Abstract

We show that the probability distributions \( P_n(q,p;y) := |\langle n|p,q;y \rangle|^2 \), which are obtained from squeezed states, obey an interesting partial differential equation, to which we give two intuitive interpretations: as a wave equation in one space dimension and also as a pseudo-diffusion equation. We also study the corresponding Wehrl entropies \( S_n(y) \), and show that they have minima at zero squeezing, \( y = 0 \).

1 Introduction

This talk is based mainly on a work which was done in collaboration with Salomon Mizrahi from Brazil.

Squeezed oscillator states are defined in terms of the bosonic creation and annihilation operators, \( a^\dagger := \frac{1}{\sqrt{2}}(x - \frac{p}{\hbar}) \), and \( a := \frac{1}{\sqrt{2}}(x + \frac{p}{\hbar}) \), as follows:

\[
|z;\xi) = |p,q;\xi) := D(q,p)S(\xi)|0\rangle, \quad \text{where} \quad z := (q + ip)/\sqrt{2},
\]

(1)

and \( |0\rangle \) is the ground state of the harmonic oscillator. Both \( D \) and \( S \) are unitary operators. \( D \) creates the coherent state, and is defined by

\[
D(q,p) := \exp[za^\dagger - z^*a] = \exp[ipx - q\frac{\partial}{\partial x}],
\]

(2)

and \( S(\xi) \) is the squeezing operator:

\[
S(\xi) := \exp[\frac{1}{2}(\xi a^\dagger a^2 - \xi^* a^2 a^\dagger)],
\]

(3)

where \( \xi \) is a complex variable. For \( \xi = 0 \), we recover the ordinary (unsqueezed) coherent states.

The squeezed states satisfy the completeness relation, \( \int |p,q;\xi)\langle p,q;\xi'| \frac{dpdq}{2\pi} = 1 \), for every \( \xi \).

Therefore,

\[
\int P_n(q,p;\xi) \frac{dpdq}{2\pi} = 1, \quad \text{where} \quad P_n(q,p;\xi) := |\langle n|p,q;\xi|n \rangle|^2,
\]

(4)

where \( |n\rangle \) is the number state. If we interpret the real parameters \( q \) and \( p \) as the position and momentum variables, then (4) allows us to interpret the non-negative functions \( P_n \) as probability distributions in the \((q,p)\)-phase plane, for every \( n \) and \( \xi \).
In this talk, I shall consider these $P_n$ for real values of the squeezing parameter $\xi$, which will be denoted by $y$. In particular, I shall show that the $P_n(q, p; y)$ satisfy the interesting partial differential equations (9) and (12), to which two intuitive interpretations can be given. Finally, I shall show that the Wehrl entropy $S_n(y)$ (14) of the $P_n$ must have their minima at zero squeezing, $y = 0$.

2 Explicit Form of the Distributions $P_n$

The distribution $P_n(q, p; \xi) := |\langle n| p, q; \xi \rangle|^2$ gives the probability of finding $n$ bosons (photons) in the squeezed states $|q, y; \xi \rangle$. It is a physically important quantity, and it has been calculated by different methods. The dependence of $P_n(q, p; \xi)$ on $n$ was studied by Schleich and Wheeler [2]. For $\xi = y$, the $P_n$ is given by the following complicated expression [1,3,7]:

$$P_n(q, p; y) := |\langle n| p, q; y \rangle|^2 = \frac{2\sqrt{\gamma}}{2^n n!(\gamma + 1)} |\hat{H}_n(2, \eta; w)|^2 \exp \left[ -\frac{q^2 + \gamma p^2}{1 + \gamma} \right], \quad n \geq 0,$$

where

$$\gamma := e^{2y}, \quad \eta := \frac{1 - \gamma}{1 + \gamma}, \quad \text{and} \quad w := \frac{q + iy}{\gamma + 1},$$

and where $\hat{H}_n(2, \eta; w)$ are the generalized Hermite polynomials ($GHP$), which are defined in terms of the raising operators $R(\alpha, \beta; z) = \alpha z - \beta \frac{\partial}{\partial z}$, as follows [1]:

$$\hat{H}_n(\alpha, \beta; x) = R^n(\alpha, \beta; x) \cdot 1 = \sum_{s=0}^{[n/2]} \frac{n!}{(n-2s)! s!} \left( \frac{-\alpha \beta}{2} \right)^s (\alpha x)^{n-2s}.$$

