NEW SQUEEZED LANDAU STATES

C. Aragone
Departamento de Física, Univ. S. Bolívar
Apartado 89000, Caracas 1080A

Abstract

We introduce a new set of squeezed states through the coupled two-mode squeezed operator. It is shown their behavior is simpler than the correlated coherent states introduced by Dodonov, Kurmyshev and Man'ko in order to quantum mechanically describe the Landau system, i.e. a planar charged particle in a uniform magnetic field. We compare results for both sets of squeezed states.

A planar charged particle moving in a uniform magnetic field is a very interesting quantum mechanical system. It is not trivial, needs the two spatial dimensions to describe it, it has some reminiscence of the two-dimensional oscillator, but requires in addition the peculiar presence of the angular momentum operator which play a role as important as the Hamiltonian. As recently it has been pointed out [1], the system has an Osc(1) dynamical degeneracy group. It seemed to us the system has a physics rich enough and mathematically particularly well understood in terms of the holomorphic (and antiholomorphic) coordinates that deserved to be revisited.

A planar particle of charge e, mass m, moving in a uniform magnetic field \( \vec{B} = B\hat{k} \) can be described by the classical first order action

\[
S = \langle \vec{p} \cdot \vec{r} - (2m)^{-1}[\vec{p}^2 - 2^{-1}eB(i\vec{r})]^2 \rangle \equiv \langle \vec{p} \cdot \vec{r} - H \rangle .
\]

(1)

\( \vec{r} \) is the two-dimensional vector position of e, \( \vec{p} \) its canonical momenta (which in the presence of the vector potential \( \vec{A} = 2^{-1}B(i\vec{r}) \) does not coincide with \( m\vec{r} \)), and the linear operator \( i \) indicates a positive \( \pi/2 \) rotation, i.e. \( (i\vec{v})_j = -\epsilon_{jkl}v_l \). We choose \( B \) such that \( eB \equiv m\omega \) is always positive, without losing generality.

The Landau system \( \Phi_L \equiv \{ \vec{r}, \vec{p}, H, \Lambda \equiv - (i\vec{r}) \cdot \vec{p} \} \) is quantized by imposing

\[
[i_j, p_j] = i\hbar \delta_{ij} \quad i, j = (1, 2).
\]

(2)

As shown in ref. [1] it is convenient to introduce two sets of additional, momentum-like variables

\[
\vec{\pi} \equiv \vec{p} - 2^{-1}m\omega(i\vec{r}), \quad \vec{\omega} \equiv \vec{p} + 2^{-1}m\omega(i\vec{r}).
\]

(3)

\( \vec{\pi} \) is the q-operator representing the observable \( m\vec{r} \). In terms of these quantities the Hamiltonian and the angular momentum take the form

\[
H = (2m)^{-1}\{ \vec{p}^2 + 4^{-1}m^2\omega^2 \vec{r}^2 + m\omega\Lambda \} = (2m)^{-1}\vec{\pi}^2 ,
\]

(4)
\[ \Lambda = (-i\pi') \cdot \vec{p}' = (2m\omega)^{-1} \{ \pi'^2 - \omega'^2 \}. \]  

Observe the interesting chiral aspect of \( \Lambda \) in terms of \( \pi' \) and \( \omega' \).

It is immediate to notice that \( \omega' \) commutes with \( \pi_j \),

\[ [\omega', \pi_j] = 0. \]  

Consequently \( \omega' \) and \( \Lambda \) commute with \( H \). Since

\[ [\Lambda, \omega_i] = -i\hbar \epsilon_{ij} \omega_j = i\hbar (i\omega')_i \]  

\[ [\omega_i, \omega_j] = -i\hbar \omega \epsilon_{ij}. \]

we see that \( \{1, \omega', \Lambda\} \) constitute a dynamical symmetric group (which will be easily recognized, when represented by its holomorphic components \( \omega_z, \omega_T \) to be Osc (1)), i.e. commutes with \( H \).

