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GENERAL CONSIDERATIONS ON THE FLOW OF COMPRESSIBLE FLUIDS

By L. Prandtl

Paper presented at Volta meeting in Italy
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By L. Prandtl

Preliminary Remark

The Royal Academy of Italy has conferred upon me the honor of opening the theoretical division of this year's Volta Congress by a lecture which shall serve as an introduction to the subject of the flow of compressible media. In view of the limited time available, I shall entirely omit any description of the historical development of this branch of the science and shall also pass over the very familiar phenomena of hydraulics that receive the usual one-dimensional treatment as, for example, the discharge from orifices, the Laval nozzle, etc. I shall confine myself, rather, to a consideration of the most important properties of these flows, from a modern point of view, starting from the differential equations of compressible flow.

I. INTRODUCTORY CONSIDERATIONS

The problem of the motion of fluids which is already sufficiently involved even when considered as incompressible, becomes still further complicated and more difficult when the property of compressibility is taken into account. In the majority of cases, therefore, when the compressibility is to be allowed for, we are forced to make simplifying assumptions in some other direction. Thus in our discussion we shall have to neglect viscosity and so assume our fluid to be frictionless and compressible. We shall further assume that the density of the compressible fluid depends on the pressure only and such inhomogeneities as, for example, the heat conducted to the fluid from the outside or arising from inner combustion, are excluded

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*"Allgemeine Ueberlegungen über die Strömung zusammendrückbarer Flüssigkeiten." (Paper on High Speed in Aviation, read at the 5th Volta Congress, at Rome, September 30 to October 6, 1935.) Reale Accademia d'Italia, Classe delle Scienze Fisiche, Matematiche e Naturali.

from our discussion. We assume that the relation between the pressure p and the density ρ is uniquely determined.

For this frictionless, homogeneous, compressible fluid, just as for the frictionless, homogeneous, incompressible fluid, the law of Lagrange applies, namely, that a fluid without initial circulation continues to move without circulation. It may be observed that this law holds true for steady motion only. Where the velocities are higher than that of sound, the motion begins to be unsteady with the occurrence of irreversible compressions, leading to an increase in entropy. In this way homogeneity is in general destroyed and the Lagrangian law is no longer applicable.

Since the state of rest is a special case of irrotational motion, any compressible fluid flow starting from rest, whether steady or unsteady, is an irrotational flow, and as such may therefore be represented by a velocity potential whose gradient is the velocity.

$$V = \text{grad } \Phi \tag{1}$$

the components being

$$u = \frac{\partial \Phi}{\partial x}, \quad v = \frac{\partial \Phi}{\partial y}, \quad w = \frac{\partial \Phi}{\partial z} \tag{1a}$$

For such potential flows of homogeneous frictionless fluids the Bernoulli equation may be applied:

$$\frac{\partial \Phi}{\partial t} + \frac{V^2}{2} + P - U = f(t) \tag{2}$$

In this equation $P = \int \frac{dp}{\rho}$ denotes the pressure function and U the force function; in the case where gravity is the force considered $U = - gz$; $f(t)$ is an arbitrary function of time. Another equation to be considered is the equation of continuity which is an expression of the constancy of the mass. We may express it either by saying that a definite element of volume of the same particle of fluid continues to maintain a constant mass or that the mass of an element of fixed volume changes with time so that more flows in than flows out. From either point of view we arrive at the equation

$$\frac{\partial \rho}{\partial t} + \text{div } (\rho V) = 0 \tag{3}$$

where it is possible to put $\operatorname{div}(\rho V) = \rho \operatorname{div} V + V \cdot \operatorname{grad} \rho$ (where the dot denotes the scalar product). Equations (2) and (3), together with (1), define our problem. Except where meteorological applications are considered, gravity generally plays a subordinate part in problems connected with compressible flow, so that it is always possible to neglect the term containing U. In such cases as are considered in acoustics where the velocities are small and rapidly changing, the quadratic terms ($V^2/2$ in equation (2) and $V \cdot \operatorname{grad} \rho$ in equation (3)) may be neglected. By equation (1) and the relation $\operatorname{div} \operatorname{grad} \Phi = \Delta\Phi$ equation (3) may then be written

$$\frac{\partial \rho}{\partial t} + \rho \Delta \Phi = 0 \tag{3a}$$

We now differentiate equation (2) with respect to time and write

$$\frac{\partial P}{\partial t} = \frac{1}{\rho} \frac{dp}{d\rho} \cdot \frac{\partial \rho}{\partial t} = \frac{c^2}{\rho} \frac{\partial \rho}{\partial t} \tag{2a}$$

The dimensions of $dp/d\rho$ are those of a velocity squared and it may therefore be set equal to c^2, where the velocity c is still a function of ρ; when the velocity is zero at infinity, $f(t)$ may be set equal to a constant. The differentiated equation (2) thus becomes

$$\frac{\partial^2 \Phi}{\partial t^2} + \frac{c^2}{\rho} \frac{\partial \rho}{\partial t} = 0 \tag{4}$$

From the relations (3a) and (4) the equation is further reduced to

$$\frac{\partial^2 \Phi}{\partial t^2} = c^2 \Delta \Phi \tag{5}$$

which is the familiar differential equation of sound. A well-known solution of this differential equation is the plane sound wave, corresponding to the equation

$$\Phi = F(x - ct) \tag{6}$$

and which travels with velocity c, the form of the wave being given by the quite arbitrary function F. Another solution is the spherical wave whose equation is

$$\Phi = \frac{1}{r} f (r - ct) \tag{7}$$

where r denotes the distance from a center. The potential for the cylindrical wave is less simple. It may be written

$$\Phi = \int_{\xi_1}^{\xi_2} \frac{f(\xi)\, d\xi}{\sqrt{(ct - \xi)^2 - r^2}} \tag{8}$$

In all these examples c clearly signifies the velocity with which the waves travel outward. It is customary to call this velocity, which we have assumed above as a function of the density, the velocity of sound. The fact that in the above equations we have treated it as a constant clearly shows that the solutions are only approximations for very small sound amplitudes where the density nowhere differs much from the initial density ρ_0. For finite amplitudes it is necessary to go back to the exact equations (2) and (3) and we thus arrive at the result first obtained by Riemann,* that the regions of condensation move forward at a greater velocity than the regions of rarefaction and therefore the wave front becomes steeper and steeper until an unstable wave is formed which is known as a "shock wave." Since in the condensed region the gas moves forward, and moves backward in the rarefied region, while the impact wave moves with velocities that do not differ much from the sound velocity, we have the result that the relative velocity between the shock wave and the condensed region is smaller, and that between the impact wave and the rarefied region larger, than the sound velocity.

II. STEADY POTENTIAL FLOW

The above qualitative discussion will be sufficient for the case of the unsteady sound motions with small flow paths. I shall now turn to the main subject of my lecture, namely, steady potential flow. If we again neglect gravity equations (2) and (3) assume the form

$$\frac{V^2}{2} + P(\rho) = \text{const} \tag{2b}$$

and

$$\operatorname{div} V + \frac{1}{\rho} V \cdot \operatorname{grad} \rho = 0 \tag{3b}$$

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*See reference 1.

