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THE NONLINEAR EVOLUTION OF MODES ON UNSTABLE STRATIFIED SHEAR LAYERS

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ABSTRACT

The nonlinear development of disturbances in stratified shear flows (having a local Richardson number of value less than one quarter) is considered. Such modes are initially fast growing but, like related studies, we assume that the viscous, non–parallel spreading of the shear layer results in them evolving in a linear fashion until they reach a position where their amplitudes are large enough and their growth rates have diminished sufficiently so that amplitude equations can be derived using weakly nonlinear and non–equilibrium critical–layer theories. Four different basic integro–differential amplitude equations are possible, including one due to a novel mechanism; the relevant choice of amplitude equation, at a particular instance, being dependent on the relative sizes of the disturbance amplitude, the growth rate of the disturbance, its wavenumber and the viscosity of the fluid. This richness of choice of possible nonlinearities arises mathematically from the indicial Frobenius roots of the governing linear inviscid equation (the Taylor–Goldstein equation) not, in general, differing by an integer. The initial nonlinear evolution of a mode will be governed by an integro–differential amplitude equations with a cubic nonlinearity but the resulting significant increase in the size of the disturbance's amplitude leads on to the next stage of the evolution process where the evolution of the mode is governed by an integro–differential amplitude equations with a quintic nonlinearity. Continued growth of the disturbance amplitude is expected during this stage, resulting in the effects of nonlinearity spreading to outside the critical level, by which time the flow has become fully nonlinear.

1 Introduction

This paper extends the theories of Churilov & Shukhman (1988) and Goldstein & Leib (1989) to the case of (non–marginally) unstable stratified flows. Such flows, in which the vertical density variation is insufficient to overcome the strongly destabilising effects of the vertical velocity variation, occur naturally in many situations. The mathematical formulation of this problem has great similarity with, amongst others, that for the related problem of buoyancy–driven instabilities above a heated plate and with the, at first sight, less physically related problem of vortex disturbances driven by wall curvature occurring in 3D boundary layers. Thus, whilst here we concentrate solely on a particular model stratified shear flow, the mathematical theory has much wider applications to both geophysical and aerodynamical flow situations.

The stratification of a shear flow is usually characterised by a physical parameter called the Richardson number (herein after denoted J); it is a measure of the ratio of vertical density variation to the scale of vertical velocity shear. The Richardson number can be both positive and negative; the former case corresponding to lighter fluid lying above heavier fluid, and the latter case corresponding to the lighter fluid lying below. The stability of a stratified shear flow was first considered independently by Taylor (1931) and Goldstein (1931). They concluded that a multi–layer system of homogeneous fluids can not be used to approximate the stability of a heterogeneous fluid, and that for the flow models considered, the flow is stable for Richardson numbers greater than one quarter.

The flow model adopted in this paper was introduced by Drazin (1958); the vertical velocity and density distribution are represented by hyperbolic tangent and exponential profiles, respectively. This model enables an exact solution of the governing linear stability equation, the Taylor–Goldstein equation, for neutral disturbances and the same upper bound on the Richardson number is found. Soon after Drazin's paper appeared, Miles (1961) presented several theorems concerning properties of solutions to the Taylor–Goldstein equation. In particular, he proved the two important results that in general flows are stable for $J > 1/4$ and that neutral modes are proportional to just one of the associated Frobenius series near the critical level. The continuation of the linear eigenfunctions in the neighbourhood of the critical levels was considered by Koppel (1964).

Several linear results for the alternative Holmboe model were computed by Hazel (1972), for a variety of boundary conditions; however, our interest herein is in the nonlinear critical–layer theory and not with a discussion about which model is superior. Around the same time, nonlinear studies started to appear eg. Maslowe (1972) using the equilibrium critical layer theory of Benney & Bergeron (1969). Subsequent nonlinear studies were also mainly for the marginal instability case ($0 < 1/4 - J \ll 1$) eg. by Maslowe (1977) who considered a viscous critical layer, and by Brown & Stewartson (1978) who considered a time–dependent critical layer for small time. These two theories were unified by Brown, Rosen & Maslowe (1981). The latter paper was corrected to some extent by Churilov & Shukhman (1987); soon after, Churilov & Shukhman (1988) essentially extended and corrected the work of Brown & Stewartson (1978)

and Churilov & Shukhman (1987). Recently, Troitskaya (1991) has considered nonlinear effects for $J > 1/4$.

There have been many more studies of (non–stratified) shear layers, in which there is no vertical density variation. The linear stability of such flows is, in general, governed by the familiar Rayleigh equation, to which the Taylor–Goldstein equation reduces for zero Richardson number. The nonlinear stability of such flows has received much attention over the past couple of decades. Benney & Bergeron (1969) developed the so–called equilibrium critical layer theory; here the mode is treated as 'quasi–steady' inside the critical layer as well as outside it. Nonlinearity affects the jump imposed across the critical layer and hence leads to modified results for the neutral (equilibrium) modes. Haberman (1972) extended the theory to include critical layers where viscosity is also significant. The first studies of non–equilibrium critical layers include the papers by Stewartson (1978), Warn & Warn (1978), Hickernell (1984) and Gajjar & Smith (1985). The key paper by Hickernell (1984) concerned a shear layer affected by Coriolis (rotational) effects; here the weakly nonlinear theory leads to an integro–differential equation rather than the (previously) more familiar Landau equation with 'polynomial' nonlinear terms. In fact, such integro–differential equations result naturally from non–equilibrium nonlinear critical layer theories when the shear layer is coupled with other physical factors such as, for instance, Coriolis effects (eg. Hickernell, 1984; Shukhman, 1991); compressibility effects (eg. Goldstein & Leib, 1989); three–dimensionality effects (eg. Goldstein & Choi, 1989; Wu et al, 1993); and buoyancy effects (eg. Churilov & Shukhman, 1988). Moreover, in a related paper (Blackaby, Dando & Hall, 1993), we demonstrate that such an integro–differential type of equation can be derived to describe the nonlinear evolution of the inviscid Görtler modes studied by Bassom & Hall (1991). However, the case of a 'simple' shear layer, not affected by any additional physical factors, is a special case in the sense that it does not lead to an integro–differential equation; instead, Goldstein & Leib (1988), found that the nonlinear evolution of a disturbance was governed by the full unsteady nonlinear critical–layer equations. This difference is due to the additional physical factors, of the former cases, resulting in stronger singularities of the inviscid disturbance quantities at the critical level.

At first sight, it appears that weakly nonlinear theories can only be usefully applied to marginally unstable flows; they rely on small growth rates and so the unstable disturbance of concern must be near to a neutral state. Thus, it was believed that such theories are of no use in describing the initial evolution of 'far–from–neutral' unstable modes. However, several recent studies have derived integro–differential equations, using weakly nonlinear theories, to describe the nonlinear evolution of (general) unstable modes on a variety of shear layers (see the previous paragraph). These studies are based on the assumption that, in actual physical flow situations, shear layer spreading or other external changes/effects would result in the otherwise relatively unstable modes having their growth rates diminished in real terms, so that a weakly–nonlinear critical–layer theory becomes appropriate; in fact, we adopt the same argument for this present study. The general proposal/argument is supported by the findings of Michalke (1964), Crighton & Gaster (1976) and Hultgren (1992).

In this study, we use the unsteady critical–layer and weakly–nonlinear theories to describe the nonlinear development and evolution of unstable linear disturbance modes on stratified shear flows where the Richardson number J takes values less than one quarter. In particular,

we consider the temporal evolution of two–dimensional (2D) modes on Drazin's (1958) model flow; we note however that the analysis can easily be modified for the spatial evolution case and/or a different choice of model flow. In addition to not complicating the analysis, the choice of considering only 2D disturbances is justified by noting (i) the model flow is also only 2D, and (ii), that Squire's theorem holds for such flows where Boussinesq's approximation is used (see Koppel, 1964). Perhaps the main assumption made, in this first such study for $J < 1/4$, is that the Reynolds number of the flow is large enough such that the critical layer is not significantly affected by viscous effects to the orders considered. Such an assumption appears justified based on our results.

Although this paper is solely concerned with the nonlinear evolution of (non–marginally) unstable stratified shear flow which has important geophysical applications, the actual initial motivation for the study was the authors' desire to develop a theory to describe the nonlinear evolution of the inviscid Görtler modes considered initially by Bassom & Hall (1991), and lately by Dando (1992), Blackaby & Choudhari (1993). The first two of the above papers demonstrate that in the presence of a relatively weak cross flow, longitudinal vortex disturbances of all wavelengths are stabilised such that the inviscid modes possess some of the largest growth rates whilst also being neutral at certain other wavenumbers. Their governing equation is similar to the Taylor–Goldstein equation which, as mentioned earlier, governs the linear stability of stratified shear flows. In fact Blackaby & Choudhari (1993) have illustrated the close connection between the two problems and propose a definition of a generalised Richardson number for such centrifically–driven instabilities. The ideas developed here in this paper also have obvious applications to inviscid modes in flow above a heated plate, similar to those considered by Hall & Morris (1992).

Whilst different from the approach adopted in this study, there are alternate/complementary nonlinear theories that have been developed recently in which two or more of the flow disturbances mutually interact. Such theories generally require smaller disturbance amplitudes but may also need the disturbances to exist in specific 'configurations'. These other theories are generally referred to as wave/wave and vortex/wave interactions. For a discussion of wave/wave interactions and resonant–triads the reader is directed to the book by Craik (1985) and the papers by Goldstein & Lee (1992), Wu (1992). Strongly nonlinear vortex/wave interactions were first looked at by Hall & Smith (1991) and their ideas were clarified and extended by Brown, Brown, Smith & Timoshin (1993). Dando (1993) has looked at this type of interaction in a heated boundary layer where both streamwise vortices and inviscid travelling waves are present.

The format of the rest of the paper is as follows. In the next section we present some background details of the flow concerning us in this paper, namely the stratified shear flow model due to Drazin (1958), as well as introducing some notation and concepts ready for the proceeding sections. In §3 the flow outside the critical layer is considered; whilst in §4 we consider the flow inside the critical layer. In §5 we discuss the range of validity and solution properties of the evolution equations. Finally, in §6 we draw some conclusions.

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2 The stratified shear flow model

Following Churilov & Shukhman (1987,88) we choose to consider the simple but realistic stratified shear flow model of Drazin (1958), where the unperturbed velocity in the x direction, u_0, and the density, ρ_0, are dependent on the vertical coordinate y (in fact $u_0 = \tanh y$ and $\rho_0 = \rho_{0c} - y$). The governing equations of motion in the Boussinesq approximation can then be written in the form

$$\frac{\partial}{\partial t} \Delta \psi - J \frac{\partial \rho}{\partial x} + \{\Delta \psi, \psi\} = \eta \kappa \Delta^2 \psi,$$

$$\frac{\partial \rho}{\partial t} + \{\rho, \psi\} = \kappa \Delta \rho, \tag{2.1a, b}$$

where $\{a, b\} \equiv (\partial a/\partial x)(\partial b/\partial y) - (\partial a/\partial y)(\partial b/\partial x)$; ψ is the stream function; ρ is the density; η the Prandtl number; κ the thermometric conductivity and J the Richardson number.

The shape of the stream function of a two–dimensional disturbance mode of the inviscid linear problem, denoted $\psi_1^{(1)}$ say, is found to satisfy a Taylor–Goldstein equation of the form

$$L_1 \psi_1^{(1)} = 0, \tag{2.2}$$

where the operators L_l have the form

$$L_l \equiv \frac{\partial^2}{\partial y^2} - \left((lk)^2 - \frac{2 \sinh y \operatorname{sech}^3 y}{(\tanh y - c)} - \frac{J}{(\tanh y - c)^2} \right). \tag{2.3}$$

Here k and c are respectively the wavenumber and wavespeed of the infinitesimal disturbance, whilst the index l corresponds to the harmonic being considered (see later). Note that the L_l are singular where $\tanh y = c$; at such locations, usually referred to as critical levels, the assumptions employed in the linear inviscid approach are no longer valid and special attention must be paid. Thin regions around such levels, usually referred to as critical layers, must be introduced into the mathematical model; such layers are generally the first to 'feel' the effects of increasing disturbance amplitude.

Drazin (1958) found neutral eigensolutions of the form

$$\psi_1^{(1)} = B_\pm \psi_a(y), \quad \psi_a(y) = \sinh^{1-k^2} |y| \operatorname{sech} y,$$

$$\text{with} \quad c = 0 \quad \text{and} \quad J = k^2(1 - k^2), \tag{2.4a - d}$$

satisfying (2.2) coupled with the boundary conditions $\psi_1^{(1)} \to 0$ as $y \to \pm\infty$. Here the $\pm$ on B are related to vertical position with respect to the critical level at $y = 0$ i.e. $y > 0$ and $y < 0$ respectively. The relationship between the amplitudes B_+ and B_- will follow from matching to the inner problem (cf. Miles, 1961). In fact, in §4 we shall find that $B_- = i^{-1-2\nu} B$ where we have written $B \equiv B_+$ and the Frobenius root ν is defined below.

For later convenience, we introduce the quantity

$$\nu = \pm \frac{1}{2} \sqrt{1 - 4J}; \tag{2.5}$$

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note that, unlike Miles (1961), we choose to include the "$\pm$" in our actual definition of ν — this is also done for convenience. It easily follows from (2.4,5) that for Drazin's model flow $\nu = 1/2 - k^2$ and thus the appropriate root is immediately determined. However, for general flows, an analytical solution is not possible and the relevant choice of sign for given k, c and J can only be determined by inspecting the shape of the eigenfunction obtained from a numerical solution; however, there does appear to be a common pattern between location on neutral curve and which sign in (2.5) is appropriate (see Blackaby & Choudhari, 1993). Note that 'half' of Drazin's neutral curve corresponds to the positive choice of sign and the other 'half' to the negative choice (see Figure 2.1); moreover, note that Drazin's solution is also valid for $J < 0$ which corresponds physically to buoyancy driven flows. For $J < 0$, it appears that the "$-$" option is always relevant; thus the eigenfunction also is singular at the critical level.

A study of the fundamental and other low harmonics outside and inside the critical layer is necessary to derive the desired amplitude equations; in the next section we consider the flow outside of the critical layer. Here the details are dependent on the flow model under consideration, but the method is quite general and can be applied to other flows. In §4 we shall see that the critical–layer analysis is almost entirely independent of the stratified shear flow being considered.

3   Outside the critical layer

We consider a wave of sufficiently small amplitude in the neighbourhood of a *general* point (J, k) on the neutral curve of Drazin (1954) (see Figure 2.1). Note that, since we are not considering the initial evolution of a mode on a marginally unstable flow, we are not restricted to the sole case $J \simeq 1/4$. In fact, we begin by allowing J to take any value $< 1/4$ (i.e. we choose to exclude the special case $J = 1/4$), and then see which values of $\nu \equiv \nu(J, k)$ require special attention later. Note that $J \neq 1/4$ for all but one point on each of Drazin's and Hazels' (1972) neutral curves; moreover, the generalised Richardson number is always negative for neutral inviscid longitudinal vortices in 3D boundary layers. In fact, the theory of this and the next section can be regarded as the extension of the work of Churilov & Shukhman (1988) to the case $\nu \neq 0$, i.e. to a far wider range of problems. At the outset, we also choose to exclude the special case $J = 0$, which corresponds to a flow with no stratification (this case is considered by Goldstein & Leib, 1988).

Following Churilov & Shukhman (1988), we introduce the small parameter ϵ, characterising the magnitude of the mode. It is also necessary to introduce the 'slow' evolutionary time $\tau = \mu t$, with μ also small i.e. the amplitudes $B_\pm$ appearing in (2.4a) are considered to be functions of τ. Note that although the time–scale τ is slow with respect to the inviscid timescale t, it is still much 'faster' than the timescale $t \sim \kappa^{-1}$ of the viscous spreading of the base shear flow, provided

$$\kappa \ll \mu \ (\ll 1);$$

this restriction is not severe due to the large size of the Reynolds number $Re \sim \kappa^{-1}$ for flows of practical interest. Later we shall have to relate the small parameters κ, ϵ and μ to one another

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so that we can balance the linear terms of the amplitude equation with possible nonlinear ones.
We write

$$J = J_0 + \mu J_1, \tag{3.1}$$

where J_0 corresponds to a general point on the neutral curve and $J_1 < 0$ is order one. The streamfunction and density are expanded as Fourier series in x,

$$\psi = \ln(\cosh y) + \sum_{l=-\infty}^{\infty} \psi_l(\tau, y)e^{iklx}, \quad \rho = \rho_{0c} - y + \sum_{l=-\infty}^{\infty} \rho_l(\tau, y)e^{iklx}, \tag{3.2a,b}$$

with boundary conditions

$$\psi_l, \rho_l \to 0 \quad \text{as} \quad y \to \pm\infty. \tag{3.3}$$

In the present study, we need to consider the fundamental, the zeroth harmonic, the second harmonic and the third harmonic. Equations (2.1a,b) are solved both outside and inside the critical layer and then these solutions are matched to obtain the evolution equations. Outside the critical layer the fundamental harmonic dominates the perturbation and has an amplitude of order ϵ. The other harmonics are the result of self–interactions, with the zeroth and second having amplitudes of order ϵ^2 and the third of order ϵ^3.

