

63/02
1N-02-CR
187794
100P

NASA Contractor Report

**Guidance Law Development for Aeroassisted
Transfer Vehicles Using Matched Asymptotic Expansions**

Anthony J. Calise and Nahum Melamed
Georgia Institute of Technology
School of Aerospace engineering
Atlanta, Georgia 30332

October, 1993

Prepared for
NASA Langley Research Center
under Grant NAG-1-1139

(NASA-CR-194523) GUIDANCE LAW
DEVELOPMENT FOR AEROASSISTED
TRANSFER VEHICLES USING MATCHED
ASYMPTOTIC EXPANSIONS Final
Technical Report, 1 May 1990 - 1
Oct. 1993 (Georgia Inst. of Tech.)
100 p

N94-14538

Unclas

63/02 0187799



Table of Contents

Section	Page
1. Introduction	1
1.1 Foreword	1
1.2 Relationship to Earlier Results	2
1.2.1 Numerical Optimization	2
1.2.2 Analytical Studies	2
1.2.3 MAE Studies	3
1.2.4 MAE Analysis of the HJB Equation	5
1.3 Contributions	6
1.4 Outline of the Report	7
Nomenclature	9
2. Matched Asymptotic Expansion Analysis Procedure	11
2.1 Conceptual Layout	11
2.2 A Simple Example	17
2.2.1 Problem Formulation	17
2.2.2 Outer Solution	17
2.2.3 Inner Solution	18
2.2.4 Zeroth Order Matching Condition	19
2.2.5 Zeroth Order Composite Solution	19
2.2.6 Enforcing Boundary Conditions	20
2.2.7 Location of the Initial Condition	21
3. Inclination Change With Minimum Energy Loss	23
3.1 Problem Formulation	23
3.2 Zeroth Order Outer Solution	27
3.3 Zeroth Order Inner Solution	28
3.4 Matching Conditions	32
3.4.1 State Matching Conditions	33

Table of Contents (cont.)

Section	Page
3.4.2 Costate Matching Conditions	34
3.5 Zeroth Order Composite Solution	37
3.6 Enforcing Boundary Conditions	40
3.7 Zeroth Order MAE Numerical Solution	42
3.7.1 Solution Procedure	42
3.7.2 Numerical Results	46
3.8 Zeroth Order Guided Solution	50
3.8.1 Solution Procedure	50
3.8.2 Numerical Results	51
4. Matched Asymptotic Expansion of the Hamilton-Jacobi-Bellman Equation	 57
4.1 Singularly Perturbed Hamilton-Jacobi-Bellman Equation	57
4.1.1 First Order Solution for the Case $t_0 \geq 0$	62
4.1.2 A Simple Example	65
4.1.3 First Order Solution for the Case $t_0 \leq 0$	68
4.2 Application to Aeroassisted Plane-Change	71
4.3 First Order Guided Solution	73
5. Conclusions and Recommendations	75
5.1 Conclusions	75
5.2 Recommendations	77
5.2.1 Completion of the First Order Correction analysis of the Aeroassisted Plane Change Problem	 77
5.2.2 Aerodynamic Heating Requirements	77
5.2.3 Atmospheric and Parametric Uncertainties	78
5.2.4 Expansion of the Euler Equations	79
5.2.5 Other Problem Formulations	79
5.2.6 Parametric Studies	80
A. Relationship for the Zeroth Order Control σ	81

Table of Contents (cont.)

