

Disturbance, the Uncertainty Principle and Quantum Optics

Hans Martens

*Dept. of Theoretical Physics, Eindhoven University of Technology
PO Box 519, 5600 MB Eindhoven, The Netherlands*

Willem M. de Muynck

*Dept. of Theoretical Physics, Eindhoven University of Technology
PO Box 519, 5600 MB Eindhoven, The Netherlands*

Abstract

It is shown how a disturbance-type uncertainty principle can be derived from an uncertainty principle for joint measurements. To achieve this, we first clarify the meaning of "inaccuracy" and "disturbance" in quantum mechanical measurements. The case of photon number and phase is treated as an example, and applied to a quantum non-demolition measurement using the optical Kerr effect.

1 Introduction

One of the most appealing aspects of quantum optics is that within its domain of application experiments can be realized that used to be confined to the domain of *Gedanken* experiments. The proposed [1] quantum non-demolition (QND) schemes for photon number measurement are such fundamental measurements. In fig. 1 we have sketched a simple setup [1, 2]. A signal beam *S* is mixed with a probe beam *P* in a non-linear Kerr medium. The refraction index of this medium is intensity dependent. Accordingly, the probe's phase will depend on the number of photons in the signal beam. By coupling the outgoing probe beam with a reference beam, the probe phase can be detected and thus the signal photon number can be deduced. However, this is not the only consequence of the interaction between signal and probe beams. Also the *S*-phase will be influenced.

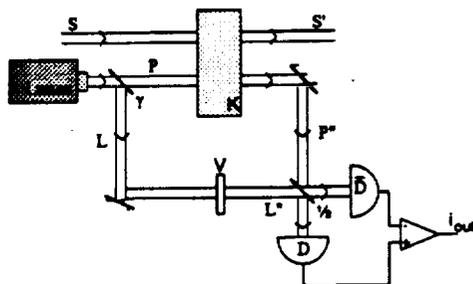


FIG. 1 Basic QND scheme, using the Kerr effect.

The experiment can be seen as an analog of Heisenberg's γ microscope experiment [3]. There a particle's position is measured in a non-destructive way. In the γ -microscope, it is argued momentum is disturbed by an amount Dp as a result of measuring position, satisfying ($\hbar = 1$)

$$\delta q Dp \gtrsim 1, \quad (1)$$

δq representing the microscope's resolution, i.e., its inaccuracy in determining position. Analogously, in the Kerr device, where photon number N is measured, the probe effect on signal phase can be expected to take the form of a disturbance, in size reciprocally related to the \hat{N} -measurement inaccuracy. In order to avoid certain ambiguities (cf. [4]) we shall give a formal definition of this disturbance notion. In particular we show how relations like (1) can be derived in a precise way from uncertainty relations for the inaccuracy, achievable in joint measurements of incompatible observables. Such relations have become available relatively recently [1, 6, 7].

2 Inaccuracy

We represent measurements by *positive operator-valued measures* (POVM's) [5, 8], a notion generalizing von Neumann's projection-valued measures (PVM's). For a discrete set of outcomes K , a POVM $\mathcal{M} = \{\hat{M}_k, k \in K\}$ generates the probability of outcome k by $\text{Tr} \hat{\rho} \hat{M}_k$, when the object is in state $\hat{\rho}$. Hence \mathcal{M} must satisfy $\sum_{k \in K} \hat{M}_k = \hat{1}$, $\hat{M}_k \geq \hat{0}$. A second POVM, $\mathcal{O} = \{\hat{O}_l\}$, is then said to represent a *non-ideal* measurement [7] of \mathcal{M} if there is a stochastic matrix λ_{lk} ($\sum_l \lambda_{lk} = 1$; $\lambda_{lk} \geq 0$) such that

$$\hat{O}_l = \sum_k \lambda_{lk} \hat{M}_k. \quad (2)$$

We use the shorthand $\mathcal{M} \rightarrow \mathcal{O}$ for this relation. The \mathcal{O} -distribution is a smeared version of the \mathcal{M} -distribution. Finally we need to characterize the amount of inaccuracy by a real number. Clearly, if $\lambda_{lk} = \delta_{lk}$, the Kronecker-delta, \mathcal{O} is equal to \mathcal{M} : then a measurement of \mathcal{O} is a perfect measurement of \mathcal{M} . Thus we need to quantify how much λ_{lk} deviates from δ_{lk} . Consider again the QND scheme of fig.1. Given that the incoming probe beam is described by a coherent state $|\beta\rangle$ and the signal beam by $\hat{\rho}$, it can be shown that the outcome probabilities $P(q)$ of the outgoing probe phase measurement are given by [2, 9]

