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ON THE THEORY OF HYDROFOILS AND PLANING SURFACES

By F. Weinig

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ON THE THEORY OF HYDROFOILS AND PLANING SURFACES*

By F. Weinig

The present article describes the application of the results of airfoil theory to hydrofoils and planing surfaces with consideration of the boundary conditions of the free upper surface.

1. BOUNDARY CONDITIONS

In the theoretical investigation of flows on a hydrofoil or on a planing surface, it is probably legitimate to restrict the treatment to their steady motion. The variations in running speeds are, after all, so inferior generally that even such nonsteady motions vary, on the whole, negligibly little from the steady motion.

Furthermore, the effects of friction and compressibility may also be disregarded for the present. For such ideal steady flows the pressure equation:

$$p + \frac{\rho}{2} w^2 + \gamma H = II = \text{constant} \quad (H \text{ positive upward})$$

is applicable. The air pressure at water level (subscript o) being $p_o = p_L = \text{constant}$ and because $\frac{II - p_a}{\gamma} = \text{const.}$, the boundary condition in this case reads:

$$\frac{w_o^2}{2g} + H_o = \text{constant}$$

This is the general boundary condition of the free water surface with allowance for gravity. Solutions with strict allowance for it are unknown. Exact compliance is, for example, important in the so-called overflow weirs in hydraulics.

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*"Zur Theorie des Unterwassertragflügels und der Gleitfläche." Luftfahrtforschung, vol. 14, no. 6, June 20, 1937, pp. 314-324.

Within small finite changes in height ΔH_0 and speed Δw_0 along the surface, the simplification

$$\Delta w_0 = - \frac{g \Delta H_0}{w_0} \quad \text{or} \quad w_{0_2} - w_{0_1} = g \frac{H_{0_1} - H_{0_2}}{w_{0_1}}$$

holds true.

This is the boundary condition of the free water surface with consideration to gravity at small speed changes along the surface. It can, in general, be prescribed in those cases rather than the exact boundary condition, where the inclination of the free water surface may be considered as being small, at least over greater zones, as for instance, in the case of wave-making resistance of ships. And for planing surfaces and hydrofoils, the simplification can frequently be extended even farther. That is to say, if the speed is very great, height changes ΔH_0, if not too great, produce only vanishingly minute changes in w_0 along the surface. We therefore confine ourselves for the time being to such high speeds that the speed along the water surface can be considered constant and then the boundary condition for the free surface becomes:

$$w_0 = \text{constant}$$

This is the boundary condition of the free water surface with gravity effect not taken into account; i.e., to prescribe it, is to ignore the effect of gravity on the flow process. This boundary condition is identical with that on the boundary of a free jet relative to a still, dead water, which already could be taken into account in many problems of hydrodynamics.

But for many investigations it is admissible and therefore expedient to carry the simplifications a step farther. In the case of minor disturbances, the departure of the shape of the free surface from the undisturbed surface may also be overlooked without incurring inadmissibly great errors.

In this case the velocity components w_x in motion direction x must be equal to the velocity at infinity w_∞, and the admitted small disturbance velocities $\Delta \underline{w} = \underline{w} - \underline{w}_\infty$ must be perpendicular to the undisturbed free surface, so that

$$w_x = w_\infty$$

$$\Delta w \perp w_\infty$$

becomes the boundary condition of the free water surface with gravity disregarded for small disturbances.

This condition is the same as that on the boundary of an assumedly slightly disturbed free jet, as used, for example, as basis for investigations of the effect of the finite dimensions of a wind-tunnel jet on airfoil measurements.

2. THE PLANE PROBLEM OF THE HYDROFOIL AND OF THE PLANING SURFACE

We first treat the plane problem of the hydrofoil - i.e., we consider its span as being very great, while bearing in mind that the flow around an airfoil in unlimited fluid can be obtained by assuming at the airfoil boundary a corresponding vortex distribution and then superposing on the ensuing interference flow - the absolute flow - the parallel flow corresponding to the uniform motion - the transport flow. The flow resulting therefrom - the relative flow - then contains the airfoil as streamline. Before a similar method can be applied to the hydrofoil itself, compliance with the boundary conditions on one single bar vortex (fig. 1) at $\underline{z} = -ih$, that is, in the abscissa $x = 0$ and at the depth $y = -h$ below the water level, must be attempted. If this depth h is not too shallow, and the circulation Γ of the vortex is not too great, the disturbances created thereby on the water surface may be considered as being small, and we need only provide for their perpendicularity to the undisturbed water level.

This, however, is accomplished at once (fig. 2) by reflection of this vortex through a similarly rotating vortex of the same circulation on the undisturbed surface - i.e., by placing a vortex of circulation Γ in the image point $\underline{z} = + ih$. The interference velocities produced by these two vortices are perpendicular to the water surface, except for a point where the interference velocity disappears altogether, so that in its vicinity the boundary conditions are more than ever complied with.

But the reflected vortex (fig. 3) creates at $z = + ih$ in place of the initial vortex at $z = - ih$ a velocity

induced velocity at $z = +ih$ due to vortex at $z = -ih$

$$w_{zus} = \frac{\Gamma}{2\pi} \frac{1}{2h}$$

which, while preserving the relative velocity of this vortex in direction, modifies it in magnitude

$$w = w_{\infty} - w_{zus}$$

This diminishes the lift A of the lower vortex filament in relation to the lift A_{∞} of the same vortex filament in infinite flow. This lift referred to span b is, according to Joukowski's theorem:

$$A = \rho \, \Gamma \, w \, b$$

whence

$$A = \rho \, \Gamma \, w_{\infty} b \left(1 - \frac{w_{zus}}{w_{\infty}}\right) \text{ against } A_{\infty} = \rho \, \Gamma_{\infty} \, w_{\infty} \, b$$

If the wing chord t is small in relation to the depth of immersion h, the effect of the reflected vortices uniformly distributed over the reflected surface of the profile may be visualized as being replaced by a single vortex of the same total circulation in the reflection point of the aerodynamic center of the hydrofoil which, meanwhile, may itself be visualized as being exchanged for a single vortex in its aerodynamic center. As the circulation for equal airfoil and equal effective trim is proportional to the relative flow velocity, we have:

$$\Gamma = \Gamma_{\infty} \left(1 - \frac{w_{zus}}{w_{\infty}}\right) \text{ and hence } A = \rho \, \Gamma_{\infty} \, w_{\infty} \, b \left(1 - \frac{w_{zus}}{w_{\infty}}\right)^2$$

with

$$\frac{w_{zus}}{w_{\infty}} = \frac{\Gamma}{4\pi \, h \, w_{\infty}} = \frac{\Gamma_{\infty}}{4\pi \, h \, w_{\infty}} \left(1 - \frac{w_{zus}}{w_{\infty}}\right)$$

we have:

$$\frac{w_{zus}}{w_{\infty}} = \frac{\dfrac{\Gamma_{\infty}}{4\pi \, h \, w_{\infty}}}{1 + \dfrac{\Gamma_{\infty}}{4\pi \, h \, w_{\infty}}} = \frac{1}{1 + \dfrac{4\pi \, h \, w_{\infty}}{\Gamma_{\infty}}} = \frac{1}{1 + \dfrac{8\pi \, h}{t} \dfrac{\frac{\rho}{2} \, b \, t \, w_{\infty}^2}{A_{\infty}}}$$

and consequently,

$$A = \rho\,\Gamma_\infty\,w_\infty\,b\left(1 - \frac{1}{1 + \dfrac{4\pi\,h\,w_\infty}{\Gamma_\infty}}\right)^2 =$$

$$= A_\infty\left(1 - \frac{1 \quad c_a\,\alpha}{c_a\,\alpha \quad 1 + \dfrac{8\pi\,h}{t}\,\dfrac{\frac{\rho}{2}\,b\,t\,w_\infty^2}{A_\infty}}\right)^2$$

with $$\frac{A_\infty}{\frac{\rho}{2}\,w_\infty^2\,b\,t} = c_{a_\infty},\quad \frac{A}{\frac{\rho}{2}\,w_\infty^2\,b\,t} = c_a$$

the lift coefficient of the hydrofoil c_a then becomes in comparison with that of the airfoil in infinite flow c_{a_∞}

$$\frac{c_a}{c_{a_\infty}} = \left(1 - \frac{c_{a_\infty}}{c_{a_\infty} + \dfrac{8\pi\,h}{t}}\right)^2 \quad (h \gg t)$$

However, this approximation holds only for such t as are small compared to h. On approaching the free water surface $(h \rightarrow 0)$ and at small trim

$$\frac{c_a}{c_{a_\infty}} \rightarrow \left(\frac{1}{2}\right)\left(\frac{h}{t} \rightarrow 0\right)$$

$-2-\sqrt{2}$

$\dfrac{-2-\sqrt{2}}{4+4\sqrt{2}+2}$

becomes independent of c_a, as will be shown later. To allow for this (fig. 4), it should be quite permissible to write

$$\frac{c_a}{c_{a_\infty}} = \left(1 - \frac{c_{a_\infty}}{(2+\sqrt{2})\,c_{a_\infty} + \dfrac{8\pi\,h}{t}}\right)^2 = \left(1 - \frac{1}{2+\sqrt{2}}\right)^2 = 1 - \frac{2}{2+\sqrt{2}} + \frac{1}{2(3+2\sqrt{2})}$$

$$\frac{2+\sqrt{2}-2+1}{(2+\sqrt{2})^2} = \frac{1+\sqrt{2}}{2(3+2\sqrt{2})}$$

For great h/t in relation to c_{a_∞}, it results in no change in the original result, while for very small h/t, it probably suffices also, since it is precisely correct - at least in the extreme case $h = 0$. For greater trims $\dfrac{c_a}{c_{a_\infty}}$ becomes, of course, a little smaller, although it always remains positive.