<u>Section</u>	<u>Page</u>
B. Initial Guess and Evaluation of the Inner Integration Constants	83
B.1 Initial Guess Procedure	83
B.2 Inner Integration Constants	84
C. Relationship Between the Zeroth and the First Order Matching Conditions	86
Bibliography	88



List of Illustrations

Figure	Page
2.1 Aeroassisted Orbit Transfer	11
2.2 Typical Skip Trajectory	12
2.3 Illustration of Outer, Inner and Composite Solutions	13
2.4 Exact, Composite, Outer and Inner State Solutions for initial Condition $x(0.0)=0.0$	22
2.5 Exact, Composite, Outer and Inner State Solutions for initial Condition $x(0.2)=0.5$	22
3.1 Coordinate Systems	24
3.2 Example of Converged Inner Newton Search for w as a Function of ψ	44
3.3 Summary of the Numerical Solution Procedure	45
3.4 Zeroth Order MAE Inner, Outer and Composite γ Solution	47
3.5 Zeroth Order MAE Inner, Outer and Composite Velocity Solution	47
3.6 Zeroth Order MAE Inner, Outer and Composite P_γ Solution	48
3.7 Zeroth Order MAE Inner, Outer and Composite P_u Solution	48
3.8 Zeroth Order MAE Normalized Lift Control	49
3.9 Zeroth Order MAE Bank Angle Control	49

List of Illustrations (cont.)

Figure	Page
3.10 Zeroth Order MAE Altitude History	50
3.11 Optimal and Guided γ Solutions	52
3.12 Optimal and Guided Velocity Solutions	53
3.13 Optimal and Guided P_γ Solutions	53
3.14 Optimal and Guided P_u Solutions	54
3.15 Optimal and Guided Normalized Lift Control	54
3.16 Optimal and Guided Bank Angle Control	55
3.17 Optimal Loh's Term	55
3.18 Optimal and Guided Altitude Solution	56
3.19 Optimal and Guided Performance	56
4.1. Inner and Outer Expansions of P in the Left and Right Regions	68
4.2. Exact and zeroth Order Guided Solutions of the Costates and first Order Calculation	74

Guidance Law Development for Aeroassisted Transfer Vehicles Using Matched Asymptotic Expansions

Anthony J. Calise* and Nahum Melamed**
Georgia Institute of Technology, Atlanta, GA 30332

Summary

This report addresses and clarifies a number of issues related to the Matched Asymptotic Expansion (MAE) analysis of skip trajectories, or any class of problems that give rise to inner layers that are not associated directly with satisfying boundary conditions. The procedure for matching inner and outer solutions, and using the composite solution to satisfy boundary conditions is developed and rigorously followed to obtain a set of algebraic equations for the problem of inclination change with minimum energy loss.

A detailed evaluation of the zeroth order guidance algorithm for aeroassisted orbit transfer is performed. It is shown that by exploiting the structure of the MAE solution procedure, the original problem, which requires the solution of a set of 20 implicit algebraic equations, can be reduced to a problem of 6 implicit equations in 6 unknowns. A solution that is near optimal, requires a minimum of computation, and thus can be implemented in real time and on-board the vehicle, has been obtained. Guidance law implementation entails treating the current state as a new initial state and repetitively solving the zeroth order MAE problem to obtain the feedback controls.

Finally, a general procedure is developed for constructing a MAE solution up to first order, of the Hamilton-Jacobi-Bellman equation based on the method of characteristics. The development is valid for a class of perturbation problems whose solution exhibits two-time-scale behavior. A regular expansion for problems of this type is shown to be inappropriate since it is not valid over a narrow range of the independent variable. That is, it is not uniformly valid. Of particular interest here is the manner in which matching and boundary conditions are enforced when the expansion is carried out to

* Professor, School of Aerospace Engineering.

** Graduate Research Assistant.

first order. Two cases are distinguished - one where the left boundary condition coincides with, or lies to the right of, the singular region, and another one where the left boundary condition lies to the left of the singular region. A simple example is used to illustrate the procedure where the obtained solution is uniformly valid to $O(\epsilon^2)$. The potential application of this procedure to aeroassisted plane change is also described and partially evaluated.

