis of the Mach cone.

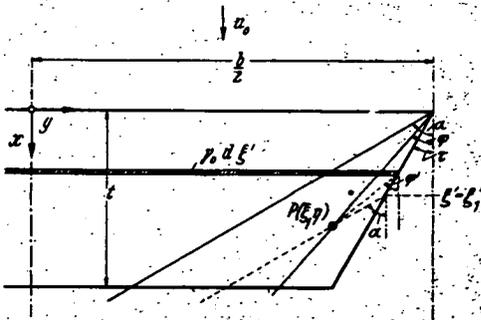


Figure 9.- Trapezoidal wing with constant lift distribution.

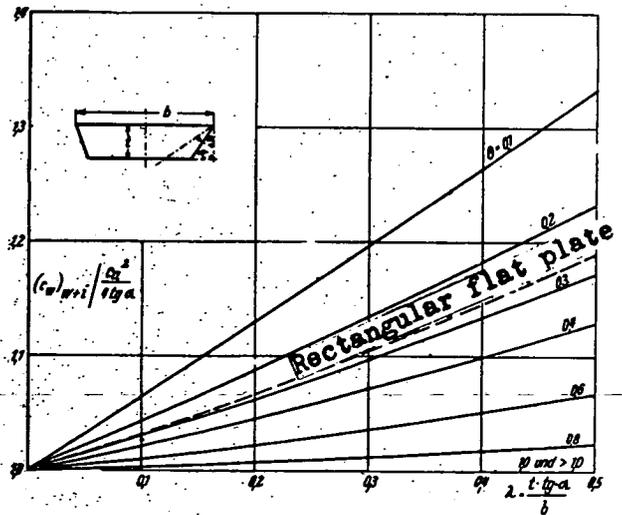


Figure 10.- Trapezoidal wing with constant lift distribution.

Coefficients of the wave plus induced drag $c_w / 4 \tan^2 \alpha$ as a function of the "reduced aspect ratio" $\lambda = t \tan \delta / b$ for various trapezoid shapes $\theta = \tan \gamma / \tan \delta$.

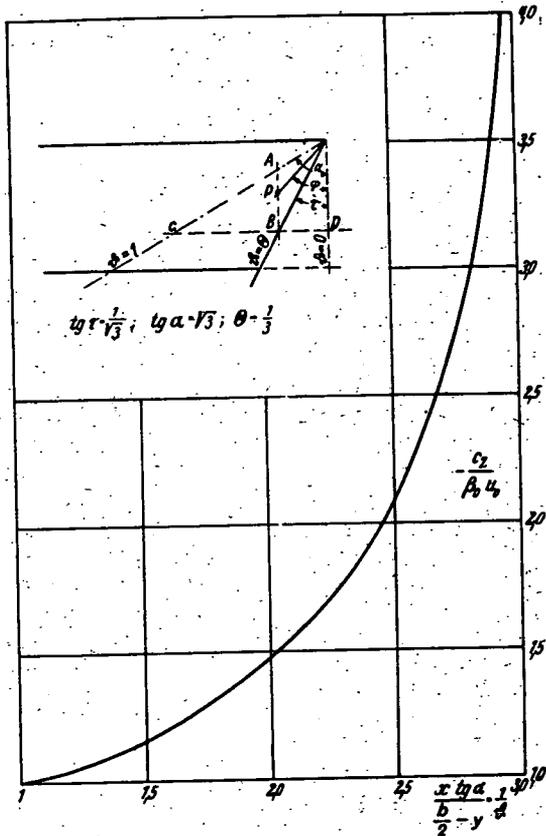


Figure 11.- Trapezoidal wing with constant lift distribution. Induced downwash velocity in section AB (in direction of flow) ($\tan \tau = 1/\sqrt{3}$; $\tan \alpha = \sqrt{3}$).