These polynomials are equal to the standard Hermite polynomials for $\alpha = 2$ and $\beta = 1$. In the limit, $\beta \to 0$, these $\hat{H}_n(x)$ becomes simple powers of $x$: $\hat{H}_n(0, 0, x) = \alpha^n x^n$. Therefore, in the limit of zero squeezing, $\gamma \to 1$, we have $\eta \to 0$, so that the above $GHP$'s become simple powers of $w$. Thus, for $y \to 0$, equation (5) gives the following well-known Poisson distribution of the unsqueezed coherent states:

$$P_n(q, p; 0) = \frac{\rho^{2n}}{2^n n!} \exp \left[ -\frac{\rho^2}{2} \right], \quad n \geq 0, \quad \text{where} \quad \rho^2 := q^2 + p^2,$$

When discussing probability distributions, it is useful to think of the regions that are surrounded by the equipotential curves, $P_n(q, p; y) = \text{const.}$; I shall call these regions potential regions. Thus, the potential regions of the above Poisson distribution $P_n(q, p; 0)$ are concentric circles in the $(q, p)$-plane. But for $y \neq 0$, these regions will have approximately elliptical shapes, whose major axes lie along the $p$-axis for $y < 0$ and along the $q$-axis for $y > 0$. These regions become more elongated in one direction and narrower in the other, as $|y|$ increases.

3 The Partial Differential Equation for the $P_n$

Since the integral (4) of the distributions $P_n(q, p; y)$ over the whole $(q, p)$-space remains constant under squeezing, it is useful to think of the change of $P_n(q, p; y)$ as functions of $y$ as a
redistribution of probability densities in phase space, which maintains the positivity condition \( P_n(q,p;y) \geq 0 \) for all \( y \). This redistribution of the \( P_n(q,p;y) \) is governed by the following interesting and amazingly simple partial differential equation:

\[
\frac{\partial}{\partial \gamma} P_n(q,p;\gamma) = \frac{1}{4} \left( \frac{\partial^2}{\partial q^2} - \frac{1}{\gamma^2} \frac{\partial^2}{\partial p^2} \right) P_n(q,p;\gamma) , \quad \text{where} \quad \gamma := e^{2\nu} . \tag{9}
\]

This equation was originally obtained [1] by straightforward but lengthy differentiation of the expression (5), and by using the following property of the \( G_P \):

\[
- \frac{1}{4} \frac{\partial^2}{\partial w^2} G_P(a,\eta,w) = -\frac{1}{4} \frac{\partial^2}{\partial w^2} G_P(a,\eta,w) . \tag{10}
\]

However, we can now derive it by two other more general methods [5], as reported in the summary section.

4 Interpretation as Wave and Pseudo-Diffusion Equations

I shall now present two possible intuitive interpretations of the above differential equation:

(I) D'Alembert or Wave Equation: The following is a new interpretation, which was not discussed in [1]: For a fixed squeezing parameter \( y \), equation (9) looks like the wave equation for one space dimension \( q \), if we think of the \( p \) variable in (9) as the time variable \( t \):

\[
\left( \frac{\partial^2}{\partial q^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \Phi(q,t;y) = -4\pi \rho(q,t;y) , \quad \text{where} \quad \rho(q,t;y) = -\frac{1}{\pi} \frac{\partial}{\partial \gamma} P_n(q,t;\gamma) . \tag{11}
\]

In this interpretation, the parameter \( \gamma \) would then play the role of the speed of light \( c(n) \) in matter, which depends on the parameter \( y \), similar to the dependence of \( c(n) \) on the index of refraction index \( n \). If the \( P_n \) are thought of as electromagnetic potentials \( \Phi(q,t;y) \), then

\[
4 \frac{\partial}{\partial \gamma} P_n(q,p;\gamma) \]

will play the role of a time-dependent charge distributions

\[
-4\pi \rho(q,t;y) .
\]

(II) Pseudo-Diffusion Equation: By substituting \( \frac{\partial}{\partial \gamma} = 2e^{2\nu} \frac{\partial}{\partial \gamma} \) into (9), we obtain a more symmetric differential equation for the \( P_n \):

\[
\frac{\partial}{\partial \gamma} P_n(q,p;y) = \frac{1}{2} \left( e^{2\nu} \frac{\partial^2}{\partial q^2} - e^{-2\nu} \frac{\partial^2}{\partial p^2} \right) P_n(q,p;y) . \tag{12}
\]

This equation is also new and permits a more pertinent intuitive understanding of the redistribution process of the \( P_n \), by comparing (12) with the diffusion equation in two dimensions [6]:

\[
\frac{\partial}{\partial t} T(q,p;t) = \sigma \left( \frac{\partial^2}{\partial q^2} + \frac{\partial^2}{\partial p^2} \right) T(q,p;t) , \tag{13}
\]
where $\sigma$ is the diffusion coefficient. Equations (12) and (13) are similar, if we interpret the squeezing parameter $y$ as the time variable. However, the two equations differ in two interesting aspects:

1. The sign in front of $\frac{\partial^2}{\partial q^2}$ in (12) is negative rather than positive. Such a "negative diffusion coefficient" leads to "infusion" rather than diffusion in the $p$-direction. Consequently, as $y$ increases, the equi-probability curves, $P_n(q,p;y) = \text{const.}$, move towards the origin along the $p$-axis, but away from the origin along the $q$-axis. Therefore, we expect the probability regions to be concentric elongated "quasi ellipses" which are extended along the $p$-axis for $y \rightarrow -\infty$. They become more and more circular as $y$ approaches zero, and then stretch outwards along the $q$-axis, as $y \rightarrow \infty$. For the above reasons, we shall call equations (9) and (12) "pseudo diffusion equation".

2. The "diffusion coefficients" $\exp[2y]/2$ and $-\exp[-2y]/2$ and in front of $\frac{\partial^2}{\partial q^2}$ and $\frac{\partial^2}{\partial p^2}$ in (12) depend on $y$. For $y \rightarrow +\infty$, the term $\frac{1}{2} \exp[2y] P_n$ dominates the r.h.s. of (12), whereas for $y \rightarrow -\infty$, the second term dominates. This dependence on $y$ can be given an interesting intuitive explanation: Let us consider the redistribution process when $y$ is very large: In this case the probability densities $P_n(q,p;y)$ are extended in the $q$-direction and tightly squeezed or compressed in the $p$-axis, which makes it difficult to compress them further along the $p$-axis. For this reason the "infusion coefficient" becomes so small, namely $\propto \exp[-2y]$. In contrast, the diffusion along the $q$-axis must become faster and faster, in order to diffuse all the incoming density flux from the other orthogonal $p$-direction, which is entering the cigar-shaped potential regions through their lengthy boundaries.

5 The Wehrl Entropy for the $P_n$

A useful measure for the information content of the probability distributions $P_n(q,p;y)$ is the Gibbs or Wehrl entropy [7], which is defined by

$$S_n(y) := -\int P_n(q,p;y) \ln P_n(q,p;y) \frac{dpdq}{2\pi}.$$  \hspace{1cm} (14)

Because of the symmetry $P_n(q,p;-y) = P_n(p,q;y)$, the entropy (14) is even in $y$: $S_n(-y) = S_n(y)$. Therefore, at $y = 0$ each $S_n(y)$ must have either a maximum or a minimum. We shall now argue that $S_n(0)$ should correspond to a minimum: We assume that $S_n(y)$ does not oscillate as a function of $y$. Therefore, it is enough to argue that $S_n(y)$ grows with $|y|$ for large values of $|y|$. For large positive $y$, equation (12) behaves essentially like a one-dimensional diffusion equation in the $q$-variable. But it is well-known that the solutions of diffusion equations lead to entropies which increase with time [6]. Therefore, the $S_n(y)$ must increase as $y \rightarrow \infty$. But since the $S_n(y)$ are even in $y$, they must also grow as $y \rightarrow -\infty$. Hence, the $S_n(0)$ must lie at the bottom of the curves $S(y)$ vs. $y$.

Finally, we note that the von Neumann entropy $S_{\text{VN}}(\rho) := -\text{Tr}(\rho \ln \rho)$ for the pure states $\rho := |n\rangle \langle n|$ must vanish. In contrast, explicit calculations of the Wehrl entropies of the Poisson
distributions (8) shows that $S_n(0) \geq 1$ for all $n$, in accordance with a conjecture by Wehrl [7], which was proved by Lieb [8].

To summarize this section: in contrast to diffusion equations, where the entropies of their solutions always increase with time, the entropies $S_n(y)$ for the solutions of the above pseudo-diffusion equation first decrease monotonically as $y$ grows from $-\infty$ to zero, but then increase monotonically as $y$ grows from zero to $+\infty$.

6 Summary and Outlook

Two equivalent partial differential equations (9) and (12) were presented and then interpreted, as wave and as pseudo-diffusion equations. The probability densities $P_n(q, p; y)$ (5) provide an infinite number of their solutions.

By the time of writing the present lecture notes, we succeeded in proving, by two general methods, that the expectation values $\langle q, p; \xi \mid O \mid q, p; \xi \rangle$ of an arbitrary operator $O$, satisfy a generalized version of the above partial differential equations, which also include rotations, i.e. for the general squeezing $\xi = r e^{i\phi}$. Interesting examples of $O$ are the number operators $N$ and $N^2$; their expectation values provide the simplest solutions of (9) and (12). Also the projection operator $\langle q, p; \xi \rangle \langle q, p; \xi \rangle$, and consequently its Wigner function, satisfy these equations.

References