Incidentally, it may be pointed out that the shape of the profile produced by superposition of the reflected vortices, is not the same as that of the initial profile, nor is the vortex distribution substituting for the hydrofoil, apart from the proportionality factor $\left(1 - \frac{w_{zus}}{w_\infty}\right)$, altogether the same as that on the same profile in infinite flow because, aside from the mean additional velocities, allowance would have to be made for their distribution and change in curvature due to the reflected flow as well. This, however, is without the scope of this paper, as it would essentially involve a representation of the theory of the two-dimensional biplane problem, to which the problem treated here is closely related. The position of the aerodynamic center, particularly, is changed by the additional disturbances. The two-dimensional biplane theory (reference 1) can also take into account, aside from changes in relative flow direction and angle of attack, acceleration and curvature of the flow induced by the wing as well as the change in acceleration and curvature along the streamlines.

Since, according to Joukowski's theorem, the lift is perpendicular to the direction of the relative wind, and it agrees in the two-dimensional problem with the direction of motion of the hydrofoil, the drag is zero: $W = 0$.

As the hydrofoil continues to shift toward the surface of the water, it becomes - on transition to the boundary of infinitely little immersion - a planing surface. Then, however, the disturbances are no longer negligible, and the boundary condition becomes: w_0 = constant.

So long as the trim is such that the free surface merges into the leading edge of the profile (fig. 5), the conditions, of course, are practically the same as on the lower surface of an airfoil in infinite flow; so that, since in this case the positive pressures on the lower surface contribute, in first approximation, just as much to the lift as the positive pressures on the upper surface,

$$\frac{c_a}{c_{a_\infty}} = \frac{1}{2} \quad (h \longrightarrow 0)$$

At greater trim (fig. 6), however, the flow branches out, exactly as on the lower surface of the profile in infinite flow. On the planing surface the upper portion of the water is flung off forward since a flow around the leading edge, as on the submerged airfoil, is no longer possible.

To this spray corresponds, according to the momentum theory on two-dimensional profiles, a resistance of the planing surface amounting to:

$$W = \text{in } d_W = \rho\, d w_\infty^2\ (1 + \cos\alpha)$$

and a corresponding lift of

$$A = W \cot\alpha = \rho\, d w_\infty^2\ (1 + \cos\alpha) \cot\alpha$$

One method of solving the two-dimensional problem of the planing surface with arbitrary camber and trim, is as follows: Visualize the plane of flow z transformed on an image plane ζ, that is, the planing surface profile assumedly of infinite length forward transformed on the lower half of the unit circle of the ζ plane $\zeta = e^{i\alpha}$. Further, let $\frac{dz}{d\zeta} = 1$ for the infinitely remote point in both planes, which is readily attainable by suitable selection of the scale.

The front part of the free surface of the z-plane is to merge into the part of the ξ axis $(\xi > + 1)$ lying before the semicircle, the rear part of the free surface of the z-plane, the flow-off line, in the part of the ξ axis $(\xi < - 1)$ lying behind the semicircle. The forward end point of the image semicircle is to correspond to the infinitely remote point of the spray. The point $\xi = + 1$ then must contain a sink which carries away from the true image flow - i.e., from the corresponding quadrant the quantity πE, if the planing speed is $w_\infty = 1$ and the spray depth $d = \pi E$. As the angle between the surface of the circle and the image of the free surface is $\pi/2$, the point $\zeta = + 1$ must contain a sink with the total absorption capacity of $Q = - 4\pi E$. Then, in order that the circle in the image flow may become streamline, the image center $\zeta = 0$ must contain a source $Q = + 2\pi E$. To the parallel flow itself corresponds a double source in image center, after which the equation of the image of the flow in the ζ plane reads:

$$\chi(\zeta) = \underbrace{\zeta}_{\text{Parallel flow}} + \underbrace{\frac{1}{\zeta}}_{\substack{\text{Double source}\\ \text{in } \zeta = 0}} + \underbrace{2E \ln(\zeta - 1)}_{\substack{\text{Sink at}\\ \zeta = +1\\ Q = +4\pi E}} - \underbrace{E \ln \zeta}_{\substack{\text{Sink at}\\ \zeta = 0\\ Q = +2\pi E}}$$

Assume that the stagnation points of this flow lie, aside from at $\zeta = -1$, the image of the trailing edge, at ζ_{st} the image of the branching-off point and, for reasons of symmetry, at $\bar{\zeta}_w$, the reflection point of the image of the forward branching-off point.

The flow in the z-plane is to follow the equation

$$\chi = \chi(z)$$

which, after differentiation and logarithmic calculation, gives:

$$\ln \frac{d\chi}{dz} = \ln \frac{w}{w_\infty} + i\nu$$

where w is the absolute value of the velocity at any point in the field of flow, w_∞ the planing speed, and ν the angle of the direction of the velocity to the negative x axis in clockwise direction.

Our boundary condition now demands that $\dfrac{w}{w_\infty} = 1$ on the free surface; or, in other words, that

$$\ln \frac{w}{w_\infty} = 0$$

become the boundary condition in the z-plane on the free surface. Since $\zeta = \zeta(z)$ after transformation of the z-plane in the ζ plane, $\ln \dfrac{\bar{w}}{w_\infty}$ can be considered as being a function of the complex ζ. On the image of the free surface $\xi > 1$, $\eta = 0$, and $\xi < -1$, $\eta = 0$

$$\ln \frac{w}{w_\infty} = 0$$

must also be the boundary condition in the ζ plane on the image of the free surface; i.e., on the ξ axis. It is further postulated that

$$\ln \frac{\overline{w}}{\overline{w}_\infty}(\zeta)$$

is regular outside of the image circle. For the exploration of a flat planing surface only, it would give:

$$\ln \frac{\overline{w}}{\overline{w}_\infty} = \ln \frac{\zeta - \zeta_{st}}{\zeta - \overline{\zeta}_{st}}$$

For the imaginary part ν of this term is for $\zeta = e^{i\alpha}$; that is, for the image of the planing surface constant for the part of the planing surface forward of the stagnation point, and differing by π in the rearward lying part, and in both parts as should be. For $\eta = 0$, it is $\frac{w}{w_\infty} = 0$, and so the boundary condition itself is fulfilled.

If δ is the planing angle, it must be

$$\zeta_{st} = e^{-i\delta}$$

But if the planing surface is not flat, it can be accounted for by expanding the above equation for the flat planing surface with additional terms and writing

$$\ln \frac{\overline{w}}{\overline{w}_\infty} = \ln \frac{\zeta - \zeta_{st}}{\zeta - \overline{\zeta}_{st}} + i \sum_{n=1}^{\infty} A_n \frac{1}{\zeta^n}$$

Through the selections of A_n, which are real, any kind of planing surface and the planing surface flow can be obtained by logarithmic solutions and integration:

$$z = \int \frac{\frac{d\chi(\zeta)}{d\zeta}}{\frac{\overline{w}}{\overline{w}_\infty}(\zeta)} d\zeta$$

or with

$$\frac{d\chi(\zeta)}{d\zeta} = 1 - \frac{1}{\zeta^2} + \frac{2E}{\zeta - 1} - \frac{E}{\zeta}$$

$$\frac{\overline{w}}{\overline{w}_\infty} = \frac{\zeta - \zeta_{st}}{\zeta - \overline{\zeta}_{st}}\, e^{i\Sigma \frac{A_n}{\zeta^n}}$$

$$z = \int \frac{1 - \frac{1}{\zeta^2} + \frac{2E}{\zeta - 1} - \frac{E}{\zeta}}{\frac{\zeta - \zeta_{st}}{\zeta - \overline{\zeta}_{st}}\, e^{i\Sigma \frac{A_n}{\zeta^n}}}\, d\zeta$$

For $\zeta = e^{i\alpha}$ one obtains the contour of the planing surface, for $\zeta = \xi > +1$ the points of the free surface from the planing surface and of the spray; for $\zeta = \xi < -1$, the points of the flow-off line.