3.1   The fundamental harmonic

In order to derive the evolution equation, the following terms of the expansion

$$\psi_1 = \epsilon\psi_1^{(1)} + \epsilon\mu\psi_1^{(2)} + \epsilon\kappa\psi_1^{(3)} + \cdots, \tag{3.4}$$

need to be considered. Here $\psi_1^{(1)}$ is the neutral mode of the inviscid linear problem; $\psi_1^{(2)}$ takes into account the τ-dependence of the solution; and $\psi_1^{(3)}$ is a correction to $\psi_1^{(1)}$ for dissipative (viscous) effects. The corresponding ρ_1 expansion includes analogous terms. Thus

$$\psi_1^{(1)} = B_\pm(\tau)\psi_a(y) \quad \text{where} \quad \psi_a(y) = \sinh^{\frac{1}{2}+\nu}|y|\,\text{sech}\,y; \tag{3.5a,b}$$

we note that

$$\psi_1^{(1)} \sim B_\pm(\tau)|y|^{\frac{1}{2}+\nu}, \quad \text{as} \quad y \to 0. \tag{3.6}$$

The second term in the expansion (3.4) satisfies

$$\text{L}_1\psi_1^{(2)} = Q_1, \qquad \psi_1^{(2)} \to 0 \quad \text{as} \quad y \to \pm\infty, \tag{3.7a}$$

where

$$Q_1 = -\frac{J_1 B_\pm \psi_a}{\tanh^2 y} - \frac{2i\psi_a}{k\tanh y}\frac{\partial B_\pm}{\partial \tau}\left(\frac{1}{\cosh^2 y} + \frac{J_0}{\tanh^2 y}\right). \tag{3.7b}$$

The solution can be considered to be the sum, $\psi_1^{(2)} = \psi_{1PI}^{(2)} + \psi_{1CF}^{(2)}$, of a particular integral of (3.7), $\psi_{1PI}^{(2)}$ say, and the complementary function, $\psi_{1CF}^{(2)}$ say. As $y \to 0$, it follows from Taylor (1931) that

$$\psi_{1CF}^{(2)} = B_\pm a_{1\pm}^{(2)}|y|^{\frac{1}{2}+\nu}\left(1 + O(|y|^{-1})\right) + B_\pm b_{1\pm}^{(2)}|y|^{\frac{1}{2}-\nu}\left(1 + O(|y|^{-1})\right), \tag{3.8}$$

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where $a_{1\pm}^{(2)}$ and $b_{1\pm}^{(2)}$ are constants as yet undetermined. Note that if the Frobenius roots $\frac{1}{2} \pm |\nu|$ differ by an integer then (3.8) is no longer appropriate (logarithms are needed). As such cases ($\nu = \frac{1}{2}m$; m integer) are isolated, we choose not to concern ourselves with them (and their immediate neighbourhood) in this paper.

A solvability condition for the above boundary–value problem (3.7) is required. Note that: (i) the operator L_1 is self–adjoint away from the critical level $y = 0$, and (ii) the right–hand side is singular at $y = 0$. Rather than closely follow the method of Churilov & Shukhman (1987) for deriving the *modified* solvability condition, for $J_0 \neq 1/4$ it is more convenient to adopt the related approach employed by, for instance, Hickernell (1984).

The solvability condition is derived by multiplying both sides of equation (3.7a) by ψ_a and integrating over all y, *excluding* the (sole) critical layer at $y = 0$. After integrating by parts; imposing the boundary conditions at $y = \pm\infty$; and the asymptotic forms of ψ_a and $\psi_{1CF}^{(2)}$ as $y \to 0$, we find that

$$\fint_{-\infty}^{\infty} \psi_a Q_1 \, dy = - \left[\psi_{1CF}^{(2)} \psi_a' - \psi_{1CF}^{(2)\prime} \psi_a \right]_{0-}^{0+} \equiv -\frac{2\nu}{B} \left(B_+^2 b_{1+}^{(2)} + B_-^2 b_{1-}^{(2)} \right), \qquad (3.9a)$$

where the bar through the integral sign indicates that the finite part of the integral should be taken. After substituting for Q_1 and using the relations $B_- = i^{-1-2\nu}B$, $B_+ \equiv B$ (to be derived in the next section), the solvability condition becomes

$$-\frac{4i^{1+2\nu}I_1}{k} \cos(\pi\nu) \left[\frac{\partial B}{\partial \tau} + \frac{J_1 I_2 k}{2I_1} \tan(\pi\nu) B \right] = -2\nu B (b_{1+}^{(2)} - i^{-4\nu} b_{1-}^{(2)}), \qquad (3.9b)$$

where

$$I_1 = \fint_0^\infty \frac{\sinh^{2\nu} y}{\cosh y} \left(\frac{1}{\cosh^2 y} + \frac{J_0}{\tanh^2 y} \right) dy \quad \text{and} \quad I_2 = \fint_0^\infty \sinh^{2\nu-1} y \, dy. \qquad (3.9c,d)$$

Four relations involving $b_{1+}^{(2)}$ and $b_{1-}^{(2)}$ are determined from the inner problem considered in the next section, thus determining the possible evolution equations for the wave amplitude $B(\tau)$.

3.2 The zeroth, second and third harmonics

The presence of these terms is due to the process of harmonic generation, i.e. due to nonlinearity the fundamental of $O(\epsilon)$ generates the zeroth and second of $O(\epsilon^2)$ and so on. The zeroth and second harmonics are expanded in the form

$$\psi_0 = \epsilon^2 \psi_0^{(1)} + \epsilon^2 \mu \psi_0^{(2)} + \epsilon^2 \kappa \psi_0^{(3)} + \cdots, \qquad (3.10)$$

$$\psi_2 = \epsilon^2 \psi_2^{(1)} + \epsilon^2 \mu \psi_2^{(2)} + \epsilon^2 \kappa \psi_2^{(3)} + \cdots, \qquad (3.11)$$

and similarly for ρ_0 and ρ_2. It is only necessary to consider the first term of the third harmonic, namely

$$\psi_3 = \epsilon^3 \psi_3^{(1)} + \cdots, \qquad (3.12)$$

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and the similarly for ρ_3. The resulting sets of equations from expansions (3.10-12) do not have simple exact solutions when $J_0 \neq 1/4$; however it is, in general, a relatively simple process, if a little tedious, to deduce their asymptotic forms as the critical level is approached. The significant terms of these asymptotes are quoted in the next subsection.

However, the first term of the second harmonic requires extra attention as it can lead to the largest nonlinearity in certain circumstances when ν is positive (see §4). The necessary analysis is simply the generalisation of that for the case $\nu = 0$ ($J = 1/4$) presented by Churilov & Shukhman (1987) in their unpublished Appendix B. Since this important work is unfortunately not readily available for reference, we take the liberty of presenting, in Appendix A of this paper, more details of their method than we would otherwise.

3.3 The asymptotic expansions as $y \to 0$

In terms of the new variable $Y = \mu^{-1}y$ and the functions

$$\phi_l = \psi_l - (1 + 2\nu)p_l, \tag{3.13}$$

where $\rho_l = -2\partial p_l/\partial y$, the asymptotes for the fundamental, second, zeroth and third harmonics as $y \to 0$ are

$$\psi_1 = \epsilon\mu^{\frac{1}{2}+\nu}\left[B_\pm|Y|^{\frac{1}{2}+\nu} - \frac{i(1+2\nu)}{2k}\frac{\partial B_\pm}{\partial\tau}\frac{|Y|^{\frac{1}{2}+\nu}}{Y}\right] + \epsilon\mu^{\frac{3}{2}-\nu}b_{1\pm}^{(2)}B_\pm|Y|^{\frac{1}{2}-\nu}$$

$$+ \epsilon\mu^{\frac{3}{2}+\nu}\left[a_{1\pm}^{(2)}B_\pm - \frac{J_1 B_\pm}{2\nu}\ln|\mu Y|\right]|Y|^{\frac{1}{2}+\nu}$$

$$+ \epsilon\kappa\mu^{-\frac{5}{2}+\nu}\frac{i(1-4\nu^2)}{48k}\{\eta(5-2\nu) + (1-2\nu)\}B_\pm\frac{|Y|^{\frac{1}{2}+\nu}}{Y^3} + \cdots$$

$$+ d_\pm|Y|^{\frac{1}{2}+\nu} + \cdots,$$

$$\phi_1 = \epsilon\mu^{\frac{1}{2}+\nu}.0 + \epsilon\mu^{\frac{3}{2}-\nu}\left[\frac{-4\nu B_\pm b_{1\pm}^{(2)}}{(1-2\nu)}|Y|^{\frac{1}{2}-\nu}\right] + \epsilon\mu^{\frac{3}{2}+\nu}\left[\frac{-J_1 B_\pm}{\nu(1+2\nu)}|Y|^{\frac{1}{2}+\nu}\right]$$

$$+ \epsilon\kappa\mu^{-\frac{5}{2}+\nu}\left[\frac{i(\eta-1)(1-4\nu^2)B_\pm}{8k}\frac{|Y|^{\frac{1}{2}+\nu}}{Y^3}\right] + \cdots, \tag{3.14a, b}$$

$$\psi_2 = \epsilon^2\mu^{-1+2\nu}\left[\frac{(1+2\nu)B_\pm^2}{2(3-2\nu)}\frac{|Y|^{1+2\nu}}{Y^2}\right] + \cdots + \mu^{\frac{1}{2}-\nu}c_\pm\left(|Y|^{\frac{1}{2}-\nu} + \mu^{2\nu}q_\pm|Y|^{\frac{1}{2}+\nu}\right) + \cdots$$

$$+ \epsilon^2\mu^{\frac{1}{2}+\nu}B_\pm^2 g_{2\pm}|Y|^{\frac{1}{2}+\nu} + \epsilon^2\mu^{\frac{1}{2}-\nu}B_\pm^2 h_{2\pm}|Y|^{\frac{1}{2}-\nu} + \cdots,$$

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$$\phi_2 = \epsilon^2 \mu^{-1+2\nu}.0 \quad - \epsilon^2 \mu^{2\nu} \frac{J_1 B_\pm^2}{2\nu(3-2\nu)} \frac{|Y|^{1+2\nu}}{Y^2}$$

$$- \epsilon^2 \kappa \mu^{-4+2\nu} \frac{i(\eta-1)(1-4\nu^2)}{4k} B_\pm^2 \frac{|Y|^{1+2\nu}}{Y^5} + \cdots - \mu^{\frac{1}{2}-\nu} \frac{4\nu c_\pm}{(1-2\nu)} |Y|^{\frac{1}{2}-\nu}$$

$$- \epsilon^2 \mu^{\frac{1}{2}+\nu} B_\pm^2 h_{2\pm} \frac{4\nu}{(1-2\nu)} |Y|^{\frac{1}{2}-\nu} + \cdots, \qquad (3.15a,b)$$

$$\frac{\partial \psi_0}{\partial \tau} = \epsilon^2 \mu^{-1+2\nu} \left[\frac{(1+2\nu)}{2} \frac{\partial |B_\pm|^2}{\partial \tau} \frac{|Y|^{1+2\nu}}{Y^2} \right] + \cdots,$$

$$\frac{\partial \phi_0}{\partial \tau} = \epsilon^2 \mu^{-1+2\nu}.0 - \epsilon^2 \mu^{2\nu} \frac{J_1}{2\nu} \frac{\partial |B_\pm|^2}{\partial \tau} \frac{|Y|^{1+2\nu}}{Y^2}$$

$$+ \epsilon^2 \kappa \mu^{-4+2\nu} \left[\frac{3}{8}(1-4\nu^2)(1-2\nu)(\eta-1)|B_\pm|^2 \frac{|Y|^{1+2\nu}}{Y^4} \right] + \cdots, \qquad (3.16a,b)$$

$$\psi_3 = -\epsilon^3 \mu^{-\frac{5}{2}+3\nu} \frac{(1+2\nu)(1-2\nu)}{2(3+2\nu)^2} \frac{B_\pm^3}{|Y|^{\frac{5}{2}-3\nu}} + \cdots ;$$

$$\phi_3 = \epsilon^3 \mu^{-\frac{5}{2}+3\nu}.0 + \cdots , \qquad (3.17a,b)$$

Note that above, for the sake of brevity, we have only included the more important and illustrative terms with regard to our aim of deriving amplitude-equations for B. It should be noted that the above asymptotes, in general, only contain the leading term (for $Y \to \infty$) at each order. Further, it should also be noted that these (incomplete) asymptotes may, depending on the relative sizes of ϵ, κ and μ, be disordered as written; it is sensible to postpone a discussion concerning the relative sizes of these quantities until §5, when the sizes of the competing nonlinearities are known. Here $d_\pm, c_\pm$ and $q_\pm$ are constants, as introduced by Churilov & Shukhman (1988); the first two are in fact expansions, each term of an order yet to be determined. The constants $g_{2\pm}$ and $h_{2\pm}$ are defined in Appendix A of this paper.

It is worthwhile to consider these asymptotes a little further. We see that the leading terms in the ϕ–expansions are all zero; in fact, such behavior is in full agreement with the theorem of Miles (1961). In the next section, we shall see that this fact results in the 'biggest' cubic-nonlinearity not contributing to the amplitude equation; the same result is also found when $J = 1/4$ ($\nu = 0$) and is responsible for the quintic nonlinearities occurring in the evolution equations

derived by Brown & Stewartson (1978), Churilov & Shukhman (1988) and in this paper. As pointed out by these previous authors, for a 'contributing' jump, it is in general necessary for the ϕ–asymptotes to be non–zero. They note that this condition is satisfied if the complementary-function terms in the ϕ_2 asymptote are non–zero (thus resulting in a 'contributing' quintic nonlinearity). Churilov & Shukhman (1988) also note that if viscosity effects are included in in the original formulation of the problem, then the above condition is satisfied as long as the Prandtl number $\eta \neq 1$ (thus resulting in a 'contributing' cubic nonlinearity). In addition to these two so–called symmetry breaking mechanisms, it is clearly obvious from the third term of (3.14b) that we have an *additional* mechanism for non–zero ν, namely non–zero J_1. This novel mechanism is made possible by the two Frobenius roots, $1/2 \pm |\nu|$, *not* differing by an integer. As far as the authors are aware, such a nonlinear, non–equilibrium critical–layer has not been the subject of any previous studies. This 'new' mechanism is essentially responsible for the 'non–viscous' cubic jump term in our later amplitude equation when ν is negative (later referred to as the 'J_1–cubic' term).

The fourth symmetry–breaking mechanism possible here corresponds to that cleverly identified by Churilov & Shukhman (1987) in the unpublished Appendix B of their paper. Ironically, there is no mention of it in the sequel paper (Churilov & Shukhman, 1988); however, it is of lesser importance for disturbances of marginally unstable flows (see later for an extended discussion on where each of the four possible nonlinearities in the evolution equation for B are the most significant). The important thing to note at this stage is that the term involving $h_{2\pm}$ in the ϕ_2 asymptote is not at one of the orders to be considered, in §4, for the preceding term involving $c_{2\pm}$. In the next section, we shall see that this symmetry–breaking mechanism provides the 'non–viscous' cubic jump when ν is positive (later referred to as the outer–complementary-function [OCF] cubic term).

