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COMPLETENESS PROPERTIES OF THE MINIMUM UNCERTAINTY STATES

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Abstract

The completeness properties of the Schrödinger minimum uncertainty states (SMUS) and of some of their subsets are considered. The invariant measures and the resolution unity measures for the set of SMUS are constructed and the representation of squeezing and correlating operator and SMUS as superpositions of Glauber coherent states on the real line is elucidated.

1 Introduction

In the present paper we consider the completeness properties of the set (and some subsets) of the states, which minimize the Schrödinger–Robertson uncertainty relation [1]

$$\sigma_q^2 \sigma_p^2 \geq \frac{1}{4}(1 + 4c^2), \tag{1}$$

where σ_q and σ_p are the dispersions of the quadrature operators Q and P ($[Q, P] = i$),

$$\sigma_q = \langle X^2 \rangle - \langle X \rangle^2, \qquad X = Q,\, P,$$

and c is their covariation,

$$c = (1/2)\langle QP + PQ \rangle - \langle Q \rangle \langle P \rangle.$$

We call such states Schrödinger minimum uncertainty states (SMUS). In fact they were introduced by Dodonov, Khurmyshev and Man'ko [2] and studed as correlated states (see Ref. [3] and references therein). When the covariation is zero, $c = 0$, one gets the Heizenberg minimum uncer tainty states (HMUS) and when in addition to this the dispersions are equal, $\sigma_q = \sigma_p$, the corresponding MUS are the Glauber coherent states (CS) [4].

From the group–theoretical point of view SMUS are equivalent [5] to the group–related CS [6] with maximal symmetry [7], the group in this case being the semidirect product $H_w \otimes SU(1,1)$ (see also [8]) of the Heizenberg-Weyl group H_w and the quasiunitary group $SU(1,1) \sim Sp(2,R)$. Up to a phase factor they coinside [5] with the Stoler states [9], known also as squeezed states or two–photon CS [10] widely used in quantum optics (see for example the review papers [11, 12] and references therein). The stable time–evolution of SMUS, which is important for the squeezing and correlating processes, is considered in [5]. In other notations it was in fact obtained in [13].

SMUS are continuous set of states, which is clear from the definition. For such sets of states the completeness properties (in the Hilbert space $\mathcal{H}$) are very important for the applications in mathematical and theoretical physics. In the weak sense [6] the completeness of a continuous set of states $|x\rangle$ is defined as a dense subset in $\mathcal{H}$, while in the strong sense it is defined as the (integral) resolution of the unity operator

$$1 = \int |x\rangle\langle x|\, d\mu(x), \tag{2}$$

where $d\mu(x)$ is a positive measure in the label space $X \ni x$. Such complete set of states $|x\rangle$ is called (in general sense) CS [6]. The group-related CS are always complete in the weak sense, while the resolution of unity has to be proved in every case. A sufficient conditions is the square integrability of the corresponding representation of the group involved against the invariant measure.

In this paper we consider the resolution of unity (2) for the set of SMUS and for some of their subsets. First we construct the corresponding invariant measures and check the square integrability against them. Since the latter failed to be valid we look and find the noninvariant measures, which provide the resolution (2). We call such measures the resolution unity measures (RUM). In other notations (i.e. in no relation to SMUS) for the $H_w \circledS SU(1,1)$–CS RUM were considered in [8],

According to the definition of CS they are always over complete (at least in the weak sense [6]) in $\mathcal{H}$ family of states. Then it worth looking for a more simple subset of CS which is also complete in $\mathcal{H}$ or in some subspace (or even subset) of $\mathcal{H}$. We consider this problem in the last section. In particular we construct the squeezing and correlating operators as integral along the real line of projectors on the Glauber CS and reproduce the result of Janszky and Vinogradov [14] for the superpositions of Glauber CS along the real axis.