The essential problem now consists in ascertaining A_n and $\zeta_{st} = e^{-i\delta}$ for a given planing surface and trim.

For small curvature and spray-free wash, the problem finds solution in the theory of thin slightly cambered airfoils. For planing with spray formation at any trim but small curvature under the assumption of $A_3 = A_4 = A_5 = \ldots = 0$, Franke's approximation (fig. 7) is excellent (reference 2). He was also able to derive the planing surface equations corresponding to Blasius' equations for the calculation of the lift and the moment of an airfoil in two-dimensional unlimited flow.

While lift and drag are easily and explicitly representable for small A_1 and A_2 and the calculation of the moment for the flat planing surface is still fairly simple, it becomes exceedingly complicated for $A_1 \neq 0$, $A_2 \neq 0$, and graphical methods are therefore preferable because the pressure distribution is always comparatively easy to compute. Another method, aside from this - what may be termed the isotach-isocline method - is the Schwarz-Christoffel method in its original form. But its practicability appears to be restricted to the treatment of flat planing surfaces, as employed by Wagner (reference 3) for gliding in a stream of infinite depth, and by Green (reference 4) for gliding of a plate in a stream of finite depth.

It is the usual method for computing the discharge jets as described earlier by Besant and Ramsay (reference 5), for instance, and in which the case of planing surface appears as a result of specialization of the conformal parameter.

Still another method (reference 6) which, while being more explanatory than that of Schwarz- Christoffel, is also largely restricted to the case of flat planing surface - that is, the hodograph method (fig. 8). Its extension to include any chosen planing surface could, in principle, be effected in somewhat similar form as its application for the determination of airfoils or turbine grid profiles with prescribed pressure distribution over the surface. Further details may, however, be dispensed with, particularly as the previously indicated isotach-isocline method is more convenient.

One important finding of the investigations is that the pressure distribution behind the stagnation point of a planing surface (fig. 9) even if cambered, is in very good agreement with the pressure distribution behind the branching-off point of a thin airfoil of the same shape in infinite flow and that, therefore, the pressure distributions in the zone, where the airfoil is as yet free from negative pressure, is still practically comparable - as a result of which the pressure distribution on planing surfaces can be closely approximated with the aid of the flow on thin airfoils.

Superficially, the formation of spray on the planing surface appears to involve a steady transition from hydrofoil to planing surface. But that, in fact, is not altogether so. The flow around the leading edge of an airfoil with corresponding setting is accompanied by great negative pressures. An eventual forming of empty space on the surface of the planing surface, whether due to evaporation or infiltration of air from the side, then resembles the transition to planing surface. This cavitation occurs so much more readily as the depth is smaller. In this manner a steady transition of the flow conditions from hydrofoil to planing surface, with decreasing depth of immergence h, is assured. A steady transition between the adhering flow around the hydrofoil and the separated flow with cavitation is, of course, not feasible. Figure 10 illustrates such a transition (reference 7). The solution is, in itself, of many meanings which we have avoided, however, by assuming the planing surface to be of infinite length forward - which, of course, is not admissible on transition from hydrofoil to planing surface. How the multiplicity of solutions is attained can be most easily seen from the hodograph (fig. 11). The depth of immersion increases or decreases as point D shifts toward A or more toward C'. The solutions are valid, after all, only in the vicinity of the planing surface, owing to the fact that the undisturbed level for gliding on infinite water depth logarithmically lies at infinite depth below the planing surface and, consequently, also below its trailing edge, so that at greater distance, at least, the effect of gravity on the velocity distribution at the surface and on its shape, can no longer be disregarded.

The trailing edge of the planing surface may still lie considerably above the undisturbed level, even for finite water depth (fig. 12). As the reflection on the assumedly horizontal background changes the planing-surface problem into the plane-flow problem of a free jet through a channel (fig. 13), we are face to face with the paradox, that it is not necessary that a jet of a certain width need pass completely and undisturbed through such a channel of greater width or, in other words, that it affords a second solution aside from the trivial solution of the flow problem. This must also be assumed in the three-dimensional problem and for large channel angles

$$0 \lneqq \alpha \lneqq \pi$$

From this follows, for example, the possibility that a wind-tunnel jet, even for assumedly frictionless flow, need not discharge directly through a larger exit cone and that repeated change from one flow condition to another would cause difficulties in tunnel operation unless the flow is stabilized by suitable measures.

It may also be mentioned that even with freedom from friction, there appears, aside from the solution with the outgoing spray for the planing surface, another possible one, namely, that a roller extends in front of the branching point, as the flow pattern and the relevant hodograph (fig. 14) indicate. But as this solution affords no equilibrium of the flow forces, it is not steadily possible.

The arguments concerning the plane problem of the hydrofoils and planing surfaces with gravity effect disregarded, close with a reference to the combined effect of planing surface and hydrofoil. If it involves an auxiliary wing near the trailing edge of the planing surface, the mutual interferences can be readily treated graphically (reference 8). The flow is then similar to that on a wing flap in burbling condition when an auxiliary wing is fitted below this flap. The result of such an investigation is illustrated in figure 15.

3. THE HYDROFOIL OF FINITE SPAN ACCORDING TO THE THEORY OF THE LIFTING VORTEX CURVE AND THE INDUCED DRAG

By observance of the boundary conditions, the hydrofoil of finite span may be treated exactly as the airfoil in infinite fluid; that is, the method of the lifting vortex curve can be applied to it.

The free vortex surface remaining downstream from the lifting vortex curve consists, on the assumption of small interferences, of bar vortices parallel to the free surface. Their reflection as vortices of the same sense of rotation fulfills the boundary conditions for their share of the interference flow. The lifting vortex curve need not be parallel to the free surface (fig. 16). Visualizing it to be replaced by a stepped vortex curve - one part of the lifting vortex curve consisting of parts parallel to the surface, the other of parts perpendicular to the surface - the share parallel to the surface reflected with the same sense of rotation again complies with the boundary conditions. The perpendicular shares must also be reflected on the surface. But as an upward vortex filament particle of the bound vortex curve of the hydrofoil shows in reflection a downward particle, and vice versa, the sense of rotation of the image in plan form is inverse to that of the original vortex filament particle. As a result, these particles cause no interference on the free surface after reflection, and so fulfill the boundary conditions for this share even more.

The reflection reduces the hydrofoil problem to that of a nonstaggered biplane of equal lift distribution over both wings (fig. 17) and with a wing gap 2h equal to twice the depth of immersion h.

The induced drag W_i of the hydrofoil becomes half as great as the induced drag W_{i_D} of this biplane. But, owing to the diminished flow velocity on the lower wing due to the image, the lift A of the hydrofoil is slightly less than half of the total lift A_D of this biplane; that is,

$$W_i = \frac{1}{2} W_{i_D}$$

$$A = \frac{1}{2} A_D \left(1 - \frac{w_{zus}}{w_\infty}\right)$$

whereby, if the aspect ratio is not too small and Γ_m is the mean value of the circulation over the span, in good approximation:

$$\frac{w_{zus}}{w_\infty} = \frac{\Gamma_m}{4\pi\ h\ w_\infty}$$

or with

$$\frac{1}{2} A_D = \rho\ \Gamma_m\ w_\infty\ b, \qquad c_{a_D} = \frac{A_D/2}{\frac{\rho}{2}\ w_\infty^2\ b\ t_m}$$

$$\frac{w_{zus}}{w_\infty} = \frac{c_{a_D}}{8\pi\ \frac{h}{t_m}}$$

and with

$$\frac{c_a}{c_{a_D}} = 1 - \frac{c_{a_D}}{8\pi\ \frac{h}{t_m}}$$

For wings of linear axis and optimum span loading and in first approximation even for other not too divergent lift distributions, the coefficient of induced drag $c_{w_i} = c_{w_{i_D}}$ is

$$c_{w_i} = \kappa\ \frac{c_{a_D}^{\ 2}}{\pi\ \frac{b^2}{F_D}}, \qquad W_{i_D} = \kappa\ \frac{A_D^{\ 2}}{\pi\ q\ b^2}$$

the value κ to be read from the appended table:

h/b = 0	0.1	0.2	0.3	0.4	0.5	0.6	∞
κ = 1	0.890	0.827	0.779	0.742	0.710	0.684	0.500

