ELASTIC PLASTIC FRACTURE MECHANICS
METHODOLOGY FOR SURFACE CRACKS

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1. *Introduction*

1.1 *General*

The safety and reliability of structures has always been a matter of vital concern to the aerospace industry. In this respect, fracture mechanics (FM) is a specially useful technology, since it can provide a quantitative description of the capability of structural parts to tolerate flaws. Initially, FM concepts covered quasi-linear elastic conditions (LEFM). Later, these methods were further developed to cover more general situations. Specifically, there was a need to extend these concepts to include cases where yielding was not necessarily contained in very small regions, for the case of new and tougher materials, higher loads, thinner sections, et cetera. This led to the development of the so-called Elastic Plastic Fracture Mechanics (EPFM) Methodology.

To apply these methods, two pieces of information are needed: the so-called material/specimen response to deformation, and the material response to crack extension. The former, obtained by finite element analysis or experimental calibration, consists of two expressions connecting the J-integral, load P, load-point displacement v, and crack length a for the specimen geometry of interest; the latter consists of a characterization of the way the material resists crack extension for the type of load applied: J, or a similar parameter, versus crack extension, for monotonic load, da/dN versus ΔK or ΔJ for cyclic loading, da/dt versus K, C* or C_t for creep crack growth, et cetera. It
is assumed that within some limitations, these curves are specimen geometry independent, i.e., the curve obtained from a small laboratory specimen applies to the structural part under consideration, as well.

A simple computer program can be developed to combine the two pieces of information mentioned and assess the structural reliability of the structural part of interest.

It is very important to devote effort to guarantee that the curve of material response to crack extension is, in fact, geometry independent. That is, it is important to understand the limitations of the parameters and/or approaches used, identify clearly their limits of validity, and eventually improve the characterization of the phenomenon, proposing new parameters and methods to extend the range of applicability of existing models.

1.2 Elastic Plastic Fracture Mechanics

Specifically, for the case of EPFM applied to monotonic load, the mentioned limitations are expressed in terms of the amount of crack extension to ligament ratio, \( r \), the ratio of ligament to applied \( J \) over the yield strength, \( m \), and the ratio of logarithmic increase of \( J \) to logarithmic decrease in ligament, \( \omega \).

To overcome some of these limitations, particularly the one on \( r \), Ernst [1.1] proposed a modified version of \( J \) called \( J_M \). Resistance (R) curves plotted in terms of \( J_M \) were not subjected to the same limitations as those using \( J \), and in general, showed
a better correlation between specimens of different size and geometry.

More recently, this methodology was further extended: general formulas were developed for \( J_M \) and \( J_D \) for growing cracks, criteria were proposed to identify the limits of applicability of both parameters, methods were presented to make use of the information of experimental points beyond this limit, and several schemes were proposed to extrapolate small laboratory specimen resistance curves to large amounts of crack extension, using \( J_M \), \( J_D \), or other parameters [1.2].

Although the progress made has been significant, and understanding has been gained on how to represent the R curve [1.2-1.6], there are still several very important points that need to be addressed before the method can be safely applied.

Among the most important ones, the need to extend this whole methodology to include cases involving three dimensions (3D) must be mentioned.

Specifically, it is mandatory to know how specimen thickness, constraint, and the possible dependence of the fracture mechanism on specimen thickness may affect the fracture resistance.

Ultimately, this knowledge gained from "2D" planar specimens should be used to explain and predict the behavior of real life 3D defects found in structures, i.e., surface or embedded cracks, et cetera.
1.3 The Leak Before Burst (LBB) Criterion

Pressure vessels containing surface flaws are often required to comply with the so-called LBB criterion. LBB is understood as the condition in which an assumed initial flaw will grow through the wall of a pressure vessel and cause leakage rather than bursting.

In particular, pressure vessels of interest to NASA have to comply to MIL-STD-1522A Standard General Requirements for Safe Design and Operation of Pressurized Missile and Space Systems. This document requires that: (1) $a/2c$ (crack depth to total width ratio) needs to be in a range from 0.05 and 0.5, and (2) LBB will occur if $K_{IC}/\sigma_{op} > 2\alpha B^{0.5}$ with $\alpha \sigma_{op} < \sigma_{YS}$ and $\alpha > 1$.

The rationale behind this expression is that the initial semi-elliptical flaw will grow in a self-similar manner, i.e., keeping $a/2c$ constant until the crack depth $a$ is exactly equal to the thickness $B$, as shown in Figure 1.1. At that time, it is considered that the flaw becomes a through crack with a total length of $2c$, with $2c = (2c/a)_oB$, where the subscript 'o' stands for 'initial'. Finally, to prevent the crack from running unstably in the longitudinal direction, i.e., bursting, it is then required that the toughness $K_{IC}$ be bigger than the applied $K$ given by $K_{app} = \sigma_{op}(\pi c)^{0.5}$.

The weaknesses of this Standard are these: (1) the above equation only holds for $a/2c = 0.5$, (2) the flaw shape is considered to always remain elliptical with constant $a/2c$, and (3) the whole analysis is based on LEFM concepts.
On the other hand, for the real materials, thicknesses, and typical flaws of interest, the situation is markedly different from the one assumed above, as can be seen in Figure 1.2. The cracks, clearly, do not grow in an elliptical self-similar manner, but rather in a very complex shape, with a dimension in the direction parallel to the surface, longer in the interior than on the surface. Moreover there is no guarantee that this dimension can be conservatively estimated by taking the original \((2c/a)_0\) and multiplying by \(B\).

1.4 This Project

The EPFM Methodology has evolved significantly in the last several years. Nevertheless, some of these concepts need to be extended further before the whole methodology can be safely applied to structural parts. Specifically, there is a need to include the effect of constraint in the characterization of material resistance to crack growth and also to extend these methods to the case of 3D defects.

As a consequence, this project was started as a 36 month research program with the general objective of developing an elastic plastic fracture mechanics methodology to assess the structural reliability of pressure vessels and other parts of interest to NASA containing defects.

The project is divided into the following tasks.
Task 1. **Constraint and Thickness Effects**

This task includes the study of the problem of constraint and thickness effects, in different specimen sizes and geometries in materials of interest.

Specifically, the following subtasks will be performed:

a) The large body of available data from centers around the World will be gathered to study this effect in specimens of different size and geometry.

b) Resistance to crack growth tests will be conducted using specimens of different size and geometry, on at least one material of interest. The material will be provided by NASA; Georgia Tech will machine the lab specimens.

c) Characterization of fracture surfaces to determine mechanisms of fracture, and typical surface dimensions will be performed using modern quantitative metallographic techniques.

d) Using the information obtained, models will be developed to describe the effect of constraint on the growth of cracks under elastic plastic conditions.
Task 2. Three Dimensional Cracks

The problem of applicability of EPFM concepts to 3D crack problems, in materials of interest, will be studied in this task. Specifically, the following subtasks will be performed:

a) Plates containing surface cracks with different initial crack aspect ratios and relative crack to plate geometry dimension will be tested. The evolution of the crack shape (planar) and the crack surface displacement with loading will be determined.

b) Analytical and numerical efforts will be devoted to determine values of J and constraint along the crack front.

c) The models and information obtained from Task 1 will be used here to predict the behavior of these 3D cracks.

d) Predictions and experimental results will be compared and, if necessary, refinement of the models will be made.
Task 3. **Leak Before Burst (LBB) Criterion**

Finally, the body of information obtained in the previous tasks will be organized in a Methodology format to assess the structural integrity of parts containing defects, in the spirit of the current LBB criterion.

1.5 **This Report**

This report covers the activities of the period March 1993 through August 1993. In this period, full advantage was taken from the experience and knowledge gained in previous projects [1.6-1.9]. In particular, some efforts were devoted in this project to complete and extend previously obtained results.

The report is organized as follows: In Chapter 2, a computer modelling algorithm used to simulate the growth of a semi-elliptical surface crack is explained in detail. This is an excerpt from the thesis of D.W. Boatwright [1.9].

In Chapter 3, a finite element investigation is presented. This investigation, an excerpt of the thesis of W.J. Curtin [1.10], compared the theoretical (HRR) stress field to that produced by elastic and elastic-plastic models. The difference in these stress fields is the constraint effect.

In Chapter 4, experimental efforts to characterize three dimensional aspects of fracture present in "two dimensional", or planar configuration specimens have been continued. This discussion specifically contains a preliminary discussion
associated with the determination of, and use of, crack face separation data.
1.6 References


1.7 Figures

Figure 1 Simple Leak Criterion

Figure 2 Growth of a Part-Through Crack to Critical Size
CHAPTER III

COMPUTER MODELLING OF SURFACE CRACK GROWTH

Background

The best way to determine the physical response of a cracked body to a particular set of loading conditions is to run a test on the structure in question. Unfortunately, this is not a practical solution in many cases. One alternative to testing is to use a mathematical model that can predict the behavior of a test specimen in any given situation.

In order to use a mathematical model, it is first necessary to run a series of mechanical tests to characterize to behavior of the specimens. Once these tests have been run, it is then possible to use the test data to develop a valid model for the specimen in question. Today, libraries of functions exist that can be used to model most common fracture specimens given the material properties \[18,22\].

After developing a mathematical model that accurately predicts test results, the key parameters of the test can be changed to observe their influence on test results.
Currently, the only way to model a surface crack specimen is to run a three-dimensional finite element model. This modelling process is computationally intensive and very time consuming since a new finite element mesh must be constructed for each iteration in the solution process.

It would be far more efficient to develop a special purpose fracture mechanics based computer program designed to model surface crack behavior. One of the goals of this work was to do just that. This chapter covers the development and testing of a fracture mechanics based computer program for modelling the behavior of surface cracks.

The intent in the development of this computer program was to develop a program general enough to model any surface crack geometry in any material that is under J-dominant conditions. Given the material deformation and fracture properties, the program should accurately predict the distribution of crack growth along the front.

The program described in this chapter represents a continuation of the work begun by Sheldon and Ernst [14,15] in computer modelling of surface crack growth. The new version of the program attempts to overcome some of the problems that surfaced in the first program. There are also concepts integrated into the current program that were not in the original program.
One new feature that has been introduced into the current version of the program is displacement control rather than load control. The current version of the program uses plastic displacement as the evolutionary variable. In the first program, the load was increased in small increments to simulate a mechanical test. One problem that arose from the use of this method was that virtually all the crack growth occurred in the last few load steps. Use of displacement control corrected this problem because with displacement control the load increases very quickly at first and very slowly around maximum load which is where the majority of crack growth occurs. Another feature of displacement control is that the simulation can continue past the maximum load; that is, the applied load can decrease. This feature would be useful in leak-before-break analyses.

The program uses the key curve to calculate the load at each displacement step given the plastic displacement and the current crack geometry. The load and the crack geometry are then used to calculate the fracture variables along the crack front.

The incrementing of plastic displacement is not intended to directly parallel an actual mechanical test. Plastic displacement was chosen because it was a nondecreasing variable in the key curve equation. The other possible
evolutionary variables were load, the crack dimension a, and the crack dimension c.

The program does not use any component of displacement in the calculation of the conditions along the front. Although, it is possible to calculate the elastic and plastic components of displacement in a real mechanical test, through unloading compliance measurements, in order to relate the simulation to an actual test.

The original modelling technique used by Sheldon [15] divided the crack front into a collection of discrete points. Sheldon assigned values of a and c to each point along the front by fitting an ellipse to the point in question by using the position of the point, the position of the center of the ellipse, and the slope at that point as determined from the neighboring points. Sheldon then used these variables in a J calculation method as developed by McCabe, Ernst, and Newman [16] which was based on the Newman-Raju equations [7]. A value of J was calculated for each point along the front. A value of crack extension at each point was obtained from the material resistance curve, and the point was then advanced in the direction normal to the simulated crack front.

