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METHOD OF CURVED MODELS AND ITS APPLICATION TO THE STUDY OF CURVILINEAR FLIGHT OF AIRSHIPS

PART I

By G. A. Gourjienko

Central Aero-Hydrodynamical Institute, Moscow, 1934

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PREFACE

The idea of the desirability of testing curved models for studying the aerodynamics of airplanes in curvilinear flight (for example, in a loop), I have expressed in my lectures on the dynamics of flight a very long time ago, as early as 1918, or even earlier, but I did not intend to publish it.* And this idea did not materialize until Comrade Gourjienko devoted about a year to the treatment of this question.

The possibility of substituting for one phenomenon - the flight of a straight model along a circular path - another phenomenon, the flight of a curved model along a rectilinear path, is based on the following proposition:

If two flows under the influence of bodies immersed in them experience the same geometrical distortion of the lines of flow, the forces acting on the bodies may be regarded as approximately equal.

In simple cases, the correctness of this assumption is fully apparent. For example, the rectilinear motion of a flat plate and the circular motion of a plate bent along the circumferential arc, along which the motion occurs (figs. 1 and 2).

The analogy holds true if in place of a thin plate, we use a symmetrical streamlined form (the so-called "rudder form"), as is shown in figures 3 and 4.

Passing to theoretical forms of wings, consisting of a basic arc, <u>bent along a circumferential arc</u>, and enclosed by a streamlined form, for example, the inversion of a parabola or an ellipse, we see that the theoretical lifting force of such wings in plane-parallel flow is algebraically composed of two components: the first depending on the angle of attack, and the second depending on the concavity of the basic arc and proportional to it.

For the small arcs of one span, which differ according to their concavity, the second component of the lifting force is proportional to the curvature of the basic arc.

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*Since I occupied myself with aerodynamic questions only in conjunction with other questions (reliability of design [designed stability ?], dynamics), and the ideas that occurred to me I expressed in lectures or discussions, in order that persons, working in this particular field, might make use of them.

Causing the curved theoretical wing to move under a zero angle of attack in three ways: (1) rectilinearly, (2) along the circumferential arc, coinciding with the basic arc, and (3) along the circumferential arc of the somewhat larger radius, we find in the first case (fig. 5) a lifting force proportional to the curvature of the basic arc; in the second case (fig. 4), we find a lifting force equal to zero, and in the third case, the intermediate case (fig. 6), we can expect no other lifting force than the one which is proportional to the difference between the curved basic arc and the circumferential arc along which the wing moves.

On figure 6, O is the center of curvature of the basic arc; O_1 is the center around which the wing moves.

The continuous arc AB is a basic arc, with its center at O.

The dotted arc AB is the path of the points A and B when they travel around the center O; the straight line (dotted line) is the path of the same points when they travel along the straight line.

Finally, if the wing moves under an angle of attack, we may here too extend the theorem concerning the lifting force to the case of motion along the circumferential arc, the angle of attack denoting the angle of turn of the wing with respect to its position AB (figs. 5 and 6).

With the considerations cited, the basic assumption for the method of curved models can be regarded, if not as demonstrated, then as entirely supported by evidence, at any rate for the cases when the radius of curvature of the wing trajectory is comparatively large with reference to the length of the chord of the wing.

Of course, the curvature of the path involves corrections which must be regarded more appreciable when the lifting force of the wing is directed away from the center of rotation, and less perceptible when the lift force is directed toward the center of the trajectory curvature. In aviation and in aerostatics, exactly the latter case is involved, which is more favorable for the application of the elementary theory.

It seems to me that for plane-parallel flow we can also give an exact hydrodynamic theory for the wing in circular motion, using as a basis the writings of P. A. Walter, on the theory of rotating turbine blades in a converging flow.

As regards the motion of three-dimensional bodies, analyzed by the author (dirigibles, airplane fuselages, air bombs), we must not expect here to obtain a hydrodynamic streamline theory, since such a theory is absent even in the simplest cases of rectilinear motion of a straight dirigible under an angle of attack. Our analogy is fully applicable also to this case of motion.

And, we may expect that the experiments and rechecks, according to the method of curved models, will outdistance the theory.

It should be noted here that the principle of the apparent curvature of the wing was applied by Glauert when he calculated the effect of the rotation of the wing on its lift. Having much less theoretical evidence, he nevertheless arrives at results that are in good agreement with experimental results.

In substance, the Glauert method is as follows: If a flat plate (basic arc of a symmetrical wing) moves forward and rotates upon going into a dive, the geometrical angles of attack (with relation to the flow unaltered by the wing) will be different for the different elements of the basic arc (fig. 7).

In order to produce such geometrical angles of attack for the wing in translatory motion, we must bend its basic arc as shown in figure 8.

On figure 7 the arrows A'A, b'b, c'c, ..., k'k, B'B show the direction of the lines of the unaltered flow with reference to the various elements of the plate which is moving to the left and is rotating.

Figure 8 shows a curved arc, which the lines of the unaltered flow approach at the same angles as in figure 7.

This same analogy can be established in another way. Let the plate, which is curved along the arc of the circle, move in still air along its direction, as in figure 2, at a linear velocity υ and at an angular velocity:

$$\omega = \frac{\upsilon}{r}$$

The angle of attack of each element will be equal to zero.

Now let us stop the rotation of the plate and let us cause it to move along the tangent to the circumference. The plate will then assume the position shown in figure 8. But this will last one moment, and while it moves along the circumference, when the rotation of the plate ceases: this is analogous to the application to it of the angular velocity ω in the reverse direction. But, while maintaining the movement of the center of gravity, we cannot obtain the conditions of figure 8 for any length of time.

However, the application of the theory of the "dynamic camber" to the oscillating wing gives very good results, as is found by tests on vibrations of wings in an air flow, by calculation of the damping effect of the wings and by determination of the rotary derivatives by means of the oscillation method.

We should expect the method of curved models to be much more exact, from the theoretical point of view, not to mention the simplicity of testing a fixed model in the tunnel.

And we should be grateful to Comrade Gourjienko for his initiative and labor in working out this method and making it practicable.

Professor V. P. Vetchinkin.

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METHOD OF CURVED MODELS AND ITS APPLICATION TO THE STUDY OF CURVILINEAR FLIGHT OF AIRSHIPS*

By G. A. Gourjienko

INTRODUCTION

Up to the present time, the experimental study of the aerodynamics of curvilinear flight and of the very closely related question concerning the dynamic stability of airships as well as other aircraft has been in the embryonic state. The reason therefor lies in the fact that in solving each of the questions mentioned, in addition to the usually given data on aerodynamic resistances (forces and moments) in rectilinear flight, that can be sufficiently determined by the generally recognized method of testing in wind tunnels, it is also necessary to include the forces and moments due to the presence of angular velocity.

In order to determine these forces and moments, usually expressed by the so-called resistance rotary derivatives, special equipment was hitherto required.

One of the most accurate (in the kinematic and dynamic sense) methods was the use of whirling arms, where there was reproduced to some extent the phenomenon of circular flight. However, in practice this method is not very suitable for wide application, on account of the following reasons: (a) the extreme difficulty in obtaining exact measurement of the aerodynamical forces and moments acting on the model, taking into account the inevitable influence of the centrifugal loads; (b) the complicated air stream that is created during the motion; (c) the rather complicated and expensive equipment.

Another generally recognized method is the small oscillation method. This method does not require any large special installation, and allows experiments to be carried on in ordinary wind tunnels - although a special, occasionally rather complicated apparatus, is used.

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*Report No. 182, of the Central Aero-Hydrodynamical Institute, Moscow, 1934.

We shall not dwell upon the well-known principle underlying this method, but will merely point out that by means of extremely laborious experiments and working up of data, it enables us to determine only the rotary derivatives of the moment. Besides, the accuracy of the determination of this derivative is very low, since in addition to the large number of different - only approximately correct - propositions, lying at the very basis of the theory of the method, this derivative, in practice, has to be obtained by measuring an extremely large number of quantities, the majority of which has no connection with aerodynamics, as, for example, the moment of inertia, friction in the apparatus, rigidity of the springs in the equipment, etc.

The inevitable errors in the measurement of all these "ballast" quantities, as well as the damping effect itself, produce very large errors in the final result.

As we have already indicated, we can obtain with this method only the rotary derivative of the moment. The rotary derivative of the lateral (or lifting) force can be determined from the derivative of the moment only by means of various approximate repeated calculations (English method).

But the rotary derivative of the drag, which is extremely important, for instance, for investigating the deceleration while turning, cannot be determined at all by the aid of the method of small oscillations.

Moreover, neither of the existing methods in any way answers the very important question concerning the distribution of aerodynamical loads on the various parts of aircraft (hull and tail surfaces of a dirigible, wings of an airplane) while turning and during curvilinear flight.

The practical impossibility of experimenting on the distribution of pressure on whirling arms is due to the very complicated arrangement of the pressure tubing on the battery gage with the revolving model, and is due to the complicated calculation of the centrifugal forces, acting on the columns of air in the tubing.

In view of the above, it was decided in the aeronautical section of the Experimental Aerodynamical Department of the Central Aero-Hydrodynamical Institute, upon the initiative of Professor K. K. Fedjaevsky, to work out as completely as possible both theoretically and experimentally an en-

~~ly as possible both theoretically and experimentally~~ an entirely new aerodynamic principle of testing curved models. Several allusions to the possibility of applying this principle were made in foreign literature dealing with dirigible aerodynamics. As far as we know, Professor V. P. Vetchinkin (see Preface) was the first to speak of the possibility of applying this method. But, unfortunately, up to the present, no one (either abroad or here) has developed this method. Therefore, we had to work out this method entirely independently. The work was done in 1933-34, and was financed by NIO of the dirigible factory (Scientific Research Division of the dirigible factory).

The basic premise of the method is as follows: In the ordinary wind tunnel with rectilinear flow it is necessary, for the model installed in it, to create a kinematically similar pattern of flight along a circular trajectory. This can be obtained, in the first place, by bending the model in a special way and, secondly, by creating across the tunnel a constant velocity gradient (since, during flight along a circle the velocity of motion of each point of the dirigible is a linear function of the radius of its trajectory). Thus, by testing the curved model on the conventional aerodynamic balance, we obtain the forces and moments acting on the model in the proposed motion along a circle. This will enable us to obtain the rotary derivatives of the moment, of the lateral force and of the drag completely independently, with accuracy and simplicity, by means of typical, simple experiments in the tunnel. Moreover, the simple arrangement of the pressure tubes of the curved model will enable us to study the distribution of pressure in curvilinear flight.

Both must be done.

Drag will be affected by screens also.

In the first part of this paper we shall present the theoretical side of the problem of constructing curved model forms and the method of testing the model. In the second part we shall present a detailed account of the first experiments according to the given method, carried out with a curved model of the nonrigid airship V-2, and a comparison of the experimental results with some data of full-scale flight tests made with this airship in 1933.

The author expresses his sincere appreciation to Professor K. K. Fediaevsky for his valuable and highly competent advice and assistance in the solution of various problems.

PART I

1. POSSIBILITIES OF ROTATING CIRCULAR MOTION

Before we come to the theory of constructing curved model forms, i.e., to the solution, as we shall see, of a purely geometrical problem, it is necessary to discuss the possibility of rotating circular motion, in a manner similar to that applied in rotating rectilinear motion in experimental aerodynamics.