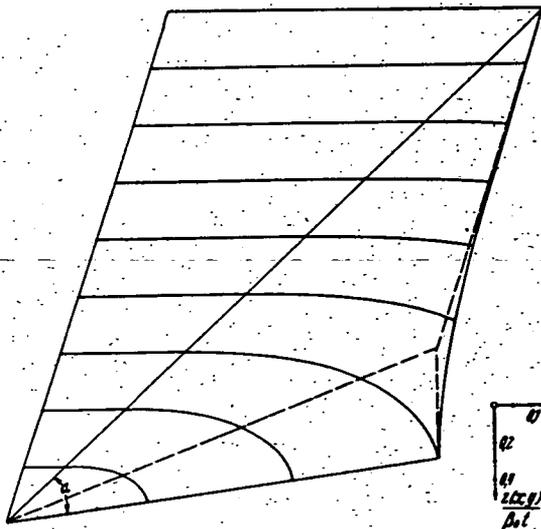


Figure 13.- Trapezoidal wing with constant lift distribution. Profile sections. ($\tan \alpha = \sqrt{3}$; $\tan \tau = 1/\sqrt{3}$).

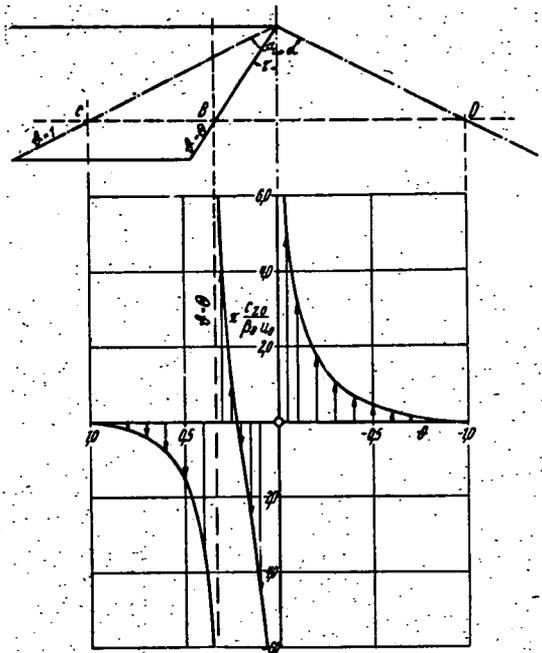


Figure 12.- Trapezoidal wing with constant lift distribution. Induced downwash velocity in section CD (at right angles to flow direction) ($\tan \tau = 1/\sqrt{3}$; $\tan \alpha = \sqrt{3}$).

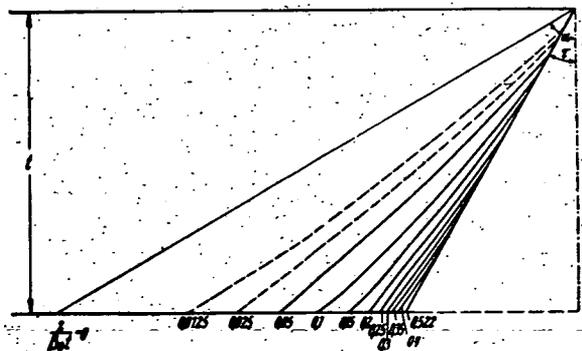


Figure 14.- Trapezoidal wing with constant lift distribution. Elevation contour lines. ($\tan \alpha = \sqrt{3}$; $\tan \tau = 1/\sqrt{3}$).

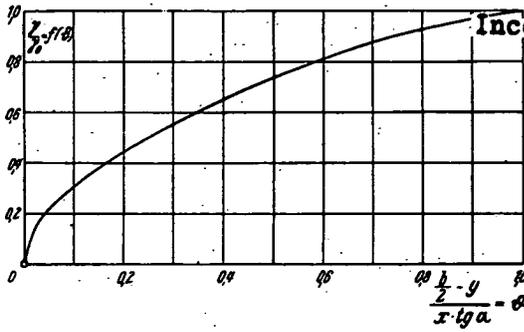


Figure 15.- Rectangular plane wing. Lift at wing edge.

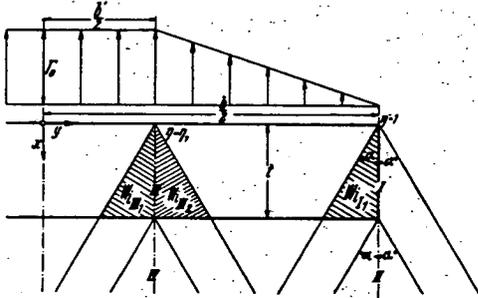


Figure 17.- Rectangular surface with trapezoidal lift distribution.

