TECHNICAL MEMORANDUMS

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

NACA-TM- No. 611

MECHANICAL SIMILITUDE AND TURBULENCE

By Th. v. Karman

Reprint from Nachrichten von der Gesellschaft
der Wissenschaften zu Gottingen, 1930

Washington
March, 1931

22

N62576

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM NO. 611

MECHANICAL SIMILITUDE AND TURBULENCE*

By Th. v. Karman

The development of hydrodynamics within the last decade has shown that skillful application of the equations from the dynamics of ideal fluids quite often brings clarity into such phenomena which in themselves are not independent of the viscosity. The vortex equations, in particular, proved themselves very useful. I may be allowed to mention the theory of the vortex street by which we are able to reproduce the mechanism of the form resistance with suitable approximation under stated conditions, although such a resistance is precluded in a fluid which is perfectly inviscid. Disregarding for the present the origination of the vortex, the stream attitude in the wake of the body may be described approximately correct by the representation of individual vortices, without transgressing the law governing the motion of such vortices in an ideal fluid. Another striking example is the theory of the induced drag of wings, which likewise shows the extent of applying the vortex equations without overstepping the bounds of the dynamics of ideal fluids.

But the prospects are ostensibly less promising for turbu-

*"Mechanische Ähnlichkeit und Turbulenz." Reprint from Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, 1930 - Fachgruppe I (Mathematik) No. 5, pp. 58-76.

lent fluid motion. It naturally is clear from the beginning that the balance between frictional and inertia forces preponderates in the problem of nascent turbulence, so that the cited method does not enter into consideration at all in this case. But there remains the topic of the "fully developed turbulence," the problem according to the theory of "hydraulic stream attitude," which is perhaps of still greater importance for the practice. Still there are some indications which hold at least some promise of success in an attack on the problem when disregarding the frictional forces, or better expressed, with the frictional forces confined to a definite zone, for instance, directly adjacent to the wall. Some time ago W. Fritsch* measured in the Aachen Aerodynamic Institute the velocity distribution in grooves between two parallel walls by constant wall distance and for varying degrees of wall roughness. The results showed that the distribution curves - apart from the immediate neighborhood of the wall - are almost exactly superimposed as soon as the shearing stress at the wall assumes the same value, regardless of whether the fluid passes by a polished, smooth wall or, with correspondingly lower velocity, past a rough wall, or even a wall being saw-like in profile. The flow resistance in these grooves with very rough walls follows the so-called "square" law, i.e., it is proportional to the fluid density and to the square of the velocity, but unaffected by the degree of fluid viscosity. Is it not feasible therefrom to surmise that

*W. Fritsch, Zeitsch. f. angew. Math. u. Mech., 1928, p. 215.

the viscosity is without predominating effect on the development of the velocity distribution? The present paper represents a method of attack in this sense, and endeavors an attempt to make the laws of turbulent flow in grooves amenable for calculation with a minimum of arbitrary assumptions.

We are indebted to O. Reynolds* for his explanation of the existence of turbulence as oscillatory motion, which - in contrast to molecular unrest, responsible for the laminary friction phenomena - he designated as molal fluctuating motion. Segregating the velocity components dependent on the time (We call them flüctuating, or oscillatory components) further the pressure variations from the mean values which correspond to the basic flow, the general hydrodynamic equations reveal the momentum components transmitted by the oscillatory motion as supplementary stresses from the standpoint of the basic motion, which are defined by mean values of products from the oscillatory velocities.

I confine myself to the case of two-dimensional flow and express the mean of velocity x in the mean flow direction by U, that of velocity y by V, and the oscillatory components by u and v. Then the supplementary stress components read as

$$\sigma_x = - \rho \overline{u^2}$$

$$\sigma_y = - \rho \overline{v^2}$$

$$\tau = - \rho \overline{u\,v}$$

*O. Reynolds, Phil. Trans. A, Vol. 186, 1894, p. 123.