We must admit that from the point of view of theoretical hydrodynamics such a rotation of motion is inadmissible. While in moving along a circle the dirigible encounters a mass of absolutely calm air, when the motion rotates (if we make the air rotate like a solid body)* the flow becomes vortical at every point in space, the intensity of the vortex being $\lambda = 2\omega_y$, where ω_y is the angular velocity of rotation. This can be easily verified by taking the value of the velocity circulation along any closed contour within the indicated hypothetical flow. If we were to consider the given assumption as an obstacle when passing from circular flight to a stationary model in circular flow and thence also to the curved model, it would be necessary to question also the validity of the use of the wind tunnel altogether - in which, as is well known, the flow is vortical, nonpotential, and nonrectilinear.

We may assume that the errors, due to the fact that rotation of circular motion does not sufficiently comply with natural law, are not larger than those which are obtained by means of the usual experiments in the tunnels.

Consequently, the method of curved models, although it is not absolutely exact, is nevertheless a close approximation to reality.

2. CONSTRUCTION OF THE CURVED MODEL PROFILE

As an initial principle for the construction of the curved model profile, let us make the following indis-

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*Incidentally, it should be mentioned that such motion of a fluid is physically impossible; that is to say, such a flow can exist for only an infinitesimal given space of time.

putable assumption: If every small section of the surface of the curved model forms - with the direction of the rectilinear flow in the wind tunnel the same intersecting angle as will be formed by a corresponding section of a straight model with the direction of its linear speed while turning along a circle, the kinematic, and consequently, the dynamic similarity will be observed. shape criterion

The similarity in the sense of the magnitude of velocity of motion of the sections considered is maintained by creating in the tunnel the constant velocity gradient referred to above. However, it does not seem possible to solve the problem in such complete form. It is not difficult to observe that by curving the model, the length of the arc of the meridional contour inevitably becomes somewhat shorter on the windward side and somewhat longer on the lee side. To introduce this degree of contraction and elongation into the solution of the problem is impossible, because, in order to determine this degree, it is necessary to have the problem already solved. velocity criterion

However, the problem can be solved with a very great degree of approximation if, instead of the surface, we examine the kinematic similarity of the motion of the axis of the model and of an infinitely large number of cross sections. After making all the calculations concerning the axis and the cross sections, we shall try to find the numerical value of that error, which results from substituting for a complete solution an approximate solution, and which, as we shall see, also is due to the indicated contraction and elongation of the arc of the meridional contour.

And so, we shall proceed to derive the equations of the curved axis and the curved cross sections.

Let us assume a dirigible turning along a circle with radius R_0 (fig. 9). Considering that its center of volume C moves along the circular trajectory A - A with a velocity v_0, the theory and the experiment show that the axis of the airship will form an angle of attack β_0 with the direction of velocity v_0.

Besides, the rudder is deviated, as shown on the diagram, at an angle δ. In addition to the basic trajectory A - A, the diagram also shows a series of other trajectories along which the points of the airship axis move.

From the center of rotation O let us drop on the axis of the model the perpendicular OB. Then the angle COB = β_0, as an angle with the sides perpendicular to ν_0C and BC. From the triangle BOC, we obtain:

$$OB = R_0 \cos \beta_0 \quad \text{and} \quad BC = x_{1_0} = R_0 \sin \beta_0$$

Now let us take on the airship axis the arbitrary point D, which moves during rotation along the trajectory m - m; that is, concentric with relation to A - A. Let us connect point D with O and draw at D the tangent to m - m. This tangent gives the direction of velocity ν of point D. The angle νDB will be the local angle of attack β of the infinitely small segment of the airship axis at point D.

From the diagram we see that $\beta > \beta_0$.

If along the axis we take point D closer to point B than is shown on the diagram, angle β begins to get smaller and becomes 0 at point B.

Designating the variable distance BD by x_1, we find $\beta = f(x_1)$. From the triangle BDO, where angle BOD = β, we obtain:

$$x_1 = R_0 \cos \beta_0 \tan \beta$$

whence

$$\tan \beta = \frac{x_1}{R_0 \cos \beta_0}$$

and

$$\beta = \text{arc} \tan \frac{x_1}{R_0 \cos \beta_0} \qquad (1)$$

it is clear that when $x_1 = x_{1_0}$, $\beta = \beta_0$.

Now, for the circular motion, let us substitute rectilinear motion, shown in figure 10.

Starting from the basic principle, it is necessary to bend the axis of the model in the direction indicated on the diagram, in such a way that the angle of attack along the axis of the model changes according to the same law (1) as in the case of curvilinear flow.

Naturally, when the axis and the cross sections are curved, it is necessary to preserve their lengths. This seems all the more necessary since, by preserving these lengths, we maintain the distribution and magnitude of the velocity circulation along the axis and along each of the cross sections, which is very essential for the similarity.

Thus, x_1 will denote the length of the arc of the curved axis from B to D.

Now, taking on figure 10 point B as the origin of the coordinates x, z',* the direction of which is indicated, we obtain the equation of the curved axis.

The angle of inclination of the tangent to the curve sought at point D must be equal to β, whence

$$\tan \beta = \frac{dz'}{dx} \tag{2}$$

On the other hand, it has already been shown that

$$\tan \beta = \frac{x_1}{R_0 \cos \beta_0}$$

where x_1 is the length of the arc, and is equal to

$$x_1 = \int_0^x \sqrt{1 + \left(\frac{dz'}{dx}\right)^2}\, dx \tag{2a}$$

Then the expression (2) is transformed as follows:

$$\frac{dz'}{dx} = \frac{1}{R_0 \cos \beta_0} \int_0^x \sqrt{1 + \left(\frac{dz'}{dx}\right)^2}\, dx \tag{3}$$

The equation obtained is also the equation of the curved axis of the model. Let us solve it. For this purpose. let us differentiate both sides and, for the sake of brevity, let us take $\frac{1}{R_0 \cos \beta_0} = n$:

$$\frac{d^2 z'}{dx^2} = n \sqrt{1 + \left(\frac{dz'}{dx}\right)^2}$$

*The direction z' coincides with the direction z, taken as a reference. Thus, the origin of the coordinates is at a distance of z'_c from the axis of the tunnel, that is to say, $z = z' - z'_c$.

let us substitute the new variable $u = \frac{dz'}{dx}$. Then

$$\frac{du}{dx} = n\sqrt{1 + u^2}$$

Dividing the variables, we obtain:

$$dx = \frac{du}{n\sqrt{1 + u^2}}$$

Let us integrate:

$$x = \frac{1}{n}\int \frac{du}{\sqrt{1 + u^2}}$$

The integral obtained is the "long logarithm" and is expressed:

$$x = \frac{1}{n} \ln \left(u + \sqrt{1 + u^2}\right) + C \tag{4}$$

The constant C we determine from the initial conditions. When $x_1 = 0$,

$$x = 0 \quad \text{and} \quad \tan \beta = \frac{dz'}{dx} = u = 0$$

Substituting, we find that $C = 0$.

Let us solve the equation obtained with reference to u, for which purpose we find from (4)

$$u + \sqrt{1 + u^2} = e^{nx}$$

and

$$\sqrt{1 + u^2} = e^{nx} - u$$

Squaring both sides, we get:

$$1 + u^2 = e^{2nx} - 2ue^{nx} + u^2$$

We reduce by u^2 and find $u = f(x)$:

$$u = \frac{e^{2nx} - 1}{2e^{nx}} \tag{5}$$

Substituting $u = \frac{dz'}{dx}$, we get:

$$\frac{dz'}{dx} = \frac{e^{anx} - 1}{2e^{nx}}$$

dividing the variables and integrating, we find:

$$z' = \int_0^x \frac{e^{anx} - 1}{2e^{nx}} \, dx \tag{6}$$

Performing division under the integral and dividing the right side into two integrals, we get:

$$z' = \frac{1}{2} \int_0^x e^{nx} \, dx - \frac{1}{2} \int_0^x e^{-nx} \, dx$$

Performing integration and substituting, we get:

$$z' = \frac{1}{2n} [e^{nx} - 1 + e^{-nx} - 1] = \frac{1}{n} \left[\frac{e^{nx} + e^{-nx}}{2} - 1 \right]$$

In analysis, the expression $\frac{e^{nx} + e^{-nx}}{2}$, is called a hyperbolical cosine and is denoted by ch nx. Thus

$$z' = \frac{1}{n} [\text{ch } nx - 1]$$

Taking $n = \frac{1}{R_o \cos \beta_o}$, we find that

$$z' = R_o \cos \beta_o \left(\text{ch} \frac{x}{R_o \cos \beta_o} - 1 \right) \tag{7}$$

This is also the equation of the curved model axis. Opening up the parenthesis, we obtain:

$$z' = R_o \cos \beta_o \text{ ch} \left(\frac{x}{R_o \cos \beta_o} \right) - R_o \cos \beta_o$$

This is no other than a "catenary" with the parameter $R_o \cos \beta_o = OB$ (fig. 9) with the origin of the coordi-

Note: It's a catenary rather than a simple circular arc due to use of exact relations (~~tan~~ β rather than β, etc), and making length along arc same as chord in curved flow, and fact β_o not zero.

nates transferred to the point of the minimum z'. Such a form is assumed by a perfectly flexible thread, suspended at two points and subjected to its own weight, which is proportional to the length of the thread. The formula obtained is very convenient for constructing the curve, since for hyperbolical functions there are available detailed tables. (See Hütte, vol. I; or Keiichi Hayashi, "Sieben- und mehrstellige Tafeln der Kreis- und Hyperbelfunktionen," [Tables of seven or more places of circle functions and hyperbolical functions], Berlin, 1926.)

Generally, the ordinates of the meridional contour of the dirigible and other data on the hull are given as functions of the distance along the axis of the dirigible from its nose. Therefore, it is useful to have the expression which enables us to find the quantity x for the construction of the curve according to formula (7), given the quantity x_1, that can be measured off along the arc of the catenary from the origin B of the coordinates.

The relation $x = f(x_1)$ is readily obtained by integrating in the expression (2a):

$$x_1 = \int_0^x \sqrt{1 + \left(\frac{dz'}{dx}\right)^2}\, dx$$

Above there was substituted $\frac{dz'}{dx} = u$, which, according to formula (5), was equal to:

$$u = \frac{e^{2nx} - 1}{2e^{nx}} = \frac{e^{nx} - e^{-nx}}{2}$$

This expression is no other than the hyperbolical sine of the argument nx; that is, to say,

$$u = \operatorname{sh} nx$$

Substituting in formula (2a), we get:

$$x_1 = \int_0^x \sqrt{1 + \operatorname{sh}^2 nx}\, dx$$

According to the basic relation between the hyperbolical functions of one argument:

$$\operatorname{ch}^2 nx - \operatorname{sh}^2 nx = 1$$

we find that $\sqrt{1 + sh^2\ nx} - sh\ nx = 1$,

wherefore,

$$x_1 = \int_0^x ch\ nx\ dx = \frac{1}{n}\left[sh\ nx\right]_0^x = \frac{1}{n}\ sh\ nx$$

(when $x = 0$ $sh\ nx = 0$).

Solving the expression obtained with reference to x_1, we have the relation sought:

$$x = \frac{1}{n}\ Ar\ sh\ nx_1 \tag{8}$$

(The symbol Ar denotes the inverse hyperbolical function and comes from the word "Area.")

Now let us proceed to the study of the curvature of the cross sections. The method of investigation is the same: We shall examine the variation of the angle of intersection along the horizontal diameter of each cross section.