From
$$\frac{c_a}{c_{a_D}} = 1 - \frac{c_{a_D}}{8\pi \frac{h}{t_m}}$$

follows
$$c_{a_D} = 4\pi \frac{h}{t_m} \left(1 - \sqrt{1 - \frac{c_a}{2\pi \frac{h}{t_m}}}\right)$$

or in first approximation for small c_a and $\frac{t_m}{h}$

$$c_{a_D} \sim c_a \left(1 + \frac{c_a}{8\pi \frac{h}{t_m}}\right) \qquad (h > t_m)$$

This equation holds for high $\frac{h}{t_m}$ only. On approaching the surface, it must be $c_{a_D} = 2c_a$ at small trim, whence it is again expedient to write:

$$c_{a_D} \sim c_a \left(1 + \frac{c_a}{c_a + 8\pi \frac{h}{t_m}}\right) \qquad \left(\frac{h}{t_m} \longrightarrow 0\right)$$

As a result,

$$c_{w_i} \sim 2\kappa \left(1 + \frac{c_a}{8\pi \frac{h}{t_m}}\right)^2 \frac{c_a^2}{\pi \frac{b^2}{F}} \qquad (h > t_m)$$

or

$$c_{w_i} \sim 2\kappa \left(1 + \frac{c_a}{c_a + 8\pi \frac{h}{t_m}}\right)^2 \frac{c_a^2}{\pi \frac{b^2}{F}} \qquad \left(\frac{h}{t_m} \longrightarrow 0\right)$$

Compared to an identical single wing in infinite fluid (subscript E for ∞) the lift, in accord with the difference for the plane problem is, respectively:

$$c_a = c_{a_E} \left(1 - \frac{c_{a_E}}{c_{a_E} + 8\pi \frac{h}{t_m}}\right)^2 \qquad (h > t_m)$$

and, for better consideration of the surface approach:

$$c_a = c_{a_E} \left(1 - \frac{c_{a_E}}{(2 + \sqrt{2})\, c_{a_E} + 8\pi \frac{h}{t_m}}\right)^2 \qquad \left(\frac{h}{t_m} \longrightarrow 0\right)$$

with
$$c_{w_{i_E}} = \frac{c_{a_E}^2}{\pi \frac{b^2}{F}}$$

it is (fig. 18)

$$c_{w_i} = 2\kappa \left(1 + \frac{c_a}{8\pi \frac{h}{t_m}}\right)^2 \left(\frac{c_a}{c_{a_E}}\right)^2 c_{w_{i_E}} \qquad (h \gg t_m)$$

or

$$c_{w_i} = 2\kappa \left(1 + \frac{c_a}{c_a + 8\pi \frac{h}{t_m}}\right)^2 \left(\frac{c_a}{c_{a_E}}\right)^2 c_{w_{i_E}} \qquad \left(\frac{h}{t_m} \longrightarrow 0\right)$$

The so-called planing ratio $\epsilon = \frac{c_{w_i}}{c_a}$ becomes, respectively,

$$\epsilon = 2\kappa \left(1 + \frac{c_a}{8\pi \frac{h}{t_m}}\right)^2 \frac{c_a}{\pi \frac{b^2}{F}}$$

and

$$\epsilon = 2\kappa \left(1 + \frac{c_a}{c_a + 8\pi \frac{h}{t_m}}\right)^2 \frac{c_a}{\pi \frac{b^2}{F}}$$

Unless the axis of the hydrofoil is straight and parallel with the surface of the water, the optimum circulation distribution Γ over the span is usually that which is equal to the distribution of the potential jump transverse to the vortex surface far downstream from the hydrofoil, when the assumedly rigid vortex surface and its reflection were moved downward at a constant speed v_{i_∞}. The circulation distribution over the span then becomes of itself equal on the hydrofoil and on its reflected image, as is necessary for compliance with the boundary conditions. The rate of downwash v_i near vicinal to the hydrofoil, is only half as great as the rate of downwash at infinite v_{i_∞}, for reasons of symmetry, that is:

$$v_i = \frac{v_{i_\infty}}{2}$$

The span loading follows from the circulation distribution as:

$$dA = \rho \Gamma w_\infty \left(1 - \frac{w_{zus}}{w_\infty}\right) db$$

and with it the distribution of the induced drag as:

$$dW_i = \frac{v_i}{w_\infty} dA = \rho \Gamma \frac{v_{i_\infty}}{2} \left(1 - \frac{w_{zus}}{w_\infty}\right) db$$

w_{zus} is defined according to the Biot-Savart law. The lift A and the induced drag W_i are then obtained after proper integration.

To circumvent these integrations in the event that the immersion h is not constant over the span, both A and W_i may be estimated with a mean value h_m, which replaces h in the derived equations.

If the sides of the hydrofoil penetrate the surface of the water, the reflection of the submerged portion reduces the problem of computing such an airfoil to that of a box plane (Kastendecker) with equal circulation distribution on top and bottom (fig. 19). As a simple illustration of such a hydrofoil, let us select a semielliptical shape for it. Through reflection, we then have an elliptical box plane (fig. 20). Let the span loading be elliptical. It is readily perceived that then the downwash distribution over the span is constant and the circulation distribution in consequence, the best for the given elliptical shape.

Since the loss in kinetic energy decisive for the induced drag on the outside of the ellipse, is equal to that on a straight single wing of equal span b in equal induced downwash, but to which is now added that within the ellipse where the downwash velocity is constant, the induced drag of the total box plane becomes:

$$W_{i_D} = \frac{\rho}{2} {v_{i_\infty}}^2 \frac{\pi}{4} b^2 \frac{2h + b}{b}$$

and its lift

$$A_D = \rho w_\infty \frac{\pi}{4} b^2 \frac{2h + b}{b} v_{i_\infty}$$

The resulting v_{i_∞} written into the equation for W_{i_D} then gives the relation of W_{i_D} to A_D.

$$W_{i_D} = \kappa \frac{A_D^2}{q b^2}$$

whereby (fig. 21):

$$\kappa = \frac{b}{2h + b} = \frac{1}{1 + \frac{2h}{b}}$$

The lift A of the hydrofoil is with a mean value of the additional velocity w_{zus_m}:

$$A = \frac{1}{2} A_D \left(1 - \frac{w_{zus_m}}{w_\infty}\right)$$

and the previously developed equations can be applied to hydrofoils with semielliptical dead rise if h is replaced by h_m. With h = maximum immersion - that is, immersion at center of span - h_m should estimate about $\frac{2}{3}$ h.

4. THE PLANING SURFACE AS LIMITING CASE OF THE HYDROFOIL AND THE NECESSITY FOR REPLACING THE CONCEPT OF CIRCULATION WITH A MORE GENERAL CONCEPT

So long as the immersion of the hydrofoil is small but still finite, the boundary condition $\Delta W \perp W_\infty$ is complied with. But, once the immersion disappears and the hydrofoil becomes planing surface, the turbulent area emanating from the trailing edge of the planing surface, becomes part of the free surface of the water (fig. 22).

However, recalling to mind that in the problem of the hydrofoil with tips protruding laterally out of the surface of the water, the boundary conditions were always fulfilled in the zone over the shedding vortex surface, even for very small but definite immersion, it becomes obvious to consider the boundary conditions in the free surface of the planing surface also as being undisturbed by the shedding vortices - as if the immersion were not zero but rather very little.

In the case of a planing surface the lift can no longer be computed with the circulation concept. From the boundary transition of the hydrofoil to vanishing immersion h, it follows that instead of the circulation, the difference of potential $\varphi(y)$ of the interference flow on the planing surface in relation to potential φ_o on the free water surface must be introduced, whence the lift follows from

$$dA = \rho \; (\varphi(y) - \varphi_o) \; w_\infty \; dy$$

and the reduced drag of the planing surface from

$$dW_i = \frac{d_i}{w_\infty} \; dA = \rho \; (\varphi(y) - \varphi_o) \; v_i \; dy$$

As the vortex area of a planing surface has substantially the shape of the trace of a trailing edge, it is proper to use it in the determination of the interference flow rather than the trace of the planing surface axis. Here also the best lift distribution is obtained with the concept of hydrofoil supplemented by reflection on a box plane. But the lift and induced drag must be computed with the value $\varphi(y) - \varphi_o$ instead of the circulation. The result is again that the best lift distribution is obtained on the assumption of fixed vortex surface and downward displacement at constant velocity. With optimum span loading and not too low aspect ratio, it is:

$$c_{w_i} = 2 \; \frac{c_a^2}{\pi \; \frac{b}{t}}$$

But the aspect ratio b/t of planing surfaces is in many cases quite bad, at least for certain operating conditions. By profile chord t is meant the distance between branching point and trailing edge. In connection herewith, we point to the result of a lift and drag study made on an airfoil with poor aspect ratio (reference 9). These results can also be applied to planing surfaces and hydrofoils.

