WAVE AND PSEUDO-DIFFUSION EQUATIONS FROM SQUEEZED STATES

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Abstract

We show that the probability distributions $P_n(q,p;y) := |\langle n|p,q;y \rangle|^2$, which are obtained from squeezed states, obey an interesting partial differential equation, to which we give two intuitive interpretations: as a wave equation in one space dimension and also as a pseudo-diffusion equation. We also study the corresponding Wehrl entropies $S_n(y)$, and show that they have minima at zero squeezing, $y = 0$.

1 Introduction

This talk is based mainly on a work which was done in collaboration with Salomon Mizrahi from Brazil.

Squeezed oscillator states are defined in terms of the bosonic creation and annihilation operators, $a^\dagger := \frac{1}{\sqrt{2}}(x - \frac{p}{\hbar})$, and $a := \frac{1}{\sqrt{2}}(x + \frac{p}{\hbar})$, as follows:

$$|z;\xi\rangle = |p,q;\xi\rangle := \mathcal{D}(q,p)S(\xi)|0\rangle, \text{ where } z := (q + ip)/\sqrt{2},$$  \hspace{1cm} (1)

and $|0\rangle$ is the ground state of the harmonic oscillator. Both $\mathcal{D}$ and $S$ are unitary operators. $\mathcal{D}$ creates the coherent state, and is defined by

$$\mathcal{D}(q,p) := \exp[z a^\dagger - z^* a] = \exp[ipx - q \frac{\partial}{\partial x}],$$  \hspace{1cm} (2)

and $S(\xi)$ is the squeezing operator:

$$S(\xi) := \exp[\frac{1}{2}(\xi a^2 - \xi^* a^2)],$$  \hspace{1cm} (3)

where $\xi$ is a complex variable. For $\xi = 0$, we recover the ordinary (unsqueezed) coherent states. The squeezed states satisfy the completeness relation, $\int |p,q;\xi\rangle \langle p,q;\xi| \frac{dp dq}{2\pi} = 1$, for every $\xi$. Therefore,

$$\int P_n(q,p;\xi) \frac{dp dq}{2\pi} = 1, \text{ where } P_n(q,p;\xi) := |\langle n|p,q;\xi|n \rangle|^2,$$  \hspace{1cm} (4)

where $|n\rangle$ is the number state. If we interpret the real parameters $q$ and $p$ as the position and momentum variables, then (4) allows us to interpret the non-negative functions $P_n$ as probability distributions in the $(q,p)$-phase plane, for every $n$ and $\xi$. 

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In this talk, I shall consider these \( P \) for real values of the squeezing parameter \( \xi \), which will be denoted by \( y \). In particular, I shall show that the \( P_n(q, p; y) \) satisfy the interesting partial differential equations (9) and (12), to which two intuitive interpretations can be given. Finally, I shall show that the Wehrl entropy \( S_n(y) \) of the \( P \) must have their minima at zero squeezing, \( y = 0 \).

2 Explicit Form of the Distributions \( P_n \)

The distribution \( P_n(q, p; \xi) := |\langle n|p, q; \xi \rangle|^2 \) gives the probability of finding \( n \) bosons (photons) in the squeezed states \( |q, y; \xi \rangle \). It is a physically important quantity, and it has been calculated by different methods. The dependence of \( P_n(q, p; \xi) \) on \( n \) was studied by Schleich and Wheeler [2]. For \( \xi = y \), the \( P_n \) is given by the following complicated expression [1,3,7]:

\[
P_n(q, p; y) := |\langle n|p, q; y \rangle|^2 = \frac{2\sqrt{\gamma}}{2^n n!(\gamma + 1)} |\tilde{H}_n(2, \eta; w)|^2 \exp \left[ -\frac{q^2 + \gamma p^2}{1 + \gamma} \right], \quad n \geq 0,
\]

where

\[
\gamma := e^{2y}, \quad \eta := \frac{1 - \gamma}{1 + \gamma}, \quad \text{and} \quad w := \frac{q + i\gamma p}{\gamma + 1},
\]

and where \( \tilde{H}_n(2, \eta; w) \) are the generalized Hermite polynomials (\( \mathcal{GHP} \)), which are defined in terms of the raising operators \( R(\alpha, \beta; x) = \alpha x - \beta \frac{\partial}{\partial x} \), as follows [1]:

\[
\tilde{H}_n(\alpha, \beta; x) = R^n(\alpha, \beta; x) \cdot 1 = \sum_{s=0}^{[n/2]} \frac{n!}{(n - 2s)!s!} \left( \frac{-\alpha \beta}{2} \right)^s (ax)^{n-2s}.
\]