The direction of one of these diameters, passing through the arbitrary point D on the axis, is represented on figure 9 by the straight line a - a.

Let us derive the equation of the curve a - a (fig. 10), which changes from the straight line a - a in passing from curvilinear flow to rectilinear flow.

It is obvious that the curve sought will also be a catenary, normal at point D to the curved axis, since the variation of the angles of attack along a - a on figure 9 does not differ theoretically in any way from the case examined in connection with the axis of the model. Let us prove it.

Taking on the straight line a - a any arbitrary point D_1, let us construct for it the local angle of attack designated by ϑ. Let us connect point D_1 with O and drop from O the perpendicular OB_1 on the direction a - a. Then the angle $D_1\ OB_1$, which is equal to ϑ, is expressed as follows:

$$\tan \vartheta = \frac{D_1 B_1}{OB_1} = \frac{D_1 B_1}{x_1}$$

whence

$$\vartheta = \text{arc tan } \frac{D_1 B_1}{x_1} \tag{9}$$

The expression obtained is completely analogous to the formula (1) derived for the axis of the model. The distance $D_1 B_1$ coordinates the distance of point D_1 from B_1, similarly to the manner in which in formula (1) the distance x_1 coordinated point D with reference to B. The quantity x_1 (formula (9)) for the entire straight line a - a plays the same role as the quantity $R_0 \cos \beta_0$ in formula (1).

It is obvious that, for the curve sought, point B_1 will perform functions analogous to the functions performed by point B for the curved axis. At B_1 the angle ϑ will be equal to zero; that is to say, B_1 will be the origin of the coordinates of the catenary sought.

In order to derive the equation for the curve sought, let us turn to figure 11. It shows the curved axis BD in terms of the coordinates x, z', and the curve a - a sought. From what we have said in making an analogy with the above, it follows that, for the curve sought, we should take B_1 as the origin of the coordinates, and draw the axes of the coordinates parallel to the axes x, z'. Denoting these axes by r and t, let us write, analogously to formula (7), the equation for the catenary a - a in terms of the coordinates r, t:

$$t = x_1 \left(\text{ch } \frac{r}{x_1} - 1\right) \tag{10}$$

It is obvious that for the line a - a, $x_1 =$ constant. When passing from one cross section to another, that is to say, when varying x_1, we naturally change the position of the origin B_1 of the coordinates. As x_1 decreases, point B_1 will travel along the curve B_1B_1'', shown on figure 11 as a dotted line. Moreover, the catenary will "contract," assuming for a certain point B_1' the shape a' - a', indicated on the diagram. This is due to the fact that the arc of the catenary (10) $\overset{\frown}{B_1D}$, which is equal in length to the distance

$$\overset{\frown}{B_1D} = R_0 \cos \beta_0 = \frac{1}{n}$$

(fig. 9), does not change when x_1 changes; that is to say,

$$B_1D = B_1'D' = B_1''B = R_0 \cos \beta_0 = \frac{1}{n} = \text{constant} \qquad (11)$$

When $x_1 = 0$, the catenary will contract into a straight line, disposed along the axis z'. On crossing the axis z', the catenary will "open up" again.

Naturally, all the chords of the cross sections, parallel to the horizontal diameter, will deflect along the segments of the same catenaries as the horizontal diameters, and all the vertical chords and diameters will remain straight. This is due to the fact that the motions examined are plane-parallel. Thus, the cross sections will curve along the cylindrical surfaces.

Now, let us express the whole family of catenaries in terms of the coordinates x, z'; this is very convenient in constructing the profile of the meridional contour of the curved model.

Let us denote the coordinates of point D by x_D and z'_D, and the coordinates of point B_1 by x_0, z_0'. Then the formulas for the change of coordinates will be written in the following form:

$$\left.\begin{aligned} r &= x - x_0, \\ t &= z' - z_0' \end{aligned}\right\} \qquad (12)$$

Let us find the values x_0 and z_0'. Let us designate the length of the arc B_1D by r_{1_D}. According to condition (11), we find that

$$r_{1_D} = \frac{1}{n}$$

On the other hand, the projection of the arc B_1D on the axis r, equal to r_D, analogously to equation (3) for the catenary $a - a$, is written:

$$r_D = x_1 \operatorname{Ar\,sh} \frac{r_{1_D}}{x_1} = x_1 \operatorname{Ar\,sh} \frac{1}{nx_1}$$

We may disregard the symbols r_D and r_{1_D}, since we are here interested only in the absolute values of the segments and arcs.

From figure 11, we see that

$$x_o = x_D + r_D$$

whence
$$x_o = x_D + x_1 \text{ Ar sh } \frac{1}{nx_1} \tag{13}$$

In order to obtain z_o', let us find the ordinate t_o of point D. Considering that

$$r_D = x_1 \text{ Ar sh } \frac{1}{nx_1}$$

according to formula (10), we find:

$$t_D = x_1 \left(\text{ch } \frac{r_D}{x_1} - 1\right) = x_1 \left[\text{ch } \left(\text{Ar sh } \frac{1}{nx_1}\right) - 1\right]$$

According to the basic formula for hyperbolic functions

$$\text{ch}^2 \varphi - \text{sh}^2 \varphi = 1$$

the preceding expression may be written as follows:

$$t_D = x_1 \left[\sqrt{1 + \text{sh}^2 \left(\text{Ar sh } \frac{1}{nx_1}\right)} - 1\right]$$

or

$$t_D = x_1 \left[\sqrt{1 + \left(\frac{1}{nx_1}\right)^2} - 1\right] = \frac{1}{n} \sqrt{(nx_1)^2 + 1} - x_1$$

But, starting from the equation of the curved axis

$$z'_D = \frac{1}{n} [\text{ch } nx_D - 1]$$

in which, according to expression (8), we may substitute

$$nx_D = \text{Ar sh } nx_1$$

we find that

$$z'_D = \frac{1}{n} \text{ ch } (\text{Ar sh } nx_1) - \frac{1}{n} = \frac{1}{n} \sqrt{(nx_1)^2 + 1} - \frac{1}{n}$$

Thus, $t_D = \frac{1}{n}\sqrt{(nx_1)^2 + 1} - x_1 = z'_D + \frac{1}{n} - x_1$

From figure 11, we see that $z_o' = z'_D - t_D$. Consequently,

$$z_o' = z'_D - z'_D - \frac{1}{n} + x_1 = x_1 - \frac{1}{n} \qquad (14)$$

Now, knowing x_o and z_o' as functions of the quantities x_1 and x_D, that is to say, of the quantities coordinating point D in the x, z' system, let us represent the formula for the change of the coordinates (12) as follows:

$$\left.\begin{aligned} r &= x - x_D - x_1 \text{ Ar sh } \frac{1}{nx_1} \\ t &= z' - x_1 + \frac{1}{n} \end{aligned}\right\} \qquad (15)$$

Then the catenary $a - a$ (formula (10)) in terms of the coordinates x, z' is expressed in the form:

$$z' - x_1 + \frac{1}{n} = x_1 \left[\left[\text{ch}\left(\frac{x - x_D - x_1 \text{ Ar sh } \frac{1}{nx_1}}{x_1}\right)\right] \quad 1\right]$$

Finding from here $z' = f(x)$ and taking x_1 out of the parenthesis, we obtain:

$$z' = x_1 \left[\text{ch}\left(\frac{x - x_D}{x_1} - \text{Ar sh } \frac{1}{nx_1}\right) - \frac{1}{n}\right. \qquad (16)$$

This is the equation of the family of catenaries sought.

It was shown above that each catenary of the family (16) must intersect the line of the curved axis at right angles. Let us verify this circumstance.

From analytical geometry we know that for mutual orthogonality of two curves, it is necessary and sufficient for the values of the derivatives of these curves at the intersecting point to be inverse in magnitude and sign. Let us find the derivative with respect to x from formula

(16):*

$$\frac{\partial z'}{\partial x} = x_1 \text{ sh} \left(\frac{x-x_D}{x_1} - \text{Ar sh} \frac{1}{nx_1}\right) \frac{1}{x_1} = \text{sh} \left(\frac{x-x_D}{x_1} - \text{Ar sh} \frac{1}{nx_1}\right) \tag{17}$$

The value of the derivative at point D is obtained by substituting $x = x_D$; we have:

$$\left(\frac{\partial z'}{\partial x}\right)_{(x=x)_D} = \text{sh} \left(- \text{Ar sh} \frac{1}{nx_1}\right) = - \frac{1}{nx_1} \tag{18}$$

According to the preceding, the value of the derivative from the equation of the curved axis will be:

$$\frac{\partial z'}{\partial x} = u = \text{sh } nx$$

substituting for x its value, expressed as x_1 according to formula (8), we find that

$$\frac{\partial z'}{\partial x} = \text{sh } [\text{Ar sh } nx_1] = nx_1 \tag{19}$$

A comparison of the results (18) and (19) shows that the catenaries (7) and (16) are orthogonal to each other at point D.

Now, it remains for us to find, analogously to the above, the expression which will enable us to lay off on the curves a - a from point D the magnitude of the local radius of the cross section of the model. Denoting this local radius by y_1, let us assume (fig. 11) that it is plotted along the curve a - a from point D above and below; that is to say,

$$\overset{\frown}{DA} = \overset{\frown}{DB} = y_1$$

Thus, points A and B belong to the meridional contour of the model. Since it is difficult to plot along the arc, let us try to find the abscissas of the points A and B in terms of the coordinates x, z' for a given y_1. Designating these abscissas by a and b, we find the length of the arc AB of the curve a - a:

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*From the theory of hyperbolic functions we know that

$$\frac{d}{d\varphi} \text{ ch } \varphi = \text{sh } \varphi \quad \text{and} \quad \text{sh } (-\varphi) = -\text{sh}\,\varphi$$

$$\overset{\frown}{AB} = 2y_1 = \int_a^b \sqrt{1 + \left(\frac{\partial z'}{\partial x}\right)^2}\, dx$$

Introducing the value of the derivative $\frac{\partial z'}{\partial x}$ according to formula (17) and substituting according to formula

$$\sqrt{1 + \mathrm{sh}^2\, \varphi} = \mathrm{ch}\, \varphi$$

we obtain:

$$\overset{\frown}{AB} = 2y_1 = \int_a^b \mathrm{ch}\left(\frac{x - x_D}{x_1} - \mathrm{Ar\ sh}\, \frac{1}{nx_1}\right) dx$$

Integrating and considering that for the given curve a - a, x_D = constant and x_1 = constant, we get:

$$\overset{\frown}{AB} = 2y_1 = x_1 \left[\mathrm{sh}\left(\frac{b - x_D}{x_1} - \mathrm{Ar\ sh}\, \frac{1}{nx_1}\right) - \mathrm{sh}\left(\frac{a - x_D}{x_1} - \mathrm{Ar\ sh}\, \frac{1}{nx_1}\right)\right] \tag{20}$$

Now let $b = x_D$, that is to say, let us assume that there was integrated not the segment of the arcs $\overset{\frown}{AB}$, but $\overset{\frown}{AD} = y_1$. Then, substituting $b = x_D$ in the preceding equation, we find:

$$\overset{\frown}{AD} = y_1 = - x \left[\mathrm{sh}\left(\frac{a - x_D}{x_1} - \mathrm{Ar\ sh}\, \frac{1}{nx_1}\right) + \frac{1}{nx_1}\right]$$

From the equation obtained, we determine the abscissa a sought:

$$a = x_D + x_1 \left[\mathrm{Ar\ sh}\, \frac{1}{nx_1} - \mathrm{Ar\ sh}\left(\frac{y_1}{x_1} + \frac{1}{nx_1}\right)\right] \tag{21}$$

Letting, in equation (20), $a = x_D$, that is to say, assuming that arc $\overset{\frown}{DB} = y_1$ was integrated:

$$\overset{\frown}{DB} = y_1 = x_1 \left[\mathrm{sh}\left(\frac{b - x_D}{x_1} - \mathrm{Ar\ sh}\, \frac{1}{nx_1}\right) + \frac{1}{nx_1}\right]$$

whence

$$b = x_D + x_1 \left[\mathrm{Ar\ sh}\, \frac{1}{nx_1} + \mathrm{Ar\ sh}\left(\frac{y_1}{x_1} - \frac{1}{nx_1}\right)\right] \tag{22}$$

In all the expressions deduced

$$n = \frac{1}{R_0 \cos \beta_0}$$

Now we proceed to derive the magnitude of the error arising from the fact that our solution is somewhat approximate as compared to the solution of the complete problem concerning the motion of the surface.