5. LIMITATION DUE TO CAVITATION OF THE LIFT

The pressure differences on the two sides of the profiles, resembling the positive and negative pressures which are produced as the hydrofoil moves through the water, are the cause of the lift and therefore necessary. Now, the desired hydrofoil action can be materially impaired in two ways: first, an all too rapid - but for ideal flow demanded - pressure rise promotes burbling, with resulting impairment of the operation of the airfoil. Proper design of the airfoils removes this risk of vortex separation, at least for the principal operating condition, the steady run. The accelerating run itself is always accompanied by vortex separation, even if the airfoils are good otherwise.

Favorable airfoil forms must have a most uniform and slowest possible pressure rise. To this end the lowest negative pressure itself must be as small as possible. But this lowest negative pressure is the cause of another departure from the desired hydrofoil action, namely, for the cavitation, which generally occurs when local conditions prolong the pressure drop until vapor pressure p_d is reached. The pressure in ideal fluid is, according to Bernoulli's equation:

$$p = p_o - \frac{\rho}{2} w^2 \qquad \left(\rho = \frac{\gamma}{g} = \text{density of water}\right)$$

Hereby p_o denotes the pressure of the fluid in a particle at rest relative to the wing in the same depth of water:

$$p_o = p_L - \gamma h + \frac{\rho}{2} w_\infty'^2$$

p_L the air pressure, γ the specific weight of water, $-h$ the depth below the surface, and w_∞ the running speed. w is the relative velocity of a water particle in relation to the wing. In proximity of the wing, w may fall below (decreased velocity) or exceed (increased velocity) w_∞. Let $w = w_\infty + w_ü$. Then

$$p = p_L - \gamma h + \frac{\rho}{2} w_\infty^2 - \frac{\rho}{2} (w_\infty + w_ü)^2$$

$$p = p_L - \gamma h - \frac{\rho}{2} (2w_\infty w_ü + w_ü^2)$$

The pressure is therefore particularly low, where $w_ü$ is high. Hence the increase of speed must be as small as possible if cavitation is to be avoided, which corresponds to thin airfoils.

To insure uniform distribution of the cavitation risk over a whole wing, $p_h \geq p_d$ must be constant. That is, with

$$K = \frac{p_a - \gamma h - p_h}{\frac{\gamma}{2g}} \qquad (p_h = \text{critical pressure})$$

$$\frac{w_{\ddot{u}}^2}{w_\infty} = \sqrt{\frac{K}{w_\infty^2} + 1} - 1$$

Here $w_{\ddot{u}}$ denotes the maximum increase of speed at the particular airfoil. The velocity distribution over an airfoil is contingent upon its form as well as upon its angle of attack.

If w_o is the velocity distribution for a profile at zero trim, it becomes at trim angle δ with respect to this lift-free relative flow direction:

$$w_\delta = w_o \cos\delta \left[1 \pm \left(\frac{1 + \Phi}{1 - \Phi}\right)^{1/2} \tan\delta\right]$$

Herein Φ denotes the potential on the airfoil border at incidence $\delta = 0$, if its scale is so chosen that it assumes the respective values +1 and −1 in the forward and rear stagnation points E and A. Transformation of the profile on a circle then yields $\Phi = \cos\vartheta$, if ϑ is the angle at the center of the image circle measured from the first profile axis, the axis of lift-free relative flow. With $\Phi = \cos\vartheta$, the transformation results in

$$\frac{w_\delta}{w_o} = \cos\delta + \cot\frac{\vartheta}{2}\sin\delta$$

$$\frac{w_\delta}{w_o} = \frac{\sin\left(\frac{\vartheta}{2} + \delta\right)}{\sin\frac{\vartheta}{2}}$$

With a view to ascertaining the relation of velocity distribution on an airfoil to its incidence and its most important parameters, camber and thickness, particularly the maximum of the increase of speed, these relations were computed on a number of Joukowski airfoils.

The results and the relevant airfoils are given in figures 23 and 24. Ordinarily, the speed is maximum at both sides of the profiles - the higher one being, of course, always decisive.

For every set of airfoils of constant camber, there is a lower envelope for the maximum speed curves. This is of special significance, as it indicates the minimum speeds which must occur for a certain camber, on Joukowski airfoils, at least, even by optimum selection of thickness parameter. There is a specific d/l for each point of those optimum speed maximums for constant f/l. By combining the points belonging to the same d/l in a diagram (fig. 25) of optimum maximum velocity curves for constant f/l, it becomes possible to ascertain the best profile thickness for a given lift c_a and a given camber.

There also is one minimum value of the maximum increase of speed for each profile. This case occurs when the maximum increases of speed are the same at both top and bottom surface of the profile. Cambered airfoils always have a lift even then; but, as both sides manifest considerable pressure rises, the profiles are not favorable at this incidence, and must therefore be more heavily loaded - i.e., be given at higher incidence than corresponds to this condition (fig. 26).

Joukowski profiles, having a sharp tip as trailing edge, are unsuitable for practical application. The velocity distribution is so closely related to the fineness of a profile form, that the results for the Joukowski airfoils are not directly applicable to other airfoils. With skillful design, it should surely be possible to obtain somewhat more favorable results than with Joukowski profiles.

On the whole, however, the results with the Joukowski profiles should approximately establish the attainable optimum values for other profiles of the same camber and thickness ratios. With increasing camber and thickness and rising lift coefficient, the attainable optimum increase of speed must definitely increase at practical settings. Even if skillful design affords some improvement, the choice of camber and thickness for a stipulated lift coefficient c_a is most assuredly possible, according to the result for Joukowski profiles - especially, in view of the quite favorable behavior of Joukowski profiles under changed operating conditions, as manifested in figure 24.

But the problem may also be to find profiles having a constant negative pressure on the suction side for a given lift coefficient. This case has been treated by Schmieden (reference 10). He found that the cavitation menace for a given wing thickness can be postponed to a certain, though not too great an extent, relative to Joukowski's profiles. Such profiles have not, however, as far as I know, been subjected to enough theoretical and practical tests, to warrant discussion.

That the lift coefficient should not be chosen too low for the sake of avoiding cavitation, probably need not be specially emphasized, as then the friction losses would become too great.

6. THE WAVE-MAKING RESISTANCE DUE TO THE EFFECT OF GRAVITY IN GENERAL

All that follows he has taken from Wagner

The arguments so far having been made without regard to gravity effect, we will now briefly touch upon the effect of gravity as it becomes conspicuous as wave-making resistance. Its smallness, in general, for the dimensions and speeds under consideration, is seen from the following.

Restricted essentially to the plane problem, we find that in sufficient water depth the wave length measured from crest to crest is

$$L = \frac{2\pi\, w_\infty^2}{g}$$

So, compared to the wave length, the usual t are for the most part, quite small. And in that case the theory of shifting pressure lines in simplified form , is applicable.

If the length element Δ_x of the planing surface (fig. 27) yields the stream force $\Delta R = p\ \Delta x\ b$, the elementary wave height is:

$$\Delta a = \frac{2\ \Delta\ R}{b\ \rho\ w_\infty^2}$$

The total wave height follows from the superposition of the elementary waves produced by the individual length elements. But, since the length t of the planing surface may be considered as being small in relation to wave length L, the wave height of the total wave follows from

$$a = \frac{2R}{b\ \rho\ w_\infty^2}$$

if R is the total stream force. The same result could be obtained by substituting a single pressure line for the planing surface. From the kinetic energy carried off by this wave, follows the wave resistance at

$$W_g = \frac{1}{4}\ g\ \rho\ a^2\ b = \frac{R^2\ g}{\rho\ w_\infty^4\ b}$$

For small inclination of the direction of the resultant R toward the direction of lift A, we may put $R = A$. With

$$R = A = c_a \frac{\rho}{2} w_\infty^2\ b\ t$$

$$W_g = c_{w_g} \frac{\rho}{2} w_\infty^2\ b\ t$$

the coefficient of the wave-making resistance for

$$\frac{t}{L/2\pi} = \frac{t}{w_\infty^2/g} = \tau$$

becomes

$$c_{w_g} = \frac{c_a^2}{\frac{2w_\infty^2}{g\ t}} = \frac{\tau}{2}\ c_a^2$$

or with

$$L = \frac{2\pi\ w_\infty^2}{g}$$

$$c_{w_g} = \tau\ \frac{c_a^2}{L/t}$$

This value is independent of the aspect ratio so long as it is not too small, and can therefore, also be used, for the present, for a finite aspect ratio.