Section I

Introduction

1.1 Foreword

Aeroassisted transfer problems concern the optimal maneuver of a space vehicle operating in vacuum around a planet with occasional passage through its atmosphere. As a rule, aerodynamic force is a dissipative force which has the effect of decreasing the vehicle's total energy. However, in terms of the fuel consumption, an aerodynamic maneuver can be inserted to achieve a transfer of the vehicle from an initial orbit to a destination orbit advantageously, compared to a purely propulsive exoatmospheric maneuver. Typical examples are a plane rotation of a space vehicle orbiting about a planet, and an aerobrake maneuver, in which the vehicle is to be transferred from a highly energetic hyperbolic orbit to a low energy elliptical or circular orbit, thereby ensuring its capture by the planet's gravitational field. Aeroassisted transfer problems are characterized by a maneuver phase in which the motion is dominated by atmospheric forces, and entry and exit phases where the motion is dominated by gravitational and inertial forces. In general, propulsive force can be used in any of the phases although in the so called aeroglide problem, it is not used in the aerodynamic part of the maneuver. The optimization of problems such as the plane change maneuver consists of guiding the vehicle using aerodynamic force, while minimizing the energy loss. Propulsive forces are used only in the exoatmospheric parts to deorbit and to compensate for the energy loss at the end of the plane change maneuver.

An objective in any guidance study related to aeroassisted orbit transfer vehicles is the development of solutions that are implementable in real time and on-board the vehicle. Therefore, solutions that are both near optimal and that require a minimum of computation are of primary interest. The ideal solution from a computational point of view, is the one which reduces the solution of the governing differential equations to a set of algebraic equations, thus eliminating the need for multiple shooting or quadrature. This solution constitutes the basis for the optimal guidance algorithm.

1.2 Relationship to Earlier Results

An extensive survey paper presented by Mease¹ gives the current status on the optimization of aeroassisted orbit transfer trajectories. In view of low cost transportation being a key to the utilization and exploration of space, important issues such as payload mass delivery capability and aerodynamic heating considerations are discussed. Aeroassisted transfer trajectories give rise to a difficult optimization problem from a guidance point of view. In general, numerical methods are required for an exact solution although approximation methods can be employed to obtain analytic solutions, or to reduce the solution to a quadrature.

1.2.1 Numerical Optimization

Examples of numerical solutions to optimal aeroassisted orbit transfer problems may be found in Ref.'s. 2-6, and in earlier works cited in these references. In Ref.'s. 2-5 a family of problems were studied in the context of optimal aeroassisted orbit transfer, including the minimization of the time integral of the flight path angle squared. The solution to this latter problem results in nearly grazing trajectories that take place in an altitude range where viscous effects are expected. It is shown that the nearly grazing solution is a useful engineering compromise between energy requirements and aerodynamic heating requirements. Optimization subject to a hard constraint on heating rate was considered in Ref. 6, by reformulation as a parameter optimization problem.

1.2.2 Analytical Studies

Approximation methods can be employed to obtain analytic solutions, or to reduce the solution to a quadrature. However, it is difficult to precisely satisfy terminal constraints using these approximate forms, because the nature of aeroassisted transfers is that the controls (typically lift and bank angle) are most effective while the vehicle is well within the atmosphere. Adjustments near the end of the maneuver to reduce terminal errors rapidly lead to control saturation.

Ref.'s. 7-11 typify the studies that have been performed on the problem of optimal aeroassisted orbit plane change, and that are directed towards obtaining analytical results. The authors of Ref.'s. 7-9 are able to integrate the state and costate equations by assuming that a quantity, known as Loh's term, $M(h,V)$, is constant over the trajectory. $M(h,V)$ represents the sum of gravitational and inertial forces, and is nearly zero for the entry phase of the maneuver, but unfortunately undergoes a rapid variation during the exit phase. In Ref. 10 a regular perturbation method is used, in which the perturbation parameter is identified as the ratio of the atmospheric scale height to the planet radius, and solutions are presented up to the first order in the perturbation parameter. The solution approach requires that quadratures be performed at each control update for the first order correction, essential to account for variations in M near the end of the trajectory. The approach applied in Ref. 11 is similar to that given in Ref. 10 except for the use of an alternate independent variable. Unfortunately, large control effort can be observed near the end of the trajectory to satisfy terminal constraints. The interesting feature in Ref.'s. 10 and 11 is that the zeroth order solution corresponds to the constant M approximation in Ref.'s. 7-9, for which complete analytic results are available.