Let us turn to figure 12, which represents a diagram of the nose of the dirigible straight model, the contour of which is given by the equation $y_1 = f(x_1)$.

Let us take on the contour any point A, the trajectory of which during the circular motion of the dirigible coincides at a given moment with the direction of its speed v. The angle between the direction of speed and the tangent to the contour we designate by β. The angle between the tangent and the cross section AD we designate by μ. Then, according to formula (9) and figure 9, we find that

$$\beta + \mu = \pi - \vartheta \tag{23}$$

Now, in accordance with the solution given above, let us curve our model and designate the angle formed by the tangent to its contour, plotted through point A, and the direction of velocity in the tunnel by β_1.

Analogously to figure 12, let us designate the angle, formed by the tangent to the contour and the tangent to the curved cross section AD, by μ_1, shown in figure 13, which represents a part of the contour of the curved model $y = f(x)$, the curved axis $z'_D = f_1(x)$ and the section AD. Then, in accordance with the principle for constructing cross sections, we get:

$$\beta_1 + \mu_1 = \pi - \vartheta \tag{24}$$

comparing formulas (23) and (24), we get

$$\beta + \mu = \beta_1 + \mu_1$$

Let us find the value of μ_1 and compare it with μ. With these values we shall be able to estimate the difference between β_1 and β. From figure 12, we see that

$$\mu = \frac{\pi}{2} - \text{arc tan}\,\frac{dy_1}{dx_1} \tag{25}$$

On figure 13 let us plot the following: Through point A let us draw the tangent MA to the cross section AD. Let us draw AN perpendicular to AM. Let us now give the argument x_1 the small increment $\Delta x_1 = DD'$. Then section DA will assume the position D'A'. From point A' let us drop the perpendicular A'F on the prolongation of the direction of velocity ν and the perpendicular A'E on the straight line AN. Let us designate by G the point of intersection of A'F and AN. Let us designate the angle between the prolongation of the chord A'A and the direction of velocity by β_1', and the angle between the same prolongation and the tangent MA by μ_1'. We readily see that

$$\mu_1' = \pi - \vartheta - \beta_1' = \text{arc tan}\,\frac{dz'}{dx} - \beta_1'$$

But

$$\beta_1' = \angle\, A'AF = \text{arc tan}\,\frac{A'F}{AF}$$

Noting that AF is an increment of x, and A'F is an increment of y for the curve of the curved contour $y = f(x)$, we may write the preceding equation as follows:

$$\beta_1' = \text{arc tan}\,\frac{\Delta y}{\Delta x}$$

Then

$$\mu_1' = \text{arc tan}\,\frac{dz'}{dx} - \text{arc tan}\,\frac{\Delta y}{\Delta x} \tag{26}$$

but $\angle\, A'AF = \angle\, A'AE + \angle\, GAF$

Hence $\text{arc tan}\,\frac{\Delta y}{\Delta x} = \text{arc tan}\,\frac{A'E}{AE} + \angle\, GAF$

but since

$$\angle\, GAF = \pi - \vartheta = \text{arc tan}\,\frac{dz'}{dx} - \frac{\pi}{2}$$

then the preceding equation may be rewritten as follows:

$$\text{arc tan}\,\frac{\Delta y}{\Delta x} = \text{arc tan}\,\frac{A'E}{AE} + \text{arc tan}\,\frac{dz'}{dx} - \frac{\pi}{2}$$

Substituting the obtained value arc tan $\frac{\Delta y}{\Delta x}$ in expression (26), we find:

$$\mu_1' = \frac{\pi}{2} - \text{arc tan}\ \frac{A'E}{AE}$$

It will be readily seen that, when Δx_1 infinitely tends toward zero,

$$\mu_1 = \lim_{\Delta x_1 \to 0} \mu_1' = \lim_{\Delta x_1 \to 0} \left(\frac{\pi}{2} - \text{arc tan}\ \frac{A'E}{AE}\right)$$

Let us designate $AE = \Delta x_1'$ and rewrite the last equation as follows:

$$\mu_1 = \lim_{\Delta x_1 \to 0} \left[\frac{\pi}{2} - \text{arc tan}\ \left(\frac{\Delta y_1}{\Delta x_1}\ \frac{A'E}{\Delta y_1}\ \frac{\Delta x_1}{\Delta x_1'}\right)\right]$$

where Δy_1 is the increment of the arc $\overset{\frown}{AD}$; that is, to say,

$$\Delta y_1 = \overset{\frown}{A'D} - \overset{\frown}{AD}$$

It is readily observed that when $\Delta x_1 \to 0$, the ratio $\frac{A'E}{\Delta y_1}$ tends toward unity, the expression $\frac{\Delta x_1}{\Delta x_1'}$ tends toward the value of the derivative $\frac{dx_1}{dx_1'}$; in this way,

$$\mu_1 = \frac{\pi}{2} - \text{arc tan}\ \left(\frac{dy_1}{dx_1}\ \frac{dx_1}{dx_1'}\right) \qquad (27)$$

By the aid of the obtained expression (27) and the value μ from formula (25), we find that the relation of the tangents μ_1 and μ will be:

$$\frac{\tan \mu_1}{\tan \mu} = \frac{\cot\left[\text{arc tan}\ \left(\frac{dy_1}{dx_1}\ \frac{dx_1}{dx_1'}\right)\right]}{\cot\left[\text{arc tan}\ \frac{dy_1}{dx_1}\right]} = \frac{dx_1'}{dx_1} \qquad (28)$$

The derivative $\frac{dx_1'}{dx_1}$ obtained characterizes that degree of contraction or elongation which is experienced by the lee side and the windward side of the model during the process of curving.

We can prove that the curved axis of the model approximates the arc of the circle with the radius $\rho = R_0 \cos \beta_0$, and the cross sections are almost plane. Making use of these circumstances for ascertaining the order of the numerical value of the ratio μ_1/μ, we find that for any point on the axis

$$\frac{dx_1'}{dx_1} = 1 - \frac{y_1}{R_0 \cos \beta_0}$$

where y_1 is the local radius of the cross section.

It will be readily seen that for the stern point and the nose point, where $y_1 = 0$, we will have $\mu = \mu_1$ and, consequently, $\beta = \beta_1$. $\frac{dx_1'}{dx_1}$ is farthest from unity in the vicinity of amidships. Let us find this value, assuming that $R_0 \cos \beta_0 \approx 2.5$ L and $y_1 \approx \frac{1}{9}$ L (which corresponds to the mean given data for the model); we obtain $\frac{dx_1'}{dx_1} \approx 0.96$; that is to say, the difference between $\tan \mu$ and $\tan \mu_1$ amounts to ~ 4 percent. But, since in the vicinity of amidships both μ and μ_1 approximate $\frac{\pi}{2}$, the 4 percent difference in the tangents creates a difference in the angles amounting to a few tenths of one percent. In the parts of the zones located between the nose and amidships and between amidships and the stern, the difference between β and β_1 is greater (due to the fact that in these zones the angles μ and μ_1 differ from $\pi/2$ more than in the case of amidships). However, according to our calculations, the difference between β and β_1 nowhere exceeds from 2 to 2.5 percent. In this way, the insignificance of the error resulting from the abovementioned alteration in the lengths of the arcs of the meridional contour has been demonstrated.

3. METHODS OF DIRECT DESIGNING OF CURVED MODEL FORMS

There are two possible methods: (1) A very exact method, which takes into account the curvature of the cross sections, and (2) a simplified method, in which the cross sections of the model are assumed to be plane and normal to the catenary of the axis.

The axis of the curved model is constructed in the same way in each method.

(1) Exact Method

a) Construction of the curved axis.- To begin with, it is necessary to find the radius of turn with which it is desired to carry out the test. In addition, we must know the angle of attack β_o formed during the rotation with the given radius.

Knowing R_o, β_o may be obtained either according to the previously constructed nomogram of the straight line $\frac{1}{R_o} = f(\beta_o, \delta)$* for the given dirigible, or from free-flight tests, or by the aid of the statistical method. The last method is the simplest, because for the straight line $\frac{1}{R_o} = f(\beta_o)$, it gives the angle coefficient $\frac{R_o\ \beta_o}{\lambda} = k$, where λ is the arm of the tail surface center of pressure. According to a whole series of foreign investigations, this coefficient preserves sufficiently well its magnitude in the case of quite a large number of airships. With k given, the angle of turn is calculated according to the formula

$$\beta_o = k\frac{\lambda}{R_o} \tag{29}$$

Here β_o is obtained in radians.**

Knowing R_o and β_o, it is necessary to find the distance from the nose of the model to point B, which is taken as the origin of the coordinates of the catenary.

Designating by $\overline{X}_1$ the distance from the nose of the model to the center of volumetric displacement, we find, according to figure 9, that the distance sought is:

$$x_{1B} = \overline{X}_1 - x_{1_o} = \overline{X}_1 - R_o \sin \beta_o \tag{30}$$

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*See our paper "Determination of Radii and Angles of Turn of the Airship at Various Rudder Angles of Deviation."

**In the experimental part of this paper we shall see that R_o and β_o can be assumed entirely arbitrarily.

If we obtain x_{1_B} in the negative, it means that in figure 10 the nose point will lie to the right of the origin of the coordinates.

For constructing the catenary of the axis $z' = f(x)$, it is necessary, for calculating z' according to formula (7), to find the magnitudes x. The simplest way to find the magnitudes x, is to determine them according to formula (8), where x_1 is the distance along the arc of the catenary from point B.

In this manner we can obtain on the curved axis of the model a series of points, the distance of which from point B and, consequently from the nose of the model along the arc is known. This is very convenient, since, in the profile of the noncurved model , the ordinates y_1 of the contour of the meridional section, as well as the position of the power cars, the tail surfaces, and other details are determined by the distance from the nose.

b) Now we have to proceed to the construction of the lines of the cross sections. These lines are constructed for each desired section according to formula (16), where the magnitudes x_D and x_1 for each point of the curved axis are known from the previous construction. The moving coordinate x we may choose according to our own judgment. Then we have to find on the constructed curves points A and B (fig. 11), belonging to the curve of the meridional contour; for these points the ordinates y_1 are known from the straight model. These constructions are made according to the formulas (21) and (22), which give the abscissas of the points A and B from the origin of the coordinates. Figure 14, showing this construction, gives an idea of how the net of the catenaries of the axis and of the cross sections looks.