$$\begin{aligned} P(q) &= \sum_{n_S} P(q|n_S) \langle n_S | \hat{\rho} | n_S \rangle, \\ P(q|n_S) &= \frac{1}{\sqrt{2\pi}\sigma} \exp \left[-\frac{1}{2} \left(\frac{q-\mu}{\sigma} \right)^2 \right], \quad \sigma = |\beta|, \quad \mu = |\beta|^2 \sin(\chi_\tau n_S), \end{aligned} \quad (3)$$

where we have taken the initial beam splitter's transmittivity $\gamma = \frac{1}{2}$. The constant χ_τ depends on the non-linearity coefficient of the medium. Defining the POVM \mathcal{O} by the requirement $P(q) = \text{Tr} \hat{\rho} \hat{O}(q)$, (2) is satisfied if $\{\hat{M}_k\}$ represents the photon number observable. The measurement inaccuracy is characterized by the width of $P(q|n_S)$, and can be interpreted as being due to excess noise inherent in the measurement. For low photon numbers the response is approximately linear: $\mu \approx |\beta|^2 \chi_\tau n_S$. Hence this measurement can be characterized by parameters quantifying *noise* (σ) and *gain* ($\partial\mu/\partial n_S$). A suitable inaccuracy measure is the ratio of these two:

$$\delta_{N_S} = \frac{\sigma}{\partial\mu/\partial n_S} = \frac{1}{|\beta| \chi_\tau}. \quad (4)$$

In deriving (3) we ignored self-phase-modulation (SPM) [1]. It can be shown that, beyond a certain probe photon number, SPM has a strongly adverse effect on the measurement quality [2]. Refining the setup, however, can compensate for SPM to a large extent [1, 9].

3 Joint Measurements and the Uncertainty Principle

For finite-dimensional Hilbert spaces a general proof has been given that joint non-ideal measurements of incompatible observables are possible, but that their quality is limited by an uncertainty relation [7]. In the present paper we will focus on the phase-number observable pair. In this infinite dimensional case no completely general result is known, but the special results obtained are nevertheless quite convincing. Consider the (*non-Hermitian*) phase observable,

$$e^{i\hat{\phi}} = \sum_n |n\rangle\langle n+1|, \quad (5)$$

given by Lévy-Leblond, $|n\rangle$ denoting the number states[10]. Not only is it incompatible with \hat{N} , but the pair forms a perfect analog of the position-momentum pair, cf.

$$e^{im\hat{\phi}} e^{i\alpha\hat{N}} = e^{i\alpha\hat{N}} e^{im\hat{\phi}} e^{i\alpha m}, \quad m \in \mathbb{Z}, \alpha \in \mathbb{R}. \quad (6)$$

Next consider a second (ancillary) system, being in state $\hat{\rho}'$, and having similar observables $\hat{\phi}'$ and \hat{N}' defined on its Hilbert space \mathcal{H}' . Then the composite observables

$$e^{i\hat{\phi}_t} := e^{i\hat{\phi}} e^{-i\hat{\phi}'}, \quad \hat{N}_t := \hat{N} + \hat{N}' \quad (7)$$

are compatible, as evaluation of their Weyl commutation relation, using (6), shows. Hence, \hat{N}_t and $\hat{\phi}_t$ can be measured jointly. Then the POVM $\{\hat{M}(\phi, n) = \text{Tr}_{\mathcal{H}'}(\hat{\rho}'|\phi, n\rangle\langle\phi, n|)\}$, $|\phi, n\rangle$ being the common eigen-states of $\hat{\phi}_t$ and \hat{N}_t , describes a joint non-ideal measurement of $\hat{\phi}$ and \hat{N} . Indeed, for the relation between the probability distributions of \hat{N}_t , \hat{N} , and \hat{N}' we find

$$P_{\hat{N}_t}(n_t) = \sum_{n=0}^{n_t} P_{\hat{N}'}(n_t - n) P_{\hat{N}}(n), \quad P_{\hat{N}'}(n') = \langle n'|\hat{\rho}'|n'\rangle. \quad (8)$$