These components are conformal to the so-called laminar stress components originating in an unsteady, viscous fluid due to the molecular motion as given by

$$\sigma_x = 2\,\mu\,\frac{\partial U}{\partial x}$$

$$\sigma_y = 2\,\mu\,\frac{\partial U}{\partial y} \qquad (2)$$

$$\tau = \mu\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right)$$

The analogy becomes particularly apparent when we include the derivation of the coefficient of viscosity μ, which is defined up to a numerical factor, the product $\mu \cong \rho\,\lambda\,c$, where ρ = density, λ = mean path length, and c = molecular velocity.

Take, for instance, the case of parallel flow in direction x, that is, $U = U(y)$, $V(y) = 0$, and consider the derivation of the shear stress - first, conformal to

$$\tau = -\,\rho\,u\,v$$

and then, according to

$$\tau = \mu\,\frac{dU}{dy} \cong \rho\,\lambda\,c\,\frac{dU}{dy}.$$

As oscillatory components in the molecular motion, we have $u = \pm\,c \pm l\,\frac{dU}{dy}$, $v = \pm\,c$, and to the extent that

$u = \pm\,c - l\,\frac{dU}{dy}$, is coupled with $v = c$, and $u = \pm\,c + l\,\frac{dU}{dy}$ with $v = -\,c$. Obviously, $\overline{uv} = -\,c\,l\,\frac{dU}{dy}$, which proves the analogy between the two expressions.

The next step to render the turbulent shear stress calculable is to formally introduce a "turbulent friction coefficient," as proposed by Boussinesq* and notably, by Stanton.** The latter, appreciating the velocity distribution developed from the center in a smooth pipe, disclosed the relation

$$U_{max} - U = f \sqrt{\frac{\tau_o}{\rho}}\, f\left(\frac{y}{r}\right) \qquad (3)$$

where τ_o = shearing stress transmitted to the wall, r = pipe diameter, and y = distance from middle of the pipe. This law of similitude is summarily adhered to by the introduction of an average turbulent friction coefficient μ_t, which is proportional to the quantity $U_{max}\, r$.***

$$\tau = \text{const.}\; U_{max}\, r\, \frac{dU}{dy}\,.$$

M. F. Treer**** recently attempted to fit Stanton's theorem to the more up-to-date test material by having resort to additional empirical assumptions. It is, however, evident that the equation is carried too far in forming the mean value and is, on the face of it, inappropriate for interpreting actual velocity distribution. Stanton's application may be adjudged as a summary averaging examination of similitude; to conceive the mechanism of turbulent flow attitude, the study of similitude

*J. V. Boussinesq, Mem. pres. par div. sav., Paris 23, 1877; 24, 1877.

**T. E. Stanton, Proc. Roy. Soc., Vol. 85, 1911, p. 366.

***To be more exact $\mu_t = \text{const.} \sqrt{\frac{\tau_o}{\rho}}\, r$; since, however, - at least, for large index figures - $\frac{\tau_o}{\rho}$ is practically proportional to U_{max}^2, the above is likewise approximately correct.

****M. F. Treer, Phys. Zeitschr., 1929, Vol. 30, p. 542.

must, to my way of thinking, be worded slightly different.

Similitude of Oscillatory Motion

Let us check the following assumption with: the mean flow to be a parallel flow in direction x; the oscillatory flow to consist in disturbances of relatively limited extent in direction y and - at least, within a certain time interval - to be carried along by the main flow approximately as steady flow configurations. Our problem shall be to find under what conditions these flows can become similar among each other, so that the flow attitudes neighboring two points, which conform to different y values, vary only by a multiplicative factor of the oscillatory velocity and in the length measure of the field of flow. In other words, we assume similitude of the oscillatory attitude, irrespective of the location of the point in whose vicinity the oscillation is examined.

We so place the coordinates that this pertinent point falls on axis $y = 0$, and develops the mean velocity according to y:

$$U = U'_0 y + U''_0 \frac{y^2}{2} + \ldots$$

Then we write the stream function adjacent to the point

$$\Psi (x,y) = U'_0 \frac{y^2}{2} + U''_0 \frac{y^3}{6} + \ldots + \psi (x,y), \qquad (5)$$

where $\psi (x,y)$ becomes the stream function of the oscillatory motion.