4 The Critical Layer

As usual in such nonlinear studies, the main purpose of this section is to calculate the second relations between $b_{1+}^{(2)}$ and $b_{1-}^{(2)}$ (the first being given by the solvability condition, (3.9b)) and thereby obtain the desired nonlinear evolution equation(s) for the disturbance amplitude $B(\tau)$. Following Brown & Stewartson (1978) and Churilov & Shukhman (1988), we define new functions Ψ, P and Φ where

$$\psi = \frac{1}{2}\mu^2 Y^2 + \Psi, \qquad \rho = \rho_0 - \mu Y - 2\mu^{-1}P_Y, \qquad \Phi = \Psi - (1 + 2\nu)P. \qquad (4.1a-c)$$

Inserting these into the governing equations (2.1), (2.2) gives

$$N_{(\frac{1}{2}+\nu)}\Psi = \frac{1}{2}(1 - 2\nu)\Phi_x - \mu^{-2}\{\Psi_Y, \Psi\}^* - \frac{2\mu J_1}{(1 + 2\nu)}(\Psi_x - \Phi_x) + \eta\kappa\mu^{-3}\Psi_{YYY}, \qquad (4.2a)$$

$$N_{(\frac{1}{2}-\nu)}\Phi = -\mu^{-2}\{\Phi_Y, \Psi\}^* - \frac{2\mu J_1}{(1 + 2\nu)}(\Psi_x - \Phi_x) + \kappa\mu^{-3}\Phi_{YYY} + (\eta - 1)\kappa\mu^{-3}\Psi_{YYY}, \qquad (4.2b)$$

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where $\{a,b\}^* = a_x b_Y - a_Y b_x$, the operator

$$N_\chi \equiv \left(\frac{\partial}{\partial \tau} + Y \frac{\partial}{\partial x} \right) \frac{\partial}{\partial Y} - \chi \frac{\partial}{\partial x}, \qquad (4.3)$$

and it is assumed that $\kappa \mu^{-3} \ll 1$ i.e. viscous effects are not large enough to affect the operator N_χ at leading order.

In the rest of this section we proceed to solve equations (4.2) for the relevant lower order terms of the lower harmonics. In fact, we need to consider the cases $\nu > 0$ and $\nu < 0$ separately; this is because the largest 'non–viscous–in–origin' cubic term in the corresponding evolution equations is different in these cases. When $\nu < 0$, this term is proportional to J_1; but when $\nu > 0$, this term is related to the complementary function of the second harmonic of the inviscid, outer problem. Note that when $\nu > 0$, the second and third terms of the ψ_1 asymptote (3.14a) are ordered and thus the cubic nonlinearity proportional to J_1 would be insignificant compared to that proportional to $b_{1\pm}^{(2)}$. Note that the other two significant nonlinearities (a cubic, due to viscosity, and a quintic, due to complementary function terms of the critical layer solution for the second harmonic) are possible in either case.

(i) The case $\nu < 0$

The solution is again constructed in the form of a Fourier series in x; we expand the fundamental, zeroth, second and third harmonics, respectively, as follows

$$\Psi_1 = \epsilon \mu^{\frac{1}{2}+\nu} \Psi_1^{(1)} + \cdots + \epsilon^3 \mu^{-\frac{5}{2}+3\nu} \Psi_1^{(2)} + \cdots + \epsilon \mu^{\frac{3}{2}+\nu} \Psi_1^{(3a)} + \cdots + \epsilon \mu^{\frac{3}{2}-\nu} \Psi_1^{(3b)} + \cdots$$

$$+ \epsilon^3 \mu^{-\frac{3}{2}+3\nu} \Psi_1^{(4)} + \cdots + \epsilon \kappa \mu^{-\frac{5}{2}+\nu} \Psi_1^{(5)} + \cdots + \epsilon^3 \kappa \mu^{-\frac{11}{2}+3\nu} \Psi_1^{(6)} + \cdots + \epsilon^5 \mu^{-\frac{11}{2}+5\nu} \Psi_1^{(7)} + \cdots,$$

$$\Psi_0 = \epsilon^2 \mu^{-1+2\nu} \Psi_0^{(1)} + \cdots + \epsilon^2 \mu^{2\nu} \Psi_0^{(2)} + \cdots + \epsilon^2 \kappa \mu^{-4+2\nu} \Psi_0^{(3)} + \cdots + \epsilon^4 \mu^{-4+4\nu} \Psi_0^{(7)} + \cdots,$$

$$\Psi_2 = \epsilon^2 \mu^{-1+2\nu} \Psi_2^{(1)} + \cdots + \epsilon^2 \mu^{2\nu} \Psi_2^{(2)} + \cdots + \epsilon^2 \kappa \mu^{-4+2\nu} \Psi_2^{(3)} + \cdots \epsilon^4 \mu^{-4+4\nu} \Psi_2^{(7)} + \cdots,$$

$$\Psi_3 = \epsilon^3 \mu^{-\frac{5}{2}+3\nu} \Psi_3^{(1)} + \cdots,$$

$$(4.4a - d)$$

and similarly for the Φ_i's. These expansions are not necessarily completely ordered (depending on the sizes of ϵ, κ and μ); moreover, only those terms crucial to deriving the evolution equations have been included. The scalings follow directly from the outer asymptotes (3.14-17) and/or by considering the process of harmonic generation.

4.1 $O(\epsilon\mu^{\frac{1}{2}+\nu})$ of the fundamental

Upon writing

$$N_{l,\chi} \equiv \left(\frac{\partial}{\partial\tau} + iklY\right)\frac{\partial}{\partial Y} - ikl\chi \; , \tag{4.5}$$

we find that at this order equations (4.2a,b) give

$$N_{1,(\frac{1}{2}+\nu)}\Psi_1^{(1)} = \frac{1}{2}(1-2\nu)ik\Phi_1^{(1)}, \quad N_{1,(\frac{1}{2}-\nu)}\Phi_1^{(1)} = 0 \; . \tag{4.6a, b}$$

A solution of these equations which matches to the outer solution (3.5) is

$$\Psi_1^{(1)} \equiv W(\tau, Y) = \frac{(1+2\nu)\Gamma(\frac{1}{2}+\nu)i^{\frac{3}{2}-3\nu}}{4\pi k^{\frac{1}{2}+\nu}} \int_C t^{-\frac{3}{2}-\nu}B(\tau-t)e^{-ikYt}dt, \tag{4.7a}$$

$$\Phi_1^{(1)} = 0, \tag{4.7b}$$

where the contour C is shown in Figure 4.1a. The function $W(\tau, Y)$ has a single asymptotic representation in the lower half-plane $(-\pi \le \arg Y \le 0)$

$$W(\tau, Y) = B(\tau)Y^{\frac{1}{2}+\nu} + O(Y^{-\frac{1}{2}+\nu}), \quad \text{as} \quad |Y| \to \infty. \tag{4.8}$$

Later we derive evolution equations for the amplitude $B(\tau)$ but for the moment we can regard it as an arbitrary function that satisfies the requirement $B(\tau) \to 0$ as $\tau \to -\infty$. Matching with the 'outer' asymptote (3.14a) yields

$$B_- = i^{-1-2\nu}B_+ \qquad (B_+ \equiv B); \tag{4.9}$$

this result has already been referred to and used in the previous sections.

4.2 $O(\epsilon^2\mu^{-1+2\nu})$ of the zeroth harmonic

At this order equations (4.2a,b) yield

$$\frac{\partial}{\partial\tau}\Psi_0^{(1)'} = ik(W\overline{W}' - \overline{W}W')', \quad \frac{\partial}{\partial\tau}\Phi_0^{(1)'} = 0, \tag{4.10a, b}$$

where the overbar again denotes the complex conjugate and prime denotes differentiation with respect to Y. A solution of the above which matches to the outer solutions is

$$\Psi_0^{(1)} = \frac{2}{(1+2\nu)}|W'|^2, \quad \Phi_0^{(1)} = 0. \tag{4.11a, b}$$

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4.3 $O(\epsilon^2\mu^{-1+2\nu})$ of the second harmonic

At this order equations (4.2a,b) yield

$$N_{2,(\frac{1}{2}+\nu)}\Psi_2^{(1)} = ik(1-2\nu)\Phi_2^{(1)} + ik(WW'' - W'^2) \quad \text{and} \quad N_{2,(\frac{1}{2}-\nu)}\Phi_2^{(1)} = 0. \qquad (4.12a,b)$$

We note that, as the right-hand sides of (4.12a,b) do not involve $\overline{W}$, $\Psi_2^{(1)}$ and $\Phi_2^{(1)}$ have unique asymptotic representations as $|Y| \to \infty$ (in the lower half plane of complex Y) of the form

$$\Psi_2^{(1)} \sim \frac{(1+2\nu)}{2(3-2\nu)}B^2 Y^{-1+2\nu} + (m_2^{(1)}Y^{\frac{1}{2}-\nu} + n_2^{(1)}Y^{\frac{1}{2}+\nu}) + \cdots, \quad \Phi_2^{(1)} \sim -\frac{4\nu m_2^{(1)}}{(1-2\nu)}Y^{\frac{1}{2}-\nu} + \cdots.$$
$$(4.13a,b)$$

Matching with the asymptotes (3.15a,b) requires that $c_{\pm} = \epsilon^2 \mu^{-\frac{3}{2}+3\nu}c_{1\pm}$, so that $c_{1+} = m_2^{(1)}$, $c_{1-} = i^{-1+2\nu}m_2^{(1)}$, $\mu^{2\nu}c_{1+}q_+ = n_2^{(1)}$ and $\mu^{2\nu}c_{1-}q_- = i^{-1-2\nu}n_2^{(1)}$. These lead to the relation $c_{1-}(q_- - i^{-4\nu}q_+) = 0$. However, as the adopted model flow is symmetric, $q_+ = q_-$ and so, for $\nu \neq 0$, $c_{1-} = 0 = m_2^{(1)} = n_2^{(1)}$. Thus

$$\Phi_2^{(1)} \equiv 0 \qquad (4.14a)$$

and the solution to (4.12a) is

$$\Psi_2^{(1)} = -\frac{i^{1-6\nu}k^{1-2\nu}(1+2\nu)^2\Gamma^2(\frac{1}{2}+\nu)}{32\pi^2}\int_C dt_1 \int_C dt_2 \int_0^\infty dt B(\tau-t-t_1)\times$$

$$B(\tau-t-t_2)(t_1 t_2)^{-\frac{3}{2}-\nu}(t_1-t_2)^2(t_1+t_2)^{\frac{1}{2}+\nu}(2t+t_1+t_2)^{-\frac{3}{2}-\nu}e^{-ikY(2t+t_1+t_2)}. \quad (4.14b)$$

4.4 $O(\epsilon^3\mu^{-\frac{5}{2}+3\nu})$ of the fundamental

From (4.2a,b),

$$N_{1,(\frac{1}{2}+\nu)}\Psi_1^{(2)} = ik\frac{1}{2}(1-2\nu)\Phi_1^{(2)} + ik(\Psi_0^{(1)''}W - \Psi_0^{(1)'}W' + 2\Psi_2^{(1)}\overline{W}'' - \Psi_2^{(1)'}\overline{W}' - \Psi_2^{(1)''}\overline{W}),$$

$$N_{1,(\frac{1}{2}-\nu)}\Phi_1^{(2)} = 0. \qquad (4.15a,b)$$

Due to the earlier Φ terms being zero, the right-hand side of (4.15b) is also zero. Thus $e_1^{(2)}V$, where $e_1^{(2)}$ is a constant and V satisfies $N_{1,(\frac{1}{2}-\nu)}V = 0$ with the asymptotic representation $V \sim Y^{\frac{1}{2}-\nu}$ as $|Y| \to \infty$. Any non-zero choice of $e_1^{(2)}$ in (4.15a) results in a $Y^{\frac{1}{2}-\nu}$ term in the asymptotic representation of $\Psi_1^{(2)}$ and this contradicts the theorem of Miles (1961), from which we know that the expansion of ψ_a as $y \to 0$ cannot involve a term proportional to $y^{\frac{1}{2}-\nu}$. Thus it follows that $\Phi_1^{(2)} = 0$.

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We also note that the right–hand side of equation (4.15a) contains $\overline{W}$ and so $\Psi_1^{(2)}$ does not have a unique asymptotic representation as $|Y| \to \infty$. However, because the asymptotic representation of $\Psi_1^{(2)}$ does not contain a term proportional to $Y^{\frac{1}{2}-\nu}$, it is impossible to obtain non-zero $b_{1\pm}^{(2)}$ at this order. Thus the strongest nonlinearity does not contribute to the evolution equation.

The solution for $\Psi_1^{(2)}$ is needed for later calculations regarding the quintic nonlinearity; the solution of (4.15a) is

$$\Psi_1^{(2)} = \frac{i^{-\frac{1}{2}-7\nu} k^{\frac{5}{2}-3\nu}(1+2\nu)^2 \Gamma^3(\frac{1}{2}+\nu)}{32\pi^3} \int_0^\infty dt \int_C dt_1 \int_C dt_2 \int_C dt_3 \, e^{ikY(t_3-t-t_1-t_2)} \times$$

$$B(\tau-t-t_1)B(\tau-t-t_2)\overline{B}(\tau-t-t_3)(t_1 t_2 t_3)^{-\frac{3}{2}-\nu}(t_3-t_1-t_2)^{\frac{1}{2}+\nu}(t_3-t-t_1-t_2)^{-\frac{3}{2}-\nu} \times$$

$$\left(t_3 \left[t_1(t_1-t_3)^2 + t_2(t_2-t_3)^2 + t_1 t_2(2t_3-t_1-t_2) \right] - t_1^{2+2\nu} t_2(t_1+t_2)^{-\frac{3}{2}-\nu} \times \right.$$

$$\left. (t_1-t_2)^{\frac{1}{2}-\nu}(t_1+t_2+t_3)(2t_3-t_1-t_2)F\left(-\frac{1}{2}-\nu, -\frac{1}{4}-\frac{\nu}{2}; \frac{3}{4}-\frac{\nu}{2}; \frac{t_2^2}{t_1^2}\right) \right) H(t_1-t_2)$$

$$(4.16)$$

Here $H(x)$ is the so-called Heaviside function:

$$H(x) = \begin{cases} 1 : x > 0 \\ 0 : x < 0, \end{cases}$$

and $F(a, b; c; z)$ is the hypergeometric function (see Erdélyi, 1953; Abramowitz & Stegun, 1964).

4.5 $O(\epsilon \mu^{\frac{3}{2}+\nu})$ of the fundamental

A major difference of our problem for general ν, compared to the $\nu = 0$ case considered by Churilov & Shukhman (1988), is that there is no linear contribution to the evolution equation from the critical layer. This can be seen from the analysis of this and the next subsection.

At order $\epsilon \mu^{\frac{3}{2}+\nu}$ terms proportional to J_1 first enter; equations (4.2a,b) yield

$$N_{1,(\frac{1}{2}+\nu)}\Psi_1^{(3a)} = \frac{1}{2}(1-2\nu)ik\Phi_1^{(3a)} - \frac{2ikJ_1}{(1+2\nu)}W,$$

$$N_{1,(\frac{1}{2}-\nu)}\Phi_1^{(3a)} = -\frac{2ikJ_1}{(1+2\nu)}W, \qquad (4.17a,b)$$

which have solutions

$$\Psi_1^{(3a)} = -\frac{J_1}{2\nu}W \, \ln(\mu Y), \qquad \Phi_1^{(3a)} = -\frac{J_1 W}{\nu(1+2\nu)}. \qquad (4.18a,b)$$

Note that for definiteness, we choose to include the complementary function alongside the nonlinear term which is going to be balanced with at this order.

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4.6 $O(\epsilon\mu^{\frac{3}{2}-\nu})$ of the fundamental

Here we find that (4.2a,b) yield

$$N_{1,(\frac{1}{2}+\nu)}\Psi_1^{(3b)} = \frac{1}{2}(1-2\nu)ik\Phi_1^{(3b)}, \quad N_{1,(\frac{1}{2}-\nu)}\Phi_1^{(3b)} = 0. \qquad (4.19a,b)$$

These have solution

$$\Psi_1^{(3b)} = 0, \qquad \Phi_1^{(3b)} = 0, \qquad (4.20a,b)$$

where once again we shall include the associated complementary function alongside the non-linear term which is going to be balanced with at this order. The solutions (4.20) imply that the whole of the linear contribution to the evolution equation comes from outside of the critical layer; note that the logarithm occurs in (4.18a) and not in (4.20a) i.e. any jump induced by the presence of the logarithm does not occur at the desired order, $O(\epsilon\mu^{\frac{3}{2}-\nu})$, necessary to affect the relationship between $b_{1-}^{(2)}$ and $b_{1+}^{(2)}$. The authors have verified that this is in fact the case by computing 'near–neutral' linear growth rates from a numerical solution of the Taylor Goldstein equation (2.2) and comparing them with those predicted analytically by equating the right–hand side of expression (3.9b) to zero.