As for it,

$$c_{w_i} = 2 \frac{c_a^2}{\pi \; b/t}$$

$\to \frac{L}{b} > \frac{\pi^2}{2}$

we have:

$$\frac{c_{w_g}}{c_{w_i}} = \frac{\pi^2}{2} \frac{b}{L}$$

$\frac{2\pi w_\infty^2}{g b} > \frac{\pi^2}{2}$

So long as $\frac{L}{b} > \frac{\pi^2}{2}$, the wave-making resistance of the planing surface is smaller than the induced resistance.

$\frac{w_\infty^2}{g b} > \frac{\pi}{4}$

$w_\infty > \sqrt{\frac{\pi}{4} g b}$

In the most important cases, the wave-making resistance of the planing surface is therefore subordinate.

Since with the hydrofoil the cause of the disturbance is farther away from the surface, it is surely permissible to assume, at least in the case of small profile chord in relation to the wave length, that the wave-making resistance of the hydrofoil is less than that of a planing surface for equal lift.

At the most there may be an additional share as a result of the finite thickness of the hydrofoil, but that effect cannot vary much from the wave-making resistance of a submerged cylinder (reference 11) of diameter d which, with

$$\underline{F} = \frac{w_\infty}{\sqrt{g\, d}}$$

is given through

$$c_{w_{g_{zus}}} = \frac{\pi^2}{2} \underline{F}^{-6} e^{-2\frac{h}{d}\underline{F}^{-2}}$$

The Froude number $\underline{F}$ is usually substantially greater than 1. So, unless the immersion h/d is too deep, this supplementary share of wave-making resistance due to profile thickness is mostly unimportant.

The effect of gravity reduces the damming up of the water (fig. 28) upstream from the planing surface or hydrofoil in relation to the motion without gravity (reference 3 and others). It corresponds to a reduction in effective trim of the planing surface or hydrofoil by $\beta_g = W_g/A$. Since the reduction in angle of trim due to the wake caused by the free vortices is $\beta_i = W_i/A$, it amounts to

$$\frac{\beta_g}{\beta_i} = \frac{W_g}{W_i} = \frac{c_{w_g}}{c_{w_i}} = \frac{\pi^2}{2}\,\frac{b}{L}$$

for the planing surface.

According to the foregoing arguments, β_g/β_i is smaller for the hydrofoil than for the planing surface.

The omission of the gravity in the investigation of planing surfaces and even more so of hydrofoils is therefore permissible in very many important practical cases or operating conditions.

Translation by J. Vanier,
National Advisory Committee
for Aeronautics.

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LEGENDS

Figures

1. In first approximation the absolute flow of an airfoil may be substituted by a single vortex. But in the case of the hydrofoil it infringes upon the boundary conditions of the free surface.

2. Compliance with boundary condition through an equably rotating vortex in the reflection of the hydrofoil.

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Figure

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22. Flow behind hydrofoil with small immersion compared to flow behind planing surface.

23. Joukowski profiles of various cambers and thicknesses.

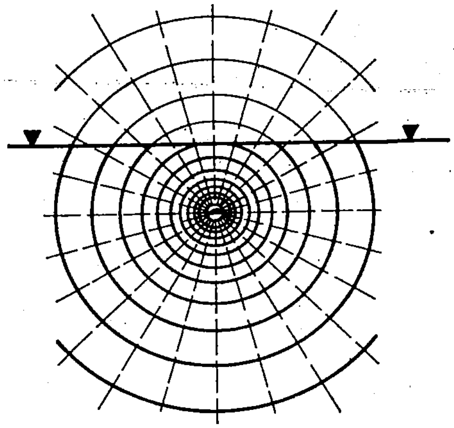
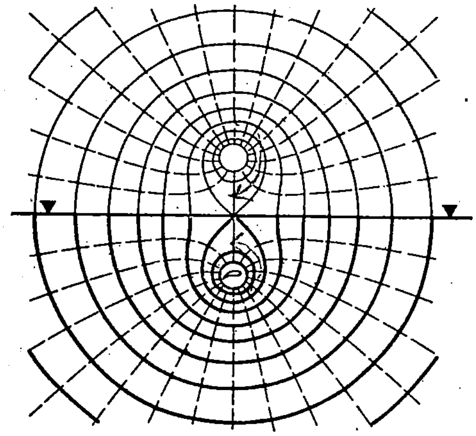


Figure 1.



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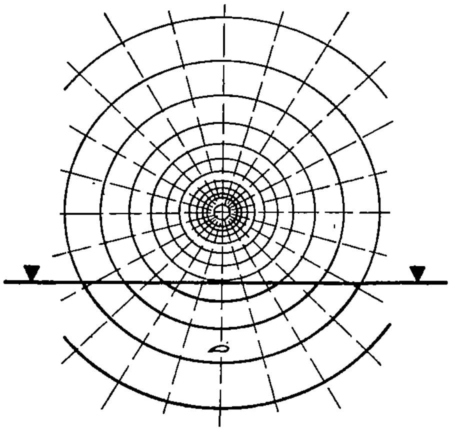

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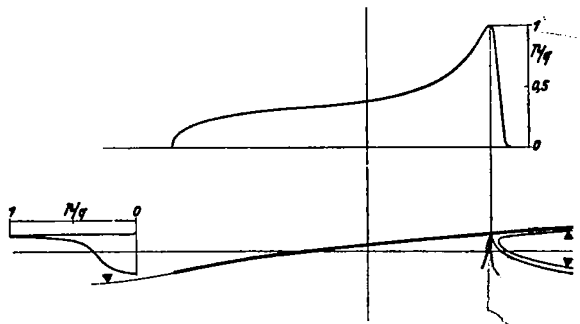

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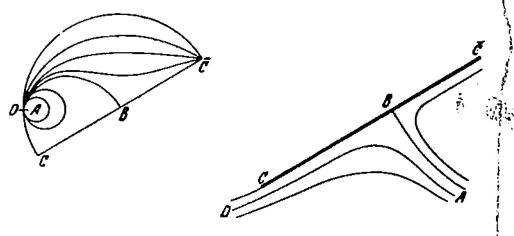

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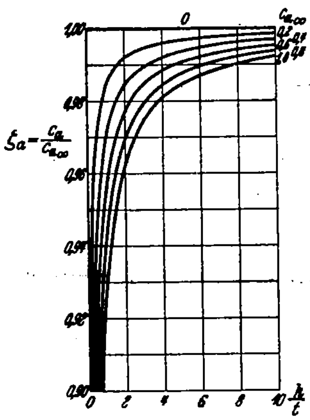

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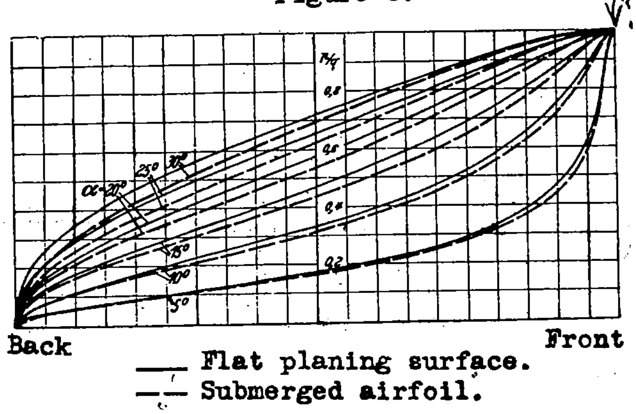

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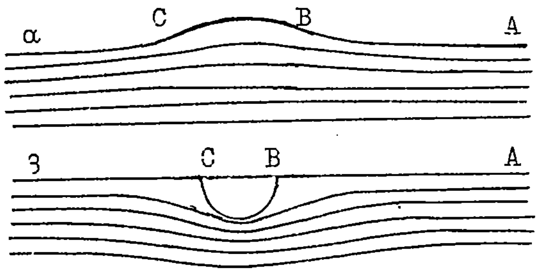