We may now set

$$\frac{1}{\rho}\,\text{grad}\,\rho = \frac{1}{\rho}\,\frac{d\rho}{dp}\,\text{grad}\,p = \frac{1}{c^2}\,\text{grad}\,P$$

and find grad P from (2b). We thus obtain

$$\text{div}\,V - \frac{1}{c^2}\,V \cdot \text{grad}\,\frac{V^2}{2} = 0 \tag{9}$$

Expressed in rectangular components u, v, w, we may write $V \cdot \text{grad}\,V^2/2$ as follows:

$$V \cdot \text{grad}\,\frac{V^2}{2} = V \cdot \Delta\,V \cdot V = u^2\,\frac{\partial u}{\partial x} + v^2\,\frac{\partial v}{\partial y} + w^2\,\frac{\partial w}{\partial z}$$

$$+ uv\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + vw\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) + wu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

Equation (9) expressed in cartesian coordinates thus assumes the form

$$\frac{\partial u}{\partial x}\left(1 - \frac{u^2}{c^2}\right) + \frac{\partial v}{\partial y}\left(1 - \frac{v^2}{c^2}\right) + \frac{\partial w}{\partial z}\left(1 - \frac{w^2}{c^2}\right)$$

$$- \frac{2}{c^2}\left(uv\,\frac{\partial u}{\partial y} + vw\,\frac{\partial w}{\partial x} + wu\,\frac{\partial w}{\partial x}\right) = 0 \tag{9a}$$

(In this equation use was made of the relation $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$ derived from the condition of irrotational motion, and the corresponding relations for the other components.)

From the form of equation (9) or (9a) it is immediately evident that in all cases where the velocity components are all small in comparison with the sound velocity, the relation reduces to

$$\text{div}\,V = 0$$

or

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

When the resulting velocity is not vanishingly small compared to the velocity of sound, but still is smaller than the latter, then the quantitative relations are changed

although the character of the motion still remains entirely similar to that of incompressible fluids. To see what equation (9) tells us when the flow velocities are of the order of magnitude of the sound velocity, we may so arrange the system of coordinates that the X axis coincides with the direction of flow. Then only the velocity component u, in the neighborhood of the origin, will be of the order of magnitude of c, while the velocity components v and w will be small compared to c. Equation (9a) then reduces to

$$\frac{\partial u}{\partial x}\left(1 - \frac{u^2}{c^2}\right) + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{10}$$

If we now assume that u increases in the direction of the flow corresponding to a fall in pressure in that direction, then $\partial u/\partial x$ is positive. From equation (10), therefore, $\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$ is negative when $u^2 < c^2$ and positive when $u^2 > c^2$ and becomes zero for $u^2 = c^2$. Expressed in words, this means that the flow converges with increasing velocity when the velocity is less than that of sound but diverges when the velocity is greater than that of sound, and moreover, that when the velocity passes through the sound velocity the streamlines are parallel. This conclusion fully agrees with what is found for the flow through a Laval nozzle from elementary considerations. In that case, too, it is found that when the pressure falls throughout the length of the nozzle the velocity in the converging part of the nozzle is smaller than the sound velocity and in the diverging part is greater. The sound velocity is exceeded just where the cross section of the nozzle is a minimum.

Equation (10), however, tells us more. If we introduce the relation given by equation (1), we obtain

$$\frac{\partial^2 \Phi}{\partial x^2}\left(1 - \frac{u^2}{c^2}\right) + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \tag{10a}$$

If we consider the immediate neighborhood of some definite point, it is sufficient to set u equal to its mean value in the region under consideration. Using this simplification, there is obtained a very well-known partial differential equation. This is of the elliptic type for the case where the expression $1 - u^2/c^2$ is positive;

that is, it is entirely related to the differential equal for incompressible flow. When $1 - u^2/c^2$ is negative, however, which is true when the velocity u is greater than the sound velocity, then the elliptic type goes over into the hyperbolic type. It is known, however, that the solutions of the elliptic type are regular within the region, whereas those of the hyperbolic type also admit of discontinuous solutions within the region, running through the range of the so-called characteristics of the differential equation. What is typical of both cases may be seen from the following consideration, which starts out from the well-known singularity in a flow source of an incompressible fluid

$$\left(\Phi = \frac{A}{r} = \frac{A}{\sqrt{x^2 + y^2 + z^2}}\right)$$

Let us inquire what is the form of the potential of such a source when the flow from the source has a constant velocity of the order of magnitude of the velocity of sound superimposed upon it. The problem may be simplified by assuming that the velocities from the source are small, thus making equation (10) or (10a) applicable. To find the singularity for this case we may apply a relative system in which the undisturbed medium is at rest and the source moves with the constant velocity $-u_0$. We may first assume an "explosion wave" (knallwelle) of spherical shape expanding in all directions, the potential of which is obtained by assuming in equation (7) a function f, which is different from zero only within a very short interval and vanishes outside this interval. This is the case when a small volume suddenly begins to increase and then maintains its new magnitude. It may then be assumed that a continuous series of such short expansions proceeds in such a manner that the center of the expansion travels forward with the velocity $-u_0$. Since we are here considering sound wave expansions obeying the differential equation (5), the potential for the whole process may, on account of the linearity of the differential equation, be built up by the superposition of the potentials of the individual waves. When the expansion has continued long enough and the center of the expansion is momentarily at the origin of coordinates, the computation leads to the following formula:

$$\Phi = \frac{A}{\sqrt{x^2 + \left(1 - \frac{u_0^2}{c^2}\right)(y^2 + z^2)}} \tag{11}$$

When u_0 is smaller than the sound velocity c, then the surfaces of constant potential instead of being spherical as is the case for incompressible fluids, are flattened ellipsoids of revolution. The more closely the velocity u_0 approaches the velocity of sound the more strongly flattened the ellipsoids become. When u_0 exceeds the velocity of sound, however, then the solution is different from zero only within a cone of angle α, which is determined by

$$\tan \alpha = \pm \frac{1}{\sqrt{\frac{u_0^2}{c^2} - 1}}$$

or

$$\sin \alpha = \pm \frac{c}{u_0} \tag{12}$$

The same result is also obtained when the momentary position of the individual waves is investigated (fig. 1). The angle α is known as the Mach angle. (It may here be remarked that when $u_0 < c$ the waves fill space in all directions, whereas for the case $u_0 > c$, they fill only the cone of fig. 1.) For $u_0 > c$, the constant potential surfaces are hyperboloids of two sheets having the given cone as asymptotic surface. Only one of these sheets has physical reality, as is easily seen from our description of the formation of the source potential from the explosion waves. Where still another potential is to be built up by the superposition of such sources, it is to be observed that everywhere outside the given cone, the potential of the individual sources is to be set equal to zero and takes on the values of formula (11) only within the cone.