The original program would divide the crack front into two halves when the program detected that the surface crack was mushrooming, that is, when the crack started growing in a non-self-similar way. The program considered the crack to be
mushrooming when it detected a point with an approximately vertical slope. The section of the front from that point to the surface was modelled as a through crack with a crack length equal to the x coordinate of the point in question. The remainder of the crack front was modelled as a semi-elliptical crack as before.

While this scheme showed promise, there were some complications that could be avoided by the use of a different model. The main problem was that calculating $J$ using two different models led to problems in establishing continuity of $J$ and $dJ/da$ along the front. The program scaled the $J$ values in order to maintain continuity of $J$, but no consideration was given to continuity of $dJ/da$. The discontinuity in $dJ/da$ led to excessive crack growth at the point of the discontinuity.

In part, the program developed in this chapter represents an evolutionary step forward. Modelling techniques that caused problems in the original program were modified. The use of two separate $J$ estimation methods has been replaced by a single method that is used for all the points along the crack front. In addition to the evolutionary modifications, new features were added to the program to better model the fracture process. The primary addition to the program was the incorporation of constraint effects. The original version of the program as written by Sheldon [15] did not consider
constraint effects. The techniques used to model crack growth will be described in the following sections.

**Computer Program Development**

This program models the evolution of the crack front in a displacement controlled test of a rectangular surface crack panel in tension. The program requires prior knowledge of the material deformation properties, the plane strain resistance curve, and the key curve for the specimen. The plane strain resistance curve is intended to represent the material fracture properties for the most constrained state of stress, and the key curve for the specimen will be used to relate the plastic displacement and current crack geometry to the load. The key curve was obtained using the same material and specimen geometry under blunt notch, or non-growing crack, conditions as explained in Chapter II.

The program simulates a growing crack test by incrementing the plastic displacement. The user must enter the final plastic displacement along with the initial semi-elliptical crack geometry. The conditions along the crack front are recalculated and updated with each displacement step. The user is free to specify the number of displacement steps in any simulation.
In this model, the crack front is represented by discrete points. The user is free to specify the number of points along the crack front. Specifying a large number of points allows the user to see how the front evolves in detail, but the program takes longer to run. The solutions calculated are insensitive to the number of points along the front; so, there is no computational advantage gained by specifying a very large number of points.

It was sufficient to only consider one half of the front due to the symmetry of the problem. Boundary conditions on crack growth direction were applied at the surface and at the deepest point in order to enforce symmetry. Modelling only half the crack front halves the time it takes to run a simulation without affecting accuracy.

This program was written in GW-BASIC. A listing of the computer program is included in the Appendix.

**J Estimation along the Crack Front**

This program models the fracture process using a J resistance curve approach that incorporates constraint effects. This methodology was based on work done by McCabe, Ernst, and Newman [16]. In the first version of this program as developed by Sheldon and Ernst [14,15], a one-parameter approach to predicting fracture was used. In that program, J was calculated at each point along the crack front; then, the
crack extension was calculated from the material R-curve. This approach did not consider constraint effects. In the current version of the program, a two-parameter approach incorporating constraint was used to calculate crack extension at each point along the crack front. The first step in modelling the crack extension across the crack front was to calculate the value of J at each point on the front.

In Chapter I, it was shown that, given two cracked bodies subjected to the same loading history that differ in crack length by a small amount, the J integral can be defined as the difference in potential energy divided by the difference in cracked area. For a body where the crack only has a length dimension, the difference in cracked area, dA, is equal to the specimen thickness multiplied by the difference in crack lengths. It is not so simple for a geometry such as a surface crack that requires two length parameters. For example, a semi-elliptical surface crack requires the two length parameters a and c.

For a semi-elliptical surface crack, Ernst [23] showed that the expression for the global value of J is linked to the way the virtual crack extension is taken. Ernst examined three different ways for taking the virtual crack extension: 1) increasing c, keeping a constant, 2) increasing a, keeping c constant, and 3) increasing a and c, keeping a/c constant.
The following equations for the plastic portion of $J$ were developed by Ernst. The only difference between the three cases is the expression for the coefficient $\eta_{pl}$.

\[ J_{pl} = \frac{\eta_{pl}}{\pi \frac{a}{c}} \int_0^{v_{pl}} P \, dv_{pl} \]  

(3.1)

Taking the key curve equation, which relates load and plastic displacement, and substituting it for the variable $P$ yields the following expression.

\[ J_{pl} = \beta_o \frac{\eta_{pl}}{(\pi/2) \frac{a}{c}} \frac{Wt^2}{N+1} \left( \frac{a}{c} \right)^{-\frac{p}{N}} \left( \frac{a}{t} \right)^{-\frac{m}{N}} \left( \frac{P}{Wt\beta_o} \right)^{\frac{N+1}{N}} \]  

(3.2)

The expressions for $\eta_{pl}$ turned out to be independent of the $a/c$ and $a/t$ ratios. These expressions are shown in Table 3.1.

<table>
<thead>
<tr>
<th>Variable Held Constant</th>
<th>$\eta$ plastic Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>$-(p+m)/2$</td>
</tr>
<tr>
<td>$c$</td>
<td>$p/2$</td>
</tr>
<tr>
<td>$a/c$</td>
<td>$-m/2$</td>
</tr>
</tbody>
</table>
At this point it was necessary to determine which method should be used to calculate the global $J$ and how that global $J$ relates to the values of $J$ along the front. The following normalization scheme was developed for this purpose.

The total system energy in a cracked body can be expressed as follows.

$$W = E + U$$

(3.3)

$W = \text{total energy supplied by external forces}$

$E = \text{energy spent growing the crack}$

$U = \text{strain energy}$

The energy release rate for an incremental crack extension can be written as follows.

$$\frac{\delta E}{\delta A} = \frac{1}{\delta A} \int_0^1 J \, ds \, dn$$

(3.4)

The differential element $ds$ represents an element along the crack front, and the differential element $dn$ represents an element normal to the crack front. The above integral expression should be integrated along the entire crack front.

In addition, taking the derivative of the energy balance with respect to the virtual increase in cracked area yields the following relationship.
Replacing the incremental quantities in the above equation with experimentally measurable global quantities yields the following equations for constant displacement conditions.

$$\frac{\delta W}{\delta A} = \frac{\delta E + \delta U}{\delta A}$$  \hspace{1cm} (3.5)

The above result is entirely independent of the geometry of the surface crack.

Now, for the case of growth of a semi-elliptical surface crack at a constant aspect ratio, Ernst shows that the following equations can be used to define $ds, dn,$ and $dA$ in terms of $a, c,$ and elliptical angle, $\phi$ [23].

$$ds = \sqrt{(c \cos \phi)^2 + (a \sin \phi)^2} \, d\phi$$  \hspace{1cm} (3.7a)

$$dn = 2 \, c \sqrt{(c \cos \phi)^2 + (a \sin \phi)^2} \, da$$  \hspace{1cm} (3.7b)
\[ dA = \pi c \, da \quad (3.7c) \]

Using these relationships allows the following definition of the energy release rate.

\[ \frac{\delta E}{\delta A} = J_{\text{avg}} = (J)_{a/c \, \text{constant}} = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} J(\phi) \, d\phi \quad (3.8) \]

This result shows that the global J obtained by taking the virtual crack extension with a/c constant is numerically equal to the linear average of J along the crack front.

Since no assumptions were made regarding material behavior, the above equation applies for the elastic portion of J, the plastic portion of J, or the total J.

This J estimation technique uses the observation that the distribution of the plastic portion of J along the crack front follows the distribution of the elastic portion of J for moderate amount on deformation [24,25]. This relationship can be expressed as follows.

\[ J_{pl}(\phi) = k \, G(\phi) \quad (3.9) \]

Therefore,

\[ J(\phi) = (1 + k) \, G(\phi) \quad (3.10) \]
Following this line of reasoning the linear averages along the crack front should also be related as follows.

\[ k = \frac{J_{pl, av}}{G_{av}} \]  

(3.11)

Out of convenience, \( k \) was incorporated into a coefficient \( D \) that is independent of applied stress.

\[ D = \frac{k}{\sigma^{(n+1)}} \]  

(3.12)

By substitution, the value of \( D \) can be shown to be equal to the following expression.

\[ D = \frac{0.224 \, n \, \beta \, \pi \, (n+1)}{\sigma^{(n+1)} \, (G_{av}/\sigma^2)} \left( \frac{a}{c} \right)^{1-p} \left( \frac{a}{t} \right)^{-2-n} \]  

(3.13)

The values of \( J \) along the crack front can be calculated using the following formula.

\[ J(\phi) = G(\phi) \left(1 + D \, \sigma^{n-1}\right) \]  

(3.14)

Now, the linear elastic portion of \( J \) as predicted by the Newman-Raju equations can be used with the above equation to perform an elastic-plastic \( J \) analysis along the crack front.

Since the Newman-Raju equations are based on a semi-elliptical crack front, it was necessary to assign values of
a/c and a/t to each point. A value of a/c was assigned to each point based on the a/c of an ellipse centered at the origin that passed through the deepest point and the position of the point in question. The value of a/t was taken to be the depth of the central point divided by the thickness. This method is illustrated in Figure 3.1.

While this method of assigning values for a/c and a/t at each point is admittedly crude, the method was chosen because of its simplicity. The fact that this method is applicable at every point along the crack front prevents some of the problems that developed in the first attempt at simulating surface crack growth as developed by Sheldon [15].

**Crack Growth Direction**

For the purposes of this computer program, crack growth was assumed to occur normal to the local crack front as defined by the points that represent the evolving crack front. The normal direction was determined by taking a numerical derivative at the point in question.

As boundary conditions, the point at the front surface was constrained to growth along the surface, and the deepest point was constrained to growth through the depth.
Figure 3.1: Procedure for Fitting an Ellipse

Crack Tip Constraint

Today, it is accepted that constraint influences the J R-Curve. ASTM acknowledged this idea in their recommendation of bend specimens for determining R-curves [17]. This recommendation was based on the fact that bend specimens exhibit more in-plane constraint than tension specimens, and, thus, produce a more conservative design specification. It should be noted here that two specimens with identical out-of-plane constraint, e.g. plane strain or plane stress, can exhibit different levels of in-plane constraint.
The typical differences in the R-Curves for bend and tension geometries illustrates how different levels of constraint affect the fracture properties of a material. The rate of change of the resistance curve slope is a function of constraint. Specimens under a high level of constraint have a slope that is lower than for low constraint configurations. Thus, the R-Curve for a high constraint situation would be below the R-Curve for a low constraint situation.

It has been widely shown that a one parameter approach to modelling surface crack growth is inadequate [26,27,28]. The incorporation of a measure of constraint into this computer program is an attempt to address the problems inherent in a one-parameter approach.

It has been shown by Ernst, Rush and McCabe [19] that there is a relationship between $dJ/da$ and the level of constraint at the crack tip which they called $(\ell_e)^{-1}$. The variable $(\ell_e)^{-1}$ has been shown to characterize constraint. This variable was defined as follows.

$$ (\ell_e)^{-1} = \frac{\left(\frac{\partial G}{\partial a}\right)_p}{G} \quad (3.15) $$

In calculating derivatives with respect to a virtual crack extension, a problem is again encountered in defining the manner that the virtual crack extension occurs. For the purpose of derivatives, the crack extension is assumed to
occur normal to the equivalent ellipse that is fit at the point in question. This convention was adopted in the interest of consistency since the values for J are calculated from the equivalent ellipses.

For the purposes of this computer simulation, dJ/da was chosen as a tool to incorporate constraint into the modelling of surface crack behavior. Briefly stated, the procedure calculates constraint effects by comparing dJ/da as calculated at a point on the crack front to dJ/da as calculated from the plane strain material resistance curve.