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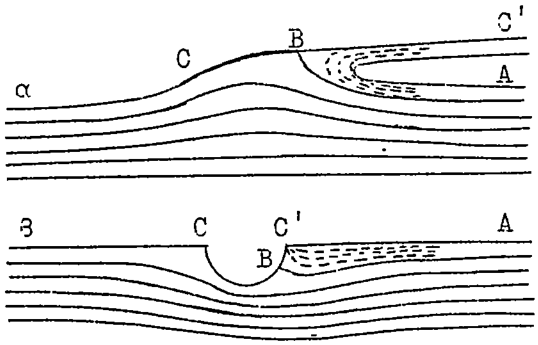

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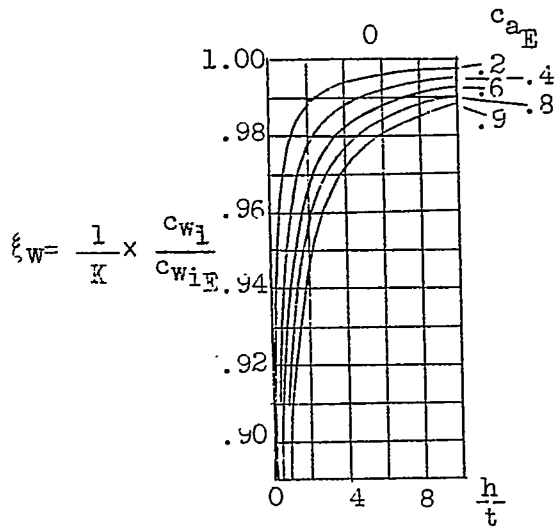

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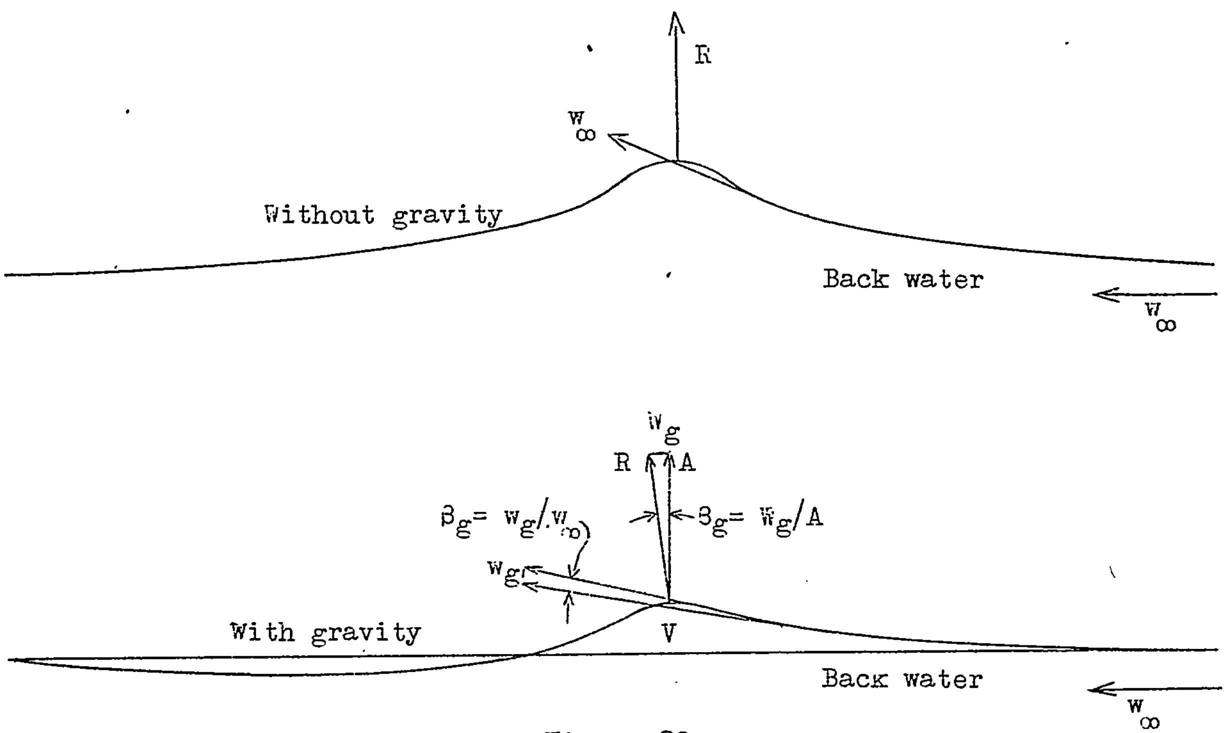

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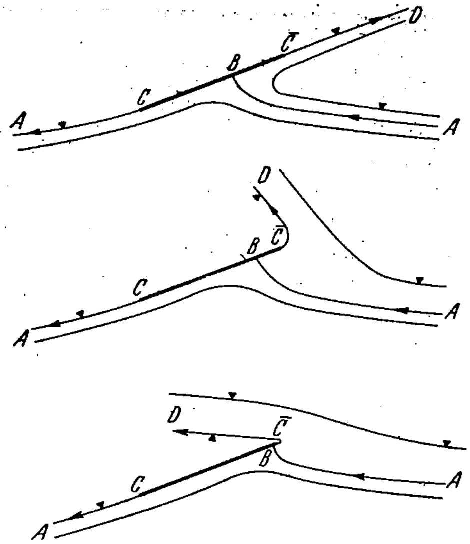

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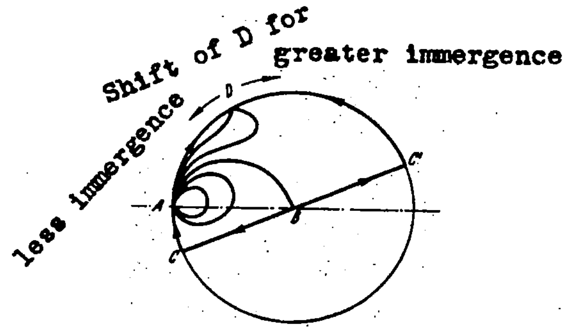

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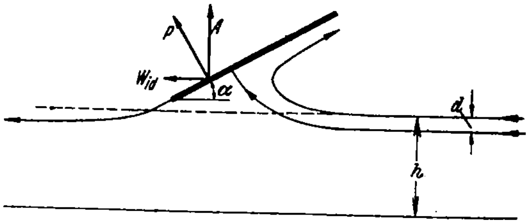

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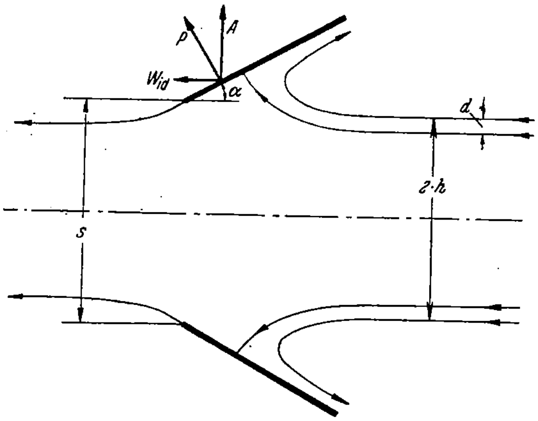

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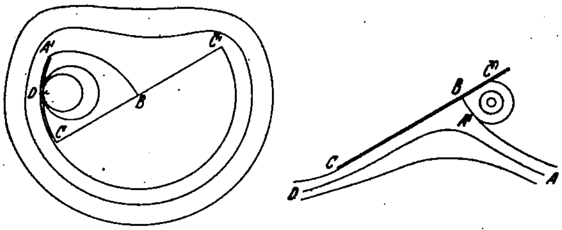

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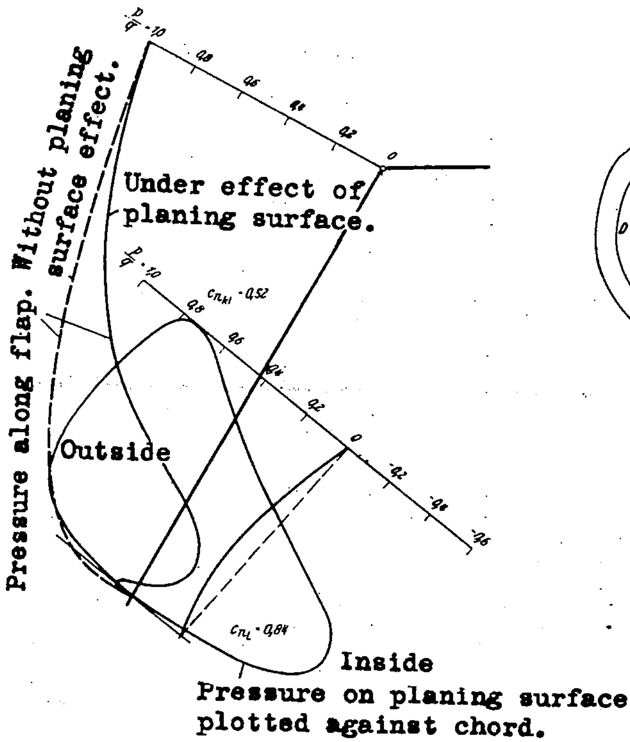