It is convenient to introduce holomorphic dimensionless variables \( z, \bar{z}, p_z, p_{\bar{z}}, \pi_z, \pi_{\bar{z}}, \omega_z, \omega_{\bar{z}} \) to analyze the system.

\[ z \equiv (2^{-1}\hbar^{-1}m\omega)^{1/2}(x + iy), \quad p_z = (2\hbar m\omega)^{1/2}(p_x - ip_y) = -i\partial_z + c.c. \]

The two momentum-like set of variables take the form

\[ \pi_z = p_z + 2^{-1}iz, \quad \pi_{\bar{z}} = p_{\bar{z}} - 2^{-1}i\bar{z} \]  

\[ \omega_z = p_z - 2^{-1}iz, \quad \omega_{\bar{z}} = p_{\bar{z}} + 2^{-1}i\bar{z} \]

while \( H \) and \( \Lambda \) become

\[ H = \hbar \omega \{ p_z p_{\bar{z}} + 4^{-1}z\bar{z} + 2^{-1}\lambda \} \equiv \hbar \omega \lambda, \]  

\[ \Lambda = i\hbar (z p_{\bar{z}} - \bar{z} p_z) \equiv i\lambda = \hbar [\pi_{\bar{z}} - z \partial_z]. \]

Heisengen commutation relations eqs. (2) change to

\[ [z, p_z] = i = [x, \pi_z] = [z, \omega_z] + c.c.. \]

The two main physical observables \( \hbar, \lambda \) have a very simple structure

\[ \hbar = \pi_z \pi_{\bar{z}} + 2^{-1}n_1 + 2^{-1}, \quad \lambda = \pi_z \pi_{\bar{z}} - \omega_z \omega_{\bar{z}} \equiv n_1 - n_2 \]

where \( \pi_z, \omega_z, \pi_{\bar{z}}, \omega_{\bar{z}} \) can be regarded as two sets of decoupled annihilation and creation operators

\[ [\pi_z, \pi_{\bar{z}}] = 1 = [\omega_z, \omega_{\bar{z}}], \]

since \([\omega_z, \pi_z, \pi_{\bar{z}}] = 0\). We emphasize the fundamental role of the both \( \hbar, \lambda(H, \Lambda) \) in determining the two-mode quantum structure of the system. The energy degeneracy is broken by the presence of \( n_2 \), the second fundamental quantum number. These two series of discretes numbers will become the origin of the two complex parameters labelling the coherent Landau states discovered long time.
ago [2] by Mal'kin and Man'ko. (Incidentally our $\pi_F$ coincides with $a$ of ref. [3] and our $\omega_z$ equals $-i\alpha_0$. To introduce coherent Landau states we introduce the state $|0,0>= \psi_{00}$

$$\psi_{00}(z\bar{z}) = \pi^{-\frac{1}{2}} e^{-\frac{1}{2} z\bar{z}}.$$  \hspace{1cm} (16)

$\psi_{00}$ belongs to the ground subspace, i.e. $\pi_F|0,0>=0$ and is unitary (using the natural measure $2^{-1}dxdy = dxdy$. The ground subspace is determined by the orthonormal set $\psi_{np} = (p!)^{-\frac{1}{2}} \omega_F^n |0,0>$

$$|0,0> = (p!)^{-\frac{1}{2}} (iz)^p |0,0>.$$  \hspace{1cm} (17)

Each level-$n$ energy eigenspace has the discrete orthonormal basis

$$\psi_{np} = (n!)^{-\frac{1}{2}} (p!)^{-\frac{1}{2}} \omega_F^n |0,0>.$$  \hspace{1cm} (18)

Equations (14) tell us $H\psi_{np} = \hbar \omega(n + 2^{-1})\psi_{np}$ and $V\psi_{np} = \hbar(n - p)$. We define the coherent Landau states [2] by

$$|w,s> = \sum e^{i\pi_F s} e^{i\pi_F s} |0,0> e^{i\pi_F s}.$$  \hspace{1cm} (19a)

$w, s \in \mathbb{C}$. They constitute an over complete unitary system of the Hilbert space $\{\psi_{np}, n; p \in 0, 1, \ldots\}$ in the usual sense (for coherent states)

$$<w_1 s_1 |w_2 s_2> = e^{-\frac{1}{2}} |w_2 - w_1|^2 - |s_2 - s_1|^2 + |w_1| |s_1| \sin(\omega\pi_F s_1) + |s_2| |s_1| \sin(\omega\pi_F s_1)$$  \hspace{1cm} (19b)

They have three basic properties: i. They are $\pi_F$ eigenstates with eigenvalue $w$, ii. they also are eigenstates of $\omega_z$ with proper value $s$

$$\pi_F |w, s> = w |w, s> \hspace{1cm} \omega_z |w, s> = s |w, s>,$$  \hspace{1cm} (20)

and iii. they propagate remaining in the family. If one starts on $|w, s>$ leaving the system to evolve, at time $t$ $\Phi_L$ will be described by

$$e^{-i\omega t} |w, s> = |w e^{-i\omega t}, s>.$$  \hspace{1cm} (21)

Eqs. (20) suggest a way to compute q-mechanical expected values for physical observables $F(p, p, z, \bar{z})$. One has to transform them to their representation in terms of the new variables $(\pi, \pi_F, \omega, \omega_z)$, then normal ordering in both types of variables and finally taking into account eqs. (20).