Now it is desired that only l and A be affected by the point, that is, by U'_o, U''_o, etc., but that function $f(\xi,\eta)$ be unaffected when

$$
\begin{aligned}
x &= l\,\xi \\
y &= l\,\eta \\
\psi &= A\, f\,(\xi,\eta)
\end{aligned}
\tag{6}
$$

The hydrodynamic equations for plane steady flow may be combined in the so-called vortex transfer equation

$$\frac{\partial \psi}{\partial y}\,\frac{\partial \Delta \psi}{\partial x} - \frac{\partial \psi}{\partial x}\,\frac{\partial \Delta \psi}{\partial y} = \nu\,\Delta\,\Delta\,\psi \tag{7}$$

where

$$\Delta\,\psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}$$

is the vortex intensity, and ν the kinematic viscosity. Disregarding the friction terms and limiting ourselves, in accord with the assumed definition of the field of oscillation in the vicinity of axis $y = 0$ to the first digits in the terms induced by the principal motion, we obtain

$$\left(U'_o\,y + \frac{\partial \psi}{\partial y}\right)\frac{\partial \Delta \psi}{\partial x} - \frac{\partial \psi}{\partial x}\,U''_o - \frac{\partial \psi}{\partial x}\,\frac{\partial \Delta \psi}{\partial y} = 0 \tag{8}$$

Introducing (6), we have

$$U'_o\,l\,\eta\,\frac{A}{l^3}\,\frac{\partial \Delta f}{\partial \xi} - \frac{A}{l}\,\frac{\partial f}{\partial \xi}\,U''_o + \frac{A^2}{l^4}\left(\frac{\partial f}{\partial \eta}\,\frac{\partial \Delta f}{\partial \xi} - \frac{\partial f}{\partial \xi}\,\frac{\partial \Delta f}{\partial \eta}\right) = 0 \tag{9}$$

where symbol Δ now pertains to the variables ξ and η. This equation is resolvable independently of A, l and U'_o, U''_o, providing

$$\left.\begin{aligned} U'_o \; l &\cong \frac{A}{l} \\ U''_o &\cong \frac{A}{l^3} \end{aligned}\right\} \tag{10}$$

or

$$l \cong \frac{U'_o}{U''_o}$$

and

$$A \cong \frac{{U'_o}^3}{{U''_o}^2} \cong l^2 \; U'_o \tag{11}$$

To denote the significance of these relations, we now compute the shearing stress τ:

$$\tau = -\rho \; \overline{u \; v} = \rho \; \overline{\frac{\partial\psi\partial\psi}{\partial x\partial y}} = \rho \; \frac{A^2}{l^2} \; \overline{\frac{\partial f}{\partial \xi} \; \frac{\partial f}{\partial \eta}} \tag{12}$$

or

$$\tau \cong \rho \; l^2 \; {U'_o}^2 \tag{13}$$

The conditions of similitude may then be combined as follows:

a) The field of oscillation retains a length indicative of the length measure of the disturbances, defined by

$$l \cong \frac{U'_o}{U''_o} = \frac{\frac{dU}{dy}}{\frac{d^2 \; U}{dy^2}}$$

b) The shearing force is proportional to the density, to the square of the characteristic length l and to the square of the velocity gradient $\frac{dU}{dy}$.

Having defined length l to only one factor, we write

$$\tau = \rho \; l^2 \left(\frac{dU}{dy}\right)^2 \tag{14}$$

and $$l = k\,\frac{U'}{U''} \tag{15}$$

where k is a nondimensional constant, dependent only upon the nature of the fluctuating mechanism, and upon the solution of the equation for $f(\xi,\eta)$. It is the sole constant entering the here developed theory of turbulence.

Prandtl's Theory of "Mixing Length"

Our data may for the present serve as corroboration of the suggestive equation put forth by Professor L. Prandtl* to demonstrate the laws of turbulent flow. He proceeded with the formula for the shearing stress

$$\tau = -\rho\,\overline{u\,v}$$

and evaluated $u = l\,\frac{dU}{dy}$ analogous of the equations of the molecular theory, by introducing the length l as the mixing length or distance (analogous to the idea of the mean free path in the kinetic theory). On the premises that a particle in the oscillatory motion travels the distance l perpendicular to the basic flow without momentum interchange $\rho\,l\,\frac{dU}{dy}$ becomes, in fact, the momentum which the particle transfers to the layer into which it was displaced by the mixing motion. Prandtl then assumed the fluctuating component v proportional to the mixing length l and to the absolute value $\left|\frac{dU}{dy}\right|$ of the velocity gradient, so that in agreement with (13), $\tau = \rho\,l^2\,\left(\frac{dU}{dy}\right)^2$.