Constraint effects were calculated in the following manner. The value of dJ/da was calculated for each point on the crack front. These values were obtained by calculating the value of J a small distance ahead of the point in question. Another equivalent ellipse was fit through the new incrementally advanced point. The a/c and a/t obtained from the three point fit were then used to calculate J. The value of dJ/da was found by dividing the difference in J values by the distance between the points.

The methodology for incorporating constraint into the crack growth model began with the R-curve for $(t_{e1})^{-1}$ equal to zero. This R-curve represents the most constrained situation possible and served as the reference for the fracture parameters calculated in the program. The following equation was used to model the R-curve.
The R-curve for the side-grooved compact tension specimen was used in the computer program because the side-grooved compact tension specimen R-curve represented the most constrained geometry tested. The bend type loading and the side-grooves contributed to the high level of constraint in these specimens.

The derivative of Equation 3.16 served a reference to be compared to the values of $dJ/da$ calculated in the program.

$$\left(\frac{dJ}{da}\right)_{\text{reference}} = \frac{1}{3 \alpha_o J^2 + 2 \beta_o J + \gamma_o}$$ (3.17)

The factor $F$ was defined as follows.

$$F = \frac{\left(\frac{dJ}{da}\right)}{\left(\frac{dJ}{da}\right)_{\text{reference}}}$$ (3.18)

The prior assumption that constraint and $dJ/da$ are related implies that constraint and the variable $F$ are related. Substituting Equation 3.17 into Equation 3.18 yields the following.
\[ \frac{dJ}{da} = \frac{F}{3 \alpha_o J^2 + 2 \beta J + \gamma_o} \]  (3.19)

Substitution of the above equation into the integral definition for crack extension yields the following expression.

\[ \Delta a = \int_{a_o}^{a} \frac{3 \alpha_o J^2 + 2 \beta J + \gamma_o}{F} \, dJ \]  (3.20)

Integrating provides the formula for crack growth which incorporates constraint effects.

\[ \Delta a = \frac{\Delta a_{\text{reference}}}{F} \]  (3.21)

Every point along the crack front had unique values for \( J \) and \( F \). Together, the two parameters were used to calculate the crack extension at each point along the front. Once every point on the front had been advanced the appropriate amount, the plastic displacement was incremented, and the process began again at the new load.
Comparison of Computer Output to Experimental Data

It was initially hoped that this computer program could be used to model the 21-6-9 stainless steel surface crack tension specimens that were tested as part of this work, but, as shown in Chapter II, a J-based approach is invalid for this material. As a consequence, this computer program is inappropriate for prediction of crack growth in these stainless steel specimens.

While this program is admittedly inappropriate for modeling the fracture behavior in these stainless steel surface crack specimens, it is convenient to use the material data developed in testing these specimens to test the general behavior of the program. The purpose for using the data from the stainless steel specimens is simply to check the program for instability, discontinuities, or any other obvious shortcomings.

The results of three computer simulations are shown in Figure 3.2, 3.3, and 3.4. These figures compare the actual test results to the computer simulation for surface crack tension specimens SC16, SC11, and SC4, respectively. These surface crack specimens were chosen because they were all tested to a point past their maximum attainable load and they all exhibited significant crack growth. More specifically, these three specimens were chosen because their different
Figure 3.2: Comparison of Simulation to Test Data for SC16
Comparison of Simulation to Test Data
SC11

Figure 3.3: Comparison of Simulation to Test Data for SC11
aspect ratios. This allows an analysis of the performance of the computer program over a wide range of aspect ratios.

Discussion and Conclusions

The results of the computer simulation using the data from the stainless steel surface crack tests shows that the program performs as expected. The new crack fronts calculated by the program do not exhibit any discontinuities or instability problems. In order to effectively evaluate the accuracy of the program, it would be necessary to compare the predicted results to the experimental results obtained from a set of surface crack specimens tested under J-dominant conditions.

The experimental results presented in Chapter II show that the growth of these cracks is a deformation process. The photographs of these same specimens in Chapter II show the extensive plastic deformation around the crack mouth and at the back surface. The fact that the maximum loads for these specimens can be accurately predicted from deformation theory serves to reinforcing the assessment that a fracture mechanics based simulation for this material is inappropriate.

It is not possible to judge the simulation results based on a comparison with the stainless steel surface crack
specimens tested here. To make a definitive judgement on the accuracy of the simulation, it would be necessary to compare results for a material that was known to be under J-dominant testing conditions.
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CHAPTER III: An Excerpt from WJ Curtin's Thesis

CHAPTER III

FINITE ELEMENT INVESTIGATION OF
CONSTRAINT PARAMETER

Introduction

Constraint is defined as the degree of crack tip stress triaxiality and is commonly quantified by the higher order crack tip stress and displacement terms (i.e. by T-stress or Q). Therefore, the elastic-plastic near tip fields are characterized by two parameters, J and a constraint parameter. The goal of the finite element analysis (FEA) investigation was to determine the coefficients of the higher order crack tip stress and displacement terms. As a subsequent step, the coefficients of the higher order terms were compared to the earlier proposed constraint parameter, \((L_\omega)^4\). The finite element investigation for this work is just the first step in a larger finite element investigation. That is, this work has started the development of an FEA data base containing the values of the higher order stress and displacement term coefficients. Future finite element investigation to determine these higher order coefficients shall include many other geometries. For example, three-dimensional finite element models of planar specimens will be developed and investigated. Upon completion of all FEA work the coefficients shall be used to compare various constraint
parameters -- including \((L_w)\), q, Q, T-stress, and others -- to the higher order terms.

The coefficients of the higher order stress and displacement terms represent one part of a larger database which shall also include information on various constraint parameters and experimental results. The FEA results are an important part of the database; however, experimental results are critical to the evaluation of a constraint parameter. Experiments are very important because of the processes which lead to fracture. Ductile crack growth occurs due to micro void nucleation, growth, and coalescence. FEA cannot account for the processes which lead to ductile crack growth. Therefore, both experimental and FEA results are required in order to judge the ability of a second fracture parameter to characterize constraint. Each parameter shall be scrutinized upon completion of the entire database.

Three types of FEA models were developed for this study -- a two-dimensional linear elastic center crack tension (CCT) model, a two dimensional elastic plastic CCT model and a three-dimensional linear elastic surface crack tension (SCT) model. Since this study has concentrated on elastic-plastic three dimensional cracks both a three dimensional model and an elastic plastic model were developed for the initial part of the finite element constraint parameter investigation.
Finite Element Models

Linear Elastic CCT Model

All finite element work done for this study utilized ABAQUS software [28]. The linear elastic CCT model consists of a two dimensional plane strain mesh using two planes of symmetry (Figure 3.1). That is, only one quarter of

![Figure 3.1: Center Crack Tension Model](image)

the specimen was modeled due to symmetry considerations. The mesh contains 8-node biquadratic hybrid elements with reduced integration. The model consisted of 392 elements and was loaded in tension. A plot of the mesh and a sample of the ABAQUS input file are shown in appendix A.* A total of five CCT meshes were developed for the analysis. The crack length over width ratio (a/w) varied from 0.25 to 0.75 for the five meshes (Figure 3.1). For consistency purposes, the same element arrangement near the crack tip was used for all five models. In order to correctly model the $1/\sqrt{r}$ stress singularity for the linear

*Appendix A not included
elastic case, quarter point node elements [29] were used at the crack tip and all crack tip nodes were tied together. The element size (radial direction) at the crack tip was $0.0006e$ for the $a/w=0.75$ mesh where $e=w-a$.

**Linear Elastic SCT Model**

The linear elastic SCT model consisted of a three dimensional mesh using two planes of symmetry (Figure 3.2). The mesh contains 20-node biquadratic displacement hybrid brick elements with reduced integration. The model consisted of 1764 elements and was loaded in tension. A plot of the mesh and a sample of the ABAQUS input file are shown in appendix B.* In order to correctly model the $1/\sqrt{r}$ stress singularity for the linear elastic case, quarter point node elements were used at the crack tip and all crack tip nodes were tied together [29]. The element size at the crack tip (radial direction) at

*Appendix B not included
\( \phi = 90^\circ \) was 0.0024 where \( d = t - a \). The modeled surface crack displays a shape of \( a/c = 1.0 \) and \( a/t = 0.5 \) (Figure 3.2). The modeled specimen thickness (t) was 1.0 inch and the half width (w) was 2.0 inches.

**Elastic Plastic CCT Model**

The elastic plastic CCT mesh was identical to the linear elastic CCT mesh except for some adjustments. In order to model the stress singularity for the elastic plastic analysis, quarter point node elements were not used at the crack tip and the crack tip nodes were not tied together. This allows for the crack tip to blunt for the elastic plastic case. The Ramberg-Osgood non-linear material deformation model was utilized for the elastic-plastic CCT model. In one dimension the Ramberg-Osgood model is,

\[
\frac{\varepsilon}{\varepsilon_0} = \frac{\sigma}{\sigma_0} + \alpha \left( \frac{\sigma}{\sigma_0} \right)^n
\]

where,

\[
\varepsilon_0 = \frac{\sigma_0}{E}
\]

and \( E \) is Young's modulus. The stress, strain, and yield strength are designated by \( \sigma, \varepsilon, \) and \( \sigma_0 \) respectively. The material dependent constants \( \alpha \) and \( n \) are the yield offset and hardening exponent. The material properties used in the model are displayed in table 3.1. These constants correspond to the properties of a typical low carbon steel. The element size (x direction) at the crack tip was 0.001e for the \( a/w = 0.75 \) mesh where \( e = w - a \) (Figure 3.1). Similar to the linear elastic model, various \( a/w \) values were analyzed for the elastic plastic CCT case. Each model was loaded in three steps. The first load step went up to 70%
of limit load. The second step went up to 85% of limit load and the third step went up to 100% of limit load. A sample of an ABAQUS input file is shown in appendix C.

<table>
<thead>
<tr>
<th>MATERIAL PROPERTY</th>
<th>VALUE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young's Modulus (E)</td>
<td>30.0 E6 [psi]</td>
</tr>
<tr>
<td>Yield Offset (α)</td>
<td>0.5</td>
</tr>
<tr>
<td>Strain Hardening Exponent (n)</td>
<td>5</td>
</tr>
<tr>
<td>Yield Strength (σy)</td>
<td>40,000 [psi]</td>
</tr>
<tr>
<td>Flow Stress (σf)</td>
<td>50,000 [psi]</td>
</tr>
</tbody>
</table>

**Theoretical Background**

In order to calculate the coefficients, the dependence on r must be determined for each of the higher order terms in the crack tip stress and displacement series (Figure 3.3). Williams [30] and Westergaard [2] have developed methodologies to determine the higher order terms dependence on r for linear elastic material properties. Sharma and Aravas [32] and Li and Wang [31] have formulated the higher order terms using an asymptotic analysis for a Ramberg-Osgood hardening material.
Linear Elastic Background

For the linear elastic case, the format of the higher order stress and displacement fields can be formulated using two early approaches developed by Williams [30] and Westergaard [2]. Williams derived expressions for the crack tip stress and displacement fields by the eigenfunction expansion method. Utilizing this method for a symmetric stress distribution (mode I) in an infinite body with traction free crack faces, the crack tip stresses at $\theta=0^\circ$ and the crack tip displacements at $\theta=180^\circ$ were derived. The results follow:

$$\sigma_{xx}(r, 0^\circ) = \left[ -\frac{3}{4} (b_1 + b_3) \right] r^{-\frac{1}{2}} + T_x + \left[ \frac{15}{4} (b_1 + b_3) \right] r^{\frac{1}{2}} + ... \quad (3.2)$$

$$\sigma_{yy}(r, 0^\circ) = \left[ -\frac{3}{4} (b_1 + b_3) \right] r^{-\frac{1}{2}} + T_y + \left[ \frac{15}{4} (b_1 + b_3) \right] r^{\frac{1}{2}} + ... \quad (3.3)$$
where $T_x$, $T_y$, $b_1$, and $b_2$ are constants that depend upon geometry and loading conditions. $H$ is defined as $v/(1+v)$. Poisson's ratio is $v$ and $\mu$ is the shear modulus. Williams [30] showed that $T_x$ and $T_y$ are zero for $\theta=0^\circ$ provided that the crack faces are traction free. However, more recent investigators have determined that $T_x$ should not be zero [21]. Notice that the $r^{\alpha}$ term in equations 3.2 and 3.3 and the $r^n$ term in equation 3.4 both depend on $b_1$ and $b_2$.