(2) Simplified Method

The simplification concerns only the construction of the cross sections, which are assumed to be plane and normal to the curved axis. That such an assumption can be made, i.e., to substitute for segments of the catenaries AB a straight line, is shown by the following consideration.

Let us find the radii of curvature of the catenaries of the cross sections at their intersection points with the axis (that is, to say, when $x = x_D$). The radius of curvature for the flat-topped curve is expressed:

$$\rho = \frac{\left[1 + \left(\frac{dz'}{dx}\right)^2\right]^{3/2}}{\frac{d^2 z'}{dx^2}}$$

Taking the expression dz'/dx according to formula (18) when $x = x_D$, we find:

$$\frac{dz'}{dx} = -\frac{1}{nx_1}$$

Differentiating formula (17), we have:

$$\frac{d^2 z'}{dx^2} = \frac{1}{x_1} \text{ ch} \left(\frac{x - x_D}{x_1} - \text{Ar sh} \frac{1}{nx_1}\right)$$

which, when $x = x_D$, gives:

$$\frac{d\ z'}{dx^2} = \frac{1}{x_1} \sqrt{1 + \left(\frac{1}{nx_1}\right)^2}$$

substituting the expressions obtained in the formula for ρ, we find that:

$$\rho = \frac{x_1 \left[1 + \left(\frac{1}{nx_1}\right)^2\right]^{3/2}}{\sqrt{1 + \left(\frac{1}{nx_1}\right)^2}} = x_1 + \frac{1}{n^2 x}$$

Substituting $n = \frac{1}{R_0 \cos \beta_0}$, we have:

$$\rho = x_1 + \frac{R_0^2 \cos^2 \beta_0}{x_1}$$

Let us show that in all the cases the value of ρ is very large. From the last expression, we can obtain:

$$\rho = \left[\frac{x_1}{L} + \left(\frac{R_0}{L}\right)^2 \left(\frac{L}{x_1}\right) L \cos^2 \beta_0\right] L$$

where L is the length of the model.

When $x_1 = 0$ $\rho = \infty$; that is to say, the cross section is a plane. When x_1 increases, the magnitude ρ decreases. ρ will have the smallest value with the maximum possible $x_1 \approx L$. On page 21 of this report it was shown that the minimum radius is equal to from 2 to 3 L - on the average, 2.5 L. The angle of attack, corresponding to the given R_0, is usually about 7 to 9°, and consequently, on the average, $\cos \beta_0 = \cos 8^\circ = 0.99$ and $\cos^2 \beta_0 = 0.978$.

Substituting the values indicated in the expression found, we get the smallest possible radius of curvature of the cross-section diameter:

$$\rho_{min} \approx 7L$$

Such a radius is obtained at the tail of the model, where the diameter of the cross section is very small, for which reason it is entirely possible to regard the cross section as a plane.

In the midship section, that is to say, when $y_1 = max$ and $x_1 \approx \frac{L}{3}$, we get:

$$\rho_{mean} \approx 21 \ L$$

This shows us that it is entirely possible to make the cross sections plane. Besides, they preserve their circular form, which is a great convenience in preparing the model.

Now let us proceed to the method of constructing the plane sections. It was shown above that these sections must be normal to the curved axis. Let us find the equation of the normal at any point on the curved axis. The equation of the normal is generally written as follows:

$$\frac{\partial F}{\partial z'} (\xi - x_D) - \frac{\partial F}{\partial x} (\eta - z'_D) = 0$$

Here F is a function in an implicit form, ξ and η are the moving coordinates of the normal line sought, and x_D and z'_D are the coordinates of the point on the curve, through which the normal passes.

According to equation (7) also $F = z'_D - \frac{1}{n}\, ch\, nx_D + \frac{1}{n} = 0$, whence

$$\frac{\partial F}{\partial z'} = 1 \quad \text{and} \quad \frac{\partial F}{\partial x} = -\, sh\, nx_D$$

Substituting in the equation of the normal the values of the derivatives and taking, according to formula (8), $sh\, nx_D = nx_{1D}$, we shall have:

$$\xi - x_D + nx_{1D}\,(\eta - z'_D) = 0$$

hence any abscissa of the normal will be equal to:

$$\xi = x_D - nx_{1D}\,(\eta - z'_D) = x_D - \frac{x_{1D}\,(\eta - z_D)}{R_o \cos \beta_o} \qquad (31)$$

It is most convenient to construct as follows (fig. 15): In the diagram there is plotted on the curved axis point D with the coordinates x_D, z'_D, through which point it is necessary to draw the normal. If we take for the second point on the normal its intersection with the axis x (that is to say, if we assume in formula (31), $\eta = 0$), this method is not very precise when the normals are constructed near the nose of the model, where the line of the axis almost coincides with the axis x. Therefore, it is more advantageous to find the intersection of the normal with some straight line parallel to the axis x and located at a sufficiently large arbitrary distance $BM = \overline{z}$ from it. Then, when $\eta = \overline{Z}$, equation (31) gives the distance ξz, from the axis z', of the points N

$$\overline{\xi z} = x_D - \frac{x_{1D}\,(\overline{Z} - z'_D)}{R_o \cos \beta_o} \qquad (31a)$$

In order to find the points A and B, belonging to the contour, we simply have to lay off on the lines of the normals from the points D the magnitudes y_1.

It is self-evident that also the tail surfaces of the airship must be somewhat deformed. Both vertical stabilizers must be bent along the segment of the catenary of the axis at the place where they are installed. But the horizontal stabilizers are constructed in the coordinate net formed (in the simplified case) by means of catenaries;

equidistant from the line of the axis, and by means of the normals to the catenaries. In this way, the right stabilizer comes out shorter than the normal, and the left stabilizer is longer than the normal. But the dimensions laid off on the normals remain unchanged.

The model is installed in the tunnel in such a way that the center of volume of the model lies on the axis of the tunnel, and the axis x (fig. 10) is placed parallel to the axis of the tunnel at a distance z'_c, which is easily determined according to formulas (7) and (8), if we know the position of the center of volume on the axis of the model.

4. THE VELOCITY GRADIENT ACROSS THE TUNNEL

It was shown above that, in order to maintain the similarity of circular flight, it is necessary to create in the tunnel, when testing the curved model, a velocity varying, across the tunnel, according to a linear law. That this law is linear is evident from figure 9. At the moment represented on the diagram, the airship turns at a constant angle of speed ω_y. Then the linear speed of its center of volume (point C) will be:

$$\upsilon_o = \omega_y R_o$$

and the linear speed of any point on the axis, for example, point D, will be:

$$\upsilon = \omega_y R$$

Subtracting one equation from the other, we get:

$$\upsilon - \upsilon_o = \omega_y (R - R_o)$$

hence

$$\upsilon = \upsilon_o + \omega_y (R - R_o)$$

When passing from curvilinear flow to rectilinear flow, it is evident from figure 10 that for point D

$$R - R_o = z' - z'_c = z$$

where z denotes the variable ordinate, read off from the axis of the tunnel crosswise to the right and to the left.

In this manner

$$\upsilon = \upsilon_o + \omega_y \ z \qquad (32)$$

The expression obtained, gives the distribution of velocity along the section of the tunnel. As is evident, this law is linear. Substituting for ω_y its expression in terms of the radius of rotation and the circumferential velocity, in the given case, the velocity of the circular flight of the airship; that is to say, $\omega_y = \frac{\upsilon_o}{R_o}$, we shall have:

$$\upsilon = \upsilon_o + \frac{\upsilon_o}{R_o} z = \upsilon_o \left(1 + \frac{z}{R_o}\right) \qquad (33)$$

Hence, the constant velocity gradient along the section of the tunnel

$$\frac{\partial \upsilon}{\partial Z} = \frac{\upsilon_o}{R_o} = \omega_y$$

that is to say, numerically it is equal to the angular velocity.

The distribution of velocity obtained has a flow which is vortical at every point. It is interesting to note that the velocity circulation along any closed contour in such a flow is equal to $\Gamma = S' \omega_y$, where S' is the area confined by the contour. In this manner, the intensity of the vortices of such a flow is obtained as $\lambda = \omega_y$, that is, half as large as in the case of circular flow. Consequently, (see p. 4), in testing the curved model, we are, in form, even closer to full-scale tests than in the case of testing in circular flow.

The law, expressed by equation (33), is diagrammatically represented in figure 10. In order to have an idea of the magnitude of the velocity gradient in that portion of the cross section of the tunnel where the curved model is installed for testing, we shall calculate the relation of the velocity at the boundary of this portion to the velocity along the axis of the tunnel. From equation (33), we get:

$$\frac{\upsilon}{\upsilon_o} = 1 + \frac{z}{R_o}$$

Analogously to the preceding, let us assume:

$$R_0 \approx 2.5 \ L$$

and the entire width of the zone, in which it is required to create the gradient,

$$2z = 0.5 \ L$$

whence $$z = 0.25 \ L$$

Such a width of the zone is entirely sufficient in order to provide ample room for any curved model. Hence,

$$\frac{\upsilon}{\upsilon_0} \approx 1 + \frac{0.25 \ L}{2.5 \ L} \approx 1.1$$

i.e., the difference between the velocity at the boundary of the portion and the velocity along the axis of the tunnel amounts to about 10 percent. The methods of creating the velocity gradient and the method accepted by us will be discussed in the experimental part of this paper.

5. METHOD OF DETERMINING ROTARY DERIVATIVES FROM WIND-TUNNEL TESTS OF THE CURVED MODEL

We have already mentioned (p. 3) that, in directing the air flow against the curved model in the tunnel, we should obtain the forces and moments acting on the model during circular motion. Let us denote the force of the drag during circular flight by ΣX, the lateral aerodynamic force during circular flight by ΣZ_1, and the moment of yawing (around the axis y) during circular flight by ΣM.

Then in the very general case, any of these quantities for any one airship will be a function of the angular velocity, of the linear velocity, of the angle of attack β_0 and of the angle of rudder deviation; i.e., for example, the force

$$\Sigma Z_1 = f(\omega_y, \ \upsilon_0, \ \beta_0, \ \delta)$$

Assuming that the motion is effected with υ_0 = const.,

β_0 = const., and δ = const., but with variable ω_y,*
we may write:

$$\Sigma Z_1 = \varphi(\omega_y)$$

Resolving the function obtained according to the Mac Laurin system which, as is well known, is as follows:

$$\Sigma Z_1 = \varphi(0) + \frac{\omega_y}{1!}\varphi'(0) + \frac{\omega_y^2}{2!}\varphi''(0) + \frac{\omega_y^3}{3!}\varphi'''(0) + \ldots$$

we find

$$\Sigma Z_1 = (\Sigma Z_1)_{\omega_y=0} + \frac{\omega_y}{1!}\left[\frac{\partial(\Sigma Z_1)}{\partial\,\omega_y}\right]_{\omega_y=0} + \frac{\omega_y^2}{2!}\left[\frac{\partial^2(\Sigma Z_1)}{\partial\,\omega_y^2}\right]_{\omega_y=0} + \ldots$$

But when $\omega_y = 0$ the force ΣZ_1 is transformed into the force Z_1, acting on the model in rectilinear motion. Thus,

$$\Sigma Z_1 = Z_1 + \omega_y \frac{\partial Z_1}{\partial \omega_y} + \frac{\omega_y^2}{2}\frac{\partial^2 Z_1}{\partial \omega_y^2} + \ldots = Z_1 + Z_{1_\omega} \qquad (34)$$

where by Z_{1_ω} we designated the <u>sum of all</u> the terms of the series, containing ω_y.