Viewing the matter from the viewpoint of the oscillation

*Compare, for example, page 62 of "Proceedings of the Second International Congress for Technische Mechanik, 1927."

theory, this assumption signifies that the correlation between u and v is supposedly unaffected by the location. This hypothesis is so obviously like our assumption of similarity, that it is not surprising to find the equation confirmed.

But now our study of similitude reveals decidedly more, for it yields an equation defining the mixing distance which in first approximation appears as ratio of the first two differential quotients of the basic velocity.

Flow between Parallel Walls

Consider the basic flow in direction x between two parallel walls $y = \pm h$ (Fig. 1). With τ_0 as shearing stress at the wall, it equals

$$\tau = \tau_0 \frac{y}{h} \tag{16}$$

at y distance away from the channel center, so that the velocity distribution becomes

$$\tau_0 \frac{y}{h} = k^2 \rho \frac{U'^4}{U''^2} \tag{17}$$

or

$$\frac{U''}{U'^2} = k \frac{\sqrt{h}}{\sqrt{\frac{\tau_0}{\rho}}} \frac{1}{\sqrt{y}} .$$

One integration yields

$$-\frac{1}{U'} = 2 k \frac{\sqrt{h}}{\sqrt{\frac{\tau_0}{\rho}}} \sqrt{y} + \text{const.}$$

or (a = a constant)

$$U' = \frac{1}{2k} \frac{\sqrt{\frac{\tau_o}{\rho}}}{\sqrt{h}} \frac{1}{a - \sqrt{y}} \tag{18}$$

The constant a, is to be defined from the relation with the conditions at the wall. For great Reynolds Numbers $\frac{dU}{dy}$ assumes a high value near the wall and finally approaches the limiting value $\frac{dU}{dy} = \frac{\tau_o}{\mu}$, which, due to the smallness of μ referred to $\frac{dU}{dy}$ at a distance away from the wall, is very high. As a result the value y, at which $\frac{dU}{dy}$ becomes infinite, may be allowed to coincide with $y = h$, so that

$$\frac{dU}{dy} = \frac{1}{2k} \frac{\sqrt{\frac{\tau_o}{\rho}}}{\sqrt{h}} \frac{1}{\sqrt{h} - \sqrt{y}} \tag{19}$$

wherefrom integration yields the velocity distribution. The maximum U_{max} is reached at $y = 0$. Then

$$U_{max} - U = \int_0^y \sqrt{\frac{\tau_o}{\rho}} \frac{1}{2kh} \frac{dy}{1 - \sqrt{\frac{y}{h}}}$$

or

$$U_{max} - U = -\frac{1}{k} \sqrt{\frac{\tau_o}{\rho}} \left(\log\left(1 - \sqrt{\frac{y}{h}}\right) + \sqrt{\frac{y}{h}}\right) \tag{20}$$

$$U = U_{max} + \sqrt{\frac{\tau_o}{\rho}} \left[\log\left(1 - \sqrt{\frac{y}{h}}\right) + \sqrt{\frac{y}{h}}\right] \tag{20a}$$

This formula was compared with two measurements of Dönch and Nikuradse.* Both defined the values $\frac{\tau_o}{\rho}$, leaving k as sole constant to be determined, which appears to have a value of about $k = 0.36$. The resistance law yields, as shown later,

*F. Donch, Forschungsarbeiten, No. 282, 1926.
J. Nikuradse, Forschungsarbeiten, No. 289, 1929.

$k = 0.38$.