1.2.3 MAE Studies

In Ref.'s. 12 and 13, a MAE analysis is performed in which the perturbation parameter is the same as that used in Ref. 10. In Ref. 12 the expressions for the matching conditions are simplified over those obtained in earlier studies by using the inner solution alone to satisfy initial conditions. In Ref. 13 the state equations are integrated under the assumption of constant controls, and the expressions for the matching conditions are simplified by using the outer solution to satisfy initial conditions. The matching procedure taken in Ref. 12 is used to obtain an optimal lift control solution of an atmospheric entry problem, and in Ref. 13 to approximate an atmospheric skip trajectory for fixed controls. However, these approximate matching approaches are not recommended here for guidance law development since they are valid only when the initial condition lies either far outside or well inside the atmosphere.

Singular perturbation methods for re-entry and aeroassisted transfer trajectories have, in more recent times, been explored in Ref.'s. 14-18. Ref.'s. 14-16 consider the re-entry problem. In Ref. 14 a singular perturbation parameter is artificially introduced on the

left hand sides of the altitude and flight path angle equations of motion. Several analytical guidance algorithms are derived for re-entry and evaluated by comparison to numerically obtained optimal solutions. While this approach appears useful for re-entry problems, it does not yield a satisfactory solution for aeroassisted transfer problems, because the boundary layer dynamics associated with satisfying the terminal constraints, is intractable. A suboptimal guidance algorithm is derived and evaluated in Ref. 15, which can be used in conjunction with the re-entry solutions in Ref. 14 to satisfy terminal constraints associated with aeroassisted transfer problems. In Ref. 16, a MAE analysis is performed in which the perturbation parameter is the same as that used in Ref. 10. The state equations are integrated under the assumption of constant controls, and the expressions for the matching conditions are simplified over those obtained in earlier studies by using the outer solution alone to satisfy initial conditions. Again, this approach is not recommended here for guidance law development, since it can only be used as an approximation when the initial condition lies far outside the atmosphere.

Aeroassisted transfer problems are characterized by a maneuver phase, in which the motion is dominated by atmospheric forces, and entry and exit phases, where the motion is dominated by gravitational and inertial forces. The method of Matched Asymptotic Expansion (MAE) analysis is a mathematical realization of this intuitive description, and it is fundamentally different from re-entry problems in that the boundary conditions are given outside the region of singularity where the inner layer occurs. This type of problem gives rise to inner layers (regions where state and control variables can exhibit rapid variation) that correspond to intervals where the maneuver is dominated by aerodynamic forces. Thus the inner solution is associated with the aeroassisted portion of the maneuver.

The application of MAE to aeroassisted transfer trajectories has been addressed in Ref.'s. 17 and 18. In Ref. 17, a general optimization problem is considered, and analytical results for the costate equations are presented. The problem is reduced to a set of constants of integration which are to be used to satisfy transversality conditions. Unfortunately, since the transversality conditions involve both states and costates, and since the state equations are not tractable in the inner layer, a multiple shooting method would have to be used to determine these parameters. Ref. 18 considers the development of an atmospheric guidance law for planar skip trajectories in which the controls are treated as constants to be updated at each guidance computation. This reference also identifies several inconsistencies that were encountered in the analysis, which are a consequence of an

incorrect approach in the application of MAE present both in Ref.'s. 17 and 18, and in earlier references taken from these papers.

