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OBSERVABLES, MEASUREMENTS AND PHASE OPERATORS FROM A BOHMIAN PERSPECTIVE

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Abstract

Bohmian mechanics is a deterministic theory of point particles in motion. While avoiding all the paradoxes of nonrelativistic quantum mechanics, it yields the quantum formalism itself—especially the role of self-adjoint operators—as a macroscopic measurement formalism. As an "application" it is shown that much of the confusion connected with the phase operator for the electromagnetic field arises from a misunderstanding of the role of operators in quantum theory.

1 Introduction

We would like to apologize for the bad title: we will try to explain why the casual use of the words "observables" and "measurements," which are on John Bell's list of bad words in his article "Against Measurement"[1], "measurement" being the worst of all, leads to much unnecessary confusion concerning the meaning of the quantum formalism. But first we introduce an even worse word: following Bell we will use the abbreviation "FAPP" for "for all practical purposes."

Quantum mechanics suffers from its irreducible reference to "observers" and "measurements": We have, for example, the fundamental rule that $|\psi(q)|^2dq$ is the *probability* of *observing* a particle in dq about q in a position *measurement*. This rule entails 1) indeterminism, because it deals with probabilities on a fundamental level; 2) subjectivity, because it refers to an observer and 3) vagueness, because the notion of measurement is vague. It has repeatedly been emphasized, however, that these are inescapable components of modern physics. The following reasons are frequently cited:

- It is *meaningless* to talk about trajectories of particles, because the uncertainty principle doesn't allow for a simultaneous measurement of position and velocity (Heisenberg).
- It leads to *contradictions* even to think that a particle might have a well-defined position and velocity at the same time.
- It is mathematically *impossible* to add "hidden variables" (e.g., actual positions) as a further specification of the quantum state (von Neumann [2]).

This is wrong! In fact it is almost trivially wrong: A counterexample has existed for more than four decades, namely Bohm's quantum theory [3], which we prefer to call "Bohmian mechanics." By trying the obvious, namely by seeking a motion of particles in space compatible with Schrödinger's equation, one is led directly to Bohmian mechanics. This theory is clear, objective and deterministic. The entire quantum formalism—operators as observables, randomness, etc.—emerges as a measurement formalism, or more precisely, as a phenomenological formalism for describing measurement-like experiments. Thus one arrives at an explanation for the quantum formalism rather than at an alternative theory which might give rise to "new predictions." We will argue, however, that Bohmian mechanics nonetheless refutes most of the approaches to the problem of the phase operator in quantum optics. It turns out, in fact, that there is no problem! But let us first give a brief review of nonrelativistic quantum mechanics.

2 The Quantum Formalism

- State: The state of an N-particle system is given by a vector $\psi \in \mathcal{H} = L_2(\mathbb{R}^{3N})$.
- Dynamics: The time evolution is given by the unitary evolution $\psi_t := e^{-\frac{i}{\hbar}Ht}\psi_0$, which is equivalent to Schrödingers equation $i\hbar\frac{\partial}{\partial t}\psi = H\psi$.
- Observables: The observables of the system are given by self-adjoint operators on $\mathcal{H}$. To find operators corresponding to classical observables one replaces the classical Poisson brackets by the commutator: $\{\ ,\ \} \mapsto \frac{1}{i\hbar}[\ ,\ ]$.
- Measurements: In a measurement of an operator $A = \sum \lambda_i |a_i\rangle\langle a_i|$ on a system in the state ψ one may find only one of its eigenvalues λ_i, with probability $\text{prob}(i) = |\langle a_i|\psi\rangle|^2$. After the measurement the system is in the corresponding eigenstate $|a_i\rangle$ (collapse rule).

3 The Fundamental Ambiguity

There can be no doubt that the predictions of quantum mechanics are of an amazing accuracy. But neither this nor the mathematical simplicity and beauty of unitary evolution in Hilbert space should hide the fact that a fundamental ambiguity enters at the very point where mathematics makes contact with reality: Measurements! Measurements of what—if the wave function ψ is really the complete state? And as J.S Bell has said [1]:

> It would seem that the theory is exclusively concerned about "results of measurement", and has nothing to say about anything else. What exactly qualifies some physical systems to play the role of "measurer"? Was the wave function of the world waiting to jump for thousands of millions of years until a single-celled living creature appeared? Or did it have to wait a little longer, for some better qualified system ... with a Ph.D.?