Westergaard [2] was the first to demonstrate that the crack tip stress fields can be derived for certain geometries by introducing an analytic function, $Z(z)$, where $z=x+iy$ and $i=\sqrt{-1}$. The normal stresses and displacements utilizing the $Z$ function are,

\[ u_x(r,180^\circ) = \frac{1}{2\mu} \left\{ \left[ \frac{3}{2} b_1 + \left( \frac{7}{2} - 4H \right) b_2 \right] r^{\frac{3}{2}} + \left[ \frac{5}{2} b_1 + \left( \frac{9}{5} - 4H \right) b_2 \right] r^2 + ... \right\} \] (3.4)

The coefficients of the $r^{\alpha}$ terms in equations 3.2 and 3.3 were defined as $\gamma_x$ and $\gamma_y$, respectively. The coefficient of the $r^n$ term in equation 3.4 was defined as $B$. The relationship between the $\gamma$'s and $B$ for a two dimensional
linear elastic stress state was determined using the Westergaard stress function. For mode I loading and \( y = 0 \), the second terms on the right hand side of equations 3.5 and 3.6 drop out. Note that \( \gamma = \gamma, \) when \( y = 0 \). After expansion of the complex numbers, evaluation of the normal stress at \( \theta = 0^\circ \), and evaluation of the displacements at \( \theta = 180^\circ \), the relationship between \( \gamma \) and \( B \) was determined for a two dimensional cracked body. This relationship was found to be,

\[
B = -\gamma \left( \frac{4}{3E'} \right) \tag{3.9}
\]

where \( E' = E \) for plane stress and \( E' = E/(1-\nu) \) for plane strain. For the linear elastic models a normalized \( B \) was defined as,

\[
B^* = B \frac{E}{K} \sqrt{\frac{\pi}{8}} \tag{3.10}
\]

and a normalized \( \gamma \) as,

\[
\gamma^* = \gamma \frac{\sqrt{2\pi}}{K} \tag{3.11}
\]

where \( K \) is the stress intensity factor. By combining equations 3.9 to 3.11, the relationship between \( B^* \) and \( \gamma^* \) was determined as,

\[
B^* = -\frac{1}{3} \gamma^* \tag{3.12}
\]

Rice [38] introduced the boundary layer approach to determine the crack tip stress and displacement fields. The boundary layer approach assumes that the boundary value stresses along the crack face are given by the extension of validity of the singular term in the elastic stress solution (Equation 1.1) to large
values of \( r \) and small scale yielding. Larsson and Carlsson [21] introduced the modified boundary layer formulation for the crack tip stress fields. They included a constant T-stress term with the singular term in the boundary layer stress field formulation (\( T_\alpha \) in equation 3.2). For a two dimensional infinite body Larsson and Carlsson determined that only the \( \sigma_\alpha \) T-stress term (\( T_\alpha \)) was non-zero. However, Nakamura and Parks [31] have determined that three T-stress terms are present in linear elastic three dimensional cracked bodies. They determined that the \( \sigma_\alpha \), \( \sigma_\beta \), and \( \tau_\alpha \) crack tip stress fields required non-zero T-stress terms for regions behind the crack tip. For a two dimensional elastic-plastic cracked body, O'Dowd and Shih have demonstrated the existence of constant higher order terms in both the \( \sigma_\alpha \) and \( \sigma_\beta \) fields [22]. They introduced these higher order stress terms using the Q parameter.

**Elastic-Plastic Background**

For the elastic-plastic analysis, the format of the higher order stress and displacement fields have been formulated by Sharma and Aravas [32] and Li and Wang [33]. These researchers considered a two-dimensional crack problem in a homogenous Ramberg-Osgood elastic-plastic material. The first two terms of the crack tip stress series and the first three terms of the displacement series were formulated by assuming an asymptotic expansion for the stress terms in the form of,

\[
\frac{\sigma_y(r, \theta)}{\sigma_0} = r^\kappa \sigma_y^{(0)}(\theta) + r^\kappa' \sigma_y^{(1)}(\theta) + \ldots
\]

(3.13)
as \( r \to 0 \) and \( s < t \). Substituting equation 3.13 into the governing equations, the first two terms in the stress and the first three terms in the displacement fields were determined as,

\[
\sigma_y(r, \theta) = \sigma_0 \left( \frac{J}{\alpha \varepsilon_0 \sigma_0 I_n} \right)^{1/(n+1)} \tilde{\sigma}_y^{(0)}(\theta) r^{-1/(n+1)} + \sigma_0 Q_0 \left( \frac{\sigma_0}{J} \right) \tilde{\sigma}_y^{(1)}(\theta) r^t + \ldots \quad (3.14)
\]

\[
u_i(r, \theta) = \alpha \varepsilon_0 \left( \frac{J}{\alpha \varepsilon_0 \sigma_0 I_n} \right)^{n/(n+1)} \tilde{\nu}_i^{(0)}(\theta) r^{1/(n+1)}
+ \alpha \varepsilon_0 \left( \frac{Q_0}{(J/\sigma_0)^t} \right)^{(n-1)/(n+1)} \tilde{\nu}_i^{(1)}(\theta) r^{(n-1)/2(n+1)} + \varepsilon^{(2)}(\theta) r^{t+1} + \ldots \quad (3.15)
\]

where \( Q_0 \) is a dimensionless constant that controls the magnitude of the second terms, \( s = -1/(n+1) \), and \( \varepsilon^{(2)} \) is an elastic strain resulting from the singular stress term. The governing equations include equilibrium, compatibility, and constitutive equations. The formulation of equations 3.14 and 3.15 neglect the effects of specimen geometry and far field loading and assume that the higher order terms are separable in \( r \) and \( \theta \). Using a Galerkin finite element technique, \( t \) was determined to be 0.055 [32] for a two dimensional crack in plane strain with a material strain hardening exponent of 5.

**Procedure**

The field values of the normal crack tip stresses \( (\sigma_{\infty} \text{ and } \sigma_{yy}) \) along \( r \) at \( \theta = 0^\circ \) and the crack mouth opening displacements \( (u_r) \) along \( r \) at \( \theta = 180^\circ \) were calculated by each of the finite element models (Figure 3.3). The coefficients
of each higher order term in the crack tip stress and displacement series were determined by curve fitting the FEA data.

**Linear Elastic Procedure**

The following formats which are similar to equations 3.2 through 3.4 were utilized to curve fit the crack tip stress and displacement fields calculated by FEA for all linear elastic models:

\[
\sigma_{xx}(r, 0^\circ) = D_x r^{\frac{1}{2}} + T_x + \gamma_x r^{\frac{1}{2}} \tag{3.16}
\]

\[
\sigma_{yy}(r, 0^\circ) = D_y r^{\frac{3}{2}} + T_y + \gamma_y r^{\frac{1}{2}} \tag{3.17}
\]

\[
u_y(r, 180^\circ) = Ar^{\frac{1}{2}} + Br^{\frac{3}{2}} \tag{3.18}
\]

where \( r \) is the distance from the crack tip. The first terms (\( r^{1/2} \) terms in equations 3.16 and 3.17 and \( r^{3/2} \) term in equation 3.18) were subtracted from the field values determined by FEA. The remaining higher order terms were curve fit using a least squares methodology. The curve fits yielded values for the coefficients \( T_x, \gamma_x, T_y, \gamma_y, \) and \( B \). Note that the modified boundary layer approach was used to curve fit the higher order terms [21].

The range near the crack tip over which the stresses and displacements were curve fit needed to be determined. For the linear elastic models, the element stresses and nodal displacements (field values) very close to the crack tip which were affected by the quarter point node elements were omitted from the curve fitted data. For the linear elastic CCT model, the curve fitting range
did not significantly affect the values of the coefficients. However, a criterion was used to determine the outer bound of the curve fitted data. The outer bound was determined using the T-stress. T-stress is the normal stress in the x-direction acting on the crack face at $\theta=180^\circ$ [21]. The linear elastic CCT finite element analysis determined that $\sigma_{\infty}$ was constant over almost the entire crack face. This T-stress value at $\theta=180^\circ$ was then compared to the $T_x$ values from curve fits extending over various distances from the crack tip at $\theta=0^\circ$. The value of $T_x$ varied, although not significantly, depending upon the distance from the crack tip over which the higher order $\sigma_{\infty}$ terms were curve fit. Therefore, the curve fit range which displayed the same $T_x$ value as the T-stress value along the crack face at $\theta=180^\circ$ was used to determine the coefficients in equation 3.16 for the linear elastic CCT model. The same curve fit range was then used for the higher order terms in equations 3.17 and 3.18.

For the linear elastic SCT model, the in-plane T-stress at $\theta=180^\circ$ was constant over much of the crack face. However, this T-stress did not coincide with the curve fitted $T_x$ values ($\theta=0^\circ$) possibly because of the three dimensional stress state. That is, the out-of-plane stresses may have affected the T-stress values. Therefore, the outer bound was determined as the distance from the crack tip over which the curve fitted coefficients best represented the FEA field values close to the crack tip. The curve fitting outer bound was determined to be approximately $0.2a$ for all elliptical angles, where a is the surface crack depth.

After the successful defense of this thesis, an error with the curve fitting format of equation 3.17 was discovered. The $T_y$ term should not be included in
the curve fits since it would invalidate the traction boundary conditions along
the crack face at $\theta = 180^\circ$. The term was found to be zero for the linear elastic
CCT model curve fits. However, $T_y$ was included in the curve fits for the linear
elastic SCT model. The inclusion of $T_y$ is incorrect. See the Discussion section
of Chapter III for further information pertaining to the incorrect format.