III. FLOWS WITH "SUBSONIC" VELOCITIES - LINEAR THEORY

Formula (11) enables us to obtain a general relation for the solution of the differential equation (10a) and this we shall discuss further on. For "subsonic" velocities it is possible to relate every solution of this differential equation to a solution of the differential equation for constant volume potential flow

$$\frac{\partial^2 \Phi}{\partial \xi^2} + \frac{\partial^2 \Phi}{\partial \eta^2} + \frac{\partial^2 \Phi}{\partial \zeta^2} = 0$$

if we write

$$\xi = x; \quad \eta = y\sqrt{1 - \frac{u_0^2}{c^2}}; \quad \zeta = z\sqrt{1 - \frac{u_0^2}{c^2}}$$

and assign the same potentials to the corresponding points of each space (reference 2). The assumption must naturally be made, as for differential equation (10a), that the velocities derived from the potential are small compared with the basic velocity u_0. The application cannot therefore be made to flows in which a stagnation point occurs, since in this case the deviation from u_0 is just as large as u_0 itself. It is permissible, however, to apply the equation to flow about very narrow shapes having a sharp entrance edge. Given the potential flow Φ_0 for an incompressible fluid, the question next arises as to what is the form of the contour that corresponds by the above rules to the potential of the compressible fluid. The slope $\frac{dy}{dx} = \frac{v}{u}$ may, to a sufficient approximation, be put equal to

$$\frac{v}{u} = \frac{1}{u_0}\frac{\partial\Phi}{\partial y}$$

Correspondingly,

$$\frac{d\eta}{d\xi} = \frac{1}{u_0}\frac{\partial\Phi}{\partial\eta}; \quad \text{with} \quad \frac{\partial\Phi}{\partial y} = \frac{\partial\Phi}{\partial\eta}\sqrt{1 - \frac{u_0^2}{c^2}}$$

we have

$$\frac{dy}{dx} = \frac{d\eta}{d\xi}\sqrt{1 - \frac{u_0^2}{c^2}}$$

We then obtain the result that the entire contour must be made thinner than that corresponding to the incompressible fluid with equal potential values approximately in the ratio* $\sqrt{1 - \frac{u_0^2}{c^2}}$ and similarly the angle of attack must

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*It may be pointed out that the points on the contour do not correspond to the above transformation equations for the potential, which require an increase in the y ordinates for points of equal potential. The above conclusions are nevertheless applicable since the difference in the flow direction for points lying near each other differ only by a second degree order from the result given above on account of the slenderness of the contour under consideration.

be made smaller in the same ratio.*

We must still consider the pressure distribution on the surface. This is determined according to the Bernoulli equation by the term $\rho u \frac{\partial u}{\partial x}$ for which we may put $\rho u_0 \frac{\partial^2 \Phi}{\partial x^2}$. Since we had set $\xi = x$ and Φ was to have the same value at corresponding points, this magnitude remains constant in the transformation, and the same holds true for the pressures themselves. The tendency toward flow separation may therefore also be expected to be the same for both cases. The conclusion follows that in order to avoid separation of flow it is necessary to make the profiles of the airfoils, etc., flatter and the angles of attack must be made correspondingly smaller, as the velocity u_0 approaches the velocity of sound. The maximum lifts attainable according to this approximate theory will be the same as those that may be expected for the incompressible flow. Actually, it is found that on approaching the velocity of sound, the relations are considerably less favorable than indicated by this approximate theory. The chief reason for this is the fact that the superimposed velocities are not actually small compared to the basic velocity u_0 and consequently it is possible for points on the suction side to exist at which the velocity of sound is either reached or exceeded, so that considerable deviations from our computations are to be expected.

It may still be asked how the fact is to be explained that as the velocity of sound is approached, the same lift is obtained with a less cambered profile and at a smaller angle of attack. The explanation lies in the fact that

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*It is also possible to coordinate the points in such a manner that the potentials Φ in the xyz space will be a multiple of the potentials at the corresponding points of the $\xi\eta\zeta$ space. If the multiple chosen is $1/\sqrt{1 - \frac{u_0^2}{c^2}}$ then $\frac{du}{dx} = \frac{\partial\eta}{\partial\xi}$, that is, the profiles and angles of attack will now agree. The differences in pressure (see below) will now be raised in the ratio $1/\sqrt{1 - \frac{u_0^2}{c^2}}$ and the tendency to separation will therefore be increased.

according to the transformation given above, the potential field and therefore also the velocity field in the direction at right angles to the basic velocity extends further in the ratio $1/\sqrt{1 - \frac{u_0^2}{c^2}}$ and so the vertical velocities, which are smaller in the ratio $\sqrt{1 - \frac{u_0^2}{c^2}}$ produce in this extended region a total impulse of the same magnitude as would be the case for incompressible flow.

For basic velocities above that of sound this analogy cannot naturally be applied, since the transformation formula (13) would in this case give an imaginary result. Here other methods must be used. (See sections 5 and 6.)

IV. FLOWS WITH SUBSONIC VELOCITIES - HIGHER APPROXIMATIONS

If it is desired to obtain a better agreement with fact than is afforded by the linear theory, it is possible, starting either from the theory of potential flow for incompressible fluids or from the solutions of the differential equation (10a), by a step-by-step process to obtain closer approximations resulting from the application of the exact equation (9a). Computations of the first kind were carried out by many investigators (references 3 and 4) but, to the best of my knowledge, no computations of the second kind.* In practice these computations are rather laborious and do not agree very well with each other. It is therefore a noteworthy fact that it is also possible to find good approximations on the basis of electrical analogy, whereby a good approximate solution may be obtained - only for the case, however, where the velocity of sound is not attained at any point of the region considered. Since this method has been worked out by Professor G. I. Taylor (reference 7), who will give a more detailed report on it at this session, I shall not touch on this subject any further.

Through three short notes by Riabouchinsky (reference 8) and Demtchenko (references 9 and 10), in the Comptes

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*G. Braun (reference 6) has obtained solutions in the neighborhood of the sound velocity by applying a variation principle of Bateman (reference 5).

Rendus of the Paris Academy, 1932, and old work by Tchaplygin (reference 11) written in the Russian language in 1904 became known, wherein it was shown that for the case of two-dimensional flow, the problem may, by transformation to new coordinates (rectangular or polar), be presented in such a form that a potential flow between two plates at a predetermined variable distance apart, may be represented by an electric flow in an electrolyte of variable depth. One coordinate χ will be a function of the ratio of the local velocity to the maximum (which may naturally also be written as a function of the ratio of the density to the maximum density); the other coordinate θ is the direction angle of the velocity of the flow, the variable distance between the two plates being a function of the first coordinate only. Since it is possible to compute mathematically the solutions for such regions whose boundaries are determined by the lines χ = constant and θ = constant, it becomes possible to solve those problems of the Helmholtz-Kirchhoff type where flat walls (direction θ = constant) and free boundaries (constant velocity, therefore χ = constant) are considered for the case where compressibility is taken into account. For one particular condition which, though not occurring in nature, may be applied approximately when the difference in density is moderate, the problem may even be formulated as a usual type of potential flow in the coordinates χ and θ. Demtchenko (reference 10) showed among other things that for the Kirchhoff flow against a flat plate, when the velocity of the flow is half that of sound, the resistance is about 7 percent higher than the Kirchhoff resistance.