4.7 $O(\epsilon^2\mu^{2\nu})$ of the zeroth harmonic

At this order we only need to calculate $\Phi_0^{(2)}$; (4.2b) gives

$$\frac{\partial}{\partial\tau}\Phi_0^{(2)\prime} = -\frac{ikJ_1}{\nu(1+2\nu)}(W\overline{W}'-\overline{W}W')', \qquad (4.21)$$

with solution

$$\Phi_0^{(2)} = -\frac{2J_1}{\nu(1+2\nu)^2}|W'|^2 \equiv -\frac{J_1}{\nu(1+2\nu)}\Psi_0^{(1)}. \qquad (4.22)$$

4.8 $O(\epsilon^2\mu^{2\nu})$ of the second harmonic

Equations (4.2a,b) yield

$$N_{2,(\frac{1}{2}+\nu)}\Psi_2^{(2)} = ik(1-2\nu)\Phi_2^{(2)} + ik(W''\Psi_1^{(3a)} - 2W'\Psi_1^{(3a)\prime} + W\Psi_1^{(3a)\prime\prime}) - \frac{4ikJ_1}{(1+2\nu)}\Psi_2^{(1)},$$

$$N_{2,(\frac{1}{2}-\nu)}\Phi_2^{(2)} = -\frac{J_1 ik}{\nu(1+2\nu)}(WW'' - W'W') - \frac{4ikJ_1}{(1+2\nu)}\Psi_2^{(1)}. \qquad (4.23a,b)$$

For further calculations only $\Phi_2^{(2)}$ is needed; using the identity $N_{2,(\frac{1}{2}+\nu)} \equiv N_{2,(\frac{1}{2}-\nu)} - 4ik\nu$ and comparing the right–hand sides of (4.12a) and (4.23b) it is easy to deduce that

$$\Phi_2^{(2)} = -\frac{J_1}{\nu(1+2\nu)}\Psi_2^{(1)} \quad . \qquad (4.24)$$

4.9 $O(\epsilon^3 \mu^{-\frac{3}{2}+3\nu})$ of the fundamental

At this order the cubic-jump proportional to J_1 emerges. The governing equations yield in concise form

$$N_{1,(\frac{1}{2}+\nu)}\Psi_1^{(4)} = \frac{1}{2}(1-2\nu)ik\Phi_1^{(4)} - \frac{2ikJ_1}{(1+2\nu)}\Psi_1^{(2)} + R_1^{(4)},$$

$$N_{1,(\frac{1}{2}-\nu)}\Phi_1^{(4)} = -\frac{2ikJ_1}{(1+2\nu)}\Psi_1^{(2)} + R_2^{(4)}, \tag{4.25a, b}$$

where, in particular,

$$R_2^{(4)} = ik(\overline{\Phi}_1^{(3a)'}\Psi_2^{(1)'} + 2\overline{\Phi}_1^{(3a)''}\Psi_2^{(1)} - \Phi_1^{(3a)'}\Psi_0^{(1)'} + \Phi_0^{(2)''}W - \Phi_2^{(2)''}\overline{W} - 2\Phi_2^{(2)'}\overline{W}')$$

$$\equiv \frac{ikJ_1}{\nu(1+2\nu)}(-2\overline{W}''\Psi_2^{(1)} + W'\Psi_0^{(1)'} - W\Psi_0^{(1)''} + \Psi_2^{(1)''}\overline{W} + \Psi_2^{(1)'}\overline{W}') \ . \tag{4.26}$$

To derive an evolution equation the asymptotic representation of $\Psi_1^{(4)}$ is needed. This is

$$\Psi_1^{(4)} = C_\pm^{(4)}Y^{\frac{1}{2}+\nu} + D_\pm^{(4)}Y^{\frac{1}{2}-\nu} + O(Y^{-\frac{1}{2}+\nu}) \quad \text{as} \quad Y \to \pm\infty \ , \tag{4.27}$$

where, in particular,

$$D_+^{(4)} - D_-^{(4)} = -\frac{i^{\frac{1}{2}-\nu}k^{\frac{1}{2}-\nu}}{2\nu\Gamma(\frac{1}{2}-\nu)}\int_{-\infty}^{\infty}dY\int_0^{\infty}dt \ \ t^{\frac{1}{2}-\nu}R_2^{(4)}(\tau-t,Y)e^{-ikYt}. \tag{4.28}$$

Note that the term proportional to $\Psi_1^{(2)}$ in (4.25b) only contributes to the jump $C_+^{(4)} - C_-^{(4)}$. After substituting for $R_2^{(4)}$ and some calculation, we find that

$$D_+^{(4)} - D_-^{(4)} = \frac{i^{1-4\nu}k^{3-4\nu}J_1(1+2\nu)\Gamma^3(\frac{1}{2}+\nu)}{32\pi^2\nu^2\Gamma(\frac{1}{2}-\nu)}\int_C dt_1 \int_C dt_2 \int_C dt_3(t_1t_2t_3)^{-\frac{3}{2}-\nu}\times$$

$$\left(B(\tau+t_1-t_3)B(\tau+t_2-t_3)\overline{B}(\tau+t_1+t_2-2t_3)t_2t_3(t_3-t_2)(t_3+t_1-t_2)\times \right.$$

$$(t_3-t_1-t_2)^{\frac{1}{2}-\nu}H(t_3-t_1-t_2)$$

$$+\frac{(1+2\nu)}{4}\int_0^{\infty}dt B(\tau+t_1+t-t_3)B(\tau+t_2+t-t_3)\overline{B}(\tau+t_1+t_2+2t-2t_3)\times$$

$$(t_1-t_2)^2(t_1+t_2)^{\frac{1}{2}+\nu}(2t+t_1+t_2)^{-\frac{3}{2}-\nu}(2t_3-2t-t_1-t_2)\times$$

$$\left. (2t+t_1+t_2+t_3)(t_3-2t-t_1-t_2)^{\frac{1}{2}-\nu}H(t_3-2t-t_1-t_2) \right) \ . \tag{4.29}$$

Matching the 'inner' asymptote (4.27) with the 'outer' asymptote (3.14a) leads to the relations

$$D_+^{(4)} = b_{1+}^{(2)}B_+, \qquad D_-^{(4)} = i^{1-2\nu}b_{1-}^{(2)}B_-,$$

which combine to give

$$D_+^{(4)} - D_-^{(4)} = B(b_{1+}^{(2)} - i^{-4\nu}b_{1-}^{(2)})_{J_1}, \tag{4.30}$$

where the subscript J_1 denotes the cubic part of the total jump which is proportional to J_1.

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4.10 $O(\epsilon^3 \mu^{-\frac{5}{2}+3\nu})$ of the third harmonic

At this order the governing equations yield

$$N_{3,(\frac{1}{2}+\nu)}\Psi_3^{(1)} = \frac{3ik}{2}(1+2\nu)\Phi_3^{(1)} + ik(\Psi_2^{(1)''}W - 3\Psi_2^{(1)'}W' + 2\Psi_2^{(1)}W''),$$

$$N_{3,(\frac{1}{2}-\nu)}\Phi_3^{(1)} = 0. \qquad\qquad (4.31a,b)$$

The right–hand sides do not contain $\overline{W}$ and so have a unique asymptotic representation. Matching with the solution outside the critical layer gives $\Phi_3^{(1)} = 0$, whilst solving (4.30a) gives

$$\Psi_3^{(1)} = \frac{i^{\frac{5}{2}-11\nu}k^{\frac{5}{2}-3\nu}(1+2\nu)^2\Gamma^2(\frac{1}{2}+\nu)}{16\pi^2\Gamma(\frac{1}{2}-\nu)} \int_0^\infty dt \int_0^\infty dt_1 \int_C dt_2 \int_C dt_3 B(\tau-t-t_1)\times$$

$$B(\tau-t-t_2)B(\tau-t-t_3)t_1^{\frac{1}{2}+\nu}t_2^{-\frac{1}{2}-\nu}t_3^{-\frac{3}{2}-\nu}(t_1+t_2)^{-\frac{3}{2}-\nu}(t_1-t_2)^{\frac{1}{2}-\nu}\times$$

$$(t_1+t_2+t_3)^{\frac{1}{2}+\nu}(3t+t_1+t_2+t_3)^{-\frac{3}{2}-\nu}(t_1+t_2-t_3)(t_1+t_2-2t_3)\times$$

$$F\left(-\frac{1}{2}-\nu,-\frac{1}{4}-\frac{\nu}{2};\frac{3}{4}-\frac{\nu}{2};\frac{t_2^2}{t_1^2}\right)e^{-ikY(3t+t_1+t_2+t_3)}H(t_1-t_2). \qquad (4.31c)$$

4.11 $O(\epsilon^4 \mu^{-4+4\nu})$ of the zeroth harmonic

At this order it is only necessary to determine $\Phi_0^{(7)}$. Equation (4.2b) gives

$$\frac{\partial}{\partial\tau}\Phi_0^{(7)'} = 0, \qquad\qquad (4.32)$$

which has the solution $\Phi_0^{(7)'} = f(Y)$. However, choosing a non-zero $f(Y)$ is merely equivalent to taking a different density profile for the unperturbed flow and thus, without any loss of generality, we choose to take $f(Y) = 0$ and $\Phi_0^{(7)} = 0$.

4.12 $O(\epsilon^4 \mu^{-4+4\nu})$ of the second harmonic

At this order the equations (4.2a,b) can be written in the form

$$N_{2,(\frac{1}{2}+\nu)}\Psi_2^{(7)} - ik(1-2\nu)\Phi_2^{(7)} = ikR_N \equiv ik(R_{11} + R_{02} + R_{31}),$$

$$\dot{N}_{2,(\frac{1}{2}-\nu)}\Phi_2^{(7)} = 0, \qquad\qquad (4.33a,b)$$

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where

$$R_{11} = (W\Psi_1^{(2)''} - 2W'\Psi_1^{(2)'} + W''\Psi_1^{(2)}), \quad R_{02} = 2(\Psi_0^{(1)''}\Psi_2^{(1)} - \Psi_0^{(1)'}\Psi_2^{(1)'})$$

$$\text{and} \quad R_{31} = (3\overline{W}''\Psi_3^{(1)} - 2\overline{W}'\Psi_3^{(1)'} - \overline{W}\Psi_3^{(1)''}). \tag{4.34a-c}$$

Following Churilov & Shukhman (1988), $\Psi_2^{(7)}$ is written as the sum $\Psi_2^{(7)} = \Psi_{2N}^{(7)} + \Psi_{2L}^{(7)}$ of a particular solution $(N_{2,(\frac{1}{2}+\nu)}\Psi_{2N}^{(7)} = ikR_N)$ and a general solution, $\Psi_{2L}^{(7)}$, of the associated homogeneous equation. Defining

$$c_{00}(\chi, k) = \frac{\chi k^{-\chi}\Gamma(\chi)}{2\pi} i^{3(1-\chi)},$$

we find that

$$\Phi_2^{(7)} = \mu^{-2\nu} c_{00}(1/2 - \nu, 2k) \int_C t^{-\frac{3}{2}+\nu} C(\tau - t)e^{-2iktY} dt,$$

$$\Psi_{2L}^{(7)} = c_{00}(1/2 + \nu, 2k) \int_C t^{-\frac{3}{2}-\nu} D(\tau - t)e^{-2iktY} dt$$

$$- \mu^{-2\nu}\frac{(1-2\nu)}{4\nu} c_{00}(1/2 - \nu, 2k) \int_C t^{-\frac{3}{2}+\nu} C(\tau - t)e^{-2iktY} dt, \tag{4.35a,b}$$

where $C(\tau)$ and $D(\tau)$ are arbitrary functions which tend to zero as $\tau \to -\infty$, and the $\mu^{-2\nu}$ factors have been added for later convenience. As $Y \to \pm\infty$,

$$\Psi_{2N}^{(7)} = M_\pm Y^{\frac{1}{2}+\nu} + O(Y^{-\frac{1}{2}+\nu}),$$

where

$$M_+ - M_- = \frac{i^{\frac{3}{2}+\nu}(2k)^{\frac{3}{2}+\nu}}{(1+2\nu)\Gamma(\frac{1}{2}+\nu)} \int_{-\infty}^{\infty} dY \int_0^{\infty} dt R_N(\tau - t, Y)t^{\frac{1}{2}+\nu} e^{-2iktY} \neq 0. \tag{4.36}$$

Thus the asymptotic representations of $\Psi_2^{(7)}$ and $\Phi_2^{(7)}$ have the forms

$$\Psi_2^{(7)} = [D(\tau) + M_\pm(\tau)]Y^{\frac{1}{2}+\nu} - \mu^{-2\nu}\frac{(1-2\nu)}{4\nu} C(\tau)Y^{\frac{1}{2}-\nu} + \cdots,$$

$$\Phi_2^{(7)} = \mu^{-2\nu} C(\tau)Y^{\frac{1}{2}-\nu} + \cdots.$$

$$\tag{4.37a,b}$$

To match with the solution outside the critical layer, in (3.15) it is necessary to set $c_\pm = \epsilon^4 \mu^{-\frac{8}{2}+3\nu} c_{2\pm}$, leading to the relations

$$c_{2+} = i^{1-2\nu}c_{2-} = C(\tau), \quad c_{2+}q_+ = D(\tau) + M_+(\tau) \quad \text{and} \quad c_{2-}q_- = i^{1+2\nu}[D(\tau) + M_-(\tau)].$$

Upon setting $q_+ = q = q_-$, we find that

$$C(\tau) = -\frac{i^{-1-\nu}}{2q}\csc\left(\frac{\pi\nu}{2}\right)(M_+ - M_-), \tag{4.38}$$

and, as $M_+ - M_- \neq 0$, this fully determines $\Phi_2^{(7)} \neq 0$. Thus, the symmetry $\Phi = 0$ has been broken and a 'jump' will occur at the next order in which $\Phi_2^{(7)}$ appears in the right–hand side of the equations. The explicit form of $M_+ - M_-$ is given in Appendix B of this paper.

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4.13 $O(\epsilon^5 \mu^{-\frac{11}{2}+5\nu})$ of the fundamental

At this order, the governing equations are

$$N_{1,(\frac{1}{2}+\nu)}\Psi_1^{(7)} = \frac{1}{2}ik(1-2\nu)\Phi_1^{(7)} + R_1^{(7)},$$

$$N_{1,(\frac{1}{2}-\nu)}\Phi_1^{(7)} = -ik\mu^{2\nu}\left(\Phi_2^{(7)''}\bar{\Psi}_1^{(1)} + 2\Phi_2^{(7)'}\bar{\Psi}_1^{(1)'}\right) \equiv -ikR_2^{(7)}. \qquad (4.39a,b)$$

The asymptotic representation of $\Psi_1^{(7)}$ has the form

$$\Psi_1^{(7)} = C_\pm^{(7)}Y^{\frac{1}{2}+\nu} + D_\pm^{(7)}Y^{\frac{1}{2}-\nu} + O(Y^{-\frac{1}{2}+\nu}),$$

where, in particular,

$$D_+^{(7)} - D_-^{(7)} = \frac{i^{\frac{3}{2}-\nu}k^{\frac{3}{2}-\nu}}{2\nu\Gamma(\frac{1}{2}-\nu)}\int_{-\infty}^{\infty}dY\int_0^{\infty}dt\, t^{\frac{1}{2}-\nu}R_2^{(7)}(\tau-t,Y)e^{-ikYt}. \qquad (4.40)$$

From (4.7a) and (4.35a),

$$R_2^{(7)} = -\frac{i^{-2\nu}k(1-4\nu^2)\cos(\pi\nu)}{2^{\frac{5}{2}}\pi}\int_C dt_1\int_C dt_2 t_1^{-\frac{1}{2}+\nu}t_2^{-\frac{3}{2}-\nu}(t_1-t_2)e^{-ikY(2t_1-t_2)}C(\tau-t_1)\overline{B}(\tau-t_2),$$
$$\qquad (4.41)$$

and simple manipulations give

$$D_+^{(7)} - D_-^{(7)} = c_q\int_0^{\infty} t^{\frac{1}{2}-\nu}C(\tau-t)\overline{B}(\tau-2t)dt, \quad c_q = \frac{2i^{\frac{3}{2}-3\nu}k^{\frac{3}{2}-\nu}(1-2\nu)\pi^{\frac{3}{2}}}{\nu\Gamma(\frac{1}{2}-\nu)\Gamma(\frac{1}{4}+\frac{\nu}{2})\Gamma(\frac{1}{4}-\frac{\nu}{2})}. \quad (4.42a,b)$$

The *quintic* contribution to $b_{1\pm}^{(2)}$ is found by matching the above asymptote with the outside expansion of the fundamental near the critical layer. This process yields

$$D_+^{(7)} - D_-^{(7)} = B(b_{1+}^{(2)} - i^{-4\nu}b_{1-}^{(2)})_q, \qquad (4.43)$$

where the subscript q denotes the 'quintic' part of the total jump. From (4.38),(4.42) and (4.43) we obtain

$$B(b_{1+}^{(2)} - i^{-4\nu}b_{1-}^{(2)})_q = -\frac{i^{-1-\nu}c_q}{2q}\csc\left(\frac{\pi\nu}{2}\right)\int_0^{\infty} t^{\frac{1}{2}+\nu}[M_+(\tau-t)-M_-(\tau-t)]\overline{B}(\tau-2t)dt. \quad (4.44)$$