Elastic-Plastic Procedure

The following formats which are similar to the asymptotic solutions of
equations 3.14 and 3.15 were utilized to curve fit the crack tip stress and
displacement fields for the elastic-plastic models:

$$\sigma_{\alpha}(r, 0^\circ) = D_{\alpha}\frac{1}{r^{n+1}} + Q_{\alpha}r^t$$

(3.19)

$$\sigma_{\gamma}(r, 0^\circ) = D_{\gamma}\frac{1}{r^{n+1}} + Q_{\gamma}r^t$$

(3.20)

$$u_y(r, 180^\circ) = A_{\gamma}r^{1/(n+1)} + B_{\gamma}r^{(n-1)s+1} + C_{\gamma}r^{r+1}$$

(3.21)

where $r$ is the distance from the crack tip, $s = -1/(n+1)$, and $t = 0.055$ for a two-
dimensional crack in plane strain with $n=5$. Note that the strain hardening
exponent does affect the exponent of $r$ in all of the terms in equations 3.19 to
3.21. The HRR terms ($r^{1/(n+1)}$ terms in equations 3.19 and 3.20 and $r^{(n+1)}$ term in
equation 3.21) were subtracted from the field values determined by FEA. The
remaining higher order terms were curve fitted using a least squares
methodology to determine the values of the coefficients $Q_{\alpha}$, $Q_{\gamma}$, $B_{\gamma}$, and $C_{\gamma}$. 
In developing a one or two parameter fracture methodology, one has to assume that the asymptotic solution on which the criterion is based provides an accurate description of the near tip stresses over distances that are sufficiently larger than the fracture process zone [32]. Hutchinson [34] suggests that one condition for a valid fracture methodology is,

\[ R > 3 \delta, \]

where \( R \) is the radius of the zone of dominance of the methodology's crack tip stress solution and is sufficiently larger than the fracture process zone. The crack tip opening is designated by \( \delta \). Shih [35] has shown that the crack tip opening can be determined by,

\[ \delta_i = d_n \frac{J}{\sigma_0} \]

where \( d_n \) is approximately equal to 0.5 for \( n=10 \) and 0.2 for \( n=3 \) [35]. By interpolation the approximate value of \( d_n \) for \( n=5 \) is 0.3. By substitution of \( \delta_i \), Hutchinson's condition becomes,

\[ R > 3d_n \frac{J}{\sigma_0} \]

Therefore, \( R \) extends a distance \( r/(J/\sigma_o) \approx 1.0 \) from the crack tip when \( n=5 \). The radial distance from the crack tip is designated by \( r \). The data inside \( R \) was omitted when curve fitting the elastic-plastic CCT stress and displacement data. The outer curve fit bound was \( r/(J/\sigma_o) = 15 \). The outer bound is sufficiently larger than \( R \) and is the approximate extent of the crack tip plastic zone for all models.
Constraint Parameter Procedure

In order to compare with the higher order term coefficients \( (L_a)^{+1} \) was calculated at constant load by,

\[
(L_a)^{+1} = \frac{\Delta G}{G_{\text{ave}} \Delta a} = \frac{(G^{i+1} - G^i)}{G_{\text{ave}} (a^{i+1} - a^i)}
\]  (3.22)

where \( G \) is the energy release rate and \( a \) is the crack length. Incremental steps are designated by \( i+1 \) and \( i \). That is, \( a^{i+1} \) corresponds to an in-plane crack size slightly larger than \( a^i \). In order to attain values for \( G^{i+1} \), all FEA models were modified so that the crack length was slightly larger than the original model. \( G_{\text{ave}} \) was defined as the average value of \( G^{i+1} \) and \( G^i \) or,

\[
G_{\text{ave}} = \frac{G^{i+1} + G^i}{2}
\]  (3.23)

The values of \( G \) were calculated by ABAQUS [28] using the internal J-integral subroutine which calculates \( G \) using the domain integral method for all linear elastic models. For the elastic plastic CCT models \( G \) was calculated as \( J_a \) using the EPRI [36] estimation scheme (see Equations 3.32-3.35).
Results

The coefficients were plotted as a function of \( a/w \) for the CCT models and elliptical angle (\( \phi \)) for the SCT model. In addition, the coefficients were compared with \((L_e)^{-1}\).

Linear Elastic CCT Model Coefficients

\( G \) was calculated by ABAQUS using the domain integral method for each of the linear elastic finite element models. The ABAQUS value of \( G \) was within 1\% of the handbook [24] value for all linear elastic CCT models.

The crack tip displacement field values at \( \theta=180^\circ \) were determined using the linear elastic finite element models for \( a/w \)'s ranging from 0.25 to 0.75. A sample of the crack tip displacements along with the singularity displacements (\( Ar^{1/2} \) term only) is shown in figure 3.4. After subtracting the singularity term in the displacement series, where for plane strain,

\[
A(\theta=180^\circ) = \frac{K}{\mu} \sqrt{\frac{2}{\pi}} (1 - \nu)
\]

(3.24)

the value of \( B \) was determined by curve fitting for each crack size, where \( \mu \) is the shear modulus. Note that \( B \) is the coefficient of the \( r^{1/2} \) term in the displacement series defined by equation 3.18. \( B \) was then normalized as \( B^* \) by use of equation 3.10. The results show an increasing trend with increasing \( a/w \) (Figure 3.5). The normalized coefficient values are shown in table 3.2.
Figure 3.4: Crack Tip Displacements for Linear Elastic CCT Model (a/w=0.25)
Figure 3.5: Variation of $B^*$ with Crack Length for Linear Elastic CCT Model
Table 3.2: Higher Order Term Coefficients For Linear Elastic CCT Model

<table>
<thead>
<tr>
<th>a/w</th>
<th>B*</th>
<th>T_x*</th>
<th>γ_x*</th>
<th>γ_y*</th>
<th>(L_o)^-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>-0.5275</td>
<td>-0.6152</td>
<td>1.396</td>
<td>1.415</td>
<td>2.328</td>
</tr>
<tr>
<td>0.375</td>
<td>-0.3482</td>
<td>-0.3962</td>
<td>0.939</td>
<td>0.972</td>
<td>1.894</td>
</tr>
<tr>
<td>0.50</td>
<td>-0.2615</td>
<td>-0.2820</td>
<td>0.671</td>
<td>0.748</td>
<td>1.832</td>
</tr>
<tr>
<td>0.625</td>
<td>-0.2157</td>
<td>-0.2117</td>
<td>0.462</td>
<td>0.603</td>
<td>1.951</td>
</tr>
<tr>
<td>0.75</td>
<td>-0.1737</td>
<td>-0.1576</td>
<td>0.079</td>
<td>0.372</td>
<td>2.520</td>
</tr>
</tbody>
</table>

The crack tip normal stress values in both the x and y directions at θ=0° were also recorded from the linear elastic CCT finite element analysis. The finite element values along with the singularity values are shown in figure 3.6. The differences between the FEA and singularity values of the crack tip stresses are due to the low constraint CCT geometry. That is, the crack tip stresses deviate from the singular values for low constraint geometries. The singularity terms, D_xr and D_yr, where

\[
D_x(\theta = 0°) = D_y(\theta = 0°) = \frac{K}{\sqrt{2\pi}} \quad (3.25)
\]

were subtracted from the FEA field values of σ_x and σ_y, respectively. The form of equations 3.16 and 3.17 were used to curve fit the remaining higher order terms in order to determine T_x, T_y, γ_x, and γ_y values. γ_x and γ_y were normalized using equation 3.11. T_x and T_y were normalized using,

\[
T_x^* = T_x \frac{\sigma_{app}}{K^2} \quad (3.26)
\]
Figure 3.6: Crack Tip Stresses for Linear Elastic CCT Model (a/w=0.25)
where \( \sigma_{ap} \) is the applied tensile stress. The curve fits determined that \( T_x \) was zero for all crack lengths. This is consistent with the Larsson and Carlsson modified boundary layer approach for a two dimensional linear elastic stress state. That is, Larsson and Carlsson defined \( T_x \) as zero in equation 2.19. The values of \( T_x^*, \gamma_x^* \), and \( \gamma_x^* \) are plotted versus crack length in figure 3.7 and their values are shown in table 3.2.

**Linear Elastic SCT Model Coefficients**

\( G_i \) values for the SCT model were calculated along the crack front by ABAQUS. These values of \( G_i \) compared well with the numerical equations for \( G_i \) developed by Newman and Raju (Equation 1.6). Figure 3.8 compares these values of \( G_i \).

The displacements and stresses calculated by FEA were transformed to the local in-plane coordinate systems at elliptical angles (\( \phi \)) ranging from \( 0^\circ \) to \( 90^\circ \) (see Figure 1.4). The local in-plane coordinate system is shown in figure 3.3 and is always perpendicular to the crack front.

The crack tip displacement field values at \( \theta=180^\circ \) were determined using the three dimensional linear elastic finite element model for elliptical angles of \( 0^\circ, 22.5^\circ, 45^\circ, 67.5^\circ, \) and \( 90^\circ \). An example of the FEA displacement field plotted versus distance from the crack tip is shown in figure 3.9. The first (singularity) term could not be subtracted from the FEA displacements because it is dependent on the unknown out-of-plane stress state (i.e. plane stress or plane strain). Along the front of a semi-elliptical surface crack the out-of-plane stress state varies from approximately a state of plane stress at the surface (\( \phi=0^\circ \)) to an
Figure 3.8: Energy Release Rate for 3-D SCF Model

Elliptical Angle (Degrees)

- Newman Rayl Equations
- Finite Element Results
Figure 3.9: Crack Tip Displacements for Linear Elastic SCT Model (φ = 45°)
approximate state of plane strain at $\phi=90^\circ$. Therefore both $A$ and $B$ were determined at each elliptical angle by curve fitting where Note that $A$ and $B$ are displacement series coefficients defined by equation 3.18. The values of $A$ determined by curve fitting were close to the plane stress value of $A$ at $\phi=0^\circ$ and to the plane strain value of $A$ at $\phi=90^\circ$. The coefficient, $B$, was then normalized as $B^*$ by use of equation 3.10. The results show an increasing trend with increasing elliptical angle (Figure 3.10). The normalized coefficient values are shown in table 3.3. A possible source of error is introduced when determining the value of $A$ by curve fitting. Error may be introduced because the $A$ term is the dominant term in the two term series. Therefore, relatively small changes in $A$ can introduce significant changes in $B$.

The crack tip normal stress values in both the local $x$ and $y$ directions at $\theta=0^\circ$ were also recorded from the linear elastic SCT finite element analysis. An example of these stresses versus distance from the crack tip is shown in figure 3.11. The singularity terms, $D_{x1/2}$ and $D_{y1/2}$ were subtracted from the FEA field values of $\sigma_{xx}$ and $\sigma_{yy}$, respectively. Local values of the stress intensity factor, $K_p$, were used to calculate the singularity terms of equation 3.25. The form of equations 3.16 and 3.18 were used to curve fit the remaining higher order terms in order to determine $T_x$, $T_y$, $\gamma_x$, and $\gamma_y$ values. The coefficients, $\gamma_x$ and $\gamma_y$, were normalized using equation 3.11 and $T_x$ and $T_y$ were normalized using equation 3.26. The values of $T_x^*$, $T_y^*$, $\gamma_x^*$, and $\gamma_y^*$ are plotted versus elliptical angle in figure 3.12 and are shown in table 3.3. Note that after the successful defense of this thesis the curve fitting format used to determine $T_y^*$ and $\gamma_y^*$ was found to be incorrect. See the Discussion at the end of Chapter III for more information.
Figure 3.10: Variation of $B^*$ with Elliptical Angle for Linear Elastic SCT Model
Figure 3.11: Crack Tip Stresses for Linear Elastic SCT Model ($\phi=45^\circ$)
Figure 3.12: Higher Order Stress Term Coefficients for the Linear Elastic SCT Model
Table 3.3: Higher Order Term Coefficients for Linear Elastic SCT Model

<table>
<thead>
<tr>
<th>φ</th>
<th>B*</th>
<th>T₀*</th>
<th>T₁*</th>
<th>γ₀*</th>
<th>γ₁*</th>
<th>(Lₙ)⁻¹</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>-0.2672</td>
<td>-1.1519</td>
<td>-0.2457</td>
<td>3.6601</td>
<td>4.4192</td>
<td>3.055</td>
</tr>
<tr>
<td>22.5°</td>
<td>-0.2175</td>
<td>-0.3677</td>
<td>0.1258</td>
<td>-0.8967</td>
<td>1.0986</td>
<td>2.621</td>
</tr>
<tr>
<td>45°</td>
<td>-0.1614</td>
<td>-0.4080</td>
<td>0.1700</td>
<td>-0.5552</td>
<td>1.3473</td>
<td>2.275</td>
</tr>
<tr>
<td>67.5°</td>
<td>-0.1163</td>
<td>-0.4295</td>
<td>0.1849</td>
<td>-0.7787</td>
<td>1.1209</td>
<td>2.014</td>
</tr>
<tr>
<td>90°</td>
<td>-0.0920</td>
<td>-0.4360</td>
<td>0.1640</td>
<td>-1.0449</td>
<td>1.2546</td>
<td>1.910</td>
</tr>
</tbody>
</table>

Accurate stress values at φ=0° were not obtained from the SCT model because the finite element mesh was not fine enough in the z direction to account for surface effects. Therefore, the values of T* and γ* were extrapolated between the elliptical angles of 0° and 22.5° in figure 3.12. In addition, a possible source of error was introduced during the evaluation of the singular stress terms. The local values of Gᵢ were calculated by ABAQUS. In order to determine Kᵢ from Gᵢ, knowledge of the out-of-plane stress state is required, but is unknown. Therefore, the condition of plane stress was assumed during the evaluation of the singular stress terms for the SCT model.