As will be seen, the full force, acting in circular flight, is the algebraic sum of the effects of the linear and rotary displacements.

Analogously, resolving in the series ΣX and ΣM, we finally find:

$$\left.\begin{aligned} \Sigma X &= X + X_\omega \\ \Sigma Z_1 &= Z_1 + Z_{1_\omega} \\ \Sigma M &= M_y + M_\omega \end{aligned}\right\} \qquad (35)$$

Hence we can obtain the terms X_ω, Z_{1_ω}, and M_ω:

$$\left.\begin{aligned} X_\omega &= \Sigma X - X \\ Z_{1_\omega} &= \Sigma Z_1 - Z_1 \\ M_\omega &= \Sigma M - M_y \end{aligned}\right\} \qquad (36)$$

---

*Naturally, such a motion appears forced (i.e., the forces and moments acting on the airship are not in equilibrium) and corresponds either to motions occurring at different times along circles of different radii, or to the motion along the noncircular trajectory.

which, as will be seen, are defined as the difference between the results of tests with the curved model and the straight model. Naturally, in addition to this, both for the curved and the straight models, we must have β_o = const., v_o = const., and δ = const., which was the initial assumption in deriving the expressions (35) and (36). The forces and the moment X_ω, Z_{1_ω}, and M_ω, similarly to the usual aerodynamic factors, can be put in the form of nondimensional coefficients.

Thus, as will be seen (considering the indicated nondimensional coefficients as functions of the angular velocity), the method of curved models permits us, for the solution of many dynamic problems, to dispense with the measurement of the rotary derivative.

But, paying tribute to the generally accepted method of studying curvilinear flight by the aid of rotary derivatives, we shall find the latter.

As is well known, by rotary derivatives we understand the increment of the rotary moment or force per unit of angular velocity, that is to say,

$$\frac{\partial X}{\partial \omega_y} = \frac{X_\omega}{\omega_y}; \quad \frac{\partial Z_1}{\partial \omega_y} = \frac{Z_{1\omega}}{\omega_y}; \quad \frac{\partial M}{\partial \omega_y} = \frac{M_\omega}{\omega_y} \qquad (37)$$

Of course, it would have been more exact to write in the left members of the equations, not ∂X, ∂Z_1 and ∂M_y, but $\partial \Sigma X$, $\partial \Sigma Z_1$, $\partial \Sigma M$. But the designations for the derivatives here given are simpler and have already been used by us in our earlier work.

The rotary derivatives are generally referred to the unit of speed of flight. Writing in place of X_ω, Z_{1_ω}, and M_ω their values from (36), we divide the right and left members of the preceding equations by v_o:

$$\left[\frac{1}{v_o}\frac{\partial X}{\partial \omega_y}\right] = \frac{\Sigma X}{\omega_y v_o} - \frac{X}{\omega_y v_o}$$

$$\left[\frac{1}{v_o}\frac{\partial Z_1}{\partial \omega_y}\right] = \frac{\Sigma Z_1}{\omega_y v_o} - \frac{Z_1}{\omega_y v_o}$$

$$\left[\frac{1}{v_o}\frac{\partial M_y}{\partial \omega_y}\right] = \frac{\Sigma M}{\omega_y v_o} - \frac{M_y}{\omega_y v_o}$$

Substituting in the right members $\omega_y = \frac{\nu_o}{R_o}$, we get:

$$\left[\frac{1}{\nu_o}\frac{\partial X}{\partial \omega_y}\right] = R_o\left(\frac{\Sigma X}{\nu_o^2} - \frac{X}{\nu_o^2}\right)$$

$$\left[\frac{1}{\nu_o}\frac{\partial Z_1}{\partial \omega_y}\right] = R_o\left(\frac{\Sigma Z_1}{\nu_o^2} - \frac{Z_1}{\nu_o^2}\right)$$

$$\left[\frac{1}{\nu_o}\frac{\partial M_y}{\partial \omega_y}\right] = R_o\left(\frac{\Sigma M}{\nu_o^2} - \frac{M_y}{\nu_o^2}\right)$$

In the parentheses of the right members were obtained the relations of the aerodynamic factors to the squares of the velocities, which it is customary to designate by the symbol R with the subscripts x, z_1, and m_y. Introducing the designations

$$\frac{\Sigma X}{\nu_o^2} = R_{\Sigma X}, \quad \frac{\Sigma Z_1}{\nu_o^2} = R_{\Sigma Z_1}, \quad \frac{\Sigma M}{\nu_o^2} = R_{\Sigma M}$$

we finally get:

$$\left.\begin{aligned} \left[\frac{1}{\nu_o}\frac{\partial X}{\partial \omega_y}\right] &= R_o\left(R_{\Sigma X} - R_X\right) \\ \left[\frac{1}{\nu_o}\frac{\partial Z_1}{\partial \omega_y}\right] &= R_o\left(R_{\Sigma Z_1} - R_{Z_1}\right) \\ \left[\frac{1}{\nu_o}\frac{\partial M_y}{\partial \omega_y}\right] &= R_o\left(R_{\Sigma M} - R_{m_y}\right) \end{aligned}\right\} \qquad (38)$$

The expressions obtained enable us to find the rotary derivatives from wind-tunnel tests of curved models by measuring on the balance and on the moment apparatus the aerodynamic force and moment when the model is installed as shown above. The factors R_X, R_{Z_1}, and R_{m_y}, in the case of the angle β_o, are known from the normal tests.

It is clear that, strictly speaking, there is no sense whatever in measuring the forces and moments for the curved model at its various angles of attack with respect to the flow (as is done with straight models), since for each angle β_o there is a corresponding degree of curvature of the model;* this may be seen, for example, from formula (7).

However, we must remember that the value we obtain for the rotary derivative of the drag is a true value only for the model, since, being determined by the coefficients of drag, the value will depend to a large extent on the Reynolds Number; this cannot be said of the rotary derivatives of the lateral force and moment.

AND THE VELOCITY SCREENS USED.

In this manner, the method of curved models gives us (as will be seen from the expression (37)) the values of the rotary derivatives - in the first place, independently one from the other and, in the second place, taking into account all the terms of the series (34); that is to say, the generally accepted assumption that the rotary derivatives are independent of the angular velocity not only was nowhere assumed, but may be verified experimentally.

This constitutes the colossal advantage of the method analyzed.

6. VARIANTS OF THE APPLICATION OF THE METHOD OF CURVED MODELS

Now, let us see in what variants can the method of curved models be applied in the aerodynamical investigations of an airship.

We have already shown above that the values R_o and β_o, which are necessary for constructing the curved model profile, can be taken from the nomogram of the straight line:

$$\frac{1}{R_o} = f(\beta_o, \delta)$$

constructed according to the method set down in our work on radii of turn. In such a case the testing of the curved model will be an excellent verification for the values of the rotary derivatives, determined by the method of small oscillations.

*In the experimental part of this paper it will be shown that this strict condition need not be adhered to, at the expense of a rather small error, thus extending the limits of application of the method of curved models.

Thus, the first variant in the application of the curved model is the verification of the values which we have for the rotary derivatives. Let us examine this variant in detail.

The initial data (R_0 and β_0) for the construction of the curved model profile are taken from graph $\frac{1}{R_0} = f(\beta_0, \delta)$. This graph, constructed on the basis of the experiments with the model of the V-2 airship, is represented in figure 16. It shows the intersection of two families of curves, one of which gives the value of $1/R_0$ in the presence of the condition of equilibrium of the forces only, acting on the airship during rotation.* The other family gives the values of $1/R_0$ in the presence of the condition of equilibrium of the moments only.

Naturally, the line (straight line), joining the points of intersection of the corresponding curves of the first and second families, gives the values of $\frac{1}{R_0} = f(\beta_0, \delta)$ when there is equilibrium both of the forces and of the moments; i.e., it gives the values of R_0, β_0, and δ, corresponding to the flight along circles which is established.

Now, let us assume that we are preparing six curved models, each of which will be constructed by the aid of R_0 and β_0 obtained from the points of intersection of the corresponding curves of the first and second families. During the tests in the tunnel, the rudder of each of these models should be deviated at that angle δ, to which, according to figure 16, correspond the points of intersection of the curves of both families.

In the result of wind-tunnel tests we obtain the force $\Sigma Z_1 = Z_1 + Z_{1\omega}$, which will be equal numerically to the centrifugal force (from the condition of equilibrium). According to the force ΣZ_1, we find the rotary derivative (formula (37)).

Thus, the aerodynamic moment of each of the curved models, when the models are properly installed in the tunnel, will be equal to zero since, under the condition of equilibrium of moments, the aerodynamic moment is extin-

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*The centrifugal force also participates in the equilibrium.

guished by the rotary effect, and according to the third formula of the system (38) the rotary derivative of the moment, when taking $R_{\Sigma M} = 0$, will be equal to

$$\left[\frac{1}{\nu_o}\frac{\partial M_y}{\partial \omega_y}\right] = - R_o \, R_{m_y}$$

In this manner, the very result $R_{\Sigma M} = 0$ shows the accuracy of the rotary derivative of the moment, determined by the method of damped oscillations. In order to obtain therefrom the generally accepted relations of the rotary derivatives to the angle of attack, it is necessary to construct the rotary derivatives, obtained from the wind-tunnel tests of the entire series of curved models, as a function of the angle β_o (which, for each model, was a basic factor in the construction of its profile).

If we extrapolate the obtained graphs of the rotary derivatives up to $\beta_o = 0$, we obtain the values of the rotary derivatives which are necessary for calculating the criteria of dynamic stability.

Now let us see how the method of curved models can be applied as an independent method.

Let us assume that for a given whole series of values of R_o and β_o, we construct as many models as the combinations that can be made from R_o and β_o. Naturally, each of the models will have its own degree of curvature. Each model must be tested on ΣM and ΣZ_1 for different values of the rudder angle of deviation δ. After this, grouping the obtained values ΣM and ΣZ_1 for the various δ and plotting them as a function of R_o for each angle β_o, it is necessary, by means of graphical interpolation, to select on the curves $\Sigma Z_1 = f\ (R_o, \delta)$ those values of R_o and β_o for which, from the condition of equilibrium of forces, ΣZ_1 is equal to the centrifugal force $\frac{m\,\nu_o^2}{R_o}\cos\beta_o$ (by intersecting the net of curves $\Sigma Z_1 = f\ (R_o, \delta_o)$ with the net of hyperbolas $\frac{m\,\nu_o^2}{R_o}\cos\beta_o = \varphi(R_o)$). Thus, having those values of R_o and β_o with which, when δ is given, equilibrium of the forces is obtained, we obtain that family of curves which in figure 16 characterizes this equilibrium.

From the curves $\Sigma M\ (R_0, \delta)$ we select those values of R_0 and β_0 for which, according to the condition of equilibrium of moments, $\Sigma M = 0$.

In this manner we obtain the second family of curves (fig. 16), as a result of the intersection of which with the first family we obtain the sought nomogram $\frac{1}{R_0} = f(\beta_0, \delta)$, which is determined with incomparably greater accuracy (that is to say, taking into account the influence of all the variables), than with the method set forth in our work on radii of turn.

It is obvious that we can also obtain the graphs of the change of the rotary effects of the moment, of the lateral force and of the drag as a function of the angle of attack, of the rudder angle of deviation and of the angular velocity.