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4.14 The cubic nonlinearity due to viscosity

The remaining terms in the expansions (4.4), proportional to κ, are driven by dissipative (viscous) effects. They are not of major concern in this paper but are included for completeness; the equations which they satisfy, their solutions and further analysis can be found in Dando (1993) (see also Churilov & Shukhman, 1988). Providing the Prandtl number is not unity, a non–zero jump, $D_+^{(6)} - D_-^{(6)} = [D_+^{(6)} - D_-^{(6)}]_0 + [D_+^{(6)} - D_-^{(6)}]_2$ say, arises at $O(\epsilon^3 \kappa \mu^{-\frac{11}{2}+3\nu})$ where

$$[D_+^{(6)} - D_-^{(6)}]_0 = -\frac{i^{1-4\nu} k^{5-4\nu}(\eta-1)\Gamma^3(\frac{1}{2}+\nu)(1+2\nu)^2}{32\nu\pi^2(3-2\nu)\Gamma(\frac{1}{2}-\nu)} \int_C dt_1 \int_C dt_2 \int_C dt_3$$

$$B(\tau - t_1 + t_2)(t_1 t_2 t_3)^{-\frac{3}{2}-\nu}(t_1 - t_2 - t_3)^{\frac{1}{2}-\nu}(t_1 - t_2)^2 \left[\frac{1}{2}\int_0^\infty dt_4 \overline{B}(\tau - 2t_1 + t_2 + t_3 - t_4)\times\right.$$

$$B(\tau - t_1 + t_3 - t_4)\left((1+2\nu)(t_1^4 + t_2^4) - 2(3-2\nu)t_1 t_2(t_2 - t_1)^2\right)$$

$$\left.-\overline{B}(\tau - 2t_1 + t_2 + t_3)B(\tau - t_1 + t_3)t_1 t_2\left((t_1 - t_2)^3 - 2t_3(t_1 - t_2)^2 + 2t_3^3\right)\right]H(t_1 - t_2 - t_3),$$

and

$$[D_+^{(6)} - D_-^{(6)}]_2 = -\frac{i^{1-4\nu} k^{5-4\nu}(\eta-1)\Gamma^3(\frac{1}{2}+\nu)(1+2\nu)^3}{256\nu\pi^2(3-2\nu)\Gamma(\frac{1}{2}-\nu)} \int_C dt_1 \int_C dt_2 \int_C dt_3 \int_0^\infty dt$$

$$\overline{B}(\tau - 2t_1 + t_2 + t_3 + 2t)B(\tau - t_1 + t_3 + t)B(\tau - t_1 + t_2 + t)(t_1 t_2 t_3)^{-\frac{3}{2}-\nu}\times$$

$$(t_1 - t_2 - t_3 - 2t)^{\frac{1}{2}-\nu}(t_3 - t_2)^2(t_2 + t_3)^{\frac{1}{2}+\nu}(2t + t_2 + t_3)^{-\frac{3}{2}-\nu}(2t_1 - t_2 - t_3 - 2t)\times$$

$$\left(2t_1^4 + (2t + t_2 + t_3)^{1+2\nu}(t_2 - t_3)^2(t_2 + t_3)^{1-2\nu} + (2t + t_2 + t_3)^4\right)H(t_1 - t_2 - t_3 - 2t). \quad (4.45a,b)$$

Matching with the outer inviscid solution requires that

$$D_+^{(6)} - D_-^{(6)} = B(b_{1+}^{(2)} - i^{-4\nu} b_{1-}^{(2)})_v, \qquad (4.46)$$

where the subscript v denotes the cubic part of the total jump which is due to dissipative effects.

It is worthwhile to consider the effects of viscosity a little further at this point. At the start of this section, it was pointed out that we assume $\kappa \ll \mu^3$, so that the effects of viscosity do not enter the crucial critical–layer operator N_χ at leading order. Instead, the effects of viscosity occur as inhomogeneities at certain lower orders of the hierarchy of critical–layer equations. This approach is entirely rational as long as $\kappa/\mu^3 \ll 1$. However, to derive amplitude equations valid over a larger range of κ values, one must follow the approach, introduced by Haberman (1972), of introducing a new parameter, λ_H say, where

$$\lambda_H = \kappa/\mu^3, \qquad (4.47)$$

is taken to be order one during the critical–layer analysis. The operator N_χ would need modifying:

$$N_\chi \longrightarrow N_\chi - \eta \lambda_H \frac{\partial^3}{\partial Y^3} \qquad (4.48)$$

and hence all the critical–layer solutions would be modified. We have chosen not to adopt this approach here as it would complicate the presentation of our theory and analysis; whilst it is not really necessary so long as λ_H is reasonably small (see Goldstein & Leib, 1989).

(ii) The case $\nu > 0$

Clearly the viscous–cubic and the quintic nonlinearities are still possible when ν is positive. Moreover, the analysis and the evolution equations are exactly the same as for the negative ν case. However, when ν is positive $\epsilon \mu^{\frac{3}{2}-\nu} > \epsilon \mu^{\frac{3}{2}+\nu}$ and the ordering of terms in expansion (4.4a) strictly needs to be changed so that the term $\Psi_1^{(3a)}$ is lower order than the term $\Psi_1^{(3a)}$. It is now no longer rational to balance the cubic term formed from the $\Psi_1^{(3a)}$ term, with the $\Psi_1^{(3b)}$ term. Thus, the J_1–cubic nonlinearity considered for negative ν is no longer a possibility.

There are two other possibilities that we can consider for positive ν. Firstly there is the cubic formed by the $\Psi_1^{(3b)}$ term, balancing with the $\Psi_1^{(3b)}$ term. However, this among other things, necessarily leads to the fully nonlinear critical layer problem. Alternately, there is another cubic nonlinearity (referred to here as the outer–complementary–function [OCF] cubic term) similar to that considered by Churilov & Shukhman (1987) (for unity Prandtl number) in the unpublished Appendix B of their paper. This cubic nonlinearity was not considered for negative ν as the J_1–cubic nonlinearity is always larger. It arises from considering the part of the complementary function term in the asymptote of the second harmonic outside the critical layer (equation (3.15a)) to be at an order fixed by the outside, inviscid problem; *rather* than just considering it at orders fixed by the process of harmonic generation inside the critical layer.

It is necessary to consider two additional terms in the (slightly re-ordered) expansions (4.4), namely terms $\epsilon^2 \mu^{\frac{1}{2}-\nu}(\Psi_2^{(8)}, \Phi_2^{(8)})$ in the expansions of the second harmonic and terms $\epsilon^3 \mu^{-1}(\Psi_1^{(8)}, \Phi_1^{(8)})$ in the expansion of the fundamental.

4.15 $O(\epsilon^2 \mu^{\frac{1}{2}-\nu})$ of the second harmonic

At this order equations (4.2a,b) yield

$$N_{2,(\frac{1}{2}+\nu)}\Psi_2^{(8)} = (1 - 2\nu)ik\Phi_2^{(8)} + ik(WW'' - W'^2), \quad \text{and} \quad N_{2,(\frac{1}{2}-\nu)}\Phi_2^{(8)} = 0. \qquad (4.49a - b)$$

The solution for $\Phi_2^{(8)}$ is written

$$\Phi_2^{(8)} = c_{00}(1/2 - \nu, 2k) \int_C t^{-\frac{3}{2}+\nu} n_2(\tau - t)e^{-2iktY} dt, \qquad (4.50)$$

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where the function $n_2(\tau)$ is determined below. For large $|Y|$,

$$\Phi_2^{(8)} \sim n_2(\tau)Y^{\frac{1}{2}-\nu}, \quad \Psi_2^{(8)} \sim m_2(\tau)Y^{\frac{1}{2}+\nu} - \frac{(1-2\nu)}{4\nu}n_2(\tau)Y^{\frac{1}{2}-\nu}, \tag{4.51}$$

where $m_2(\tau)$ is a suitable arbitrary function. Matching with the expansions outside of the critical layer yields

$$g_{2-} = i^{1+2\nu}g_{2+}, \quad h_{2-} = i^{1+6\nu}h_{2+} \quad \text{and} \quad B^2h_{2+} = -\frac{(1-2\nu)}{4\nu}n_2. \tag{4.52a-c}$$

From the relations (4.52a-c) and (A.5d), it follows that

$$n_2(\tau) = \frac{4i^{\frac{1}{2}-3\nu}\nu(\alpha\beta_a - \beta\alpha_a)\sin[\pi(1+2\nu)/4]}{(1-2\nu)\alpha_a\sin(\pi\nu)}B^2(\tau), \tag{4.53}$$

and thus as $\Phi_2^{(8)}$ is non–zero, it will lead to a 'contributing' nonlinear jump.

4.16 $O(\epsilon^3\mu^{-1})$ of the fundamental

The governing equations at this order are

$$N_{1,(\frac{1}{2}+\nu)}\Psi_1^{(8)} = \frac{1}{2}(1-2\nu)ik\Phi_1^{(8)} + ik(2\overline{W}''\Psi_2^{(8)} - \overline{W}'\Psi_2^{(8)'} - \overline{W}\Psi_2^{(8)''}),$$

$$N_{1,(\frac{1}{2}-\nu)}\Phi_1^{(8)} = -ik(\overline{W}\Phi_2^{(8)''} + 2\overline{W}'\Phi_2^{(8)'}) \equiv -ikR_2^{(8)}. \tag{4.54a,b}$$

The asymptotic representation of $\Psi_1^{(8)}$ as $Y \to \pm\infty$ has the form

$$\Psi_1^{(8)} = C_\pm^{(8)}Y^{\frac{1}{2}+\nu} + D_\pm^{(8)}Y^{\frac{1}{2}-\nu} + O(Y^{-\frac{1}{2}+\nu}),$$

where, in particular, $D_+^{(8)} - D_-^{(8)}$ is given by the right–hand side of equation (4.40) but with $R_2^{(8)}$ replacing $R_2^{(7)}$. Simple manipulation gives

$$R_2^{(8)} = -\frac{2^{\frac{5}{2}+\nu}ik(1-4\nu^2)\cos(\pi\nu)}{\pi}\int_C dt_1 \int_C dt_2 \overline{B}(\tau - t_1)n_2(\tau - t_2)e^{-ik(2t_2-t_1)Y}t_1^{-\frac{1}{2}+\nu}t_2^{-\frac{3}{2}-\nu}(t_1 - t_2) \tag{4.55}$$

and we find that

$$D_+^{(8)} - D_-^{(8)} = -\frac{64i^{\frac{5}{2}+\nu}k^{\frac{5}{2}-\nu}(1-2\nu)\Gamma(\frac{1}{2}+\nu)\pi^{\frac{1}{2}}}{\nu\Gamma(\frac{1}{4}-\frac{\nu}{2})\Gamma(\frac{1}{4}+\frac{\nu}{2})}\int_0^\infty t^{\frac{1}{2}-\nu}\overline{B}(\tau - 2t)n_2(\tau - t)dt. \tag{4.56}$$

Matching with the outer, inviscid solution requires that

$$D_+^{(8)} - D_-^{(8)} = B(b_{1+}^{(2)} - i^{-4\nu}b_{1-}^{(2)})_{ocf}, \tag{4.57}$$

where the subscript ocf denotes the cubic part of the total jump which is due to the outer–complementary–function terms.

5 The Evolution Equations

The four possible forms of the evolution equation for $B(\tau)$ simply correspond to matching the solvability condition (3.9b) with each of the four jump expressions (4.29), (4.43), (4.45) and (4.56) in turn; for the sake of succinctness, we do not explicitly quote these evolution equations. Instead, it is more beneficial to the discussion here to form one 'composite' evolution equation

$$-\frac{2iI_1}{k}(1+i^{4\nu})\left[\frac{\partial B}{\partial \tau} + \frac{J_1 I_2 k}{2I_1}\tan(\pi\nu)B\right] = -2\nu B(b_{1+}^{(2)} - i^{-4\nu}b_{1-}^{(2)})$$

$$= -2\nu B\left\{\epsilon^2\mu^{-3+4\nu}(b_{1+}^{(2)} - i^{-4\nu}b_{1-}^{(2)})_{J_1}H(-\nu) + \epsilon^2\mu^{-\frac{5}{2}+\nu}(b_{1+}^{(2)} - i^{-4\nu}b_{1-}^{(2)})_{ocf}H(\nu)\right.$$

$$\left. +\epsilon^2\mu^{-7+4\nu}\kappa(b_{1+}^{(2)} - i^{-4\nu}b_{1-}^{(2)})_v + \epsilon^4\mu^{-7+6\nu}(b_{1+}^{(2)} - i^{-4\nu}b_{1-}^{(2)})_q\right\}, \qquad (5.1)$$

where the explicit forms of the nonlinear quantities appearing on the right–hand side of this equation have been derived, and are quoted, in §4; again H denotes the Heaviside function. As the composite equation (5.1) contains the sum of the four (individual) nonlinearities, it has the advantage of being valid for all values of κ, ϵ and μ that lead to an unsteady (weakly nonlinear and weakly viscous) critical–layer.

In this section we shall discuss the parameter ranges of validity of each of the four possible forms of the individual 'base' evolution equations for $B(\tau)$, before presenting some numerical results for the 'J_1'–nonlinearity case. This section is concluded with a discussion of the expected solution properties of the other (base) evolution equations. Further conclusions are drawn in the next section.

5.1 Parameter ranges of validity

The range of validity/application of each of the possible base evolution equations is summarised in Figures 5.1a,b, for $\nu < 0$ and $\nu > 0$ respectively; the governing balances determining the dashed curves separating the three parts of region III follow immediately by comparing the sizes of the four terms on the right–hand side of equation (5.1). As an example, we note that (when $\nu < 0$) the nonlinearity proportional to J_1 is equal in size or larger than the quintic nonlinearity if $\epsilon^2\mu^{-3+4\nu} \geq \epsilon^4\mu^{-7+6\nu}$ i.e. if $\epsilon \leq \mu^{2-\nu}$. We see that in each case, depending on the relative sizes of κ, ϵ and μ, three of the four base evolution equations are applicable. Also plotted are the expected evolutionary paths of the disturbance for each case; this shall be discussed in more detail later but at present it suffices to note that, based on our assumptions concerning viscous–spreading effects resulting in an unstable linear disturbance mode approaching a later neutral state, initially our disturbance will lie in the bottom right–hand corner of region IIIb in each case. This partly justifies our relegation of viscosity to lower order effects, as well as indicating that the base evolution equations with the J_1–cubic and the OCF–cubic nonlinearities deserve the first attention. In the next subsection we present numerical calculations for the

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former case ($\nu < 0$); the evolution equation with OCF–cubic nonlinearity ($\nu > 0$) is discussed in the final subsection.

5.2 Numerical results for the J_1–cubic

Recall that if, and only if, $\nu < 0$ then a cubic nonlinearity proportional to J_1 is possible. In fact it is the largest nonlinearity in the (composite) evolution equation when $\mu \gg \kappa^{\frac{1}{4}}$ and $\epsilon \ll \mu^{2-\nu}$; thus, as mentioned in the previous section, it corresponds to the first nonlinear evolutionary stage for an unstable linear disturbance whose growth rate is diminished by viscous–spreading effects. In this subsection we present some numerical results for this case.