Figure 2 illustrates the comparison, which is found to be good. One more word about the behavior of mixing length l. The calculation yields

$$l = k \frac{U'}{U''} = 2\,k\,h \sqrt{\frac{y}{h}} \left(1 - \sqrt{\frac{y}{h}}\right) \tag{21}$$

Near to the wall we may write

$$y_1 = h - y$$

so that

$$l = 2kh \left\{\sqrt{1 - \frac{y_1}{h}} + \frac{y_1}{h} - 1\right\} = ky_1 \left(1 - \frac{y_1}{4h} - \frac{y_1^2}{8h^2} - \ldots\right). \tag{22}$$

The course of $\frac{l}{h} = f\left(\frac{y_1}{h}\right)$ is noted in Figure 3. It does not correspond to Dönch and Nikuradse's data (dotted line), which they arrived at with formula

$$l = \frac{\sqrt{\frac{\tau}{\rho}}}{\frac{dU}{dy}}.$$

Their chief contention is that quantity l assumes a constant value in the channel center, whereas it attains a maximum at $y_1 = \frac{3}{4} h$ and drops to zero for $y_1 = h$, according to our calculation. Our comments are:

In the first place, l is exceedingly difficult to determine on account of the uncertainty of the differentiation of the point-by-point recorded velocity curve; then they compute the mixing length in the channel center by a modified formula, the authenticity of which is not quite beyond question. It is assumed forthwith that $\frac{dU}{dy} = 0$ for $y = 0$, that is, the velocity is smoothed out at the apex. Our formula shows a hump for $y = 0$, a sure sign that the degree of approximation in the calculation is insufficient at this point, but it is strange that the unsmoothened velocity curve also shows a definite bend. I believe the observation material was not extensive enough to define the behavior of l in the channel center conclusively.

The Resistance Law

By a high velocity gradient $\frac{dU}{dy}$, as near to the wall, our similitude consideration no longer holds true, because the omission of the viscosity appears to be no longer justified. It is a question of whether $\mu \frac{dU}{dy}$ can be disregarded or not along side of the kinematic shearing stress $-\rho\, \overline{u\, v}$. It has been proved experimentally that near to the wall something like a "laminar layer" exists and, in order to define the resistance law, i.e., the U_{max} value corresponding to a certain value of $\frac{\tau_0}{\rho}$, the connection from the velocity curve to the laminar layer must be established.

Before discussing a more exact theory of the oscillatory field, we attempt two different applications which, however, yield the same plausible results:

a) We assume the mixing length l to diminish below zero to a value proportional to the thickness of the laminar layer. This minimum is supposedly reached at the boundary of the laminar layer where the laminar layer is then attached.

To carry this idea through we resort to G. J. Taylor's conception, who arrived at the conclusion that the vortex division (i.e., the mixing length l) of the turbulence at the wall can only be affected by the shearing stress τ_o, and that it must be proportional to the value $\frac{\nu}{\sqrt{\frac{\tau_o}{\rho}}}$. Writing $l = \alpha \frac{\nu}{\sqrt{\frac{\tau_o}{\rho}}}$ for this limiting value, we obtain for the relevant quantity of $y = \frac{1}{k}\alpha\nu\sqrt{\frac{\rho}{\tau_o}}$ and $U^e = U_{max} - U(y)$, that is for the velocity difference between channel center and laminar layer boundary

$$U^e = \frac{1}{k}\sqrt{\frac{\tau_o}{\rho}}\left[\log \frac{h\sqrt{\frac{\tau_o}{\rho}}}{\frac{\alpha}{k}\nu} + \text{const.}\right] \tag{23}$$

Then we estimate the difference in velocity U^{ee} between both boundaries of the laminar layer, i.e., between the wall and the free boundary. Since the thickness on one side is $\delta \cong \frac{\nu}{\sqrt{\frac{\tau_o}{\rho}}}$ and we can put $\tau_o = \mu \frac{U^{ee}}{\delta}$ on the other, we obtain $U^{ee} \cong \sqrt{\frac{\tau_o}{\rho}}$, and the difference between the wall and the channel center becomes

$$U_{max} = U^e + U^{ee} = \frac{1}{k}\sqrt{\frac{\tau_o}{\rho}}\left[\log \frac{h\sqrt{\frac{\tau_o}{\rho}}}{\nu} + A\right], \tag{24}$$

The constants k and A are independent of dimensions and Reynolds Number.