Besides, by testing only curved hulls of airships (without tail surfaces) and working up the data by the aid of methods analogous to the one described, we can obtain, simultaneously with the very interesting relation,

$$\left[\frac{1}{\upsilon_0}\frac{\partial M_y}{\partial \omega_y}\right]_{hull} = f(\beta_0)$$

(which it is extremely difficult to obtain in the tunnel with the method of small oscillations) as well as the values:

$$\left[\frac{1}{\upsilon_0}\frac{\partial Z_1}{\partial \omega_y}\right]_{hull} = f(\beta_0)$$

which, as far as we know, have not been determined anywhere experimentally with sufficient accuracy.*

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*The English, in their work, calculate for all airship hulls in general,

$$\left[\frac{1}{\upsilon_0}\frac{\partial Z_1}{\partial \omega_y}\right]_{hull} = 0.1\ m$$

where m is the mass of the airship; this requires expermental proof.

It is self-evident that such a complete investigation of curved models of airships is profitable only if we have one model which can be curved to the desired profile. The question of the possibility of producing such models (elastic) will be taken up next year by the Air Section of the Experimental Aerodynamic Department of the Central Aero-hydrodynamic Institute.

By arranging the pressure tubes of the curved models, we can find the distribution of pressure in flight along circles.

By the way, it is interesting to note that, by the aid of curved model tests on the distribution of pressure, we can find experimentally the influence on the final result of that altered length of the arc of the meridional contour which is obtained when the curvature is effected. According to our opinion, it is necessary for this purpose to integrate the distribution of pressure along the surface of the curved model and compare the obtained result with the integral of the same distribution of pressure, but with the corresponding points located along the surface of the straight model.

Thus, we think that the method of curved models herein expounded, when used on the very widest scale, can and must become the most exhaustive method of studying the dynamics of curvilinear flight and the dynamic stability of airships.

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Navy Department,
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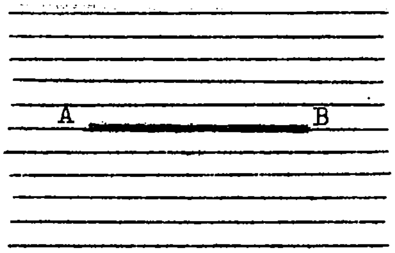
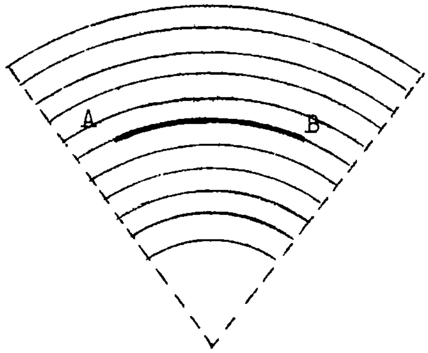


Figure 1.



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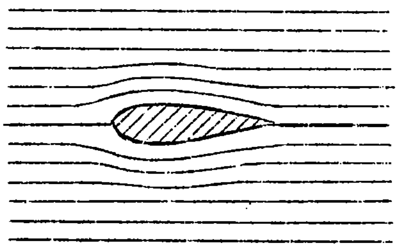

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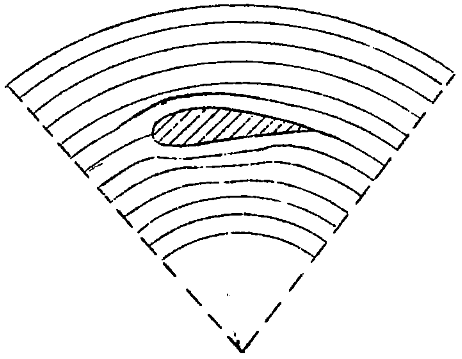

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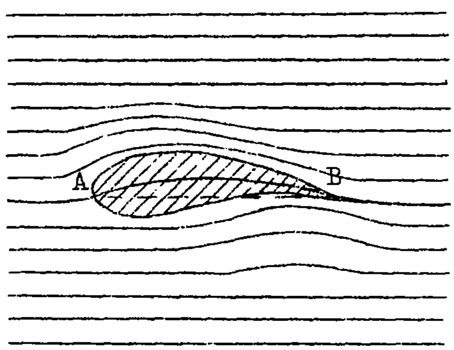

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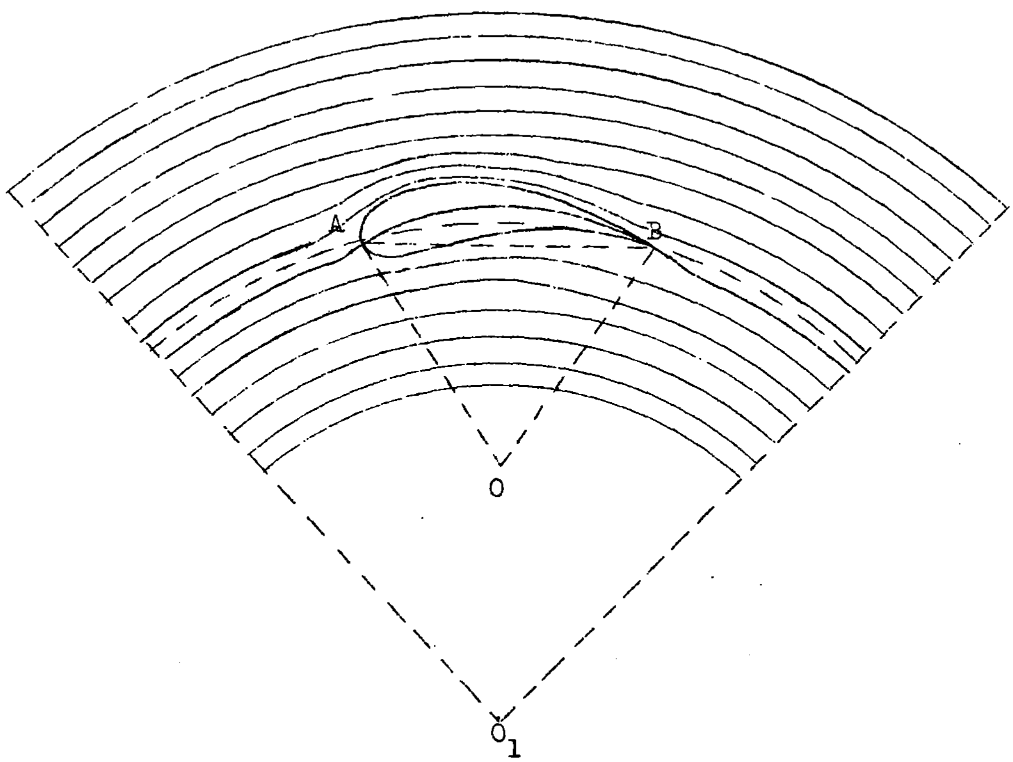

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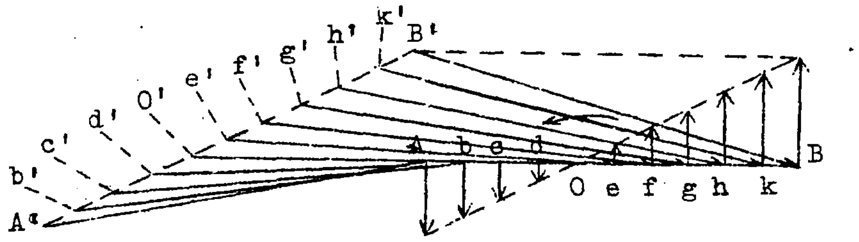

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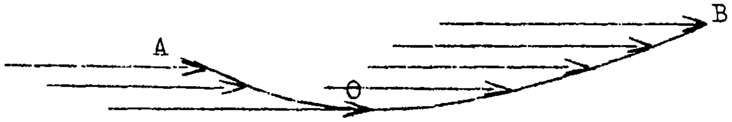

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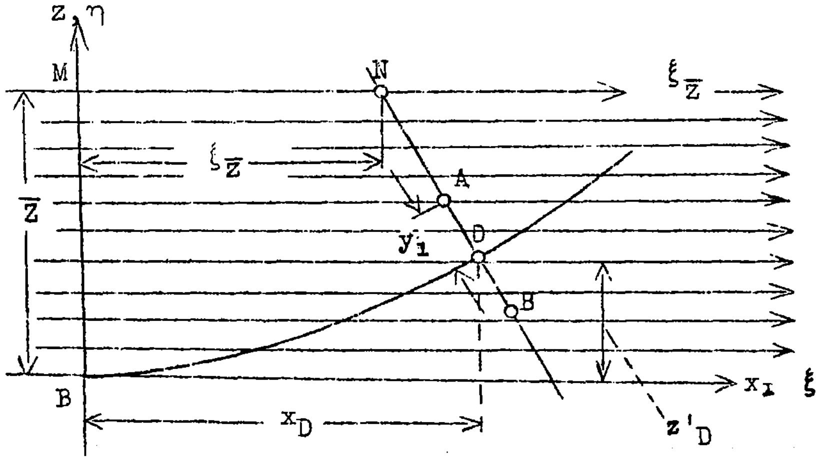

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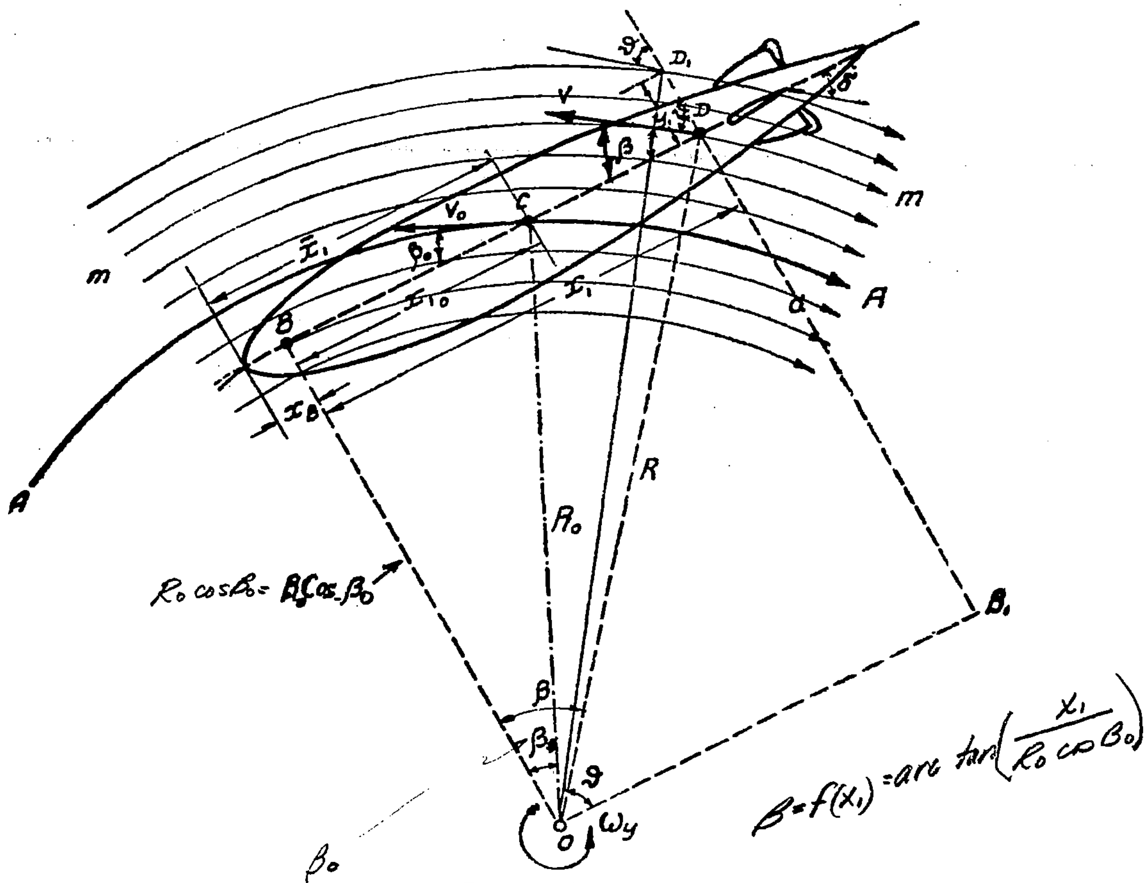