To ease numerical calculations the jump expression (4.29) is transformed into kernel form,

$$D_+^{(4)} - D_-^{(4)} = -\frac{i^{1+2\nu}k^{3-4\nu}J_1\pi(1+2\nu)}{4\nu^2\Gamma^4(\frac{1}{2}-\nu)}\int_0^\infty ds\, s^{2-4\nu}\int_0^1 d\sigma\, \sigma^{1-2\nu}G(\sigma)B(\tau-s)B(\tau-\sigma s)\overline{B}(\tau-(1+\sigma)s)$$

$$\equiv B(b_{1+}^{(2)} - i^{-4\nu}b_{1-}^{(2)})_{J_1} \tag{5.2a}$$

where

$$G(\sigma) = \left[\frac{\Gamma^2(\frac{1}{2}-\nu)(1+\sigma)^{-\frac{1}{2}-\nu}}{(1+2\nu)\Gamma(1-2\nu)}\left\{2(1+\sigma)(1-2\nu)F_1\left(\frac{3}{2}-\nu,\frac{3}{2}+\nu,\frac{1}{2}+\nu,1-2\nu;\sigma;\frac{\sigma}{(1+\sigma)}\right)\right.\right.$$

$$-(1+\sigma+\sigma^2)(3-2\nu)F_1\left(\frac{5}{2}-\nu,\frac{3}{2}+\nu,\frac{1}{2}+\nu,2-2\nu;\sigma;\frac{\sigma}{(1+\sigma)}\right)$$

$$\left.+\frac{\sigma(1+\sigma)(5-2\nu)(3-2\nu)}{8(1-\nu)}F_1\left(\frac{7}{2}-\nu,\frac{3}{2}+\nu,\frac{1}{2}+\nu,3-2\nu;\sigma;\frac{\sigma}{(1+\sigma)}\right)\right\}$$

$$+(1+\sigma)(1-\sigma)^{-2\nu}\times$$

$$\left.\int_0^1 dt\frac{t^{-\frac{1}{2}-\nu}(1-t)^{\frac{1}{2}-\nu}(1-\sigma t)^{-\frac{1}{2}+3\nu}}{(1+\sigma t)^{\frac{3}{2}+\nu}(1-\sigma^2 t)^{\frac{3}{2}+\nu}}(2-\sigma+\sigma t-2\sigma^2 t)F\left(-\frac{1}{2}-\nu,-\frac{1}{4}-\frac{\nu}{2};\frac{3}{4}-\frac{\nu}{2};\sigma^2 t^2\right)\right].$$

$$\tag{5.2b}$$

Here $F_1(a,b,c,d;x,y)$ is the hypergeometric function of two variables (see Erdélyi, 1953; Churilov & Shukhman, 1988). The evolution equation for this case can then be written in the form

$$\frac{\partial B}{\partial \tau} = \gamma_1 B + \gamma_2 \int_0^\infty ds\, s^{2-4\nu}\int_0^1 d\sigma\, \sigma^{1-2\nu}\hat{G}(\sigma)B(\tau-s)B(\tau-\sigma s)\overline{B}(\tau-(1+\sigma)s), \tag{5.3}$$

where

$$\gamma_1 = \frac{-J_1 I_2 k \tan(\pi\nu)}{2I_1}, \quad \gamma_2 = \frac{J_1 k^{4-4\nu}\pi|1+2\nu|}{8\nu\Gamma^4(\frac{1}{2}-\nu)\cos(\pi\nu)I_1} \quad \text{and} \quad \hat{G}(\sigma) = -\text{sgn}\,(1+2\nu)G.$$

$$\tag{5.4a-c}$$

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Let us consider the range $-2 < \nu < 0$; in Figures 5.2a,b $-\gamma_1/J_1$ and $-\gamma_2/J_1$ are plotted versus ν for this range. These figures illustrate that both γ_1 and γ_2 are positive in this range (recall that, since we are considering unstable modes, $J_1 < 0$). The lower value $\nu = -2$ chosen here for this numerical investigation has no special significance, it was principally chosen so that we calculate numerical results for a complete period of the trigonometric functions present in the analytical expressions. Note though that, for their study of the related neutral inviscid Görtler modes in a three–dimensional boundary layer, Blackaby & Choudhari (1993) find that this range of ν–values is appropriate. Moreover, it may also be argued that for increasingly negative values of ν (corresponding to the Richardson number becoming increasingly negative) the now destabilising effect of stratification means that the growth rates of the more unstable linear inviscid disturbances are just too large for viscous–spreading effects to damp significantly, thus rendering the weakly–nonlinear theory inapplicable.

It is convenient for numerical calculations to introduce a so–called 'logarithmic time' (see Churilov & Shukhman, 1988; Shukhman, 1991)

$$T = \frac{\gamma_2}{(2\gamma_1)^{4-4\nu}}|B_0|^2 e^{2\gamma_1\tau}, \tag{5.5a}$$

having set

$$B(\tau) = B_0 b(\tau) e^{\gamma_1\tau}, \tag{5.5b}$$

where the constant B_0 is chosen such that $b(\tau) \to 1$ as $\tau \to -\infty$. This is done in order to reduce the number of parameters in the evolution equation. Equation (5.2) now has the form

$$\frac{\partial b}{\partial T} = \int_0^1 d\sigma K(\sigma) \int_0^\infty dx x^{2-4\nu} e^{-x} b(Te^{-x/(1+\sigma)}) b(Te^{-\sigma x/(1+\sigma)}) \bar{b}(Te^{-x}), \tag{5.6a}$$

with initial condition

$$b(T = 0) = 1; \tag{5.6b}$$

where the kernel

$$K(\sigma) = \frac{\sigma^{1-2\nu}}{(1+\sigma)^{3-4\nu}} \hat{G}(\sigma). \tag{5.6c}$$

In Figure 5.3 $K(\sigma) \equiv K(\sigma; \nu)$ is plotted, versus σ, for a few representative ν values. It is interesting to note that $K(\sigma)$ is always negative ($0 \le \sigma \le 1$); this appears to be the case for all $-2 < \nu < 0$; cf. the viscous–jump kernel for the $\nu = 0$ ($J = 1/4$) case considered by Churilov & Shukhman (1988) which changes sign. Thus it is possible to deduce all the qualitative results of the solution properties of the evolution equation (5.6) from results plotted in this figure (see the discussion in §5.2). However, we still chose to solve the integro–differential equation (5.6) numerically for completeness, as well as to obtain actual quantitative results.

In Figure 5.4, we present the results of a numerical solution of the evolution equation (5.6) for two representative ν values. The results show that $b(T)$ oscillates with a fast rising amplitude as T increases. The period of these oscillations is seen to depend strongly on ν: for ν values corresponding to the smaller $x^{2-4\nu}e^{-x}K(\sigma)$ values (i.e. $\nu = -0.4$) the time T for b to attain a given large value is longer than for ν values corresponding to the larger $x^{2-4\nu}e^{-x}K(\sigma)$ values (i.e. $\nu = -1.4$). In summary, the numerical calculations indicate that, for all $0 < \nu < 2$, the

amplitude $B(\tau)$ will oscillate, with its magnitude rising sharply after a few oscillations; our results indicate that *no* 'finite–time' singularity of the evolution equation (5.6) occurs, instead the oscillations become increasingly wilder.

5.3 Discussion

We begin this discussion by continuing to consider the evolution equation (5.6) with non-linearity (solely) due to J_1; later in this subsection we consider the other (base) evolution equations. In the last section we reported that our numerical computations show that $K(\sigma)$ is everywhere negative ($0 < \sigma \leq 1$) for all $-2 < \nu < 0$. It was remarked that from these facts (and the magnitude of $K \equiv K(\sigma; \nu)$) it is possible to deduce all the qualitative results of the solution properties of the evolution equation (5.6), without needing to perform the numerical integration. To illustrate that this is so, let us consider the right–hand side of equation (5.6). Initially $b(T) > 0$ and thus the right–hand side of equation is clearly negative (as $K(\sigma) < 0$); thus this equation tells us that $b(T)$ will decrease in value until such time when the right–hand side of equation (5.6) is positive. The latter condition cannot be reached until $b(T)$ becomes negative. However as the right–hand side of equation depends on *all* previous $b(T)$ values, there is a delay until the right–hand side of equation is actually positive; during this period $b(T)$ has becomes more negative. Once the right–hand side of equation is positive, $b(T)$ grows until it becomes positive. Again there is a delay until the right–hand side of equation is negative, at which time $b(T)$ starts to decrease again, and so on. It is clear that the larger the typical magnitude of the *overall* kernel $x^{2-4\nu}e^{-x} K(\sigma)$, the more effect we can expect the nonlinearity to have i.e. larger oscillations at earlier T–values. Note that we would not expect a singularity to develop because of the smooth mechanism underlying the behavior of $b(T)$, as described above. In fact, it is possible to show analytically that the solution of the evolution equation (5.6) can only develop a singularity if $\int_0^1 K(\sigma)d\sigma > 0$ (see Churilov & Shukhman, 1988).

Thus, disturbances initially governed by the evolution equation (5.6) would soon become so large in magnitude that their evolution would move into its second stage (region IIIc of Figure 5.1a) where the largest (leading order) term in the composite evolution equation (5.1) is the quintic nonlinearity. We note that this result is not at all surprising and could have been deduced as soon as the scales and terms of the critical–layer expansions (4.4a-d) were deduced: as viscosity does not enter our analysis at leading order, the base evolution equations due to J_1–nonlinearity, the OCF–nonlinearity and the quintic–nonlinearity will all lead to unbounded amplitude growth, whether by increasing disturbance oscillations or by a singularity occurring (they *do not* permit so–called equilibrium solutions). As viscosity does not enter at leading order it certainly cannot damp out the amplitude growth. We note that even with stronger viscosity effects, equilibrium states may still not necessarily be reached (see Goldstein & Leib, 1989). Moreover, we note that for the marginal instability case considered by Churilov & Shukhman (1988), the nonlinearity due to viscosity permitted singular solutions in which the amplitude became unbounded at a finite T.

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6 Conclusion

In this paper we have considered the nonlinear development of unstable disturbances in stratified shear flows where the Richardson number J is less than one quarter in value. Although such modes are initially fast growing, we have assumed that viscous–spreading effects result in them evolving in a linear fashion until they reach a state where their amplitudes are large enough but their growth rate have diminished significantly so that amplitude equations are derived using weakly nonlinear and unsteady critical layer theories. We have found that four different (base) integro–differential amplitude equations are possible, including one due to a novel mechanism (the 'J_1'–cubic nonlinearity). The relevant choice of amplitude equation, at a particular instance, being dependent on the relative sizes of the disturbance amplitude, the growth rate of the disturbance, its wavenumber and the viscosity of the fluid. This richness of choice of possible nonlinearities arises mathematically from the indicial Frobenius roots of the governing linear inviscid equation (the Taylor–Goldstein equation) not, in general, differing by an integer. The initial nonlinear evolution of a mode will be governed by an integro–differential amplitude equations with a cubic nonlinearity but the resulting significant increase in the size of the disturbance's amplitude leads on to the next stage of the evolution process where the evolution of the mode is governed by an integro–differential amplitude equations with a quintic nonlinearity. Continued growth of the disturbance amplitude is expected during this stage, resulting in the effects of nonlinearity spreading to outside the critical level, by which time the flow has become fully nonlinear.

We finish by mentioning some further points that may be worthy of investigation; mainly these are related to the relaxation of the assumptions made here in this paper. Obviously, the inclusion of viscosity at leading order in the critical layer (i.e. treating $\lambda_H \sim O(1)$) would answer the questions posed in the previous section as to whether it can damp down the unlimited growth in the magnitude of the disturbances, allowing equilibrium solutions to exist. Another obvious extension of this work is to three–dimensional disturbances; then it may be possible to consider interactions between two or more disturbances. Comparision of our theory with experiments or large–scale numerical simulations is necessary at some point. As mentioned in the introduction, as well as geo–physical flow applications, the theory has applications to aerodynamical flow situations. In fact, in a related paper Blackaby, Dando & Hall (1993) apply the ideas contained in this paper to the problem concerning the nonlinear evolution of inviscid Görtler vortices in a three–dimensional boundary–layer; their problem is more complex than that for the model stratified flow considered here, as the longitudinal vortex problem is necessarily three–dimensional. The theory developed in this paper can also be applied to a nonlinear study of the inviscid vortex instabilities in the three–dimensional boundary–layer flow above a heated plate.

Acknowledgements

Appendix A

This appendix contains some additional results, concerning the outer inviscid problem for the second harmonic, needed for the case $\nu > 0$. It is merely a generalisation of the unpublished Appendix B of Churilov & Shukhman (1987) to the case $\nu \neq 0$.

Outside the critical layer, the first term of the second harmonic, $\psi_2^{(1)}$, satisfies

$$L_2\psi_2^{(1)} = Q_2 \equiv B_{\pm}^2 \sinh^{2\nu+1}|y| \left[\frac{2}{\cosh^4 y} + \frac{3J_0}{2\sinh^4 y} \right]. \qquad (A.1)$$

We introduce the functions f_a and f_b which satisfy the homogeneous equation $L_2\psi_2^{(1)} = 0$, such that

$$f_a = e^{-k\sqrt{k^2+3}|y|}(1 + O(e^{-2|y|})), \quad f_b = e^{k\sqrt{k^2+3}|y|}(1 + O(e^{-2|y|})), \quad \text{as} \quad y \to \pm\infty.$$

The solution of (A.1) is then written in the form

$$\psi_2^{(1)} = B_{\pm}^2(f(y) + C_{\pm}f_a(y)), \qquad (A.2)$$

where $C_{\pm}$ are constants and f is a particular solution of equation (A.1) that does not contain $e^{\pm k\sqrt{k^2+3}|y|}$ as $y \to \pm\infty$. As $y \to \pm 0$,

$$f = \frac{(1+2\nu)}{2(3-2\nu)}|y|^{-1+2\nu}(1 + O(y^2)) + \alpha|y|^{\frac{1}{2}+\nu}(1 + O(y)) + \beta|y|^{\frac{1}{2}-\nu}(1 + O(y)); \qquad (A.3)$$

where α and β are to be determined. Also

$$f_{a,b} \sim \alpha_{a,b}|y|^{\frac{1}{2}+\nu}(1 + O(y)) + \beta_{a,b}|y|^{\frac{1}{2}-\nu}(1 + O(y)), \quad \text{as} \quad y \to \pm 0, \qquad (A.4)$$

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where the $\alpha_{a,b}$ and $\beta_{a,b}$ are fully determined by the definition of $f_{a,b}$ i.e. they are determined solely from the outer, inviscid problem. Thus, we can write

$$\psi_2^{(1)} = \frac{B_\pm^2(1+2\nu)}{2(3-2\nu)}|y|^{-1+2\nu}(1+O(y^2)) + B_\pm^2\left(g_{2\pm}|y|^{\frac{1}{2}+\nu} + h_{2\pm}|y|^{\frac{1}{2}-\nu}\right)(1+O(y)), \quad (A.5a)$$

as $y \to \pm 0$, where

$$g_{2\pm} = \alpha + C_\pm\alpha_a, \quad h_{2\pm} = \beta + C_\pm\beta_a. \quad (A.5b,c)$$

It is easy to show from these relations that

$$h_{2+} = (h_{2+}/h_{2-})(\alpha\beta_a - \beta\alpha_a)\frac{[1-(g_{2+}/g_{2-})]}{\alpha_a[(g_{2+}/g_{2-}) - (h_{2+}/h_{2-})]}. \quad (A.5d)$$

To evaluate α and β, we subtract the singular term in (A.3) by introducing

$$\tilde{f}(y) = f - \frac{(1+2\nu)}{2(3-2\nu)}\sinh^{-1+2\nu}|y|\cosh^{-2}y, \quad (A.6a)$$

so that

$$\tilde{f} = \alpha|y|^{\frac{1}{2}+\nu} + \beta|y|^{\frac{1}{2}-\nu} + \cdots, \quad \text{as} \quad y \to \pm 0, \quad (A.6b)$$

and $\tilde{f}(y)$ satisfies

$$L_2\tilde{f} = \tilde{Q}_2 \equiv \frac{(1-2\nu)}{2(3-2\nu)}\sinh^{-1+2\nu}|y|\cosh^{-2}y\left(8\tanh^2 y - (1+2\nu)\right). \quad (A.6c)$$

Then

$$\tilde{f} = \int_{\pm\infty}^{y} dz\frac{\tilde{Q}_2}{W}[f_a(y)f_b(z) - f_a(z)f_b(y)], \quad (A.7)$$

where the Wronskian $W = f_a'f_b - f_af_b' = 2\nu(\alpha_a\beta_b - \alpha_b\beta_a) = -2k\sqrt{k^2+3}$. Considering $y \to 0$, we see that

$$\alpha = \int_0^\infty \frac{\tilde{Q}_2}{W}(\alpha_b f_a - \alpha_a f_b)dz \quad \text{and} \quad \beta = \int_0^\infty \frac{\tilde{Q}_2}{W}(\beta_b f_a - \beta_a f_b)dz. \quad (A.8)$$