2 The Invariant Measures and RUM for SMUS

Up to a phase factor SMUS can be written in the form [5] of the $H_w \circledS SU(1,1)$–CS with maximal symmetry

$$\begin{aligned} |\xi;\eta\rangle &= N(\xi;\eta)\exp\left[\frac{\xi}{2}a^{\dagger^2} + \eta a^\dagger\right]|0\rangle, \\ N(\xi;\eta) &= (1-|\xi|^2)^{1/4}\exp\left[-\frac{1}{2}\frac{|\eta|^2 + \mathrm{Re}(\bar{\xi}\eta^2)}{1-|\xi|^2}\right], \end{aligned} \tag{3}$$

where $a^\dagger = (1/\sqrt{2})(Q - iP)$ is the boson creation operator, $[a, a^\dagger] = 1$, η is arbitrary complex number and ξ belongs to the unit disk, $|\xi| \le 1$. One also has the relation to the Stoler states $|z;\alpha\rangle$ (i.e. the squeezed states or the two–photon CS)

$$|\xi;\eta\rangle = |z;\alpha\rangle = \exp\left[\frac{1}{2}\left(za^{\dagger^2} - \bar{z}a^2\right)\right]|\alpha\rangle,$$

where $|\alpha\rangle$ is the Glauber CS and

$$\xi = e^{i\phi}\tanh|z|, \qquad \eta = \alpha' - \xi\bar{\alpha}', \qquad \alpha' = \alpha\cosh|z| + \bar{\alpha}e^{i\phi}\sinh|z|.$$

The second momenta σ_q, σ_p and c are expressed in terms of ξ in [5, 15] and in terms of z in [8].

The $H_w \ni\!\!\!\!\!\supset SU(1,1)$-CS (3) are related to the representation $T(g)$, generated by the semidirect sum algebra $h_w \ni\!\!\!\!\!\supset su(1,1)$ (known also as the one mode two-photon algebra)

$$\begin{aligned} h_w &= \text{lin. env. } \{1, a, a^\dagger\}, \\ su(1,1) &= \text{lin. env. } \left\{ K_- = \frac{1}{2}a^2, \quad K_+ = \frac{1}{2}(a^\dagger)^2, \quad K_0 = \frac{1}{2}(a^\dagger a + \frac{1}{2}) \right\}; \\ T(g) &= \exp(\gamma K_+ - \bar{\gamma} K_- + i\omega K_0)\exp(it + \alpha a^\dagger - \bar{\alpha} a) \\ &\equiv T(\gamma,\omega)\,T(t,\alpha), \qquad g \equiv g(\gamma,\omega;t,\alpha). \end{aligned} \tag{4}$$

In terms of the above group parameters the invariant measure is a product of the $SU(1,1)$- and the H_w-invariant measures,

$$d\mu(\gamma,\omega;t,\alpha) = \frac{\sinh^2\Lambda}{\Lambda^2}\, d^2\gamma\, d^2\omega\, d^2\alpha\, dt, \qquad \Lambda^2 = 4|\gamma|^2 - \frac{1}{4}\omega^2. \tag{5}$$

But the representation (4) is not square integrable against the invariant measure (5) on the group manifold. Then we have to look for the invariant measure $d\mu(\xi;\eta)$ on the factor space $G/K \ni (\xi;\eta)$, which is a label space for the SMUS $|\xi;\eta\rangle$, Eq. (3),

$$d\mu(\xi;\eta) = \frac{d^2\gamma\, d^2\eta}{(1-|\xi|^2)^3}. \tag{6}$$

This measure is not a product of the $SU(1,1)$-invariant measure on the label space $\mathbb{D}_1 \ni \xi$ and the H_w-invariant measure $d^2\eta$ on the label space $\mathbb{C} \ni \eta$. And we still do not have the square integrability, i.e. the right hand side of the Eq. (2) with $|x\rangle = |\xi;\eta\rangle$ and $d\mu(x) = d\mu(\xi;\eta)$ goes to infinity.