This fundamental ambiguity, connected with "measurement" and collapse is also responsible for the familiar paradoxes associated with orthodox quantum mechanics such as Schrödinger's cat paradox or the measurement problem. In the following we shall show that these difficulties simply evaporate by giving up the unquestioned assumption that ψ alone provides a complete description of the state of a system. Bohmian mechanics will permit an understanding of quantum phenomena in a language everybody is using anyway: a theory of particles moving in space.

4 Bohmian Mechanics

- State: $(q, \psi), q \in \mathbb{R}^{3N}, \psi \in L_2(\mathbb{R}^{3N})$, i.e., the state of an N-particle system is given by its wave function *and* the actual positions $q = (\mathbf{q}_1, \ldots, \mathbf{q}_N)$ of the particles *which the theory is about.*

- Dynamics: The time evolution is given by a first-order differential equation for the positions of the particles, with ψ evolving in the usual way:

$$\frac{d}{dt}\mathbf{q}_k(t) = \mathbf{v}_k^{\psi_t}(q(t)) = \frac{\hbar}{m_k}\mathrm{Im}\frac{\nabla_k \psi_t}{\psi_t}(q(t)) \tag{1}$$

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = (-\sum_{k=1}^{N}\frac{\hbar^2}{2m_k}\Delta_k + V(x))\psi(x,t) \tag{2}$$

(Note that the role of ψ is to generate a Galilean covariant vector field on configuration space which guides the motion, and this leads directly to (1).)

This is all we need! It is a crucial property of this dynamical system that it conserves the distribution $\rho = |\psi|^2$, which we call the equivariant measure. The quantum formalism, randomness, Born's rule—"If a system has wave function ψ then its configuration has distribution $|\psi|^2$"—and all the rest emerges from a detailed analysis of these equations. No further axioms about measurements are necessary nor is there room for any such axioms. That this is so was already sketched by David Bohm in his 1952 paper [3]; a more detailed analysis can be found in [4]. Let us give a summary of the main crucial features of Bohmian mechanics: In addition to being clear, objective and deterministic it also agrees with experiment. There is, however no need for collapse, no measurement paradox and no need to split the world into system and observer.

Let us look at some simple examples.

4.1 Example: the motion of a Gaussian wave packet

Consider the time dependent one-particle wave function $\psi_t(x)$ of a freely evolving Gaussian, which starts at the origin with velocity v_0 and width σ. From (1) one obtains the velocity vector field and easily solves the differential equation for the positions, obtaining the solution flow $\Phi_{q_0}(t) := q(t) = v_0 t + q_0\sqrt{1 + t^2/\sigma^4}$. Note that the motion is clearly non-Newtonian. Only in the limit of large times, the particles move with constant velocity $v_\infty(q_0) := v_0 + q_0/\sigma^2$, which means v_∞ is a random variable with a Gaussian distribution, centered around v_0. Now let us define the momentum as the random variable $p := mv_\infty$, which can be approximately determined by measuring the position $q(T)$ at a large time T: $p \approx mq(T)/T$. Clearly the probability distribution for p is exactly the same as the one obtained by projecting the initial state on the eigenstates of the momentum operator. It can in fact be shown quite generally that for an arbitrary freely evolving wave function $\psi_t(x)$, p is well defined, with distribution given in the usual way by the Fourier transform $|\hat{\psi}_0(p)|^2$. Note that p is not at all the same as the "classical momentum" given by m times the actual velocity.

4.2 Example: the two-slit experiment in Bohmian mechanics

The particle passes through either the upper or the lower slit. The interference pattern occurs because the wave function guiding the particle develops this pattern. Closing one slit will lead to a different wave function and therefore to different paths and a different—or no—pattern. The randomness observed in the experiment is due to uncertainty in the initial conditions, as in classical chaotic systems.

5 Measurements/Experiments

Let us sketch an analysis of measurement-like experiments; for a much more detailed analysis see [5]. We describe the combined evolution of a composite system consisting of System $\otimes$ Apparatus. Let the initial state of the apparatus be ϕ_0 and let ϕ_i denote the orthogonal apparatus wave functions corresponding to the possible outcomes. (Think of separated wave packets corresponding to possible pointer positions or patterns of ink spots on a computer printout—which may, for example, register detection by a photocounter.) We assign the values λ_i to the "pointer states" ϕ_i. It turns out [5] that if an experiment is repeatable then in the simplest case there exists a basis $\{|\psi_i\rangle\}$ of the system Hilbert space such that under the interaction with the apparatus

$$|\psi_i\rangle \otimes |\phi_0\rangle \rightarrow |\psi_i\rangle \otimes |\phi_i\rangle. \tag{3}$$

(Note that the unitarity of the time evolution together with the orthogonality of the $|\phi_i\rangle$ forces the orthogonality of the $|\psi_i\rangle$.)