Comparing this with (2) we see that the \hat{N}_t measurement is a non-ideal measurement of \hat{N} , i.e., $\hat{N} \rightarrow \hat{N}_t$, the stochastic matrix λ_{lk} being given by $P_{\hat{N}'}(n_t - n)$. Therefore the inaccuracy of the non-ideal \hat{N} -measurement is determined by the spread in the number n' present in the state $\hat{\rho}'$ of the ancillary system. Similarly the $\hat{\phi}_t$ -measurement can be seen to be a non-ideal measurement of $\hat{\phi}$: $\hat{\phi} \rightarrow \hat{\phi}_t$, the inaccuracy being determined in an analogous way by the phase spread of the ancillary system. As a measure δ_ϕ of the inaccuracy of the $\hat{\phi}$ -measurement we may take [8] $\delta_\phi^2 = -1 + |\langle e^{i\hat{\phi}'} \rangle|^{-2}$. In this way we have a formal scheme of generating joint non-ideal measurements of incompatible observables. Indeed for position-momentum this scheme has long been known (e.g. [11]). From an uncertainty relation derived for observables \hat{N}' and $\hat{\phi}'$ in state $\hat{\rho}'$ [8, 10], the following inequality now straightforwardly follows for the inaccuracies of the jointly performed \hat{N} - and $\hat{\phi}$ -measurements:

$$\delta_N \delta_\phi \geq \frac{1}{2}. \quad (9)$$

This relation is of the same kind as (1). These were termed inaccuracy relations in [7]. In this special case this relation is a consequence of the restrictions in preparing the ancillary object state.

4 From Inaccuracy to Disturbance

Neither the γ -microscope nor the QND measurement referred to in sect. 1 are joint measurements: in the first momentum is not actually measured; neither is phase in the second. Yet, with however good a measuring instrument we try to measure the signal's initial phase, we can never quite remove the inaccuracy from this measurement. It appears that there is a limiting inaccuracy present already in the outgoing signal beam in the form of a phase *disturbance* caused by the presence of a measurement arrangement for measuring photon number. In order to be able to obtain a quantitative expression for this phase disturbance we first consider the general description of measurements once again. In sect. 2 we saw that the outcome probabilities of measurement results (i.e., the determinative aspect of measurement) in general are described by POVM's. Now we also need to take into account the object state after the measurement i.e., the preparative aspect of the measurement. In the von Neumann framework a *measurement transformation of the first kind* leaves the object in an eigenstate of the measured observable. In realistic cases this should be generalized to *operation valued measures* (OVM's) [5]. If a measurement yields outcome k , the output state will be $\hat{\mu}_k(\hat{\rho})$, given that the object started out in state $\hat{\rho}$. The probability of k is then given by $Tr[\hat{\mu}_k(\hat{\rho})]$. Accordingly, the mapping $\hat{\rho} \rightarrow \hat{\mu}_k[\hat{\rho}]$ should satisfy

$$\sum_{k \in K} Tr[\hat{\mu}_k(\hat{\rho})] = Tr[\hat{\rho}], \quad \hat{\rho} > \hat{0} \rightarrow \hat{\mu}_k(\hat{\rho}) \geq 0. \quad (10)$$

The POVM $\mathcal{M} = \{\hat{M}_k\}$ corresponding to the OVM $\{\hat{\mu}_k\}$ is therefore given by

$$\forall_{\hat{\rho}} Tr[\hat{\mu}_k(\hat{\rho})] = Tr[\hat{\rho}\hat{M}_k] \Leftrightarrow \hat{M}_k = \hat{\mu}_k^\dagger[\hat{1}]. \quad (11)$$

For every OVM there is only one POVM, whereas many measurement transformations may realize a given POVM. Now consider the outgoing object. Suppose we measure some POVM $\mathcal{O} = \{\hat{O}_l\}$ on it. Then the probabilities are given by

$$P_{\mathcal{O}}(l) = Tr[\hat{\mu}_K(\hat{\rho})\hat{O}_l] = Tr[\hat{\rho}\hat{\mu}_K^\dagger(\hat{O}_l)], \quad \hat{\mu}_K = \sum_{k \in K} \hat{\mu}_k. \quad (12)$$