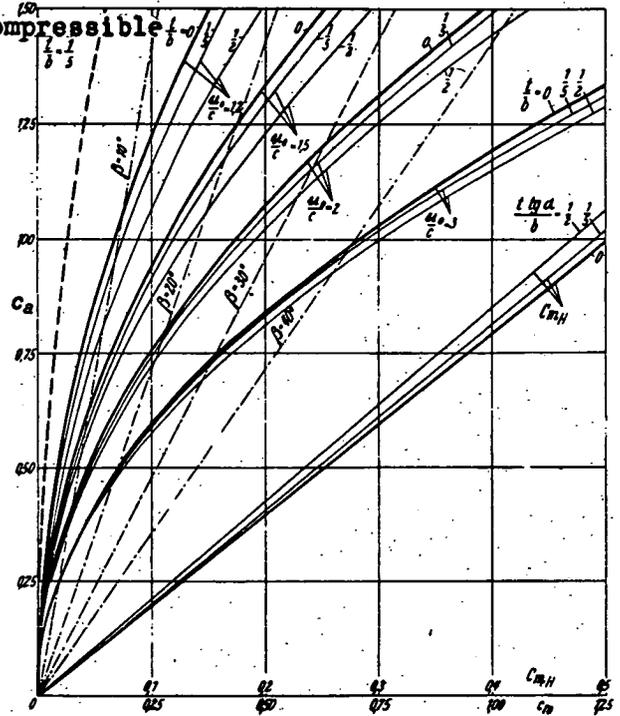


Figure 16.- Polars of plane rectangular wing for various aspect ratios.

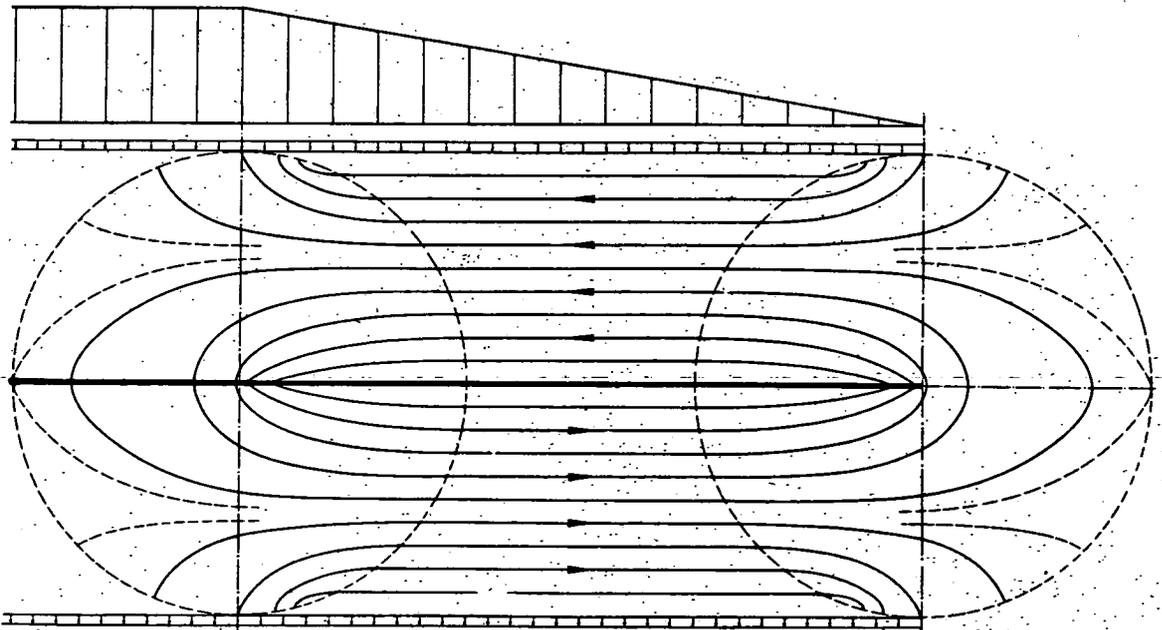


Figure 18.- Lifting line with trapezoidal lift distribution. Streamline picture of the y- and z- velocities in a plane at right angles to the axis of the Mach cone.