Elastic-Plastic CCT Model Coefficients

All coefficients were determined for the elastic-plastic CCT models at 70% of limit load (Pₗ). The total J was calculated by ABAQUS using the domain integral method for each model. The total J values were within 5% to 10% of the EPRI estimation scheme [36] values of J for all models. The plastic
zone size determined by the equivalent stress being equal to $\sigma_e$ at $\theta=180^\circ$ was observed to extend between $r/(J/\sigma_e) = 13$ and $r/(J/\sigma_e) = 20$ for all elastic-plastic models where $r$ is the distance from the crack tip.

The crack tip displacement field values at $\theta=180^\circ$ were determined using the elastic-plastic finite element models for $a/w$'s ranging from 0.25 to 0.75. An example of the crack tip displacements versus distance from the crack tip is shown in figure 3.13. After subtracting the HRR term ($A_{ep} r^{(n+1)}$ in Equation 3.21) from the FEA displacements, where

$$A_{ep}(\theta=180^\circ) = \alpha \varepsilon_0 \left( \frac{J}{\alpha \varepsilon_0 \sigma_0 I_n} \right)^{n/(n+1)} \tilde{u}_y(180^\circ)$$

(3.27)

the value of $B_{ep}$ and $C_{ep}$ were determined by curve fitting. In equation 3.27, $\alpha$ and $n$ are the yield offset and strain hardening exponent respectively. $I_n$ is an integration constant where $I_n = 5.0$ for plane strain [37]. The material yield strength is designated by $\sigma_0$ and $\varepsilon_0 = \sigma_0/E$. $J$ was calculated by ABAQUS and $\tilde{u}_y(180^\circ) = 2.3678$ [37]. $B_{ep}$ and $C_{ep}$ were then normalized where,

$$B_{ep}^* = \frac{B_{ep}}{\alpha \varepsilon_0 \left( \frac{\sigma_0}{J} \right)^{n/(n+1)} \left( \frac{J}{\alpha \varepsilon_0 \sigma_0 I_n} \right)^{n/(n+1)} \left( \frac{1}{J/\sigma_0} \right)}$$

(3.28)

$$C_{ep}^* = \frac{C_{ep}}{\alpha \varepsilon_0 \left( \frac{\sigma_0}{J} \right)^{n/(n+1)} \left( \frac{J}{\alpha \varepsilon_0 \sigma_0 I_n} \right)^{n/(n+1)} \left( \frac{1}{J/\sigma_0} \right)}$$

(3.29)

The coefficients, $B_{ep}^*$ and $C_{ep}^*$, versus $a/w$ are plotted in figure 3.14. The normalized values are shown in table 3.4. $B_{ep}$ and $C_{ep}$ were normalized by the
Figure 3.13: Crack Tip Displacements for Elastic-Plastic CCT Model (a/w = 0.25)
Figure 3.14: Higher Order Displacement Term Coefficients for the Elastic-Plastic CCT Model.
coefficients of the respective terms in equation 3.17 to obtain a dimensionless value. An additional $1/(J/\sigma_0)$ term was then included to obtain normalized units of $[1/\text{length}]$. The units of $(L_a)^{i}$ are also $[1/\text{length}]$.

Table 3.4: Higher Order Term Coefficients For Elastic-Plastic CCT Model

<table>
<thead>
<tr>
<th>a/w</th>
<th>$B_n^*$</th>
<th>$C_m^*$</th>
<th>$Q_x^*$</th>
<th>$Q_y^*$</th>
<th>$(L_a)^{i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>176.62</td>
<td>1206.2</td>
<td>-1583.8</td>
<td>-1391.9</td>
<td>2.30</td>
</tr>
<tr>
<td>0.375</td>
<td>104.21</td>
<td>1129.4</td>
<td>-1284.7</td>
<td>-1149.2</td>
<td>1.90</td>
</tr>
<tr>
<td>0.50</td>
<td>61.83</td>
<td>1265.5</td>
<td>-1311.3</td>
<td>-1213.5</td>
<td>1.90</td>
</tr>
<tr>
<td>0.625</td>
<td>51.40</td>
<td>1429.9</td>
<td>-1447.5</td>
<td>-1370.9</td>
<td>2.13</td>
</tr>
<tr>
<td>0.75</td>
<td>46.81</td>
<td>1793.5</td>
<td>-1868.8</td>
<td>-1809.7</td>
<td>2.82</td>
</tr>
</tbody>
</table>

The crack tip normal stress values in both the $x$ and $y$ directions at $\theta=0^\circ$ were also recorded from the elastic-plastic CCT finite element analysis. An example of the normal stresses versus distance from the crack tip ($r$) is shown in figure 3.15. The HRR terms, $D_{\sigma_x r^{-(n+1)}}$ and $D_{\sigma_y r^{-(n+1)}}$, where

$$D_{\sigma_r} (\theta = 0^\circ) = \left( \frac{J}{a \varepsilon_0 \sigma_0 l_n} \right)^{1/(n+1)} \tilde{\sigma}_r (0^\circ) \quad (3.30)$$

were subtracted from the FEA field values of $\sigma_x$ and $\sigma_y$, respectively. The values of the functions of theta were $\tilde{\sigma}_x (0^\circ) = 1.6836$ and $\tilde{\sigma}_y (0^\circ) = 2.2172$ [37]. The form of equations 3.19 and 3.20 were used to determine $Q_x$ and $Q_y$ by curve fitting. All $Q$'s were normalized in a similar fashion as $B_{\sigma_r}$ using,
Figure 3.15: Crack Tip Stresses for Elastic-Plastic CCT Model (a/w=0.25)
The values of $Q_x^*$ and $Q_y^*$ are plotted versus crack length in figure 3.16 and are shown in table 3.4. Both $Q_x$ and $Q_y$ were found to be non-zero quantities and similar in magnitude. This is consistent with the findings of O'Dowd and Shih [22]. These researchers introduced a $Q$ parameter in order to extend $J$ based fracture mechanics to include low constraint geometries. The $Q$ parameter quantifies constraint by representing the second term in the near tip stress fields. Their investigation determined that the higher order stress terms ($Q_x r$ and $Q_y r$ in equations 3.21 and 3.22) were approximately constant but did show a slight dependence on $r$ for stresses in the forward sector of the plastic zone ($-90^\circ < \theta < 90^\circ$). The present findings also demonstrated that the higher order stress terms showed only a slight dependence on $r$.

**Constraint Parameter Comparison**

The coefficients of the higher order crack tip stress and displacement terms were plotted with $(L_a)^{-1}$ for each of the finite element models. These figures were developed as one part of a growing database of information on crack tip constraint. The database also includes information on additional constraint parameters and experimental results of various fracture geometries. The ability of these parameters to characterize constraint shall be scrutinized upon the completion of the entire database.

$(L_a)^{-1}$ was calculated for each of the linear elastic CCT models. $B^*$ was then compared to $(L_a)^{-1}$. Figure 3.17 and 3.18 show that $-B^*$ and $-T^*$ display

$$Q_x^* = \frac{Q_x}{\sigma_0 \left( \frac{\sigma_0}{J} \right) \left( \frac{1}{J/\sigma_0} \right)} \quad (3.31)$$
Figure 3.16: Higher Order Stress Term Coefficients for Elastic-Plastic CCT Model
Figure 3.17: \((\text{Le})^{-1}\) and \(-B^*\) for Linear Elastic CCT Model
Figure 3.18: (Lei)-1 and -Tx* for Linear Elastic CCT Model.
similar trends when plotted with $(L_d)$ from $a/w=0.25$ to $a/w=0.5$, but at higher values of $a/w$ the trends diverge. Figure 3.19 displays $\gamma_\star$ and $\gamma_\star$ with $(L_d)^{-1}$.

For the linear elastic SCT model, local values of the proposed constraint parameter were calculated by incrementing the size of the crack and applying equation 3.22. Local values of virtual crack extension perpendicular to the original crack front were used to calculate $(L_d)^{-1}$. Figure 3.20 displays similar trends for $-B^\star$ and $(L_d)^{-1}$. Accurate stress values at $\phi=0^\circ$ were not obtained from the SCT model because the finite element mesh was not fine enough in the $z$ direction to account for surface effects. Therefore, the values of $T^\star$ and $\gamma^\star$ were extrapolated between the elliptical angles of $0^\circ$ and $22.5^\circ$ in figures 3.21 and 3.22. $T_x^\star$ and $-T_y^\star$ show similar trends with the proposed constraint parameter when plotted as a function of elliptical angle (Figure 3.21). Figure 3.22 compares $\gamma_x^\star$ and $\gamma_y^\star$ with $(L_d)^{-1}$, but a similar trend was not found. Note that an error was found with the higher order stress term coefficients for the SCT model which is explained in the Discussion of Chapter III.

For the elastic-plastic models, the linear elastic portion of $J$ ($G$ or $J_d$) was calculated using the EPRI [36] estimation scheme, where

$$J_{el} = \frac{(1-v^2)K^2(a_{el})}{E}$$  

(3.32)

and,

$$a_{el} = a + \frac{1}{1+(P/P_0)^2} \frac{1}{\beta \pi} \left( \frac{n-1}{n+1} \right) \left( \frac{K(a)}{\sigma_o} \right)^2$$  

(3.33)

$$P_0 = \frac{4}{\sqrt{3}} t b \sigma_f$$  

(3.34)
Figure 3.20: \((Le)^{-1}\) and \(B^*\) for Linear Elastic SCT Model
Figure 3.21: (Le)-1 and T* for Linear Elastic SCT Model
Figure 3.22: $(\lambda_{el}-1)$ and Gamma* for Linear Elastic SCT Model
\[ \sigma_f = \frac{\sigma_w + \sigma_0}{2} \]  

(3.35)

where \( \beta = 6 \) for plane strain, \( P \) is the total applied load, and \( n \) is the strain hardening exponent. The specimen thickness, length of the remaining ligament, ultimate tensile strength, and yield strength are designated by \( t \), \( b \), \( \sigma_w \), and \( \sigma_0 \) respectively. Loaded to 70% of limit load \( (P_0) \), the ABAQUS calculated total \( J \) values deviated from the EPRI \( J_u \) values by 15% to 35%. Therefore, 15% to 35% of the total \( J \) was due to plasticity \( (J_p) \). For the elastic-plastic CCT model, \( B_{\gamma} \) and \( C_{\gamma} \) were then plotted with \( (L_{\alpha})^{-1} \) in figure 3.23 and 3.24, respectively. Figure 3.25 displays \( Q_\gamma \) and \( Q_\gamma \) with the proposed constraint parameter.

**Discussion**

The coefficients of the higher order terms developed from the linear elastic CCT model were compared with higher order term relationships derived using the eigenfunction expansion method \([3,30]\) and the Westergaard \([3]\) \( Z \) function (Equation 3.12). As predicted by the eigenfunction expansion and the \( Z \) function methods, \( \gamma_\gamma \) is virtually equal to \( \gamma_\gamma \) in our numerical results. However, the values begin to deviate at higher \( a/w \)'s; additional higher order stress terms are possibly required with larger cracks since the finite body effects (at the surface) become more significant. The relationship determined by the Westergaard approach (Equation 3.12) between \( \gamma^* \) and \( B^* \) was:

\[ B^* = (-1/3)\gamma_\gamma^* = (-1/3)\gamma_\gamma^*. \]

However, the FEA results (with \( a/w \leq 0.5 \)) indicate a
Figure 3.25: \((L_e) - 1\) and the Higher Order Stress Coefficients for Elastic-Plastic CCT Model
relationship of the form: \( B^* \approx -0.37 y_z^* \approx -0.37 y_y^* \). The difference in the two results could be attributed to finite body effects.