In this way we obtain:

$$<z>_{CL} = <w_s z|w_s> = <w_s|z|w_s> = i(w - \bar{z})$$  \hspace{1cm} (22a)

$$<z^2>_{CL} = -(w - \bar{z})^2,$$  \hspace{1cm} (22b, c)

plus their respective complex (hermitian) conjugates. We also obtain

$$<p_z>_{CL} = 2^{-1} <\pi_z + \omega_z>_{CL} = 2^{-1}(\omega + s) + c.c.$$  \hspace{1cm} (23a)
Recalling definitions (8) relating $z, \bar{z}$ and real dimensionless variables $\bar{z}, y$ we can calculate physical uncertainties, which are defined for canonical sets of variables in terms of holomorphic variances $\Delta z, \Delta \bar{z} = \langle z \bar{z} \rangle - \langle z \rangle \langle \bar{z} \rangle, \Delta p_z, \Delta p_{\bar{z}}$. They turn out to be

$$< p_z^2 >_{CL} = 4^{-1} < (\pi_z + \omega_z)^2 > = 4^{-1}(w + s)^2 + h.c.,$$

$$< p_z p_r >_{CL} = 4^{-1} + 4^{-1}(w + 3)(w + s)$$

$$< \hbar >_{CL} = < w s | h | w s > = w w + 2^{-1}, < \lambda >_{CL} = w w - s \bar{s},$$

$$< \lambda ^2 >_{CL} = < \lambda ^2 >_{CL} = < \lambda ^2 >_{CL} + w w + s \bar{s}.$$ 

Recalling definitions (8) relating $z, \bar{z}$ and real dimensionless variables $z, y$ we can calculate physical uncertainties, which are defined for canonical sets of variables in terms of holomorphic variances $\Delta z, \Delta \bar{z} = \langle z \bar{z} \rangle - \langle z \rangle \langle \bar{z} \rangle, \Delta p_z, \Delta p_{\bar{z}}$. They turn out to be

$$(\Delta z)^2_{CL} = 4^{-1}(\Delta z)^2_{CL} + 4^{-1}(\Delta \bar{z})^2_{CL} + 2^{-1}(\Delta z \bar{z})_{CL} = 2^{-1} = (\Delta y)^2_{CL}, (\Delta z \bar{z})_{CL} = 0.$$ 

In a similar way, we find for the physical momenta

$$(\Delta p_z)^2_{CL} = 2^{-1} = (\Delta p_z p_{\bar{z}})_{CL} = 0.$$ 

Consequently both uncertainties attain lowest bound

$$(\Delta z)_{CL}(\Delta p_z)_{CL} = 2^{-1} = (\Delta y)_{CL}(\Delta p_{\bar{z}})_{CL}.$$ 

Coherent Landau states are minimum uncertainty states (MUS).

Squeezing can be now analysed, since the standard procedure to consider this type of states involves the squeezing of associated coherents states. Complexive decoupled squeezed Landau states have been introduced in ref. [3], where they have been called correlated coherent states.

Since squeezing is not that intuitive we face in principle four different types of squeezing: partial squeezing in $\pi, \pi, \omega, \omega$ or full, complexive squeezing in both sets of variables.

The complexive squeezing might be either decoupled or coupled in both set of variables. One might think that it could be enough to squeeze just in the dynamical constituents of the hamiltonian $\pi, \pi, \omega, \omega$ in order to obtain physically appealing results. This primary type of "squeezing" can be shown to lead to states which are irrelevant, since they are neither minimum uncertainty states nor the variances of any canonical variable can tend to zero.

We are obliged to turn our interest to more radical way of squeezing. As we said above, we must try complexive squeezing, i.e. to introduce squeeze operators which squeeze both type of quanta, the $\pi$ and the $\omega$-ones.