V. FLOW WITH SUPERSONIC VELOCITY - LINEAR THEORY - FLOW AROUND A CORNER

I shall now consider the flows for which the velocity is greater than the velocity of sound (the so-called "supersonic velocity"). We may here again start out from formula (10) or (10a). If a two-dimensional flow is considered, then equation (10a) now reads as follows:

$$\frac{\partial^2 \Phi}{\partial x^2}\left(\frac{u_0^2}{c^2} - 1\right) - \frac{\partial^2 \Phi}{\partial y^2} = 0 \qquad (10b)$$

The general solution of this linear differential

equation may immediately be set down. It is

$$\Phi = F_1 \ (y - x \tan \alpha) + F_2 \ (y + x \tan \alpha) \qquad (14)$$

where again

$$\tan \alpha = 1/\sqrt{\frac{u_0^2}{c^2} - 1} \qquad (15)$$

so that $$\sin \alpha = \frac{c}{u_0}$$

According to the reasoning that led from equation (9a) to equation (10), equation (14) is the solution for the neighborhood of a point, when the X axis is chosen to lie in the direction of the flow. The functions F_1 and F_2 are subject to the limitation only that the magnitudes appearing in the differential equation (10b) exist; otherwise they may be taken quite arbitrarily. When the functions F_1 and F_2 are taken to represent a wave form, then equation (14) represents the superposition of two wave trains, crossing each other at the angle $\pm\alpha$ with respect to the mean direction of flow. For a more general solution, the wave trains are such that for each position represented by equation (15), u_0 corresponds to the local mean velocity at that point. Since such waves may be photographed (see fig. 2), it is possible by measuring the angle 2α between the two crossing wave trains to obtain the ratio c/u_0. (It should be noted that c is not the constant sound velocity of the gas at rest but the variable sound velocity of the adiabatically cooled gas.) The direction of the flow bisects the angle.

Let us now consider the following special case of flow. In equation (14) let $F_2 = 0$, $F_1 = 0$ for positive values of the argument $y - x \tan \alpha$; for negative values let $F_1 = \lambda \ (y - x \tan \alpha)$. In the neighborhood of zero it is possible to pass to the limit zero by means of a transition arc, or the radius of curvature of the transition arc may be used. The above formula corresponds to the limiting value. To obtain the total potential, the potential of the straight streamlines $u_0 \cdot x$ must be added. In the region $x > y \cot \alpha$ the components of the velocity then become

$$\left.\begin{aligned} u &= \frac{\partial \Phi}{\partial x} = u_o - \lambda \tan \alpha \\ v &= \frac{\partial \Phi}{\partial y} = \lambda \end{aligned}\right\} \tag{16}$$

In our example, therefore, the flow is parallel in front of the boundary line $y = \tan \alpha$ and is in the direction of the X axis. Behind the boundary line it is also rectilinear but makes an angle β to the X axis given by (see figs. 3 and 4)

$$\tan \beta = \frac{v}{u} = \frac{\lambda}{u_o - \lambda \tan \alpha} \sim \frac{\lambda}{u_o} \tag{17}$$

If the density and pressure upstream from the boundary $y = x \tan \alpha$ are denoted by ρ_o and p_o, respectively, and the pressure below the boundary by p, then approximately

$$\begin{aligned} p - p_o &= - \rho_o u_o \cdot (u - u_o) \\ &= \rho_o u_o \lambda \tan \alpha \end{aligned} \tag{18}$$

Along the boundary $y = x \tan \alpha$, therefore, there is a pressure jump (which may be converted into a steady pressure rise by introducing the transition curves mentioned above in the function F_1). The fluid particles are accelerated in the direction of the pressure rise, that is, normal to the boundary line $y = y \tan \alpha$. This result may also be obtained directly from equation (16), from which the direction of the velocity vector is obtained, as follows:

$$\tan \alpha' = \frac{v}{u - u_o} = - \frac{1}{\tan \alpha}$$

Positive values of λ and β correspond to a condensation; negative values to a rarefaction. It should be stressed, however, that the above computation is derived from the approximated linear differential equation (10a) and is therefore valid for small pressure differences and small angles of deviation only.

What may be expected for larger deviations may clearly be seen if we take two pressure jumps of the same sign at a

short distance from each other and see what we get from equations (16) to (18). The result is obtained that for the case of a pressure rise the two pressure jumps converge; for a pressure drop they diverge. Not only is the direction of flow in the second field rotated by the angle β with respect to the first, but the velocity behind the first pressure wave is smaller and therefore the Mach angle α_2 is larger than α_1. For a rarefaction wave, on the contrary, the velocity behind the first wave is larger and therefore the Mach angle smaller. The two wave fronts converge or diverge accordingly by the amount $\beta + \alpha_2 - \alpha_1$.

The transition from the type of flow just considered to that of a continuously curved flow may now easily be effected by replacing the first streamline (given wall along which the flow takes place) by a polygon of very many sides. From each edge of the polygon, waves start out expanding and diverging in case of a convex wall and contracting and converging in case of a concave wall. In this latter case it is also possible that the waves meet each other completely. The compression may then assume an unsteady finite value (the so-called "compression shock"). This behavior of compression and condensation waves starting out from a curved wall is entirely analogous to the result already mentioned obtained by Riemann for plane waves of finite amplitude.

We see, moreover, from the result of a transition to a boundary of radius of curvature zero, that for a wall which forms a convex side inclined at an angle $-\beta$ (see fig. 6), the flow remains unchanged up to a surface forming an angle α_1 with the direction of flow, then expands within an angle $-\beta + \alpha_1 - \alpha_2$ maintaining the pressure and velocity constant along each ray; then in the direction $-\beta$ again passes over into parallel flow with constant velocity. The quantitative relations for a flow of this type for a gas that obeys the law $p = p_0 \cdot (\rho/\rho_0)^k$ were given in the Göttingen dissertation of Th. Meyer in 1908.* In case of a concave edge there is a compression shock which lies between the angles α_1 and $\beta + \alpha_2$. In front of and behind the condensation shock the flow shows constant velocity and constant pressure.

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*See reference 12.

VI. APPROXIMATE METHOD FOR GENERAL TWO-DIMENSIONAL SUPERSONIC FLOW

I shall now turn to the case where the flow is affected not only by a single wall but also by an oppositely lying wall. In this case the waves pass through each other from opposite sides and also deviate from their original direction. The relations are most clearly brought out by the following procedure.