After the successful defense of this thesis, the curve fitting format of equation 3.17 was found to be incorrect. The \( T_y \) term was included when curve fitting the higher order \( \sigma_{yy} \) terms for the linear elastic SCT model. However, the \( T_y \) term should not have been included in the curve fits since it would invalidate the traction boundary conditions along the crack face. The term was not found to be zero, as it was in the linear elastic CCT model, because the higher order \( \sigma_{yy} \) terms converged to a finite stress value at the crack tip as \( r \to 0 \). The most likely explanation for the finite value convergence is the inaccuracy of the singular \( \sigma_{yy} \) term calculation. An assumption on the out-of-plane stress state was made during the calculation of the singular term in order to convert the energy release rate, \( G_y \) calculated by ABAQUS [28] to the stress intensity factor, \( K_i \); after converting, \( K_i \) was then used to calculate the singular stress terms. Perhaps a better procedure to determine the singular stress term includes iteratively offsetting the higher order \( \sigma_{yy} \) terms by \( T_y \). That is, determine the higher order \( \sigma_{yy} \) term coefficients using the out-of-plane stress state assumption. Then, add \( T_y \) to the singular stress term to determine the value of an iterative singular stress term. At this point, the iterative singular stress term is subtracted from the FEA stresses to obtain iterative values of the higher order \( \sigma_{yy} \) stress terms. The iterative higher order terms can be curve fit in order to obtain iterative values for \( T_y \) and \( y_y \). Repeat the previous steps until the iterative singular \( \sigma_{yy} \) term converges and \( T_y = 0 \). After convergence, a more accurate value for \( K_i \) can be calculated using the singular stress term. This value of \( K_i \) should
then be used to calculate the singular $\sigma_\infty$ term and the first term in the
displacement series, so that the higher order $\sigma_\infty$ and $u_y$ term coefficients can be
determined. In addition, perhaps a similar iterative procedure could be utilized
to bring the higher order term coefficients of the linear elastic CCT model
($a/w=0.625,0.75$) into agreement with equation 3.12.

The Q values for the elastic-plastic CCT model were determined by
curve fitting the two-term stress series expansion (Equations 3.19 and 3.20).
The two terms yielded good representations of the crack tip stresses over the
curve fitted range $1 < r/(J/\sigma_y) < 15$. Sharma and Aravas [32] used this two term
series in their asymptotic analysis of crack tip fields. However, they also stated
that at the distances $r/(J/\sigma_y)=2$ and $r/(J/\sigma_y)=5$ from the crack tip additional
terms in the asymptotic stress expansion (Equation 3.13) may be needed for an
accurate representation of the stress field in front of the crack tip.

The second and third order terms were required for curve fitting the
elastic-plastic displacements. The Sharma and Aravas and Li and Wang
analyses both state that the second order term ($B_r r^{-n-1}$) does not account for
the effects of elasticity. The third order displacement term is a function of the
elastic strain resulting from the singular (HRR) stress term. Therefore, with the
second and third order terms included, excellent agreement between the FEA
field values and the curve fitting constants were obtained for the displacement
fields.

The similar trends observed between $C_\omega^*$ and $(L_\omega)^{-1}$ in figure 3.24 and $Q^*$
and $(L_\omega)^{-1}$ in figure 3.25 are significant developments. The similar trend
suggests that $C_\omega^*$, $Q^*$, and $(L_\omega)^{-1}$ are related. Of course, an implicit relationship
exists between the terms in the crack tip stress, strain and displacement series. Therefore, if a relationship does exist between \((L_e)^{-1}\) and the higher order terms, \((L_e)^{-1}\) can be considered a second fracture parameter (i.e. constraint parameter). However, an extensive analytical study of the higher order term coefficients is required in order to develop a relationship with \((L_e)^{-1}\).

\((L_e)^{-1}\) is a parameter associated with the higher order strain/displacement terms for a linear elastic stress state. \(C_e^*\) is a function of the elastic strain/displacement resulting from the HRR term in the stress series. Therefore, the similar trend between \(C_e^*\) and \((L_e)^{-1}\) suggests that constraint parameters may be separable in elastic and plastic components by relating these separate terms to different higher order terms in the stress, strain, or displacement series. However, finite geometry effects have not been accounted for in the comparison between \(C_e^*\) and \((L_e)^{-1}\). In addition, experimental results must be implemented in the investigation before concluding that \((L_e)^{-1}\) is a valid constraint parameter for general use. Nevertheless, the possible relationship with the higher order stress and displacement terms suggests a bright future for \((L_e)^{-1}\) as a constraint parameter.
REFERENCES


CHAPTER IV

EFFORTS TO CHARACTERIZE THREE-DIMENSIONAL EFFECTS OBSERVABLE IN TWO DIMENSIONAL FRACTURE SPECIMENS.

by D. Lambert and H. Ernst

4.1. Introduction

The underlying purpose of this research is to develop a methodology which would allow the characterization of three-dimensional (3D) effects in fracture. This characterization should include:

(1) geometric effects arising from crack front curvatures (curvatures are present, for example, with surface cracks),

(2) geometric effects related to thickness and ligament length (the gross sizing details that affect the three-dimensionality of the state of stress at the crack tip), and

(3) loading geometry effects (including three-dimensionality of the far field stress arising from the character of applied loads, and, especially the gradient of the far-field stress arising from differing ratios of bending-to-tension).

Two-dimensional (2D) or planar specimens have been observed to generate different fracture resistance curves when different
thicknesses are tested. Specifically discussed here are JMR curves that use the J-modified parameter as developed by Ernst [4.1, 4.2]. Different configurations have been shown to support a differing degree of triaxiality of the stress field in the vicinity of the crack front, where the fracture process is occurring. The degree of stress field triaxiality that is exhibited is referred to as the constraint. The JMR curves are a result of the different, averaged constraint in each specimen. Even though the configurations are considered to be planar, curvatures can develop in crack fronts that result from fracture in the presence of a gradient of the constraint within the specimen. Thus, the complexities that occur in the most general cases of fracture appear in the simplest cases of planar specimens. Ultimately, to evaluate fracture resistance retaining a planar analogy requires that the crack front fall within specific limits of straightness.

Since, 3D stress fields are present in planar configurations, an effort to map the crack face separation profiles of a variety of geometries as a function of the position within the cross-section has been proposed. The crack tip opening displacement (CTOD) is a linear function of the J-integral [4.3]. Profiling represents an extension of that functional relationship.

One goal of the overall research program is to test a wide variety of planar specimens, varying the thickness and length of the initial remaining ligament, as well as the bending-to-tension

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1 Numbers in brackets refer to the references at the end of the paper
ratio due to the nature of the applied load to produce significant changes in fracture behavior. The results will be compared using the ligament dimensions as variables, and finite-element analysis will be used to evaluate the triaxiality of the stress field. Although other parameters might be used, one parameter being considered to quantify the stress field triaxiality is $h$:

$$h = \frac{\sigma_{\text{mean}}}{\sigma_{\text{VM}}} \tag{4.1}$$

Here, $\sigma_{\text{mean}}$ is the mean stress and $\sigma_{\text{VM}}$ is von Mises equivalent stress. Ultimately, this approach is expected to produce parameters and fracture behavior that can be generalized to the 3D cases that are of the most interest.

4.2 Introduction to Profiling

The displacement of the faces of a crack are a function of the loading and of the position. Using a two dimensional analogy, the displacement of a point along a crack face within an elastic body was given by Tada, et al [4.4]:

$$v_{el} = \frac{4\sqrt{2}}{E'\sqrt{\pi}} K \sqrt{r} \tag{4.2}$$
Here, $v_{el}$ is the "y"-directed elastic displacement at position $r$, measured from the crack tip to the point in question, $E'$ is the equivalent modulus ($E' = E$ for plane stress, $E' = E/(1-v^2)$ for plane strain, $v$ is Poisson's ratio). The loading is specified in the presence of a flaw by $K$, the stress intensity parameter. This displacement relationship has a square-root of $r$ form.

Hutchinson [4.5] and Rice and Rosengren [4.6] developed a similar form for plastic response that follows Ramberg-Osgood deformation characteristics, i.e.:

$$\frac{\varepsilon}{\varepsilon_0} = \alpha \left( \frac{\sigma}{\sigma_0} \right)^n$$

(4.3)

In this equation, $\varepsilon$ and $\sigma$ are the equivalent strain and stress and $\varepsilon_0$, $\sigma_0$ and $n$ are material constants. The form of the displacement is as follows:

$$v_{pl} = k \cdot J_{pl}^{\frac{n}{n+1}} r^{\frac{n}{n+1}}$$

(4.4)

This equation is written for a non-growing crack, and $v_{pl}$ is the displacement of the body, assuming Ramberg-Osgood type deformation, and $k$ includes the functionality with regards to the constraint, i.e., plane stress or plane strain condition.

Looking at the equations (4.2) and (4.4), the constraint appears
in the coefficient $E'$ for the linear elastic case and in the coefficient $k$ for the plastic case. Thus, the separation at various points through the ligament thickness could be expected to reflect that difference in constraint that arises with the depth into the thickness. It may also provide a measure of that constraint.

Since the development above is for a non-growing crack situation, differences that occur between the theoretical elastic plus plastic displacements and the displacement profile of an actual growing crack near the crack tip might provide a fracture criterion on that local level.

4.3 **Profiling Matrix and Details**

One objective in the research was to characterize the separation between the surfaces of cracks. This separation profile is a function of the level of $J$ and of the position within the ligament. In this case the position would include the distance from the load-line, $x$, in the direction of crack growth, and the depth beneath the surface in the thickness direction, $z$.

The primary effort in the past six months has been to characterize the crack face separation of selected specimens. The specimen identities and the corresponding configurations appear in Table 4.1, below. Data was generated for a total of six compact tension (CT) specimens and three center-crack tension (CCT) specimens. The CCT configuration produces two crack
profiles per specimen, and thus the total number of crack fronts observed is thirteen. Profiles were made of the AL6061-T651 in every case: the IN718-STA1 material has proven too hard to polish in the same fashion as the aluminum. Epoxy infused into the gap of the crack was effective in producing a well-defined crack profile for the aluminum, but the profiles of the nickel were rounded and poorly-defined. Until the techniques have been modified to overcome the rounding, the profiling was suspended for the nickel.