These polynomials are equal to the standard Hermite polynomials for \( \alpha = 2 \) and \( \beta = 1 \). In the limit, \( \beta \to 0 \), these \( \tilde{H}_n(x) \) becomes simple powers of \( x \): \( \tilde{H}_n(0, 0; x) = \alpha^n x^n \). Therefore, in the limit of zero squeezing, \( \gamma \to 1 \), we have \( \eta \to 0 \), so that the above \( \mathcal{GHP} \)'s become simple powers of \( w \). Thus, for \( y \to 0 \), equation (5) gives the following well-known Poisson distribution of the unsqueezed coherent states:

\[
P_n(q, p; 0) = \frac{\rho^{2n}}{2^n n!} \exp \left[ -\frac{\rho^2}{2} \right], \quad n \geq 0, \quad \text{where} \quad \rho^2 := q^2 + p^2,
\]

When discussing probability distributions, it is useful to think of the regions that are surrounded by the equipotential curves, \( P_n(q, p; y) = \text{const.} \); I shall call these regions potential regions. Thus, the potential regions of the above Poisson distribution \( P_n(q, p; 0) \) are concentric circles in the \( (q,p) \)-plane. But for \( y \neq 0 \), these regions will have approximately elliptical shapes, whose the major axes lie along the \( p \)-axis for \( y < 0 \) and along the \( q \)-axis for \( y > 0 \). These regions become more elongated in one direction and narrower in the other, as \( |y| \) increases.

3 The Partial Differential Equation for the \( P_n \)

Since the integral (4) of the distributions \( P_n(q, p; y) \) over the whole \( (q,p) \)-space remains constant under squeezing, it is useful to think of the change of \( P_n(q, p; y) \) as functions of \( y \) as a
redistribution of probability densities in phase space, which maintains the positivity condition \( P_n(q,p;y) \geq 0 \) for all \( y \). This redistribution of the \( P_n(q,p;y) \) is governed by the following interesting and amazingly simple partial differential equation:

\[
\frac{\partial}{\partial \gamma} P_n(q,p;y(\gamma)) = \frac{1}{4} \left( \frac{\partial^2}{\partial q^2} - \frac{1}{\gamma^2} \frac{\partial^2}{\partial p^2} \right) P_n(q,p;y(\gamma)) , \quad \text{where} \quad \gamma := e^{2\nu} .
\]  

(9)

This equation was originally obtained [1] by straightforward but lengthy differentiation of the expression (5), and by using the following property of the \( G_P \) [1]:

\[
\frac{\partial}{\partial \eta} \tilde{H}_n(\alpha,\eta,w) = -\frac{1}{4} \frac{\partial^2}{\partial w^2} \tilde{H}_n(\alpha,\eta,w) .
\]  

(10)

However, we can now derive it by two other more general methods [5], as reported in the summary section.

4 Interpretation as Wave and Pseudo-Diffusion Equations

I shall now present two possible intuitive interpretations of the above differential equation:

(I) D'Alembert or Wave Equation: The following is a new interpretation, which was not discussed in [1]: For a fixed squeezing parameter \( y \), equation (9) looks like the wave equation for one space dimension \( q \), if we think of the \( p \) variable in (9) as the time variable \( t \):

\[
\left( \frac{\partial^2}{\partial q^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \Phi(q,t;y) = -4\pi \rho(q,t;y) , \quad \text{where} \quad \rho(q,t;y) = -\frac{1}{\pi} \frac{\partial}{\partial \gamma} P_n(q,t;y(\gamma)) .
\]  

(11)

In this interpretation, the parameter \( \gamma \) would then play the role of the speed of light \( c(n) \) in matter, which depends on the parameter \( y \), similar to the dependence of \( c(n) \) on the index of refraction index \( n \). If the \( P_n \) are thought of as electromagnetic potentials \( \Phi(q,t;y) \), then \( 4\frac{\partial}{\partial \gamma} P_n(q,p;y(\gamma)) \) will play the role of a time-dependent charge distributions \(-4\pi \rho(q,t;y)\).

(II) Pseudo-Diffusion Equation: By substituting \( \frac{\partial}{\partial \gamma} = 2e^{2\nu} \frac{\partial}{\partial \gamma} \) into (9), we obtain a more symmetric differential equation for the \( P_n \):

\[
\frac{\partial}{\partial y} P_n(q,p;y) = \frac{1}{2} \left( e^{2\nu} \frac{\partial^2}{\partial q^2} - e^{-2\nu} \frac{\partial^2}{\partial p^2} \right) P_n(q,p;y) .
\]  

(12)

This equation is also new and permits a more pertinent intuitive understanding of the redistribution process of the \( P_n \), by comparing (12) with the diffusion equation in two dimensions [6]:

\[
\frac{\partial}{\partial t} T(q,p;t) = \sigma \left( \frac{\partial^2}{\partial q^2} + \frac{\partial^2}{\partial p^2} \right) T(q,p;t) ,
\]  