b) The result is identical when we assume the "turbulent component" $\tau - \mu \frac{dU}{dy}$ to be proportional to $\rho\, l^2 \left(\frac{dU}{dy}\right)^2$ instead of the whole shearing stress. Then l may quietly diminish to zero and the velocity distribution near to the wall becomes

$$\tau - \mu \frac{dU}{dy} = k^2 \; y^2 \; \left(\frac{dU}{dy}\right)^2 \tag{25}$$

This is the same equation used by Mr. Wada* in formulating the resistance law, but which, owing to having been published in an inconspicuous place, has not received sufficient attention. Unfortunately, Mr. Wada held the equation valid for the whole channel which made his formulas a little too complicated, although the results in this paragraph are really contained in his report in an implied form.

To derive the resistance law, I slightly deviate from the conventional parameters and introduce the factors

$$R_m = \frac{U_{max} \; h}{\nu}$$

$$\psi = \sqrt{\frac{\tau_o}{\rho \dfrac{U_{max}^2}{2}}} \tag{26}$$

In place of the mean velocity the maximum velocity appears as reference quantity, so that (24) yields

$$\frac{k\sqrt{2}}{\sqrt{\psi}} = \log\left(R_m \sqrt{\psi}\right) + A - \tfrac{1}{2}\log 2 \tag{27}$$

or

$$\frac{k\sqrt{2}}{\sqrt{\psi}} = \log\left(R_m \sqrt{\psi}\right) + C \tag{28}$$

We derived the relation between U_{max} and ψ for plane flow.

*Journal of Japanese Soc. Naval Arch. 41, 1927, p. 103.

But it can also be proved applicable to circular pipes, where even constant k retains the same value and where constant C alone is different. In our derivation we used the mixing distance $k \times$ wall distance and estimated the thickness of the laminar layer. But it may be assumed that this is valid for a circular pipe also, for the further course of the mixing distance comes in evidence only in the C constant.

Figure 4 shows some observations of $\frac{1}{\sqrt{\psi}}$ and $\log_{10} R_m \sqrt{\psi}$ ($R_m = \frac{U_{max}\, r}{\nu}$, r = pipe diameter).* It will be noted that there is a linear relation between both quantities, extending from $R_m = 2000$ to $R_m = 1,600,000$. Constant

$$k = 0.38$$
$$C = 1.83 \qquad (29)$$

the first being perhaps universal, the second applying particularly to circular pipes.

The So-called Power Laws

It is known that the resistance law within large ranges of Reynolds Numbers can be adequately expressed by the interpolation formula

$$\psi = \frac{\text{const.}}{R^m} \qquad (30)$$

Upon this premise it can be proved by a line of reasoning advanced by Dr. Prandtl that the velocity distribution (figured as beginning at the wall) is given by formula

*The data on very large Reynolds Numbers were supplied by Mr. Nikuradse, who placed his, as yet unpublished material, at my disposal.

$$U = \text{const.} \sqrt{\frac{\tau_o}{\rho}} \left(\frac{\sqrt{\frac{\tau_o}{\rho}}\, y}{\nu}\right)^{\frac{m}{2-m}} \tag{31}$$

The validity of this power formula ceases in the immediate neighborhood of the wall (effect of laminar layer), but extends in surprising manner to the other side nearly to the channel center. The exponent n drops as the Reynolds Number increases (within about $\frac{1}{7}$ to $\frac{1}{9.5}$ in the range examined thus far). This puzzle is simple to explain. Slightly transformed, our derived formulas express the

Resistance law: $$\frac{1}{\sqrt{\psi}} = a + b \log \left(R\sqrt{\psi}\right)$$

Velocity distribution: $$\frac{U}{\sqrt{\frac{\tau_o}{\rho}}} = a' + b' \log \left(\frac{y\sqrt{\frac{\tau_o}{\rho}}}{\nu}\right) \tag{32}$$

So when we make $n = \frac{m}{2 - m}$, equations (30) and (31) can be written as

Resistance law: $$\frac{1}{\sqrt{\psi}} = \text{const.} \left(R\sqrt{\psi}\right)^n$$