After mounting the specimens in epoxy, the exposed surface was polished to provide a surface profile. After recording the profile, 0.025- to 0.035-inches was removed by grinding and polishing to produce the next profile to be recorded. This was continued for each specimen into the center of the cross-section.
REM dJ/da CALCULATION
REM TRANSFER ARRAYS SO EXISTING SUBROUTINES CAN BE USED
FOR DUM=0 TO NUM:JT(DUM)=J(DUM):XT(DUM)=X(DUM):YT(DUM)=Y(DUM):NXT(DUM)=NY(DUM):NEXT DUM
CALCULATE NORMAL TO EQUIVALENT ELLIPSE
YD(DUM)=-X(DUM)*A*(1-(X(DUM)/C(DUM))^2)^-.5/C(DUM)^2
PX=-Y(DUM):PY=1
HY=(1+PX^2)^.5
NX(DUM)=PX/HYP
NY(DUM)=PY/HYP
NEXT DUM
CALCULATE COORDINATES OF POINTS AHEAD OF FRONT
X(DUM)=XT(DUM)+NX(DUM)*(T/2500)
Y(DUM)=YT(DUM)+NY(DUM)*(T/2500)
NEXT DUM
CALCULATION OF EFFECTIVE C'S SUCH THAT NORMALS ARE EQUAL
C(DUM)=(X(DUM)*2/(1-(Y(DUM)/A)^2))^-.5
NEXT DUM
C(NUM)=C(NUM-1):REM BECAUSE C AT THE TOP IS UNDEFINED
C(0)=X(0):REM BY DEFINITION
CALCULATE ELLIPTICAL ANGLES
GOSUB 12000
CALCULATE G's
GOSUB 8000
CALCULATE J's
GOSUB 8000
CALCULATE J's
GOSUB 8000
CALCULATE dJ/da FROM INCREMENTAL GROWTH OF EQUIV. FRONTS
FOR DUM=0 TO NUM:JDA(DUM)=(JT(DUM)-J(DUM))/(T/2500):NEXT DUM
CALCULATE dJ/da FROM MASTER R-CURVE
JDAM(DUM)=3*ALPA*J(DUM)^2+2*BETA*J(DUM)+GMMA
JDAM(DUM)=JDAM(DUM)/JDAM(DUM)
NEXT DUM
FOR DUM=0 TO NUM
IF F(DUM)<1 THEN F(DUM)=1:REM BY DEFINITION F CANNOT BE LESS THAN 1
NEXT DUM
TRANSFER BACK ARRAYS
FOR DUM=0 TO NUM:J(DUM)=JT(DUM):Y(DUM)=YT(DUM):X(DUM)=XT(DUM):NX(DUM)=NY(DUM):NEXT DUM
RETURN
C(DUM) = ((X(DUM)^2 + A^2) / (A^2 - Y(DUM)^2))^0.5

NEXT DUM

C(NUM) = C(NUM-1)

FOR DUM=0 TO NUM

IF C(DUM)>W/2 THEN C(DUM)=W/2: PRINT" C > W/2"

NEXT DUM

RETURN

REM ################################################################################

REM CALCULATION OF Fs AT ONE POINT

FOR A/C>1

M1 = (C(DUM)/A)^0.5 * (1 + 0.06*C(DUM)/A)

M2 = 2*(C(DUM)/A)^4

M3 = -11*(C(DUM)/A)^4

LG = 1 + (1 + 0.35*(C(DUM)/A)^2)*(1 - SIN(TH(DUM)))^2

FT = ((C(DUM)/A)^2*COS(TH(DUM))^2 + SIN(TH(DUM))^2)^0.25

RETURN

REM ################################################################################

REM CALCULATION ELLIPTICAL ANGLE ALONG THE FRONT

FOR DUM=1 TO NUM-1

IF A/C(DUM)<1 OR A/C(DUM)=1 THEN OPP=Y(DUM); ADJ=(A^2 - OPP^2)^0.5

IF A/C(DUM)>1 THEN ADJ=X(DUM); OPP=(C(DUM)^2 - ADJ^2)^0.5

IF A/C(DUM)<0 OR X(DUM)>C(DUM) THEN PRINT"A/C OUT OF RANGE"; GOTO 500

TH(DUM) = ATN(OPP/ADJ)

NEXT DUM

RETURN

REM ################################################################################

REM CALCULATION OF Dstar

FOR DUM=0 TO NUM

XDI = 0.224*N*W

XD2 = PI*BN * (N+1) * (GAVG/SIG^2)

XD3 = (A/C(DUM))^2 * (1 - PP*N)

XD4 = (A/T)^(-2 - M*N)

XDS(DUM) = XDI * XD3 * XD4 / XD2

NEXT DUM

RETURN

REM ################################################################################

REM INCREMENT PLASTIC DISPLACEMENT

VPL = VPL + IVPL

IF VPL>-VPLM THEN 500

P = W*T*BN * (Y(NUM)/X(0)) * PP* (Y(NUM)/T) * M*(VPL/T) * (1/N)

SIG = P/(W*T)

RETURN

JDAM(DUM) = I/JDAM(DUM)

NEXT DUM

RETURN
POINT J CALCULATION

FIND GAVG

GAVG = GTOT / (NUM + 1)

DELTA-a CALCULATION

ADVANCE THE CRACK FRONT

CALCULATION OF EFFECTIVE c ALONG CRACK FRONT
PLOT THE CRACK FRONT ON SCREEN

LOCATE 1,1:PRINT:PRINT "LOAD = ";P;" lbf

VPL = ";VPL;" in "

LINE (2.6*X(0)/T,Y(0)/T)-(2.6*X(1)/T,Y(1)/T)
FOR DUM = 1 TO NUM-1
LINE (2.6*X(DUM)/T,Y(DUM)/T)-(2.6*X(DUM+1)/T,Y(DUM+1)/T)
NEXT DUM
RETURN

DIRECTION OF CRACK EXTENSION

CONstrains edge to horizontal freedom
CONstrains edge to horizontal freedom
CONstrains center to vertical freedom
CONstrains center to vertical freedom

FOR DUM=1 TO NUM-1
IY=Y(DUM+1)-Y(DUM-1)
IX=X(DUM+1)-X(DUM-1)
PX=IY
PY=-IX
HYP=(PX*2+PY*2)^.5
NEXT DUM
RETURN

G CALCULATION

FOR DUM=0 TO NUM
PHI2=(1+1.464*(C(DUM)/A)^1.65):REM PHI2
FW=(COS((PI*C(DUM)/W)*(A/T)^.5))^-5:REM FW
IF A/C(DUM)>1 THEN GOSUB 11000 ELSE GOSUB 11500:REM M1,M2,M3, LG, FT
FS=(M1+M2*(A/T)^2+M3*(A/T)^4)*LG*FT*FW
K=SIG*FS*(PI*A/PHI2)^.5
G(DUM)=K^2/E
NEXT DUM
RETURN
2350 RETURN
2398 REM
2399 REM -------------------------------------- PLATE DIMENSIONS
2400 REM --------------------------------------
2401 REM
2410 T=.25:REM PLATE THICKNESS
2420 W=2:REM PLATE WIDTH
2430 RETURN
2498 REM
2499 REM -------------------------------------- MATERIAL DEFORMATION PROPERTIES
2500 REM --------------------------------------
2501 REM
2505 PRINT:PRINT"MATERIAL: 21-6-9 SS"
2510 E=2.84E+07:REM YOUNG'S MODULUS
2520 BN=.076
2530 PF=.0383
2540 N=.575
2560 RETURN
2598 REM
2599 REM -------------------------------------- J-da CURVE INPUT
3000 REM --------------------------------------
3001 REM
3060 ALPA=7.226487E-14
3070 BETA=-6.2973138D-10
3080 GMMA=0.0000040561366#
3150 RETURN
3198 REM
3199 REM -------------------------------------- CRACK GEOMETRY
3200 REM --------------------------------------
3220 PRINT:INPUT "Initial A/T";ADT
3230 IF (ADT>I) OR (ADT=I) THEN GOTO 3220.
3240 PRINT:INPUT "Initial A/C";ADC
3250 A=ADT*T
3260 C=A/ADC
3290 RETURN
3998 REM
3999 REM -------------------------------------- SET-UP THE SCREEN
4000 REM --------------------------------------
4010 PI=3.14159265359#
4020 PIDT=PI/2
4030 FOR DUM = 0 TO NUM
4040 X(DUM)=C*COS(DUM*PIDT/NUM)
4050 Y(DUM)=A*SIN(DUM*PIDT/NUM)
4060 XO(DUM)=X(DUM):YO(DUM)=Y(DUM)
4070 NEXT DUM
4075 Y(0)=0::X(NUM)=0::YO(0)=0::XO(NUM)=0
4170 SCREEN 9
4180 WINDOW (-1,-.25)-(3.8,1.25)
4190 LINE (0,-.25)-(0,1.25)
4200 LINE (-1,0)-(0,0):LINE (2.6*C/T,0)-(3.8,0)
4210 LINE (-1,1)-(3.8,1)
4290 RETURN
CLS: CLEAR: KEY OFF: SCREEN 0

DEFINT D
DEFDBL A-C,E-Z

DIM X(50),Y(50),NX(50),NY(50),ADEL(50),TH(50),NXT(50),NYT(50)

DIM G(50),J(50),C(50),EXT(50),START(50),INCA(50),YD(50)

DIM XO(50),¥0(50),XDS(50),JT(50),XT(50),YT(50),JDA(50),JDAM(50)

REM
35 GOSUB 2000: REM MAXIMUM VPL AND VPL INCREMENT
40 GOSUB 2300: REM NUMBER OF POINTS ALONG THE CRACK FRONT
45 GOSUB 2400: REM PLATE DIMENSIONS
50 GOSUB 2500: REM MATERIAL DEFORMATION PROPERTIES
55 GOSUB 3000: REM J-R CURVE INPUT
60 GOSUB 3200: REM CRACK GEOMETRY
65 GOSUB 4000: REM SCREEN SET-UP
70 REM
71 REM ---------------- MAIN LOOP ----------------
72 REM
75 GOSUB 6000: REM PLOT THE CRACK FRONT
80 REM
81 FOR DUM=0 TO NUM
82 J(DUM)=0
83 NEXT DUM
85 GOSUB 14000: REM INCREMENT THE PLASTIC DISPLACEMENT
86 REM
90 GOSUB 7000: REM DIRECTION OF CRACK EXTENSION
91 REM
92 GOSUB 10000: REM CALCULATION OF EFFECTIVE C ALONG FRONT
93 REM
94 GOSUB 12000: REM CALCULATE ELLIPTICAL ANGLES ALONG THE FRONT
95 REM
96 GOSUB 8000: REM CALCULATE G ALONG THE CRACK FRONT
97 REM
98 GOSUB 8500: REM CALCULATE Jtot ALONG THE CRACK FRONT
99 REM
100 GOSUB 15000: REM CALCULATE dJ/da ALONG FRONT
101 REM
105 GOSUB 9000: REM DELTA-A
106 REM
110 GOSUB 9500: REM ADVANCE THE CRACK FRONT
111 REM
150 GOTO 75
500 END

REM
1998 REM
1999 REM ---------------- VPL AND VPL INCREMENT ----------------
2000 REM ---------------- NUMBER OF POINTS ALONG CRACK FRONT ----------------
2001 REM
2002 REM
2020 PRINT: INPUT "Enter the Maximum Plastic Displacement (in) ":VPLM
2170 IVPL=VPLM/50
2283 RETURN
2298 REM
2299 REM ---------------- число точек на трещине ----------------
2300 REM
2301 REM
2302 REM
2303 NUM=10
2304 NUM=NUM-1: REM THIS ACCOUNTS FOR NUMBERING BEGINNING AT 0
2305 REM
2335 TH(0)=0: TH(NUM)=3.14159265359/2
Appendix to DW Boatwright's Thesis

APPENDIX

COMPUTER PROGRAM LISTING
APPENDIX
4.4 Preliminary Results of Profiling

Two of the profiles produced has been included as figures (4.1a and 4.1b) for the sake of discussion. Figure (4.1a) is a profile of a crack taken at the surface. The precrack and the monotonic fracture regions are marked on the figure. Figure (4.1b) shows a profile of the same crack taken at the center of the cross-section. Again, the precrack, and the monotonic fracture regions are shown. The character of the two profiles is quite different. The monotonic fracture region at the surface (figure 4.1a) is at an angle to the precrack region, while that angle is not obvious for the central section (figure 4.1b). To illustrate a second observation, the net displacement has been

![Figure 4.1: Typical Crack Face Separation Profiles; (a) at surface, (b) at central cross-section.](image-url)
calculated as the difference between the y-values of the upper and the lower parts of the profile, and figures (4.2a) and (4.2b) have been included below to register this second observation.

In figures (4.2a) and (4.2b), the precrack region and the monotonic fracture region are again marked. The two separation profiles look dramatically different: a substantial region of stretch exists at the end of the precrack region and the beginning of the monotonic fracture region at the surface. This stretch is not apparent in the central cross-section.

At this time, the profile data is being developed and observations are being made.
4.5 **Future Efforts**

Profiling continues, and analysis of the profiles and synthesis of an approach to account for the constraint in the crack growth of these "planar" specimens.
4.7 References