Let us now look for the noninvariant resolution unity measure (RUM). The noninvariant RUM if exists is highly nonunique. It is clear from the definition of RUM as a measure providing the resolution (2), that if $d\mu(x)$ is a RUM for a group-related CS $|x\rangle$ then

$$d\mu_g(x) = d\mu(g \cdot x), \tag{7}$$

where $g \cdot x$ denotes the action of the group element on $x \in X$, is a set of RUM. It is an open problem whether the noninvariant RUM exists simultaneously with the invariant one. For the Glauber CS $|\alpha\rangle$ the invariant measure $d^2\alpha$ is the only RUM. In our case of SMUS the simplest noninvariant RUM reads (in Stoler parameters)

$$d\mu_0(z,\alpha) = \frac{1}{\pi^2}e^{-z\bar{z}}d^2z\, d^2\alpha, \tag{8}$$

which can be expressed in terms of ξ, η by means of the relations obtained above. The other measures $d\mu_g(\xi;\eta)$, Eq. (7), are obtained by means of the group action

$$g \cdot (\xi;\eta) = \left(\frac{\bar{u}\xi - v}{u - \bar{v}\xi}, \frac{\eta - \bar{\alpha}\xi + \alpha}{u - \bar{v}\xi} \right), \tag{9}$$

where $g = g(\gamma,\omega;t,\alpha)$ and u, v are the new $SU(1,1)$ parameters,

$$u = \cosh\Lambda - \frac{i\omega}{2\Lambda}\sinh\Lambda, \qquad v = -\frac{2\gamma}{\Lambda}\sinh\Lambda, \qquad \Lambda^2 = 4|\gamma|^2 - \frac{1}{4}\omega^2.$$

As we have already noted the RUM for the $H_w \otimes SU(1,1)$-CS were constructed in Ref. [8]. With the Note added in proof in [8] their measure should read (however we were not able to obtain the resolution of unity by means of this measure)

$$d\mu\,(\xi;\eta) = \frac{2^2\Gamma(2)\exp\left[\mathrm{Re}\,(\bar{\xi}\eta^2)/(1-|\xi|^2)\right]}{(1-|\xi|^2)^{3/2}}\, d^2\xi\, d^2\eta.$$

3 Completeness of Some Subsets of SMUS

The two parameters subset $|\xi_0;\eta\rangle$ of SMUS with fixed ξ_0 (i.e. with fixed second momenta of the quadrature operators) forms a strongly complete system in $\mathcal{H}$

$$\int_{\mathbb{C}} d\nu(\eta)\,|\xi_0;\eta\rangle\langle\eta;\xi_0| = 1, \qquad d\nu(\eta) = \frac{1}{\pi}\left(1-|\xi_0|^2\right)^{-1} d^2\eta, \tag{10}$$

which in Stoler parameters is known [16] and corresponds to the generalized Glauber CS (i.e. to the H_w-CS with the squeezed and correlated vacuum as the initial vector). Such resolution of unity was used in [16] for construction of new quasi probabilities "based on squeezed state". Note that the RUM in (10) is H_w-invariant and is obtained (up to a constant factor) from the $H_w \otimes SU(1,1)$-invariant measure (6) by fixing $\xi = \xi_0$. If we fix the other complex parameter $\eta = \eta_0$ we get the subset $\{|\xi;\eta_0\rangle\}$ (this is $SU(1,1)$-CS with Glauber CS as initial vector) which however is not complete even in the weak sense in $\mathcal{H}$ since the $SU(1,1)$ representation involved here is not irreducible. If we put $\eta_0 = 0$ we obtain the complete (but only in the weak sense) set of even $SU(1,1)$-CS $|\xi;+\rangle$ [15] in the subspace $\mathcal{H}_+$ of even functions. The state $|\xi;+\rangle$ is in fact squeezed (and/or correlated) vacuum. In the subspace $\mathcal{H}_-$ of odd states we have the strongly complete system of the odd $SU(1,1)$-CS $|\xi;-\rangle$ [15],