An arbitrary state $|\psi\rangle = \sum c_i|\psi_i\rangle$ may be expressed in this basis, with $c_i = \langle\psi_i|\psi\rangle$. The linearity of the time evolution implies that

$$|\psi\rangle \otimes |\phi_0\rangle \rightarrow \sum c_i|\psi_i\rangle \otimes |\phi_i\rangle. \tag{4}$$

Thus using Born's rule—which we remind you is a *consequence* of Bohmian mechanics—we find that $|\langle\psi_i|\psi\rangle|^2$ is the probability to find the outcome λ_i.

Let us make a remark on the "measurement problem": Certainly the wave function is in a superposition after interaction with a superposition of eigenstates, but the complete state is given by the wave function *and* the actual configuration. The trajectory will end up in but one of the different disjoint wave packets, and thus the dynamics does not lead to a macroscopic superposition of outcomes, as would be the case if we had only a Schrödinger wave function. Moreover, for the further evolution the influence of the other wave packets turns out to be FAPP negligible. In this way collapse is merely a matter of convenience.

Now let us make contact with the usual operator formalism. Define the self-adjoint operator

$$A := \sum \lambda_i |\psi_i\rangle\langle\psi_i|. \tag{5}$$

With this operator we can calculate the statistics for the outcome in the usual way.

The fact that a self-adjoint operator on the system Hilbert space *alone* suffices to describe the full statistics for the outcome of the experiment supports the misleading idea that some preexisting properties of the system have actually been "measured," the apparatus playing a purely passive role. That this is not generally the case, that we rather have to regard the result as being the joint product of the system *and* the apparatus, has been emphasized by Bohr.[1]

For the analysis of more general experiments it is convenient to introduce the following notation. The map $\Delta \mapsto P(\Delta) := \sum_{\lambda_i \in \Delta} |\psi_i\rangle\langle\psi_i|$, from subsets of $\mathbb{R}$ to projectors on $\mathcal{H}$, is what mathematicians call a projection-valued measure (PV).[2] With this notation $\langle\psi|P(\Delta)|\psi\rangle$ is the probability to find the result in the set Δ. Note that $A = \int \lambda P(d\lambda)$.

It turns out [5] that if one doesn't assume repeatability a positive-operator-valued measure (POV) $O(\Delta)$ plays the role of $P(\Delta)$. These operators need not be projectors, i.e., it may be that $O(\Delta)^2 \neq O(\Delta)$. The probability of finding the result in the set Δ is given by $\langle\psi|O(\Delta)|\psi\rangle$. Define the self-adjoint operator $B := \sum \lambda_i O(\lambda_i)$ $(= \int \lambda O(d\lambda))$. Thus the expected value of the outcome is given by $\langle\psi|B|\psi\rangle$. Note that knowledge of B alone does not provide complete information about the statistics of the outcome, as it does for repeatable experiments, because in general $B^n \neq \sum \lambda_i^n O(\lambda_i)$. Thus for nonrepeatable measurements it is not possible to cast the information about the entire statistics into a bilinear form involving a single self-adjoint operator.

POV's have been proposed as a means of providing a generalized description for "fuzzy measurements"[6]. Note, however, that POV's arise naturally from a measurement analysis in Bohmian mechanics, in which there is no "intrinsic fuzziness."

6 The Phase Problem in Quantum Optics

6.1 A brief history of the phase operator

For the following discussion it will be sufficient to focus on a single mode of the electromagnetic field, which is well-known to be equivalent to an one-dimensional harmonic oscillator. We will use the standard notation $a, a^\dagger$ for the annihilation and creation operators, and $N := a^\dagger a$ for the number operator.

For a classical harmonic oscillator the phase is a respectable observable. What is its quantum mechanical counterpart? We give a short sketch of some of the main approaches to the "phase problem." A detailed discussion can be found in [7].

- 1927 Dirac [8]: A polar decomposition of the creation and annihilation operator into $e^{i\Phi}\sqrt{N} := a$, which seems to imply $\sqrt{N}e^{-i\Phi} = a^\dagger$, "yields" $[\Phi, N] = -i$. Dirac noticed himself that this definition leads to contradictions, e.g., if one takes the expectation value of the commutator for an energy eigenstate.

- 1964: Susskind and Glogower [9] prove that there is no way to define an unitary operator U with the property $U\sqrt{N} = a$. Therefore there can be no self-adjoint operator Φ such that $U = e^{i\Phi}$, which explains the flaw in Dirac's ansatz. They conclude that a self-adjoint phase operator doesn't exist.