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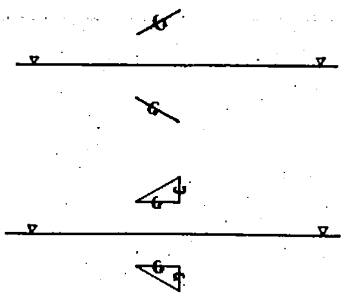

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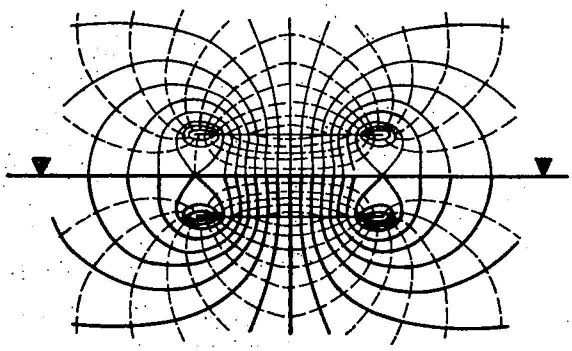

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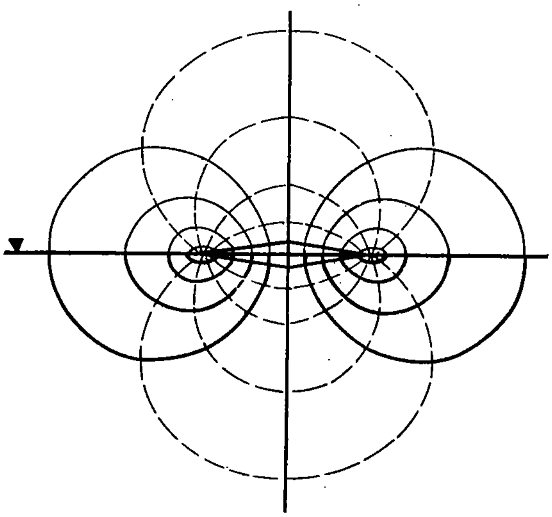

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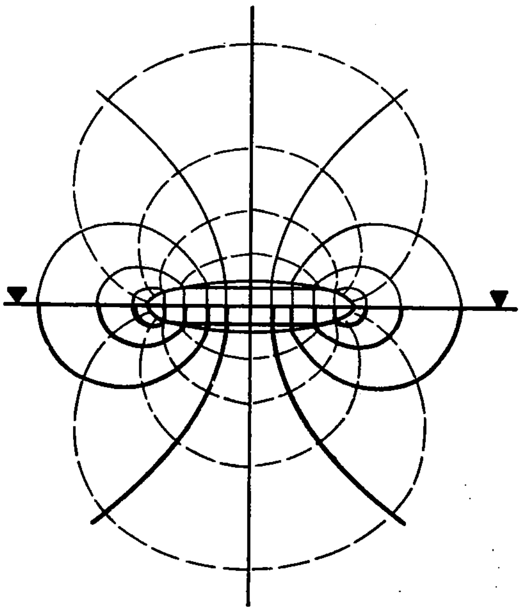
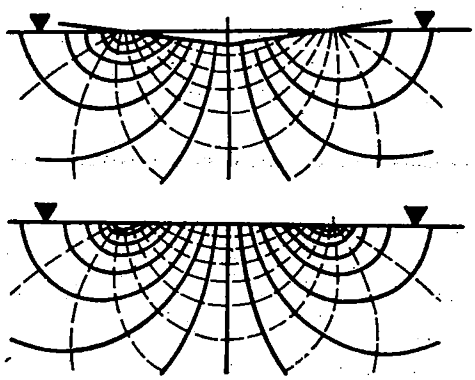


Figure 20.



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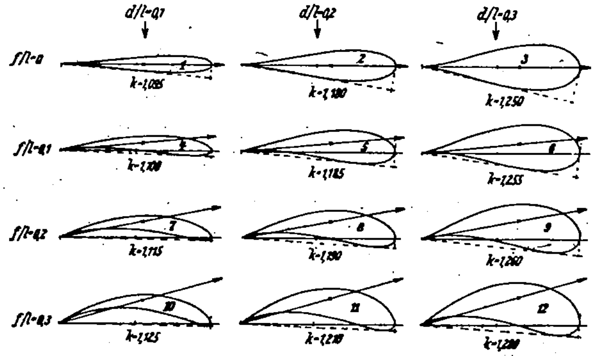

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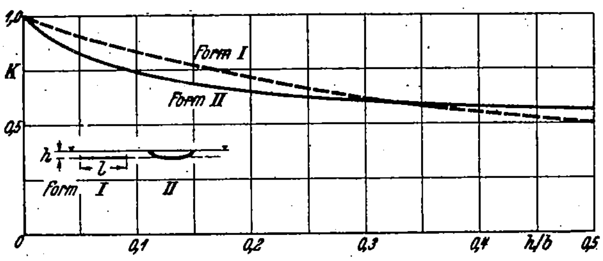

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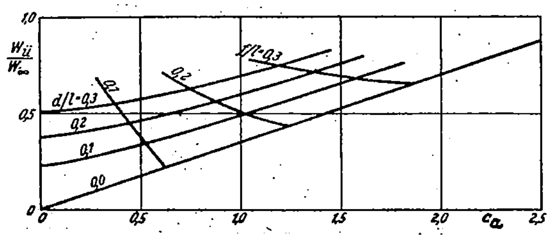

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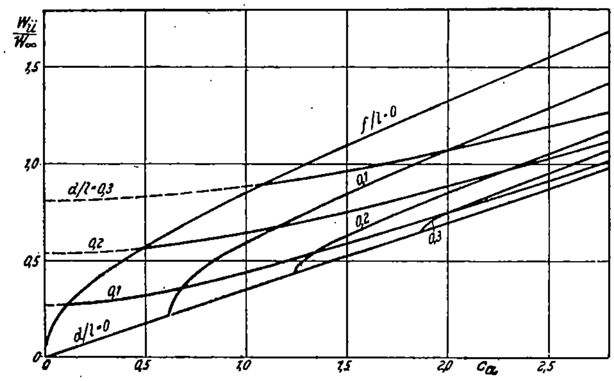

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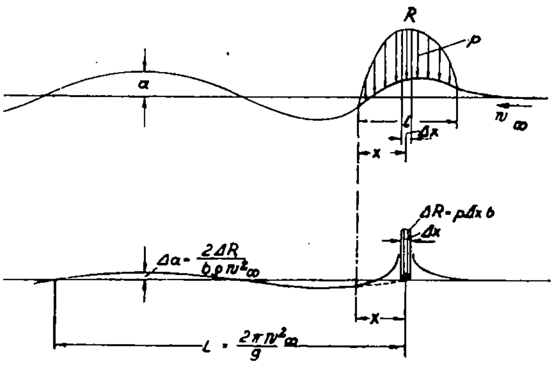

Figure 27.

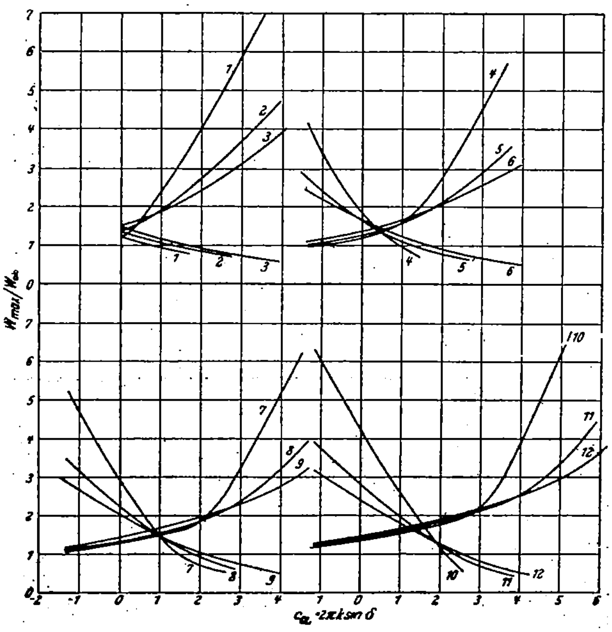

Figure 24.