$$|\xi;-\rangle \;=\; \left(1-|\xi|^2\right)^{3/4}\exp\left[\xi(a^\dagger)^2/2\right]|1\rangle, \tag{11}$$

$$\int_{\mathbb{D}_1} d\nu(\xi)|\xi;-\rangle\langle-;\xi| \;=\; 1_-, \qquad d\nu(\xi) = \frac{1}{2\pi}\frac{d^2\xi}{(1-|\xi|^2)^2}, \tag{12}$$

where $|1\rangle$ is the first exited state and $d\nu(\xi)$ is the $SU(1,1)$–invariant measure. The state $|\xi;+\rangle$ is the squeezed vacuum, and $|\xi;-\rangle$ is the squeezed one–photon state. Note that $|\xi;-\rangle$ is not SMUS. The second momenta σ_q, σ_p and c in this state obey the equality

$$\sigma_q^2\sigma_p^2 = \frac{1}{4}\left(1+4c^2+8\right), \tag{13}$$

i.e. $|\xi;-\rangle$ is another type of MUS. As in the case of squeezed ground state it is correlated when $\mathrm{Im}\,\xi \neq 0$ and $\sigma_q \to 0$ when $\xi \to 1$. In the subspaces $\mathcal{H}_\pm$ there are also strongly complete sets of even and odd CS $|\alpha\rangle_\pm$ [17], which are linear combinations of two Glauber CS $|\alpha\rangle$ and $|-\alpha\rangle$.

Let us consider the subset of SMUS $|\xi;\eta\rangle$ with fixed $\xi = \xi_0$ and $\mathrm{Im}\,\eta = \eta_{2,0}$, that is with fixed second momenta $\sigma_q = \sigma_{q,0} \equiv \sigma_0$ and $c = c_0$ and fixed first momentum $\langle P\rangle \equiv p = p_0$. This is the one parameter set of states $|q;\xi_0,p_0\rangle$, $q \sim \mathrm{Re}\,\eta \in \mathbb{R}$. It is the set of CS for the commutative subgroup generated by the unit operator and by P. It is also the subset of general Glauber CS along the real axis, the initial vector being the squeezed and correlated vacuum, displaced by p_0. The unitary representations of the group of translations (by q along the real line) are highly reducible thereby the set $\{|q;\xi_0,p_0\rangle\}$ is not complete in $\mathcal{H}$ even in the weak sense. Let for simplicity $p_0 = 0$ and consider the operators

$$B(\xi_0) = \int_{\mathbf{R}} |q;\xi_0\rangle\langle\xi_0;q|\,dq. \tag{14}$$

$B(\xi_0)$ is an unbounded (Hermitean) operator, well defined in the Hilbert space $\mathcal{H}$ with the following property: it leaves the set of SMUS stable, that is the states $|\psi'\rangle = B(\xi_0)|\psi\rangle$ is SMUS if $|\psi\rangle$ is. Moreover if $|q\rangle$ is the Glauber CS on the real line then (one can calculate that) $B(\xi_0)|q\rangle$ is an arbitrarily squeezed and correlated state. Thus $B(\xi_0)$ is an (one dimensional) integral representation of the squeezing and correlating operator. One can also get an arbitrary SMUS by means of a fixed operator

$$B = \int_{\mathbf{R}} |q\rangle\langle q|\,dq = B(\xi_0 = 0),$$

but acting on different states $|\psi\rangle$. The obtained state $B|\psi\rangle$ is clearly a superposition of the CS $|q\rangle$ with the weights $\psi(q) = \langle q|\psi\rangle$. If (but not only if) $|\psi\rangle$ is SMUS then $B|\psi\rangle$ is also SMUS with arbitrary c and $\sigma_q > 1$. The representation of squeezed states as superpositions of Glauber CS on the real line was recently considered by Janszky and Vinogradov [14] in the form $\int_{\mathbf{R}} |q\rangle G(q)\,dq$, $G(q)$ being the Gaussian weight function.

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