1.2.4 MAE Analysis of the HJB Equation

In Ref. 11 a regular perturbation analysis was performed for the aeroassisted inclination change problem and guided solutions were obtained in which the control solution was corrected to first order by regular expansion of the Hamilton-Jacobi-Bellman (HJB) equation. This work is similar to the results reported in Ref. 10, wherein the procedure was originally developed. The expansion entails quadrature along a characteristic curve, which is defined by the zeroth order Euler solution for the same problem. The quadrature must be repeated at each update of the control solution. A significant feature identified in this report is that the zeroth order solution from the regular expansion in Ref. 11 corresponds the zeroth order inner solution of the MAE formulation (but with modified boundary conditions), and that the zeroth order MAE composite solution represents a significant improvement over the zeroth order solution of Ref. 11. This can be explained by the fact that the effect of gravitational and inertial forces are accounted for in the zeroth order MAE problem, whereas they are not in the zeroth order regular expansion solution where Loh's term is modeled as zero. The variations in this term are accounted for in the zeroth order outer problem of the MAE formulation. Of particular interest in this report is the manner in which matching and boundary conditions are enforced when the expansion of the HJB equation is carried out to first order. This is significantly more complex than for the case of regular expansion in Ref.'s. 10 and 11. In this report a general procedure for constructing a matched asymptotic expansion of the HJB equation, based on the method of characteristics, is developed for the first time.

In Ref. 22, a uniformly valid power series solution to the HJB equation was obtained for a class of nonlinear, singularly perturbed systems, in which a small parameter appears on the left hand side of the equations of motion. The dynamics for this type of singular perturbation formulation is composed of slow and fast state variables, and the process of matching and forming a composition solution is considerably simplified by the fact that the left and right boundary conditions occur within the regions of singularity. For this class of singularly perturbed systems the slow and fast variables separately satisfy their respective boundary conditions in the zeroth order problem. In contrast, the MAE

formulation treated here is characterized by the fact that at any instant in the trajectory, the left boundary condition (which is the current state of the vehicle) may occur either to the left, within, or to the right of the region of singularity. For the problem of aeroassisted plane-change, this region reduces to a single point, corresponding to the lowest altitude point, in the limit as the perturbation parameter approaches zero. Moreover, it is not possible to identify separate slow and fast variables. For this class of systems, the problem is singularly perturbed due to the explicit dependence that the dynamics has on both t and t/ϵ on the right hand side of the equations of motion, where t is the independent variable and ϵ is the perturbation parameter. In the aeroassisted plane-change problem this parameter is closely approximated by the ratio of the atmospheric scale height to the radius of the Earth.

1.3 Contributions

This report addresses and clarifies a number of issues related to MAE analysis of atmospheric skip trajectories, or any class of problems that give rise to inner layers that are not associated directly with satisfying boundary conditions. A key point that has been overlooked in previous studies is that the inner solution is crucially involved in satisfying the boundary conditions as well as the outer solution. Other key issues that are either lacking, or improperly dealt with, in earlier studies are: (1) the need for left and right outer solutions, (2) the role that the inner solution plays in joining the discontinuities that occur between the outer solutions through the matching conditions, and (3) the need to properly select the reference altitude used in defining the independent variable of integration in the outer solution. The procedure for matching inner and outer solutions, and using the composite solution to satisfy boundary conditions, is rigorously followed in developing for the first time a complete algebraic solution to the problem of inclination change with minimum energy loss, uniformly valid to $O(\epsilon)$.

The motivation in the research reported here is that, by its nature, the aeroassisted transfer problem is better suited to analysis by singular perturbations rather than by regular perturbation analysis. In fact, it is shown that the regular perturbation problem in Ref.'s. 10 and 11 is actually the inner part of a two time scale analysis based on singular perturbation theory. The interesting feature here is that Loh's term variations are now accounted for in the zeroth order outer solution (where the motion is Keplerian), for which

an analytic solution can be obtained. The zeroth order inner solution corresponds to the solution presented in Ref. 9, but with important differences that pertain to the treatment of boundary conditions in a MAE approach. Moreover, it is shown that this approach is more accurate in satisfying terminal constraints than the regular perturbation approach in Ref.'s. 10 and 11.

The consequence of the above results is that the problem has been reduced to a set of algebraic equations, whose solution forms the basis for a feedback guidance algorithm. We have developed a zeroth order guidance algorithm, which is based on the MAE method, and is used to perform a detailed evaluation for the aeroassisted inclination change problem. Repeated solution of the algebraic equations resulting from the MAE analysis along the trajectory, treating each current state as an initial state, constitutes a feedback guidance algorithm.