Velocity distribution: $$\frac{U}{\sqrt{\frac{\tau_o}{\rho}}} = \text{const.} \left(\frac{y\sqrt{\frac{\tau_o}{\rho}}}{\nu}\right)^n \tag{33}$$

The decrease in exponent n with the Reynolds Number becomes readily apparent from the following:

Comparing flow attitudes in the same channel which correspond to an identical value of τ_o, and permitting the viscosity to vary, for instance, ν to decrease gradually, the velocity distribution is obviously the same up to the point where the influence of the viscosity becomes noticeable. There a kind of laminar layer is set up, although the point continues to shift along the velocity distribution curve as the viscosity decreases, i.e., as the Reynolds Number rises. The conjointed distribution curve

$$y = f\ (U_{max} - U),$$

which is, aside from the channel center, exponential (Fig. 5), is to be approximated to $y = U^{1/n}$ by a series of power curves which touch the axis $y = 0$, inducing the contact point to gradually shift toward the right; accordingly $1/n$ increases and n decreases.

Roughness

It has been repeatedly emphasized that the ratio between the thickness of the laminar layer and the mean protuberance of the roughness predominates the phenomena on rough walls. I call a wall rough when the projections are large referred to the laminar layer thickness. In this case it may be assumed that the minimum value of the mixing length l is not conditioned by the thickness, but by the size of the roughness elements. Thus if ϵ is the mean roughness projection (the characteristic length measure of the roughness), the minimum value of l may be made proportional to ϵ, which discloses as relation between velocity in channel center U_{max} and the shearing force at the wall τ_0

$$U_{max} = \frac{1}{k} \sqrt{\frac{\tau_0}{\rho}} \left(\log \frac{h}{\epsilon} + \text{const.}\right) \tag{34}$$

or for the resistance coefficient

$$\frac{k \sqrt{2}}{\sqrt{\psi}} = \log \frac{h}{\epsilon} + \text{const.}$$

In other words, the flow resistance follows the "square" law, and the resistance coefficient is dependent on the relative roughness according to (35). This equation admits of a check when comparing experiments in grooves, where the distance of the walls (2h) varies, but where the nature of the walls (quantity ϵ) is to remain constant. This, however, is to be treated in a future report.

Translation by J. Vanier,
National Advisory Committee
for Aeronautics.

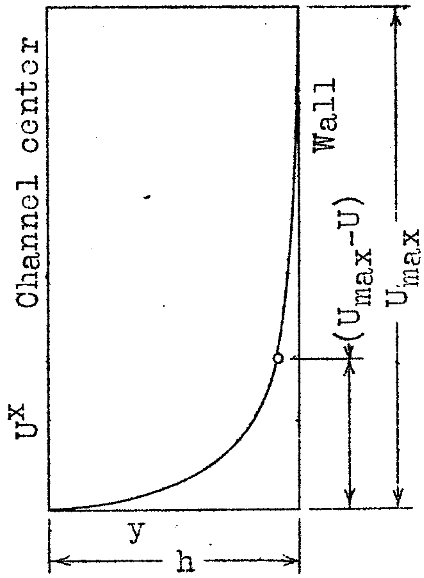
Channel center
Wall
U^x
$(U_{max}-U)$
U_{max}
y
h

Fig. 1

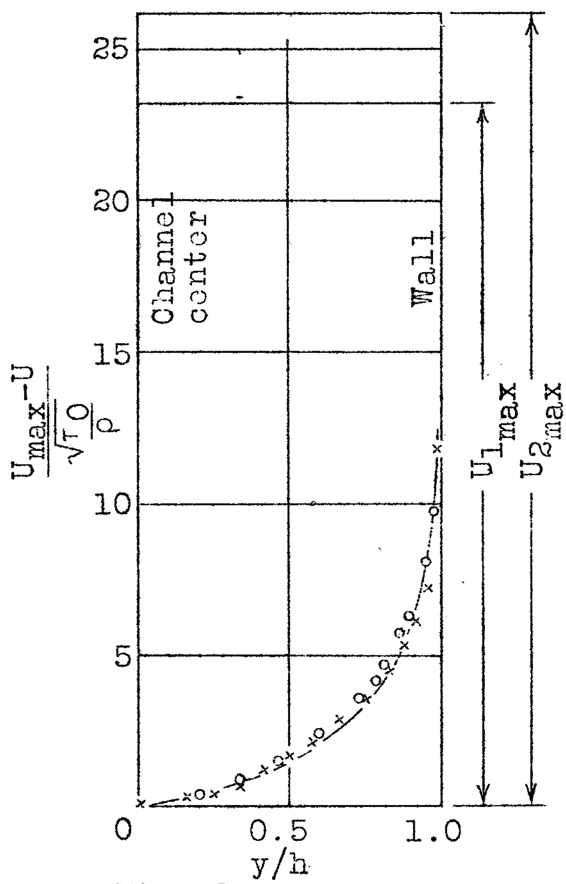
$\frac{U_{max}-U}{\sqrt{\frac{\tau_0}{\rho}}}$
25
20
15
10
5
0
0.5
1.0
y/h
Channel center
Wall
$U_{1_{max}}$
$U_{2_{max}}$