Let us first consider what we call "decoupled" squeezing, as it has been done in ref. [3]. The squeezing operator is defined as

$$S(q_1, q_2) = e^{\frac{1}{4}q_1^2 + \frac{1}{2}q_2^2 - \frac{1}{4}q_1^2 - \frac{1}{2}q_2^2} = S^\pi(q_1)S^\omega(q_2).$$

We consider the squeezed states

$$| w s, q_1, q_2 > \equiv S(q_1, q_2)| w, s >.$$ 

where both $w$ and $s$ are distorted.
Both the $\pi$ and $\omega$ variables transform non trivially here,

\[
\begin{align*}
(\pi_T)_q &= S^+_q \pi S_q = \pi_T \chi r_1 + e^{2i\omega_1} \sigma r_1 \pi
\quad \text{(31)}
\end{align*}
\]

\[
\begin{align*}
(\omega_T)_q &= S^+_q \omega S_q = \omega_T \chi r_2 + e^{2i\omega_2} \sigma r_2 \omega_T + \text{h.c.}
\quad \text{(32)}
\end{align*}
\]

The squeezed transformed of the Heinsenberg canonical variable $z$ $z_{q_1 q_2} \equiv S^+_q z S_q$, becomes in the present case

\[
\begin{align*}
z_{q_1 q_2} &= i(\pi_T \chi r_1 + e^{2i\omega_1} \sigma r_1 \pi - \omega_T \chi r_2 - e^{-2i\omega_2} \sigma r_2 \omega_T).
\end{align*}
\]

The complexive squeezed expectation values of $z$ and $p_z$ are therefore

\[
\begin{align*}
\langle z \rangle_{q_1 q_2} &= \text{i}[s_{q_1}(w) - s_{q_2}(s)],
\quad \text{(34a)}
\end{align*}
\]

\[
\begin{align*}
\langle p_z \rangle_{q_1 q_2} &= \text{i}[s_{q_1}(w) + s_{q_2}(s)],
\quad \text{(34b)}
\end{align*}
\]

where subindex $CL$ indicates the coherent Landau state $|w, s\rangle$ and $s_q(u) \equiv \chi ru + e^{2i\omega} \sigma ru$.

Quadratic complexive squeezed expectation values become

\[
\begin{align*}
\langle z^2 \rangle_{q_1 q_2} &= \overline{|s_{q_1}(w) - s_{q_2}(s)|^2 - \sigma r_1 \chi r_1 e^{2i\omega_1} - \sigma r_2 \chi r_2 e^{-2i\omega_2}}
\quad \text{(35a)}
\end{align*}
\]

\[
\begin{align*}
\langle z^2 \rangle_{q_1 q_2} &= \overline{z^2} + \sigma r_1^2 + \sigma r_2^2,
\quad \text{(35b)}
\end{align*}
\]

\[
\begin{align*}
\langle p_z^2 \rangle_{q_1 q_2} &= 4^{-1} \overline{|s_{q_1}(w) + s_{q_2}(s)|^2 + 4^{-1} \sigma r_1 \chi r_1 e^{2i\omega_1} + 4^{-1} \sigma r_2 \chi r_2 e^{-2i\omega_2},
\quad \text{(36a)}
\end{align*}
\]

\[
\begin{align*}
\langle p_z^2 \rangle_{q_1 q_2} &= 4^{-1} \overline{|s_{q_1}(w) + s_{q_2}(s)||s_{q_1}(w) + s_{q_2}(s)|^2 + 4^{-1} \sigma r_1^2 r_1 + 4^{-1} \sigma r_2^2 r_2.}
\quad \text{(36b)}
\end{align*}
\]

From this expressions for the holomorphic variables we can evaluate physical uncertainties to see how they behave for complexive decoupled squeezing. They are

\[
\begin{align*}
(\Delta \chi z)_{q_1 q_2}^2 &= (\Delta \chi z)_{q_1 q_2}^2 = 2^{-1} \chi r_1 (\chi r_1 - \sigma r_1 \cos 2\varphi_1) + 2^{-1} \chi r_2 (\chi r_2 - \sigma r_2 \cos 2\varphi_2) - 2^{-1},
\quad \text{(37)}
\end{align*}
\]

\[
\begin{align*}
(\Delta \chi z)_{q_1 q_2}^2 &= (\Delta \chi z)_{q_1 q_2}^2 = 2^{-1} \chi r_1 (\chi r_1 + \sigma r_1 \cos 2\varphi_1) + 2^{-1} \chi r_2 (\chi r_2 + \sigma r_2 \cos 2\varphi_2) - 2^{-1}.
\quad \text{(38)}
\end{align*}
\]