(13)
where $\sigma$ is the diffusion coefficient. Equations (12) and (13) are similar, if we interpret the squeezing parameter $y$ as the time variable. However, the two equations differ in two interesting aspects:

1. The sign in front of $\frac{\partial^2}{\partial y^2}$ in (12) is negative rather than positive. Such a "negative diffusion coefficient" leads to "infusion" rather than diffusion in the $p$-direction. Consequently, as $y$ increases, the equi-probability curves, $P_n(q,p;y) = \text{const.}$, move towards the origin along the $p$-axis, but away from the origin along the $q$-axis. Therefore, we expect the probability regions to be concentric elongated "quasi ellipses" which are extended along the $p$-axis for $y \to -\infty$. They become more and more circular as $y$ approaches zero, and then stretch outwards along the $q$-axis, as $y \to \infty$. For the above reasons, we shall call equations (9) and (12) "pseudo diffusion equation".

2. The "diffusion coefficients" $\exp[2y]/2$ and $-\exp[-2y]/2$ and in front of $\frac{\partial^2}{\partial y^2}$ and $\frac{\partial^2}{\partial p^2}$ in (12) depend on $y$. For $y \to +\infty$, the term $\frac{1}{2} \exp[2y]/2 P_n$ dominates the r.h.s. of (12), whereas for $y \to -\infty$, the second term dominates. This dependence on $y$ can be given an interesting intuitive explanation: Let us consider the redistribution process when $y$ is very large: In this case the probability densities $P_n(q,p;y)$ are extended in the $q$-direction and tightly squeezed or compressed in the $p$-axis, which makes it difficult to compress them further along the $p$-axis. For this reason the "infusion coefficient" becomes so small, namely $\propto \exp[-2y]$. In contrast, the diffusion along the $q$-axis must become faster and faster, in order to diffuse all the incoming density flux from the other orthogonal $p$-direction, which is entering the cigar-shaped potential regions through their lengthy boundaries.

5 The Wehrl Entropy for the $P_n$

A useful measure for the information content of the probability distributions $P_n(q,p;y)$ is the Gibbs or Wehrl entropy $[7]$, which is defined by

$$S_n(y) := -\int P_n(q,p;y) \ln P_n(q,p;y) \frac{dp dq}{2\pi}.$$  \hspace{1cm} (14)

Because of the symmetry $P_n(q,p;-y) = P_n(p,q;y)$, the entropy (14) is even in $y$: $S_n(-y) = S_n(y)$. Therefore, at $y = 0$ each $S_n(y)$ must have either a maximum or a minimum. We shall now argue that $S_n(0)$ should correspond to a minumum: We assume that $S_n(y)$ does not oscillate as a function of $y$. Therefore, it is enough to argue that $S_n(y)$ grows with $|y|$ for large values of $|y|$. For large positive $y$, equation (12) behaves essentially like a one-dimensional diffusion equation in the $q$-variable. But it is well-known that the solutions of diffusion equations lead to entropies which increase with time $[6]$. Therefore, the $S_n(y)$ must increase as $y \to \infty$. But since the $S_n(y)$ are even in $y$, they must also grow as $y \to -\infty$. Hence, the $S_n(0)$ must lie at the bottom of the curves $S(y)$ vs. $y$.

Finally, we note that the von Neumann entropy $S_{vn}(\rho) := -\text{Tr}(\rho \ln \rho)$ for the pure states $\rho := |n\rangle \langle n|$ must vanish. In contrast, explicit calculations of the Wehrl entropies of the Poisson
distributions (8) shows that $S_n(0) \geq 1$ for all $n$, in accordance with a conjecture by Wehrl [7], which was proved by Lieb [8].

To summarize this section: in contrast to diffusion equations, where the entropies of their solutions always increase with time, the entropies $S_n(y)$ for the solutions of the above pseudo-diffusion equation first decrease monotonically as $y$ grows from $-\infty$ to zero, but then increase monotonically as $y$ grows from zero to $+\infty$.

6 Summary and Outlook

Two equivalent partial differential equations (9) and (12) were presented and then interpreted, as wave and as pseudo-diffusion equations. The probability densities $P_n(q, p; y)$ (5) provide an infinite number of their solutions.

By the time of writing the present lecture notes, we succeeded in proving, by two general methods, that the expectation values $\langle q, p; \xi | O | q, p; \xi \rangle$ of an arbitrary operator $O$, satisfy a generalized version of the above partial differential equations, which also include rotations, i.e., for the general squeezing $\xi = re^{i\phi}$. Interesting examples of $O$ are the number operators $N$ and $N^2$; their expectation values provide the simplest solutions of (9) and (12). Also the projection operator $|q, p; \xi \rangle \langle q, p; \xi |$, and consequently its Wigner function, satisfy these equations.

References