Fig. 2

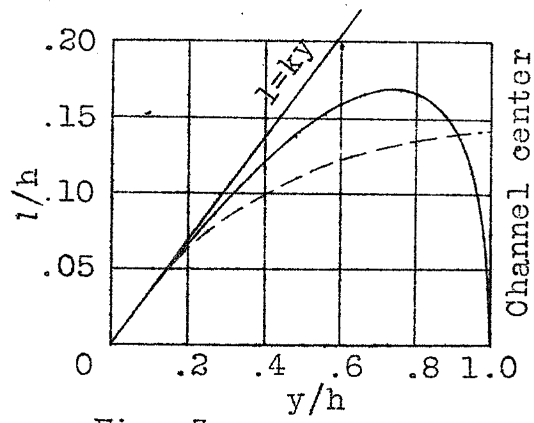

Fig. 3

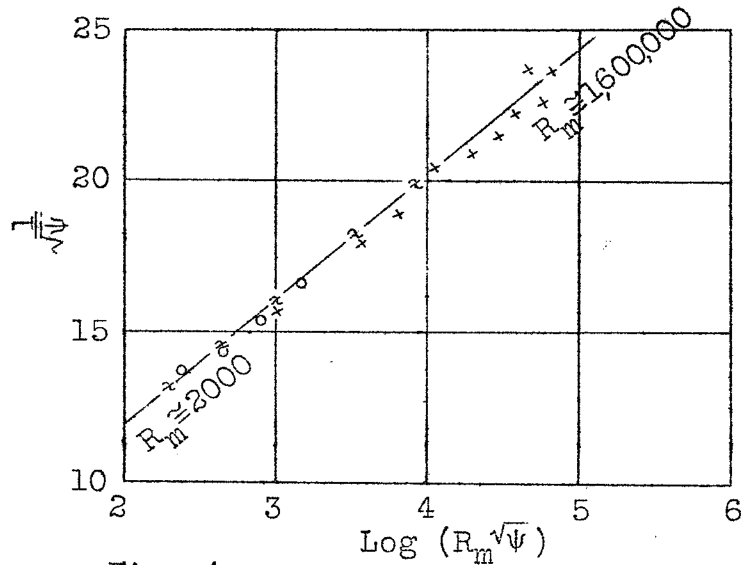

Fig. 4

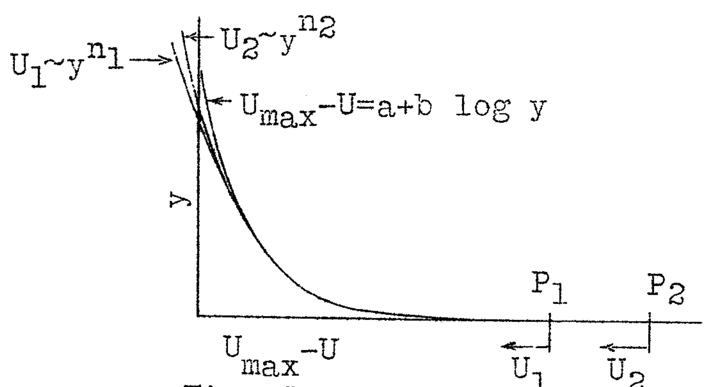

Fig. 5