Hence a measurement of \mathcal{O} in the final state can be seen as a measurement of $\tilde{\mathcal{O}} = \{\tilde{\hat{O}}_l\} = \{\hat{\mu}_K^\dagger(\hat{O}_l)\}$ in the initial state. Moreover, every repetition of the experiment yields values for *both* l and k . Therefore we have a joint measurement, characterized by the bivariate POVM $\{\hat{\mu}_k^\dagger(\hat{O}_l)\}$, of which $\tilde{\mathcal{O}}$ is one marginal and \mathcal{M} is the other one. Summarizing, we see that consecutive measurements of \mathcal{M} and \mathcal{O} may be seen as joint measurements of \mathcal{M} and $\tilde{\mathcal{O}}$.

Let us apply this to the QND scheme. Suppose we want to look at the outgoing signal beam S' in order to find out the *initial* signal phase. Then we must not measure the phase of the outgoing state $\hat{\rho}_{S'}$, i.e., *not* $\hat{\phi}_{S'} \rightarrow \mathcal{O}$ in $\hat{\rho}_{S'}$, but we must have $\hat{\phi}_S \rightarrow \tilde{\mathcal{O}}$ in $\hat{\rho}_S$. We should build the \mathcal{O} -device such that $\tilde{\mathcal{O}}$ is related to $\hat{\phi}_S$ by (2), rather than that \mathcal{O} itself is thus connected to $\hat{\phi}_{S'}$. In this way possible distortions in the medium are compensated for. Since SPM has the effect that $\hat{\phi}_S$ and $\hat{\phi}_{S'}$ are incompatible [2], this difference is not quite trivial here.

If $\hat{\phi}_S \rightarrow \tilde{\mathcal{O}}$, however, we have a joint measurement of \hat{N}_S and $\hat{\phi}_S$. The former, the QND POVM, measures \hat{N}_S , the latter we must choose so as to measure $\hat{\phi}_S$. Accordingly, (9) is applicable. The phase inaccuracy thus achievable is limited by $(2\delta_{N_S})^{-1}$.

Note that we have made no assumption about the nature of POVM \mathcal{O} whatsoever. The above reasoning holds quite generally. Define therefore

$$\epsilon_{\phi_S} := \inf_{\mathcal{O}}(\delta_{\phi_S}), \quad (13)$$

where the infimum is taken in the set of all POVM's $\tilde{\mathcal{O}}$ satisfying $\hat{\phi}_S \rightarrow \tilde{\mathcal{O}}$. Assuming $\hat{\phi}_S$ to be optimal, for all such POVM's $\tilde{\mathcal{O}}$ the bound (9) must hold, so that [12]

$$\epsilon_{\phi_S} \delta_{N_S} \geq \frac{1}{2}. \quad (14)$$

The quantity (13) does not depend on $\tilde{\mathcal{O}}$ (which is a variable in a set of POVM's), but on the meter's transformation $\hat{\mu}_K$, which is implicitly contained in the condition $\hat{\phi}_S \rightarrow \tilde{\mathcal{O}}$. Thus ϵ_{ϕ_S} is a property of the \hat{N}_S -meter, known once the OVM $\{\hat{\mu}_k\}$ has been calculated from the device's blueprint. ϵ_{ϕ_S} characterizes how much initial phase information can be retrieved from the outgoing signal. In that sense the term *disturbance* is apt [12]. If all phase information is lost (e.g., if $\{\hat{\mu}_k\}$ is a measurement of the first kind), the disturbance ϵ_{ϕ_S} is maximal. If, on the other hand, the meter measures nothing (e.g., if $\hat{\mu}_K(\hat{\rho}) = \hat{\rho}$ for all $\hat{\rho}$), there is no disturbance at all, and $\epsilon_{\phi_S} = 0$.