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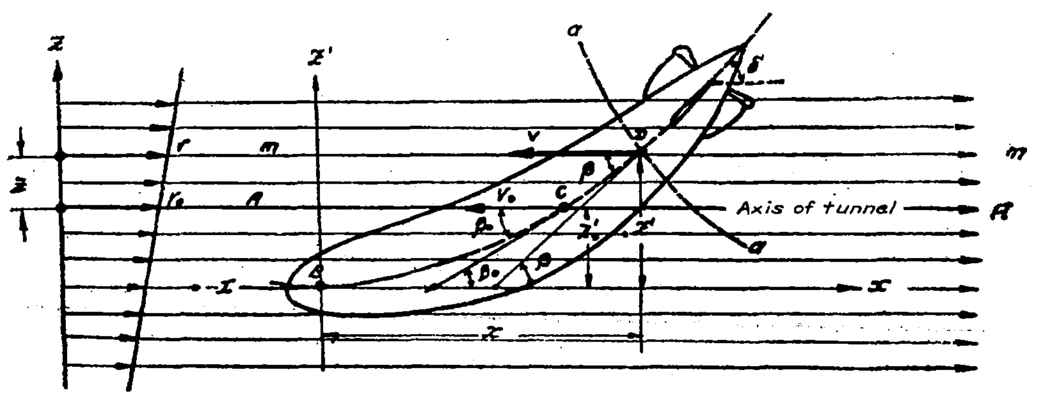

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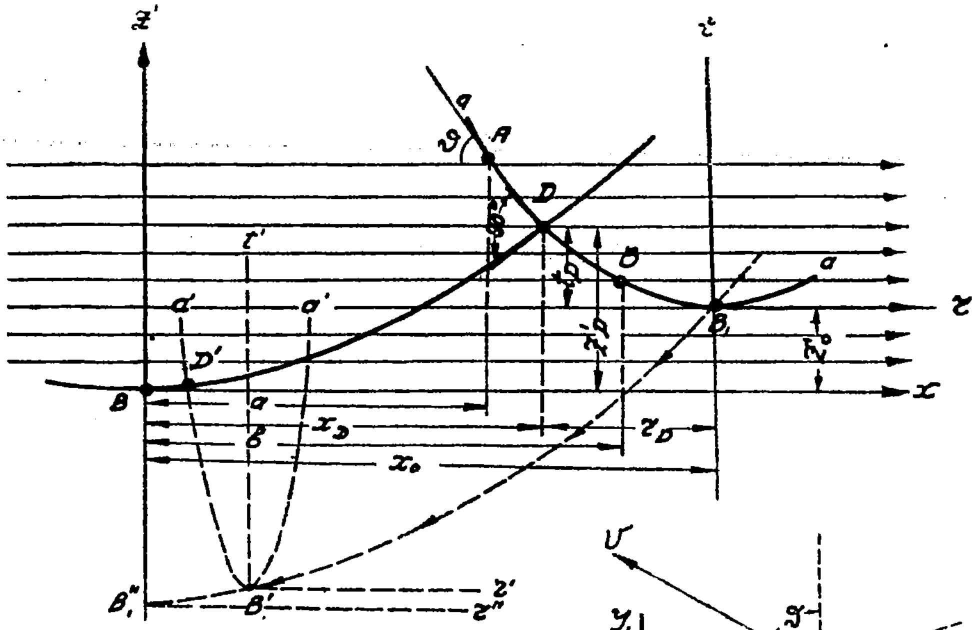
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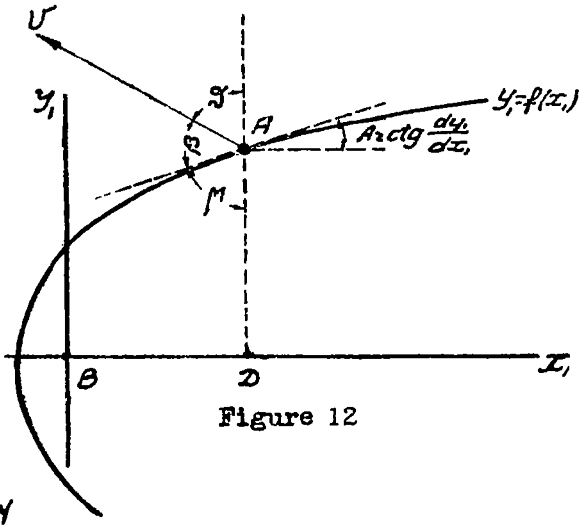
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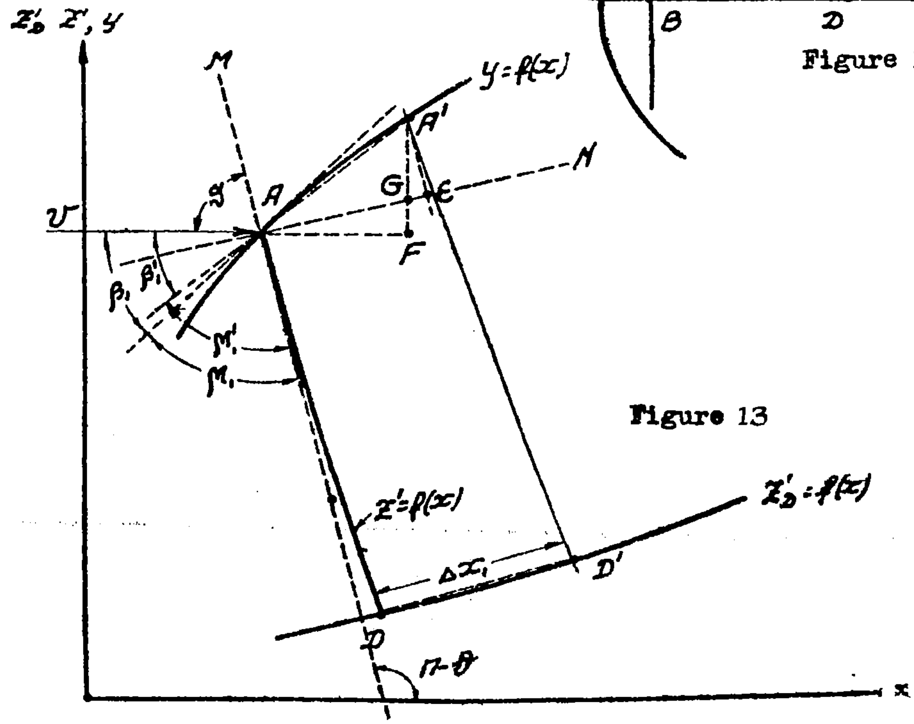
Figure 13

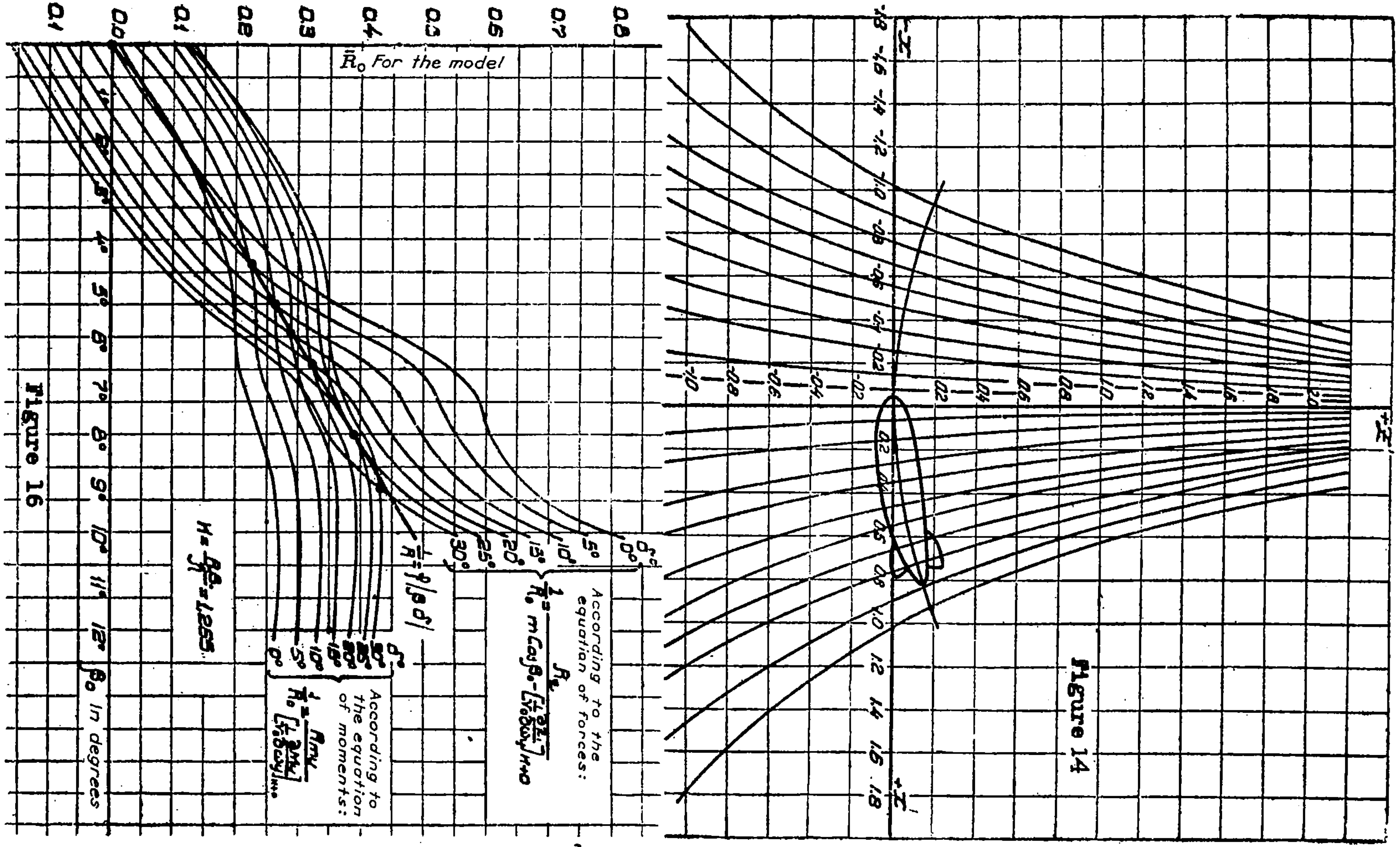

Figure 16.- Construction of the relation $\frac{1}{R_0} = f(\beta_0, \delta)$ according to the experiments with the model of the V-2 ship $\frac{1}{64.64}$HB (variant 1, λ = const.)