For $\varphi_1 = 0 = \varphi_2$, $\Delta r$ and $\Delta \chi r$ are squeezed since:

\[
\begin{align*}
(\Delta \chi z)_{q_1 q_2}^2 &= (\Delta \chi z)_{q_1 q_2}^2 = 4^{-1} e^{-2\chi r_1} + 4^{-1} e^{-2\chi r_2} \rightarrow 0^+, \quad r_1, r_2 \rightarrow \infty
\quad \text{(39)}
\end{align*}
\]

while, of course $\Delta \chi r_1$ and $\Delta \chi r_2$ increase according to eq. (38). The partial uncertainties get closer to their lowest bound,

\[
\begin{align*}
(\Delta \chi z)_{q_1 q_2}^2 &= (\Delta \chi z)_{q_1 q_2}^2 \big|_{\varphi_1 = 0 = \varphi_2} = 8^{-1}[1 + \chi r_2 (r_2 - r_1)] = (\Delta \chi z)_{q_1 q_2}^2 = (\Delta \chi z)_{q_1 q_2}^2 \big|_{\varphi_1 = 0 = \varphi_2}
\quad \text{(40)}
\end{align*}
\]

This result indicates that physical squeezing, in the sense that the squeezed states are also minimum uncertainty states, is obtained just for $r_2 = r_1$. Complexive decoupled squeezing leads to physical squeezing modes, but the two independent "a priori" parameters $q_1$ and $q_2$ have to coincide.

A nicer solution to finding squeezed states of $\Phi_1$ arises by considering the fact that we have two modes in the system. For this situation a more natural squeezed operator can be defined,
similarly to what has been done for the two photon case in ref. [4]. The "coupled" squeezing operator we postulate is given by

$$S_q = e^{i\frac{1}{2}q^2 \pi_T + \frac{1}{2}q^2 \omega_s}. \quad (41)$$

It naturally depends upon only one parameter. It is straightforward to show that the squeezed values of $\pi_T$ and $\omega_s$ respectively are

$$S_q^+ \pi_T S_q = \pi_T \cosh \frac{r}{2} + e^{i\omega_s} \sinh \frac{r}{2} \omega_s, \quad (42)$$

$$S_q^+ \omega_s S_q = \omega_s \cosh \frac{r}{2} + e^{i\omega_s} \sinh \frac{r}{2} \pi_T. \quad (43)$$

As expected this type of squeezing makes $\pi$-variables to have $\omega$-components and vice versa.

The new associates squeezed states are defined by

$$|w, s, q \rangle = S_q |w, s \rangle \quad (44)$$

where $S_q$ has been introduced in eq. (41). It is immediate to perform in this case similar calculations to what has already been done for the previous case. Results turn out to be mathematically simpler and physically interesting. We get

$$< z > = < z_q >_{CL} = i(w - \bar{s}) \cosh \frac{r}{2} + i\sinh \frac{r}{2} (\bar{s} e^{2i\omega} - We^{-2i\omega}),$$

$$< p_z > = 2^{-1}(W + s) \cosh \frac{r}{2} + 2^{-1} \sinh \frac{r}{2} (\bar{W} e^{2i\omega} + We^{-2i\omega}).$$

In addition one finds that $(\Delta z)_q^2 = (\Delta \pi_T)_q^2 = 0$. Finally the variances of the canonical variables attain the respective forms.

$$(\Delta x)_q^2 = 4^{-1} e^{r} (1 - \cos 2\varphi) + 4^{-1} e^{-r} (1 + \cos 2\varphi) = (\Delta \nu)_q^2, \quad (47a)$$

$$(\Delta p_x)_q^2 = 4^{-1} e^{r} (1 + \cos 2\varphi) + 4^{-1} e^{-r} (1 - \cos 2\varphi) = (\Delta \nu)_q^2, \quad (47b)$$

Both uncertainties coincide, their value being

$$(\Delta x)_q^2 (\Delta p_x)_q^2 = (\Delta \nu)_q^2 (\Delta \nu)_q^2 = 4^{-1} (\sinh^2 \varphi + \cosh^2 \varphi).$$

For $\varphi = k\pi/2$ we obtain squeezing and minimum uncertainty.

In conclusion we feel these coupled squeezed states (44) are the natural ones for introducing squeezing in the Landau system. We have shown they behave in a simpler way then those defined in ref. [3] while they also lead to physical squeezing.

References