In the same manner as, for the purpose of certain approximate calculations, a curve is replaced by a series of straight-line steps or a polygon, it is possible to replace the continuous deviation of the velocity direction in a supersonic flow by a series of sudden deviations of the kind shown in figures 3 and 4. If the angle β of these deviations is chosen to be of the same size in all cases, for example 2°, then only such directions occur which differ by an integral multiple of β; that is, in our example, of 2°. If we start from a definite supersonic velocity, then on the basis of the Meyer formula, only certain discrete values of the velocity may occur. In order to obtain a better idea of these velocities and directions, the method suggests itself of drawing all the possible velocity vectors of the system at each point. If in this figure we trace those conditions corresponding to the case we have just considered, namely, where waves are assumed to start out from one wall, then all the points in the velocity picture corresponding to that state lie on a single curve (the thick curve of fig. 7). The entire system of velocity vectors for the case where waves travel into the region under consideration from both sides is then evidently obtained by drawing curves of the same sort through all points lying 4° apart on a circle w = constant ($w = \sqrt{u^2 + v^2}$). All the intersection points so obtained then represent the end points of the velocity vectors possible in this system.

According to a relation shown in a previous section, the direction of the vector difference of the velocity, in front of and behind a wave, is perpendicular to the wave front. This relation enables us to construct graphically the entire flow picture for any flow when the magnitude and direction of the velocity at the entrance section, and either the pressure or direction of flow at the side edges, are given beforehand. For the permanent gases the curves

of figure 7 are epicycloids and it is possible to construct the curves easily on the drafting board by using a rotatable ellipse to draw the directions normal to the curve tangents, as Busemann has shown.

In order to maintain constant the flow direction along the wall, for the case of a straight wall, or maintain the velocity constant in the case where a constant pressure is given, in accordance with Bernoulli's law, it is necessary to reflect the expansion and compression waves that reach the boundary. As may easily be seen from figure 7, at a straight wall condensations are reflected as condensations and rarefactions as rarefactions; at a free jet boundary, however, condensations are reflected as rarefactions and rarefactions as condensations. For curved walls, the walls are replaced by a polygon of angle β and waves start out from each corner.

In order to obtain a clear construction it is advisable to number each epicycloid, one set with numbers increasing in clockwise order, the other set decreasing in clockwise order. It may then easily be seen that the difference of these sets of numbers is a measure of the angle of the flow direction, while the sum is constant on a radius; in other words a function of the velocity. It may also be remarked in this connection that, as in the treatment of the subsonic velocity by Tschapligin, the angle of the flow direction and a function of the velocity again appear as deciding factors.

It is possible by a contact transformation to transform equation (9a) so that u, v, and w become the independent variables; and in the case of two-dimensional flow in the plane of u and v, it is possible to pass over to polar coordinates and so introduce the radius and the angle as new independent variables. As the form of equation (9a) already shows, the contact transformation offers the advantage that the differential equation becomes linear in the dependent variables, since the expressions $1 - u^2/c^2$, etc., are now functions of the coordinates. By a suitable stretching of the radii - that is, by introducing a function of the radius instead of the radius - it is possible to simplify still further the differential equation as was done by Tschapligin. When the relations between the waves in the field of flow and the corresponding velocity field are somewhat more closely analyzed, it is found that to each field of flow limited by two waves in the one system and two waves of the other system there corresponds a point

in the velocity field (since according to the assumption made the velocity is to be constant in each such field). Conversely, to a point of intersection of two waves in the field of flow there corresponds a quadrilateral in the velocity field, since at the point of intersection four fields of different velocities meet. Both of these figures are therefore the reciprocals of each other, just like a truss diagram and its reciprocal force diagram. The similarity goes even further. Just as a truss together with its force plane may be represented by the Airy stress surface, so in this case there exists a corresponding function, namely, the function $ux + vy - \Phi$ obtained from the potential Φ by the contact transformation.

The graphic method here described, to which there naturally corresponds an analytical method derived on the same basis, offers a very convenient means of discussing all two-dimensional supersonic flows. Applications of this method have been made to the discharge from nozzles, the flow about streamlined bodies, airfoils, etc. See A. Busemann (reference 13) or the examples given in his article in the Handbuch der Experimentalphysik (reference 14). For rotational symmetric flows there are no such simple methods. The analogous cases to the flow around a corner, the flow at the tip of a cone of finite angle, have been solved graphically by Busemann (reference 15) and analytically by G. I. Taylor and J. W. Maccoll (reference 16). Solutions for the general case of supersonic flow about a body of rotation have been given at least by approximate linearized methods by Th. v. Kármán and N. B. Moore (reference 17), which methods are chiefly applicable to very slender bodies.

VII. PASSAGE THROUGH THE VELOCITY OF SOUND

The methods described in sections 3 and 4 for the flow with subsonic velocity approach one another only provided the velocity of sound is nowhere reached. The methods discussed in sections 5 and 6 for supersonic velocities are similarly applicable only when the velocity is everywhere greater than the sound velocity. Neither of these two methods therefore provides any information about the manner in which passage through the velocity of sound takes place. There are, however, solutions for such cases that have been obtained some time ago. The first one, given by T. Meyer (reference 12), was obtained by a power series development in the coordinates x and y

of a point lying on the curve $u = c$. It turned out that the flow potential could be developed at such a point and tests showed good agreement with the computations. Another example was given by G. I. Taylor (reference 18). This is the curved line flow in a usual potential vortex, where the velocities are exactly as large as the velocities in the potential vortex of an incompressible fluid. Both of these examples, the one with a straight mean streamline and an increasing velocity in the direction of flow, the other with curved streamlines and a constant velocity along each streamline, allow us to conclude that in going from subsonic to supersonic velocities a continuous passage through the sound velocity may be expected. It is to be noted, however, that in the supersonic velocity region it is very easy for conditions to be set up such that a series of infinitesimal condensation waves combine to form a condensation shock of finite magnitude. In the passage from large velocities to smaller such condensation shocks occur almost regularly. In this connection it should also be mentioned that, in contradistinction to potential flows with subsonic velocities, which are symmetrical forward and aft about bodies that are themselves symmetrical at both ends, a potential flow with supersonic velocities is probably never symmetrical because the condensation and rarefaction waves starting out from the walls of the body always travel obliquely toward the rear and never forward. Perhaps it is still possible by taking this lack of symmetry into account, to solve analytically more of those cases of flow where passage from subsonic to supersonic velocities occur, especially when the supersonic region is limited in extent, as may be expected, for example, in the flow about cylinders, etc.

From the analytical standpoint, for steady frictionless flow, it is possible by reversing the velocities of one solution to obtain another solution. In applying this method to a symmetrical body or channel with nonsymmetrical flow forward and aft, we thus obtain by reflection a second solution different from the first and with the same flow direction. The passage from the single valued symmetrical solution for the subsonic velocities to the two-valued solution indicates the existence of a special kind of branching position. This may perhaps be the reason for the fact that the analytical methods that have been used up to the present have ceased to converge. By the application of the direct methods of the calculus of variations, G. Braun (reference 6) was able to obtain solutions for the branching positions, which solutions, however, should be applicable to infinitely small deviations only.