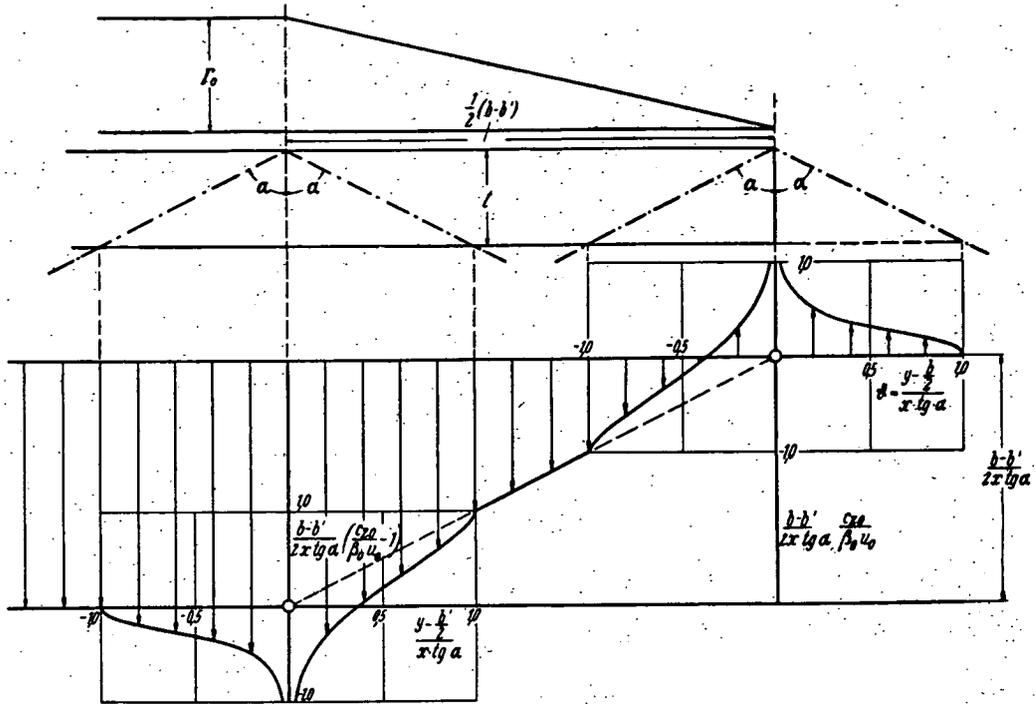


Figure 19.- Rectangular wing as lifting surface with trapezoidal lift distribution. Downwash distribution for $x < t$.

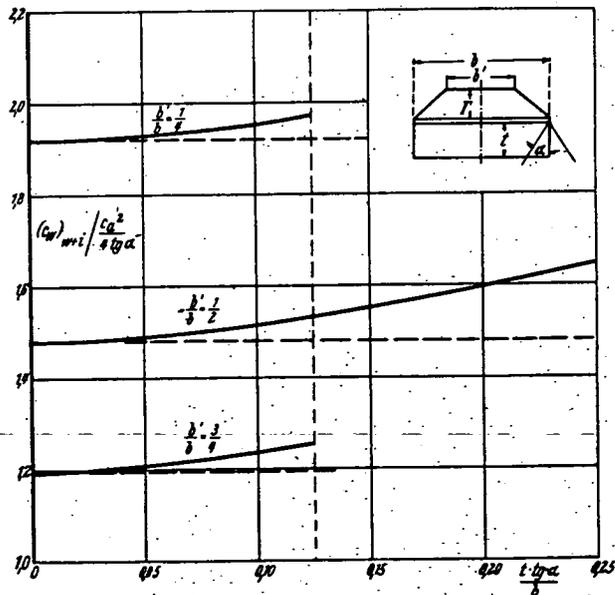


Figure 20.- Lifting surface with trapezoidal lift distribution. Coefficient of wave plus induced drag $c_w / \frac{c_a^2}{4 \tan \alpha}$ as a function of the "reduced aspect ratio" λ for various values of b'/b .