Appendix B

Again following Churilov & Shukhman (1988), in accordance with (4.34), we write

$$M_+ - M_- = (M_+ - M_-)_{11} + (M_+ - M_-)_{02} + (M_+ - M_-)_{31}, \quad (B.1)$$

and after extensive calculations we obtain

$$(M_+ - M_-)_{11} = \frac{i^{\frac{1}{2}-7\nu}k^{\frac{9}{2}-3\nu}2^{-\frac{3}{2}+\nu}(1+2\nu)\Gamma^2(\frac{1}{2}+\nu)}{\pi^2\Gamma(\frac{1}{2}-\nu)} \int_0^\infty dt \int_0^\infty dt_4 \int_C dt_1 \int_C dt_2$$

$$\int_C dt_3 B(\tau - t - t_4 - t_1)B(\tau - t - t_4 - t_2)B(\tau + t + t_1 + t_2 + t_4 - t_3)\overline{B}(\tau - t - t_4 - t_3)\times$$

$$(t_1 t_2 t_3)^{-\frac{3}{2}-\nu}t_4^{\frac{5}{2}+\nu}(t_3 - t - t_1 - t_2 - 2t_4)^{-\frac{3}{2}-\nu}(t_3 - t - t_1 - t_2)^{-\frac{3}{2}-\nu}(t_3 - t_1 - t_2)^{\frac{1}{2}+\nu}\times$$

$$\left(t_3\left[t_1(t_1 - t_3)^2 + t_2(t_2 - t_3)^2 + t_1 t_2(2t_3 - t_1 - t_2)\right] - t_1^{2+2\nu}t_2(t_1 + t_2)^{-\frac{3}{2}-\nu}\times \right.$$

$$\left. (t_1 - t_2)^{\frac{1}{2}-\nu}(t_1 + t_2 + t_3)(2t_3 - t_1 - t_2)F\left(-\frac{1}{2} - \nu, -\frac{1}{4} - \frac{\nu}{2}; \frac{3}{4} - \frac{\nu}{2}; \frac{t_2^2}{t_1^2}\right)\right)\times$$

$$H(t_1 - t_2)H(t_3 - t - t_1 - t_2 - 2t_4), \tag{B.2}$$

$$(M_+ - M_-)_{02} = \frac{2^{\frac{1}{2}+\nu}i^{\frac{1}{2}-5\nu}k^{\frac{9}{2}-3\nu}(1+2\nu)\Gamma(\frac{1}{2}+\nu)}{\pi\Gamma^2(\frac{1}{2}-\nu)} \int_0^\infty dt \int_0^\infty dt_1 \int_C dt_2$$

$$\int_C dt_3 B(\tau - t - t_1)B(\tau - t - t_2)B(\tau + t + t_1 + t_2 - t_3)\overline{B}(\tau - t - t_3)(tt_1)^{\frac{1}{2}+\nu}\times$$

$$(t_2 t_3)^{-\frac{1}{2}-\nu}(t_3 - 2t - t_1 - t_2)^{-\frac{1}{2}-\nu}(t_1 - t_2)^{\frac{1}{2}-\nu}(t_1 + t_2)^{-\frac{3}{2}-\nu}(2t + t_1 + t_2)\times$$

$$(t + t_1 + t_2)F\left(-\frac{1}{2} - \nu, -\frac{1}{4} - \frac{\nu}{2}; \frac{3}{4} - \frac{\nu}{2}; \frac{t_2^2}{t_1^2}\right)H(t_1 - t_2)H(t_3 - 2t - t_1 - t_2), \tag{B.3}$$

$$(M_+ - M_-)_{31} = \frac{2^{-\frac{1}{2}+\nu}i^{\frac{1}{2}-5\nu}k^{\frac{9}{2}-3\nu}(1+2\nu)^2\Gamma(\frac{1}{2}+\nu)}{\pi\Gamma^2(\frac{1}{2}-\nu)} \int_0^\infty dt \int_0^\infty dt_1 \int_0^\infty dt_3 \int_C dt_2$$

$$\int_C dt_4 B(\tau - t - t_3 - t_1)B(\tau - t - t_3 - t_1)B(\tau + 2t + t_1 + t_2 + t_3 - t_4)\overline{B}(\tau - t_3 - t_4)\times$$

$$(t_1 t_3)^{\frac{1}{2}+\nu}t_2^{-\frac{1}{2}-\nu}t_4^{-\frac{3}{2}-\nu}(t_4 - 3t - -t_1 - t_2 - 2t_3)^{-\frac{3}{2}-\nu}(t_1 + t_2)^{-\frac{3}{2}-\nu}(t_1 - t_2)^{\frac{1}{2}-\nu}\times$$

$$(t_4 - 3t - 2t_3)^{\frac{1}{2}+\nu}(t_4 - 2t_3)^{-\frac{3}{2}-\nu}(t_4 - 3t - 2t_1 - 2t_2 - 2t_3)(2t_4 - 6t - 3t_1 - 3t_2 - 4t_3)\times$$

$$(t_4^2 - t_3^2)F\left(-\frac{1}{2} - \nu, -\frac{1}{4} - \frac{\nu}{2}; \frac{3}{4} - \frac{\nu}{2}; \frac{t_2^2}{t_1^2}\right)H(t_1 - t_2)H(t_4 - 3t - t_1 - t_2 - 2t_3). \tag{B.4}$$

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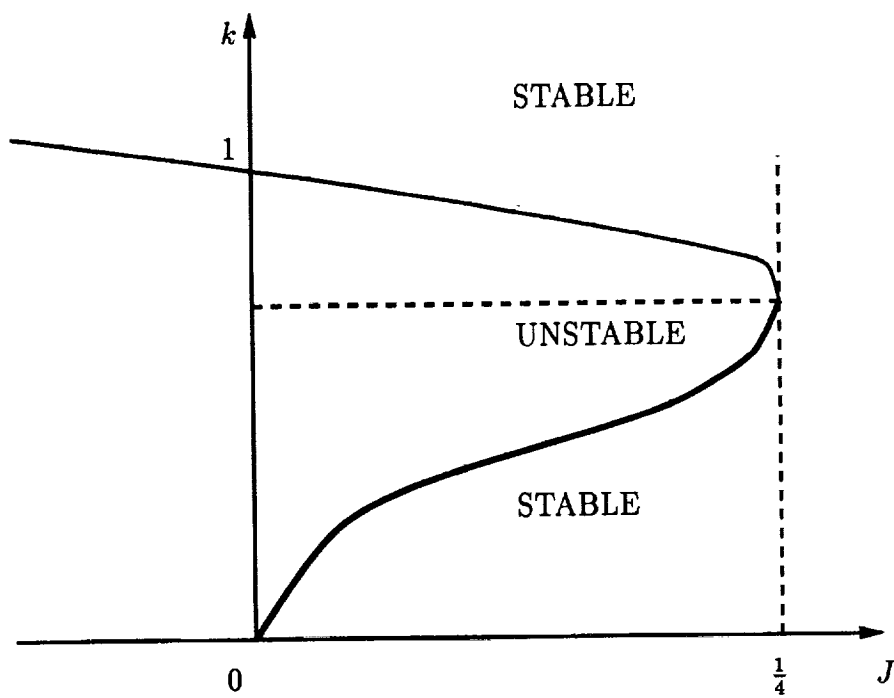
Figure 2.1. Drazin's neutral curve, $J = k^2(1 - k^2)$, with the thick line denoting the part of the neutral curve with ν +ve and the thin line that part with ν -ve.
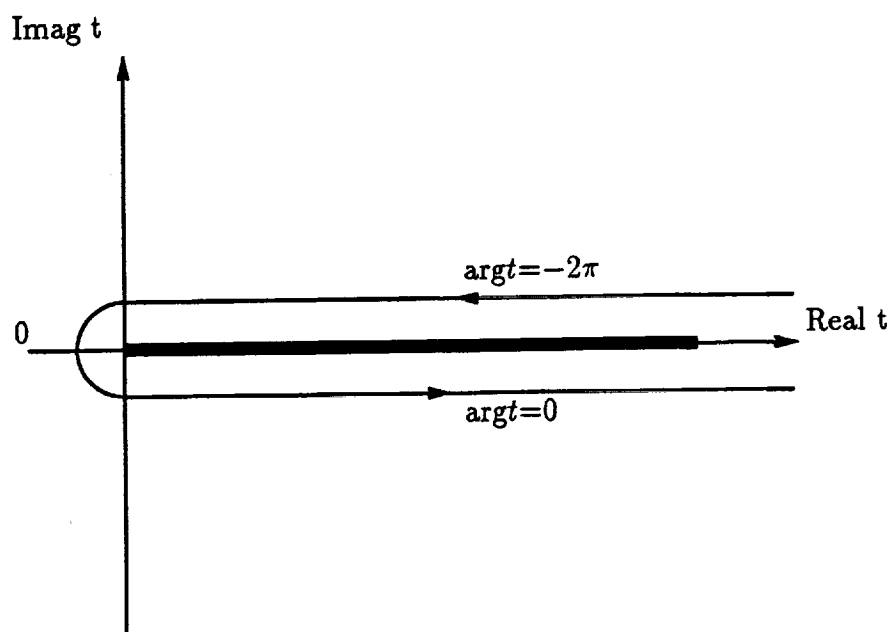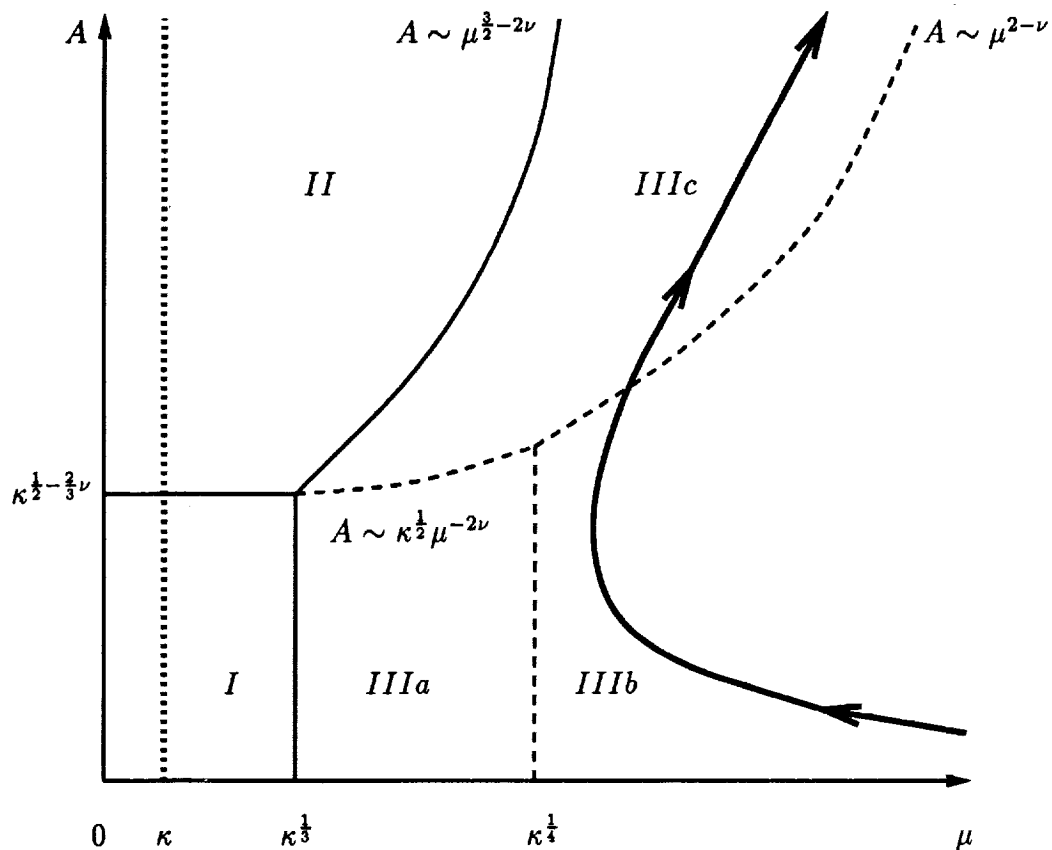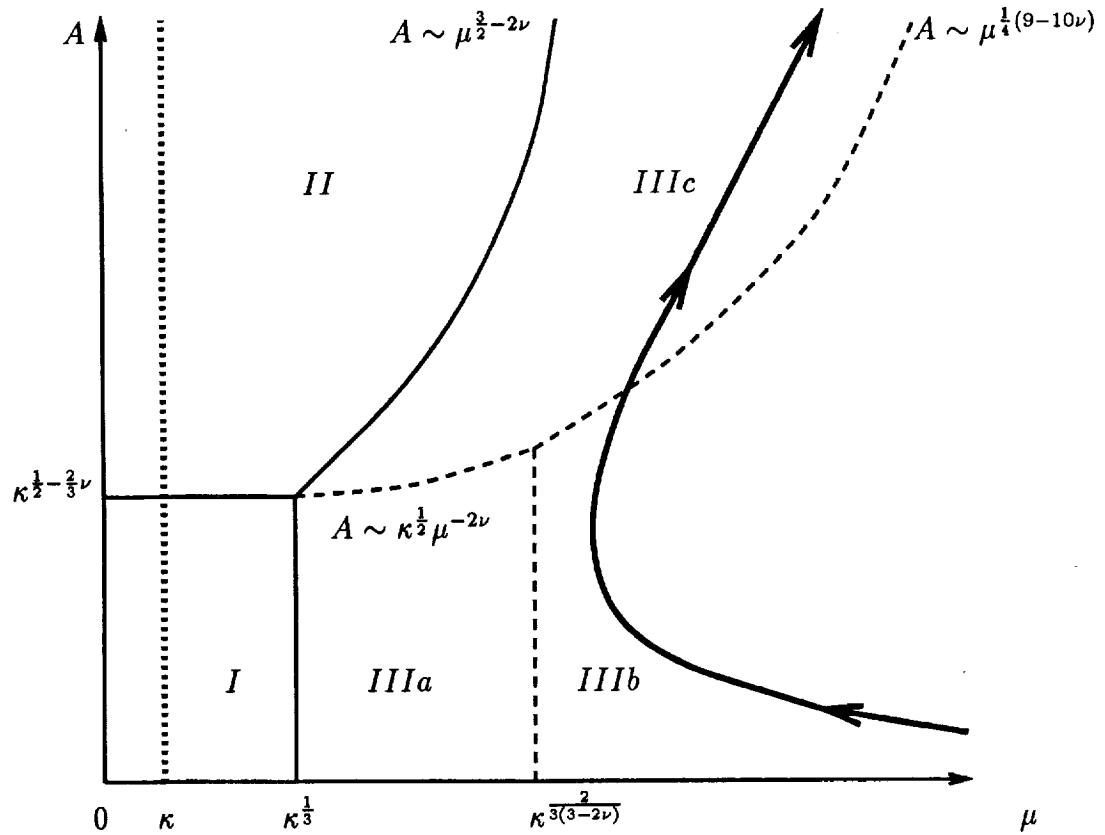
Figure 4.1 The contour C.

Figure 5.1a A diagram of the various regimes of the critical layer for $\nu < 0$.

I: viscous, steady critical layer; Landau–Stuart–Watson equation.

II: strongly nonlinear, equilibrium critical layer; Benney & Bergeron theory.

$IIIa$: unsteady critical layer; largest term in integro–differential equation (IDE) is cubic and due to viscosity.

$IIIb$: unsteady critical layer; largest term in IDE is cubic and due to J_1.

$IIIc$: unsteady critical layer; largest term in IDE is quintic.

The thick line on the diagram indicates the expected evolutionary path of the disturbance.

Figure 5.1b A diagram of the various regimes of the critical layer for $\nu > 0$.

I: viscous, steady critical layer; Landau–Stuart–Watson equation.

II: strongly nonlinear, equilibrium critical layer; Benney & Bergeron theory.

$IIIa$: unsteady critical layer; largest term in IDE is cubic and due to viscosity.

$IIIb$: unsteady critical layer; largest term in IDE is cubic and due to OCF.

$IIIc$: unsteady critical layer; largest term in IDE is quintic.

The thick line on the diagram indicates the expected evolutionary path of the disturbance.