Here I should like to say a little more about the compression shocks. The points in front of and behind the shock do not lie on an adiabatic, but rather there is an increase in entropy due to the irreversible process. The magnitude that remains constant throughout the condensation shock is the total energy = kinetic energy + heat content; per unit mass = $w^2/2 + I$ (the heat content $I = U + pV$ is also called the enthalpy). In the general case the direction of flow forms an angle with the normal to the impact surface. These relations assume their simplest form for the two-dimensional compression shock. If the normal to the impact plane is taken as the X axis and the direction of flow is in the XY plane, the following equations result:

$$\rho_1 u_1 = \rho_2 u_2 \quad \text{..... (continuity)}$$

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2 \quad \text{..... (impulse in the X direction)}$$

$$v_1 = v_2 \quad \text{..... (impulse in the Y direction)}$$

$$\frac{1}{2}(u_1^2+v_1^2) + I_1 = \frac{1}{2}(u_2^2+v_2^2) + I_2 \quad \text{.. (energy)}$$

(The index 1 for the condition before the impact, index 2 for the condition after the impact.)

The computations for this problem, using the equations of state for ideal gases as a basis, were carried out by Th. Meyer (reference 12) after Hugoniot (reference 19), and independently Stodola (reference 20) had previously explained the behavior of the normal compression shock. The relations are clearly brought out in a diagram given by Busemann (reference 21). In the velocity-field picture he draws for each given initial velocity, the "impact polar," namely, the geometrical locus of all the velocity vectors for the state after the impact and thus obtains curves like those of figure 8, where β denotes as before the deviation angle of the flow, γ the angle between the impact plane and the direction of flow. The smallest value of γ is obtained for shocks with small velocity difference and naturally agrees with the Mach angle α. The largest difference in velocity is obtained in normal impact. The velocity behind the normal impact is always smaller than that of sound. The increase in entropy practically denotes a lowering in the pressure compared to that which would follow from the Bernoulli equation. This

difference in pressure is only insignificant, however, when the difference in velocity in front of and behind the shock is not too great. The process in any case involves a loss in energy.

VIII. APPLICATION TO AIRFOILS

Before concluding we shall make a few observations on the application of the foregoing discussion to airfoils. As far as subsonic velocities are concerned, it may be pointed out that in the neighborhood of the wing tip considerable supersonic velocities are set up which for high lift coefficients attain double the values of the flight velocity. It may therefore be expected that at speeds of 170 m/s (380 miles/hr.) the velocity of sound may already be attained locally. This may explain the fact that in the region of 200 m/s (450 miles/hr.) there is already a notable decrease in the lift coefficient.

As regards the induced velocity, it should be pointed out that it is possible to obtain the wing lift as well as the induced drag from the trailing vortices behind the wing. The velocities in these vortices are, however, in each case small compared to the velocity of sound, so that the usual laws for incompressible flow may be applied without objection for the computation of the lift and induced drag. In computing the lift distribution of a wing of given form, however, it is necessary to take into account the fact that due to the compressibility, the value of $\frac{dc_a}{d\alpha}$ is increased in the ratio $1 : \sqrt{1 - \frac{u^2}{c^2}}$ (as far as the approximate formula remains applicable).

For the profile characteristics at supersonic velocities, an approximate formula may be derived from the application of the considerations of section 5. If we consider a flat plate that is inclined by the angle β to the direction of flow (fig. 9) (Ackeret reference 22), then on the pressure side the relations of figure 3 hold and there results therefore a pressure rise of an amount that is easily computed from formulas (17) and (18). On the suction side there is a corresponding lowering in pressure as required by the above-mentioned formulas for a negative angle. We thus obtain the lift coefficient

$$c_a = 4\beta / \sqrt{\frac{u^2}{c^2} - 1} \qquad (19)$$

It is a noteworthy fact that the suction force, which exists in the case of subsonic velocities at the forward edge, is entirely missing in a flow of this type and therefore the resulting force is here not perpendicular to the direction of flow, but normal to the surface, so that even for a frictionless fluid the lift-drag ratio becomes

$$\frac{c_w}{c_a} = \beta = \frac{c_a}{4}\sqrt{\frac{u^2}{c^2} - 1} \tag{20}$$

It may be seen that for a given value of c_a this ratio is favorable just above the velocity of sound, but with increasing velocity, becomes considerably less favorable. (At very large speeds the lift coefficients will likewise be very small.) These increased drags above the velocity of sound bear a connection with the waves that travel outward from the moving object. Busemann (reference 23) was able to show that this wave energy was converted into heat partly near the wing and partly at a great distance away.

As far as the induced drag is concerned, it follows different laws at speeds above the velocity of sound from those below the velocity of sound. For an unwarped rectangular airfoil the induced drag is all included in the triangular regions at each end that are limited by the Mach angle (cross-hatched areas in fig. 10). It is, moreover, not clearly separable from the wave resistance which according to the above is also proportional to c_a^2 and therefore does not have the same independent significance as the case of subsonic velocities. According to a remark of Busemann, the induced drag disappears entirely when the two triangular parts lying in the region of disturbance are removed.

Translation by S. Reiss,
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REFERENCES

1. Riemann, B.: Ueber die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite. Abh. d. Ges. d. Wiss. Göttingen, Math.-Phys. Kl., 8(1860), p. 43. Gesammelte Werke, Leipzig, 1876, p. 144.

   Riemann-Weber: Die partiellen Differentialgleichungen der Mathematischen Physik. 6. Aufl. Braunschweig 1919 (5. Aufl. 1912), vol. 2, p. 503.

2. Prandtl, L.: Ueber Strömungen deren Geschwindigkeiten mit der Schallgeschwindigkeit vergleichbar sind. Jour. of the Aeronautical Research Institute, Tokyo Imp. Univ. No. 65 (1930), p. 14.

   Glauert, H.: The Effect of Compressibility on the Lift of Aerofoils. Proc. Roy. Soc. A, vol. 118 (1928), p. 113.

3. Janzen, O.: Beitrag zu einer Theorie der stationären Strömung kompressibler Flüssigkeiten. Phys. Zeitschr., vol. 14 (1913), p. 639.

4. Rayleigh, Lord: On the Flow of Compressible Fluid Past an Obstacle. Phil. Mag., vol. 32 (1916), p. 1. Scientif. Paper, vol. 6, p. 402.

5. Bateman, H.: Notes on a Differential Equation which Occurs in the Two-Dimensional Motion of a Compressible Fluid and the Associated Variational Problems. Proc. Roy. Soc. A, vol. 125 (1929), p. 598.

6. Braun, G.: Die ebene kompressible Potentialströmung als Variations- und Eigenwertproblem. Dissertation, Göttingen, 1932, Ann. d. Phys. (5), vol. 15 (1932), p.645.

7. Taylor, G. I., and Sharman, C. F.: A Mechanical Method for Solving Problems of Flow in Compressible Fluids. Proc. Roy. Soc. A, vol. 121 (1928), p. 194.

8. Riabouchinsky, D.: Mouvement d'un fluide compressible autour d'un obstacle. Comptes Rendus, vol. 194 (1932), p. 1215.

9. Demtchenko, B.: Sur les mouvements lents des fluides compressibles. Comptes Rendus, vol. 194 (1932), p. 1218.

10. Demtchenko, B.: Variation de la résistance aux faibles vitesses sous l'influence de la compressibilité. Comptes Rendus, vol. 194 (1932), p. 1720.