- 1968: Loudon defines nonorthogonal "phase eigenstates" [10]: $|\phi\rangle := \sqrt{\frac{1}{2\pi}} \sum_{n=0}^{\infty} e^{in\phi}|n\rangle$.

[1] Position measurements are exceptions. Position plays a distinguished role in Bohmian mechanics, as it does in the real world.

[2] $P^2 = P = P^\dagger; P(\emptyset) = 0, P(\Omega) = 1; P(\cup\Delta_i) = \sum P(\Delta_i)$ for mutually disjoint sets Δ_i.

- 1976: Lévy-Leblond, using the Loudon states, constructs a POV [11]: $\Delta \to \int_\Delta d\phi|\phi\rangle\langle\phi|$, for Δ any subset of $[-\pi,\pi]$.

- 1986: Barnett and Pegg introduce "negative-photon-number" states and define the unitary operator [12]: $e^{i\Phi} := \sum_{-\infty}^{\infty} |n\rangle\langle n+1|$.

- 1988: Barnett and Pegg suggest a limiting procedure, based on the definition of phase eigenstates in a finite-dimensional Hilbert space [13]. $|\phi_n\rangle := \frac{1}{\sqrt{s+1}}\sum_{m=0}^{s} e^{im\phi_n}|m\rangle$, $\phi_n := \phi_0 + 2\pi/(s+1)n, n = 0\ldots s$, $\Phi_s := \sum_{n=0}^{s}\phi_n|\phi_n\rangle\langle\phi_n|$. The limit $s \to \infty$ is then taken at the *end* of any calculation.

- 1991: Mandel proposes an operational approach to the quantum phase [14]. He suggests an experiment, together with some procedure to derive quantities he calls the "cosine and sine of the phase difference." He finds disagreement with the predictions based on the (second) Barnett-Pegg operator or the Susskind-Glogower operator.

6.2 Discussion of the different approaches

Let us first address two questions which might now be irritating the reader: 1) How can it be the case that we have a nonexistence proof and several explicit constructions of self-adjoint phase operators at the same time? 2) What exactly is going on in this peculiar (hi)story?

The answer to the first question is easy. The nonexistence proof of Susskind and Glogower tells us that there is no polar decomposition of the annihilation operator into a positive and a unitary operator. None of the "phase operators" suggested by Barnett and Pegg provide such a decomposition (if they serve any purpose at all, it is certainly not for this). But how can one decide who is right? And, perhaps more to the point, what is the physical relevance of all these operators?

This leads us to the second question. We are often told that for every classical observable there exists a corresponding self-adjoint operator. Recipes such as "replace the classical Poisson brackets by the commutator" are used as a guide to postulate the correct commutation relations. This seems to work perfectly well for position and momentum but not for the phase. But so what? Why should it?

We have sketched in (4.1) how to describe "momentum measurements" without invoking postulated commutation relations. The analysis of the experiment shows that the momentum operator as a multiplication operator in Fourier space yields the correct statistics. Note, however, that it can be shown that for the actual velocity—certainly a classical observable—there is neither a corresponding operator nor a POV! This simply means that there is no experiment which measures the actual velocity *in the sense of section (5)*.

The POV proposed by Lévy-Leblond is an explicit example how to describe an *abstract* phase "measurement" without a self-adjoint operator. In order to decide which is the "right" description for the phase one would have to ask for the experiment which an operator or POV is supposed to describe.[3] But what is the physical relevance of pursuing the question as to which experiments are described by a given operator? Note, however, that for a given experiment, say Mandel's

[3]This has been emphasized by Lévy-Leblond. His focus, however, was more on advertising a more general formalism for describing experiments than on applying it to a special example.

experiment, it is well-known how to calculate the photocount statistics, which is all that is relevant. There is no room left for *postulating* operators or eigenstates. An analysis of the experiment at hand shows what quantities are actually "measured" and which mathematical objects, be they operators or POV's or what have you, simplify the description of the predictions. And, as is also stressed by Mandel, different experiments yield different operators. There is no unique phase operator, nor do we need one. In other words: There is no problem!

7 Conclusion

We end by quoting Bell one last time [15]:

> ..... in physics the only observations we must consider are position observations, if only the positions of instrument pointers. It is a great merit of the de Broglie-Bohm picture to force us to consider this fact. If you make axioms, rather than definitions and theorems, about the "measurement" of anything else, then you commit redundancy and risk inconsistency.

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III. THEORETICAL DEVELOPMENT