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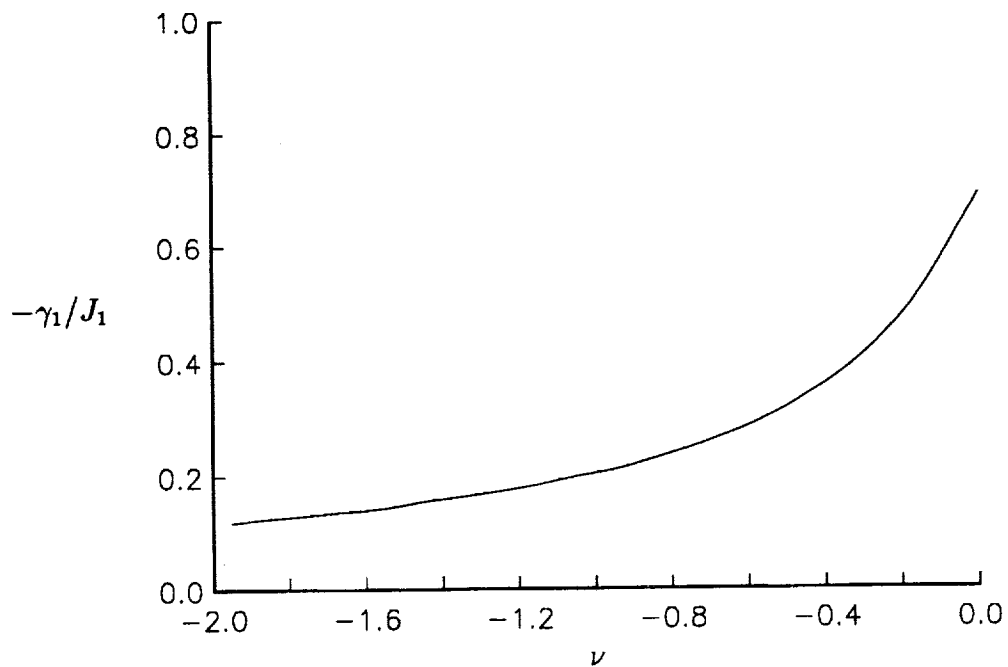
Figure 5.2a The constant $-\gamma_1/J_1$ for $-2 < \nu < 0$.
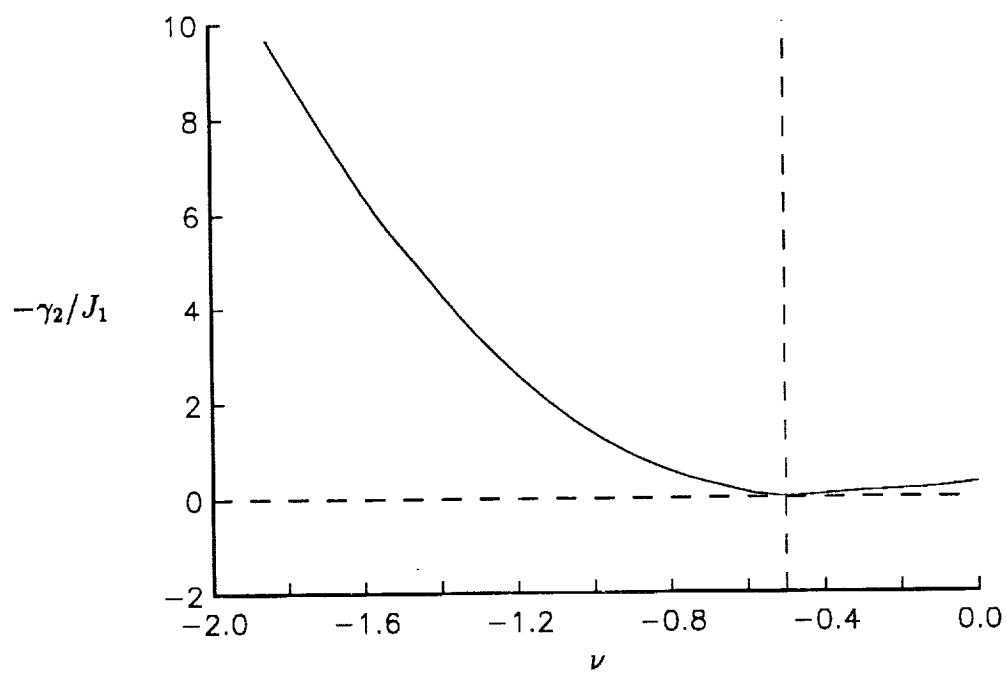
Figure 5.2b The constant $-\gamma_2/J_1$ for $-2 < \nu < 0$.

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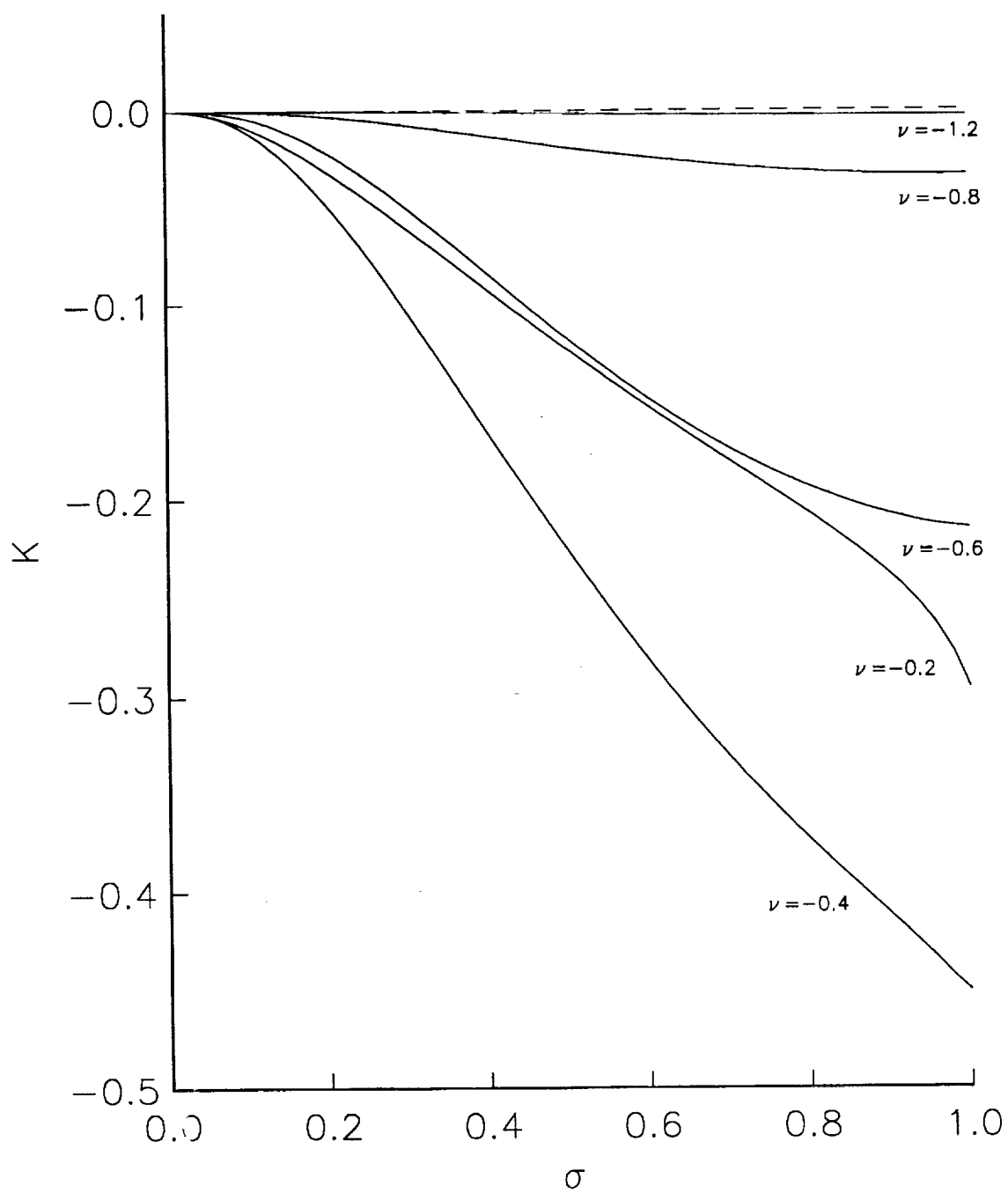
Figure 5.3 The kernel K for various values of ν.
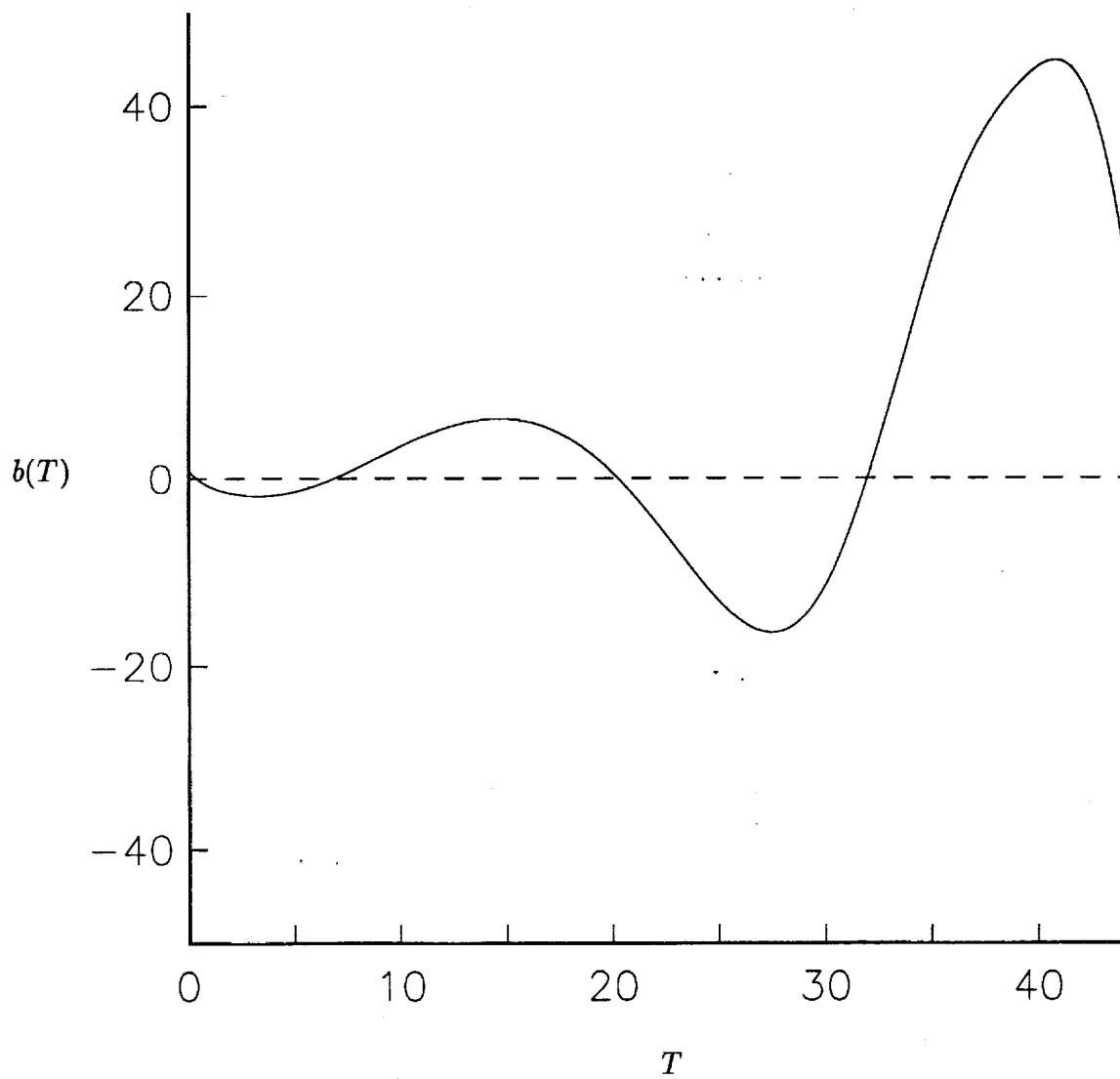
Figure 5.4a Solution of the J_1 evolution equation for $\nu = -0.4$.
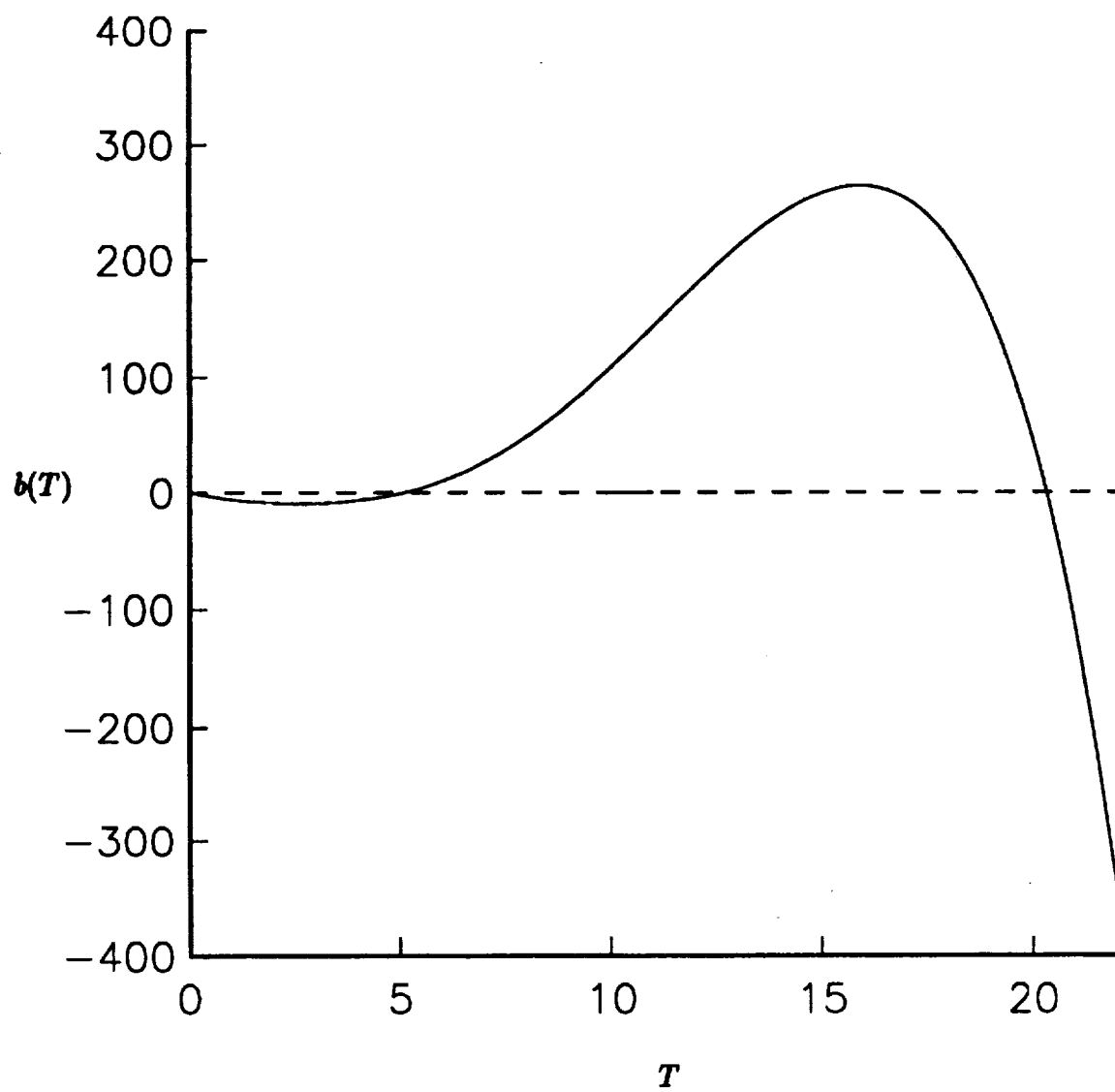
Figure 5.4b Solution of the J_1 evolution equation for $\nu = -1.4$.

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13. ABSTRACT (Maximum 200 words)

The nonlinear development of disturbances in stratified shear flows (having a local Richardson number of value less than one quarter) is considered. Such modes are initially fast growing but, like related studies, we assume that the viscous, non-parallel spreading of the shear layer results in them evolving in a linear fashion until they reach a position where their amplitudes are large enough and their growth rates have diminished sufficiently so that amplitude equations can be derived using weakly nonlinear and non-equilibrium critical-layer theories. Four different basic integro-differential amplitude equations are possible, including one due to a novel mechanism; the relevant choice of amplitude equation, at a particular instance, being dependent on the relative sizes of the disturbance amplitude, the growth rate of the disturbance, its wavenumber and the viscosity of the fluid. This richness of choice of possible nonlinearities arises mathematically from the indicial Frobenius roots of the governing linear inviscid equation (the Taylor-Goldstein equation) not, in general, differing by an integer. The initial nonlinear evolution of a mode will be governed by an integro-differential amplitude equations with a cubic nonlinearity but the resulting significant increase in the size of the disturbance's amplitude leads on to the next stage of the evolution process where the evolution of the mode is governed by an integro-differential amplitude equations with a quintic nonlinearity. Continued growth of the disturbance amplitude is expected during this stage, resulting in the effects of nonlinearity spreading to outside the critical level, by which time the flow has become fully nonlinear.

14. SUBJECT TERMS

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