5 Phase Disturbance in the QND-Scheme

Finally, we study the phase disturbance in the Kerr-setup of fig.1. Define generalized phase states

$$|\phi; \nu \rangle := \sum_n (2\pi)^{-1/2} e^{i\phi n + \frac{1}{2}i\nu n(n+1)} |n \rangle. \quad (15)$$

For $\nu = 0$ these reduce to the eigenstates of (5). Then it can be shown that we need to measure the POVM $\{|\phi; -\frac{1}{2}\chi_\tau \rangle \langle \phi; -\frac{1}{2}\chi_\tau| \}$ on S' in order to get information on $\hat{\phi}_S$. In fact, [2, 13]

$$\begin{aligned} \phi_K(|\phi; -\frac{1}{2}\chi_\tau \rangle \langle \phi; -\frac{1}{2}\chi_\tau|) &= \int_{-\pi}^{\pi} \mu(\phi - \phi') |\phi' \rangle \langle \phi'| d\phi', \\ \mu(\phi) &\approx \frac{1}{2\pi} \Theta_3[\frac{1}{2}\phi; e^{-\frac{1}{2}|\beta|^2 \chi_\tau^2}], \end{aligned} \quad (16)$$

the latter approximation being valid for low photon numbers. Here Θ_3 denotes the third of Jacobi's Θ -functions. The smearing function μ is plotted in fig.2. Note that the convolution form of (16)

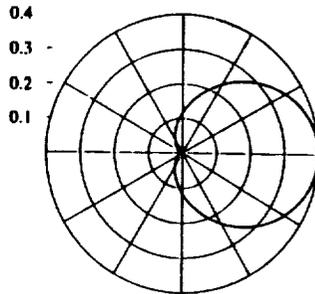


FIG. 2 Polar plot of the phase smearing function $\mu(\phi)$ (linear regime, $|\beta|^2 \chi_\tau^2 = 8$).

is in agreement with (2). We have discarded an uninteresting phase bias term in (16).

Calculating δ_{ϕ_s} from (16), we get $\delta_{\phi_s}^2 \simeq -1 + \exp(\frac{1}{4}|\beta|^2\chi_r^2)$, implying (cf. (4)):

$$\log(1 + \delta_{\phi_s}^2)\delta_{N_s}^2 \simeq \frac{1}{4}. \quad (17)$$

This is only slightly worse than the bound set by the uncertainty principle (14), indicating that the measurement procedure described by (16) is optimal in the sense that $\delta_{\phi_s} \simeq \epsilon_{\phi_s}$.

As said before, the disturbance concept evades distortions in the medium, and therefore phase disturbance is unaffected by SPM, contrary to photon number inaccuracy (but see [9]).

6 Acknowledgments

One of the authors (H.Martens) acknowledges financial support from the Foundation for Philosophical Research (SWON), which is subsidized by the Netherlands Organization for Scientific Research (NWO).

References

- [1] Y. Yamamoto, S. Machida, S. Saito, N. Imoto, T. Yanagawa, M. Kitagawa and G.Björk, *Progress in Optics* **28**, (ed. E. Wolf, North Holland, Amsterdam, 1990), p. 87.
- [2] H. Martens and W. de Muynck, *Quantum Aspects of Optical Communications*, (eds. C. Bendjaballah, O. Hirota and S. Reynaud, Springer, Berlin, 1991), p. 41.
- [3] W. Heisenberg, *Zeits. f. Phys.* **43**, 172 (1927).
- [4] K. Kraus, *Phys. Rev.* **D35**, 3070 (1987).
- [5] E. Davies, *Quantum Theory of Open Systems*, (Academic, London, 1976).
- [6] S. Ali and E. Prugovecki, *Journ. Math. Phys.* **18**, 219 (1977).
- [7] H. Martens and W. de Muynck, *Found. Phys.* **20**, 257, 355 (1990).
- [8] A. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory*, (North Holland, Amsterdam, 1982).
- [9] H. Martens and W. de Muynck, *Quantum Optics* (1992) to be published.
- [10] J.-M. Lévy-Leblond, *Ann. of Phys.* **101**, 319 (1976).
- [11] C. Helstrom, *Found. Phys.* **4**, 453 (1974).
- [12] H. Martens and W. de Muynck, *Journ. Phys. A* (1992), to be published.
- [13] H. Martens, *The Uncertainty Principle*, (PhD thesis, Eindhoven University of Technology, 1991, unpublished).