11. Tchaplygin, A.: Ueber Gasstrahlen. Wissensch. Annalen d. Univ. Moskau, Phys.-Math. Kl., vol. 21 (1904). p. 1. (russisch)

12. Meyer, Th.: Ueber zweidimensionale Bewegungsvorgänge in einem Gas, das mit Ueberschallgeschwindigkeit strömt. Diss. Göttingen. Forschungsheft, No. 62, des Vereins deutscher Ing., Berlin, 1908.

13. Prandtl, L., and Busemann, A.: Näherungsverfahren zur zeichnerischen Ermittlung von ebenen Strömungen mit Ueberschallgeschwindigkeit. Stodola-Festschrift, Zurich, 1929, p. 499.

14. Busemann, A.: Gasdynamik. Handbuch d. Exper.-Physik, vol. IV, 1, Leipzig, 1931, p. 421.

15. Busemann, A.: Drücke auf kegelformige Spitzen bei Bewegung mit Ueberschallgeschwindigkeit. Z.f.a.M.M., vol. 9 (1929), p. 496.

16. Taylor, G. I., and Maccoll, J. W.: The Air Pressure on a Cone Moving at High Speeds. Proc. Roy. Soc. A, vol. 139 (1933), pp. 278 and 298.

17. Kármán, Th. v., and Moore, N. B.: Resistance of Slender Bodies Moving with Supersonic Velocities with Special Reference to Projectiles. Trans. Amer. Soc. Mech. Eng., June 1932.

18. Taylor, G. I.: Recent Work on the Flow of Compressible Fluids. Jour. London Math. Soc., vol. 5 (1930), p. 224. Strömung um einen Körper in kompressibler Flüssigkeit. Z.f.a.M.M., vol. 10 (1930), p. 334.

19. Hugoniot, H.: Jour. éc. polyt., cah 58 (1889), p. 80.

20. Stodola, A.: Die Dampfturbinen und die Aussichten der Wärmekraftmaschinen. Z.V.D.I., vol. 47 (1903), p. 7. Dampf- und Gasturbinen 5. Aufl., Berlin, 1922, p. 68.

21. Busemann, A.: Verdichtungstösse in ebenen Gasströmungen. Vorträge aus d. Gebiete d. Aerodynamik, Aachen, 1929.

22. Ackeret, J.: Luftkräfte auf Flügel, die mit grösserer als Schallgeschwindigkeit bewegt werden. Z.F.M., vol. 16 (1925), p. 72.

23. Busemann, A.: Gasdynamik. Handbuch d. Exper.-Physik, vol. IV, 1. Leipzig, 1931, p. 443.

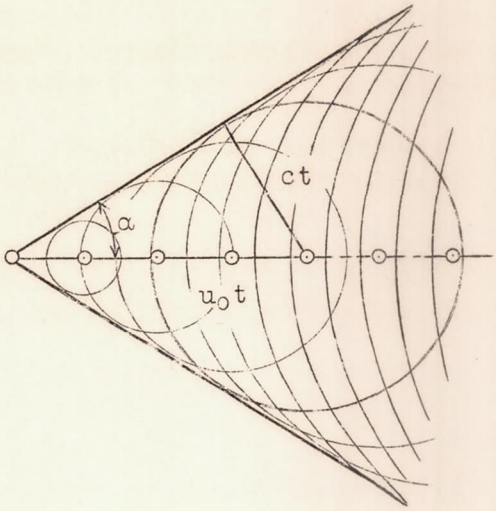


Figure 1

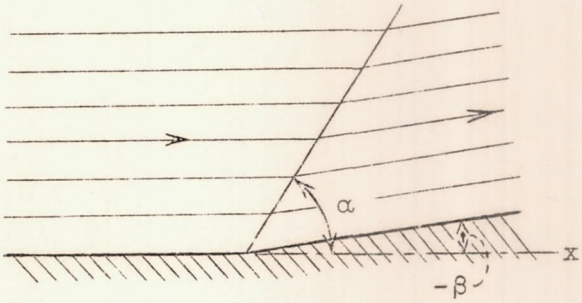


Figure 3

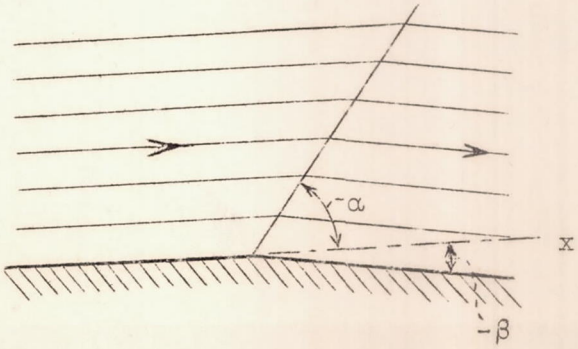


Figure 4

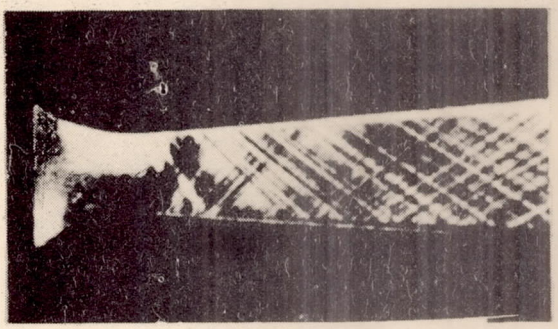


Figure 2

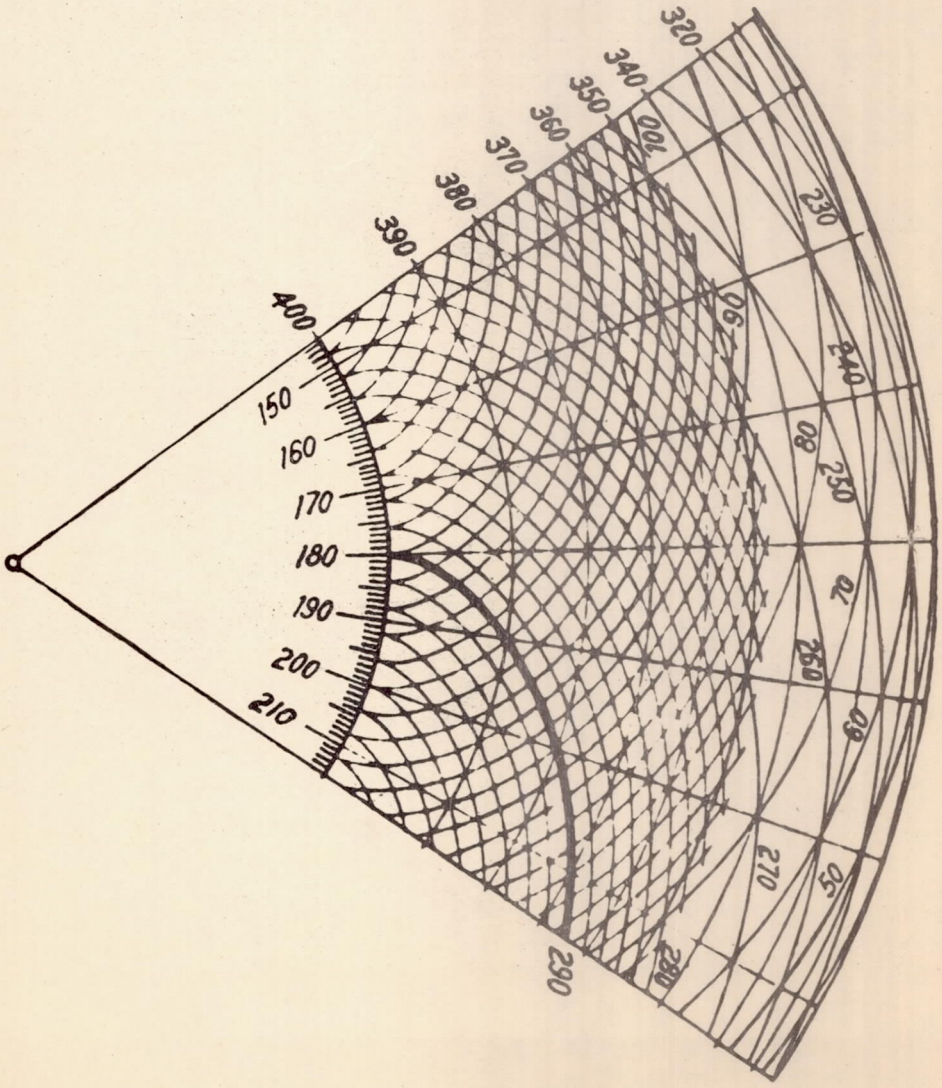


Figure 7

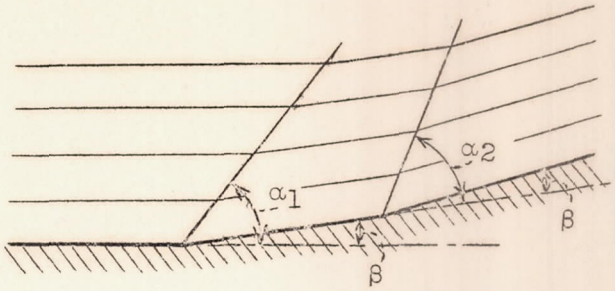
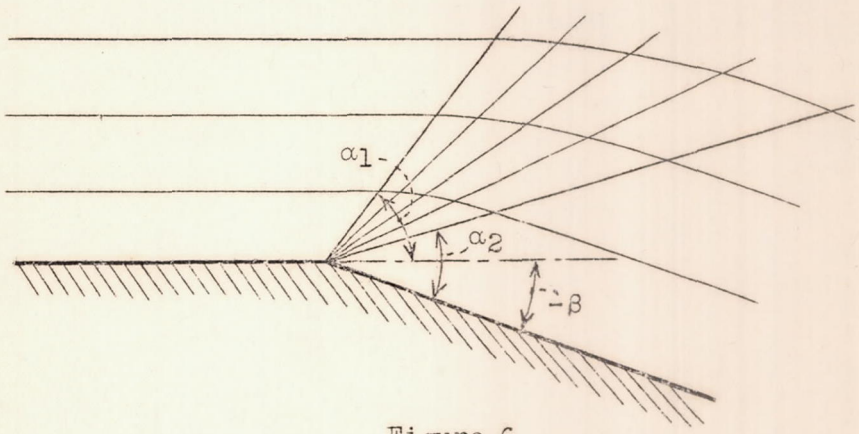


Figure 5



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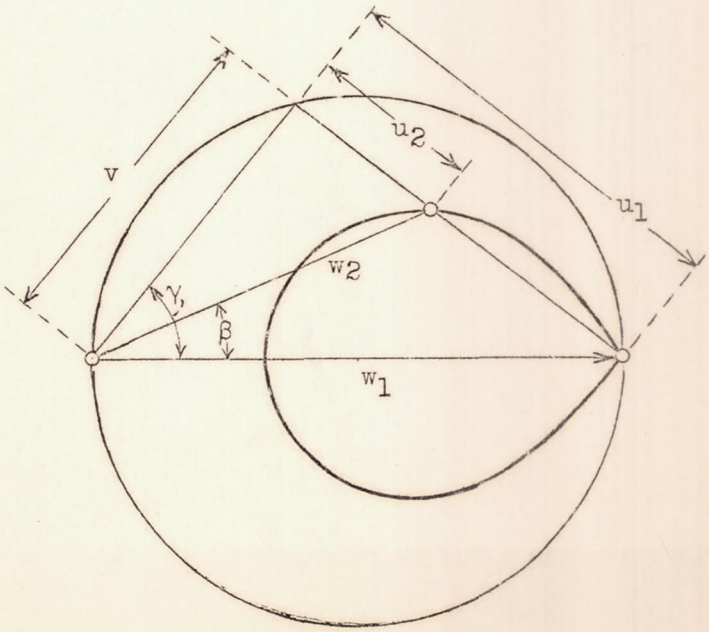

Figure 8

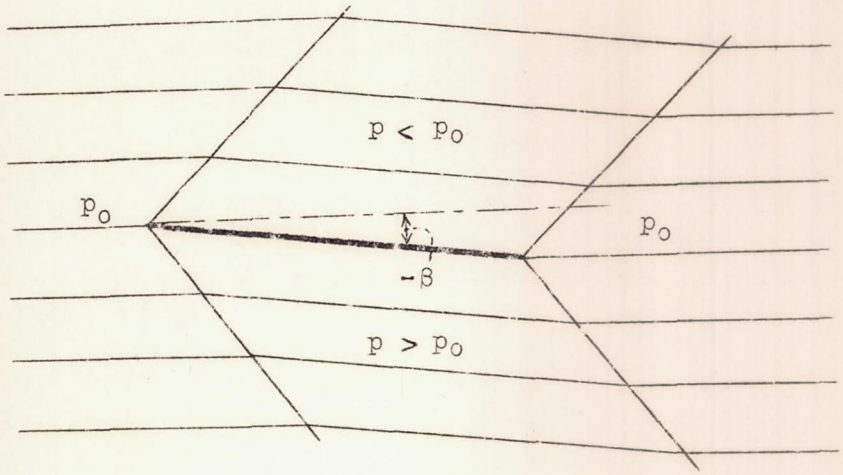

Figure 9

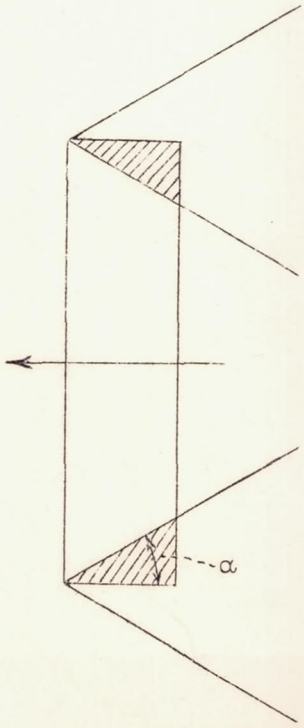

Figure 10