Finally, a general procedure is developed for constructing a MAE of the HJB equation based on the method of characteristics. Of particular interest here is the manner in which matching and boundary conditions are enforced when the expansion is carried out to first order, and how the characteristic curves are to be determined. It is shown that the MAE solution to the Euler system of equations associated with the zeroth order MAE problem provides the characteristic curves for the first order expansion of the HJB equation. Two cases are distinguished - one where the left boundary condition coincides with, or lies to the right of, the singular region and another one where the left boundary condition lies to the left of the singular region. A simple example is used to illustrate the procedure, where the obtained solution is uniformly valid to $O(\epsilon^2)$. The procedure's potential application to aeroassisted plane change is also described and partially evaluated.

Ref.'s. 34-40 are all the publications that have resulted from this research.

1.4 Outline of the Report

Section 2 gives a general description of our approach in applying MAE to aeroassisted transfer trajectory analysis, and establishes some notation to be followed throughout the report. Section 3 uses the approach outlined in Section 2 in analyzing the orbit transfer problem of inclination change with minimum energy loss in the atmosphere. The analysis makes use of the problem formulation and analytic results in Ref. 14 for the outer solution. Then a transformation of variables is made to solve the inner problem,

using the analytical results presented in Ref. 9. This is followed by a description of the zeroth order MAE solution procedure, which makes use of a unique approach for solving the resulting algebraic equations. Numerical results are given to evaluate the resulting solution, which indicate a need for a first order correction. Section 4 gives the procedure for constructing a matched asymptotic expansion of the HJB equation to first order, based on the method of characteristics. It is also shown in Section 4 how to determine the characteristic curves of the expansion for two distinct cases, using a simple example to illustrate the procedure. This is followed by a detailed description of the procedure's potential application to aeroassisted plane change. Appendices A-C give details pertaining to intermediate derivations.

Nomenclature

a	Integration Constant
c	Integration Constant
C	Aerodynamic Coefficient
E	Lift to Drag Ratio
g	Gravitational Acceleration
h	Normalized Altitude - Outer Independent Variable
H	Hamiltonian
J	Performance Index
k	Inner Integration Constants
m	Mass
\bar{M}	Loh's Term
P	Return Function
r	Radius from Earth's Center
s	Reference Area
t	Independent Variable, Time
u	Transformed Velocity - Outer Variable
v, v	Transformed Velocity - Inner Variable
V	Velocity
w	Transformed Altitude - Inner Variable
x	Dependent Variable

Greek Letters

β	Inverse Scale Height
δ	Vertical Control Component
ϵ	A Small Parameter
ϕ	Cross Range Angle
θ	Down Range Angle
η	Stretched Normalized Altitude

Nomenclature (cont.)

ψ	Heading
γ	Flight Path Angle
λ	Normalized Lift Coefficient
μ	Bank Angle
ρ	Atmospheric Density
σ	Horizontal Control Component
τ	Independent Variable, Stretched Time

Subscripts

D	Drag
f	Final Value
i	Initial Value
L	Lift
s	Radius and Acceleration at Reference Altitude
u	Partial Derivative with Respect to u
γ	Partial Derivative with Respect to γ
ψ	Partial Derivative with Respect to ψ

Superscripts

c	Composite Variable
i	Inner Variable
L	Left Side
o	Outer Variable
R	Right Side
*	Corresponds to Maximum Lift to Drag Ratio

Section II

Matched Asymptotic Expansion Analysis Procedure

This Section clarifies a number of issues related to the MAE analysis of skip trajectories, or any class of problems that give rise to inner layers that are not associated directly with satisfying boundary conditions. In addition, a simple example is presented to illustrate the procedure.

2.1 Conceptual Layout

In any aeroassisted transfer problem, the vehicle passes through two distinct regions, in terms of the dominating forces (Fig's. 2.1 and 2.2).

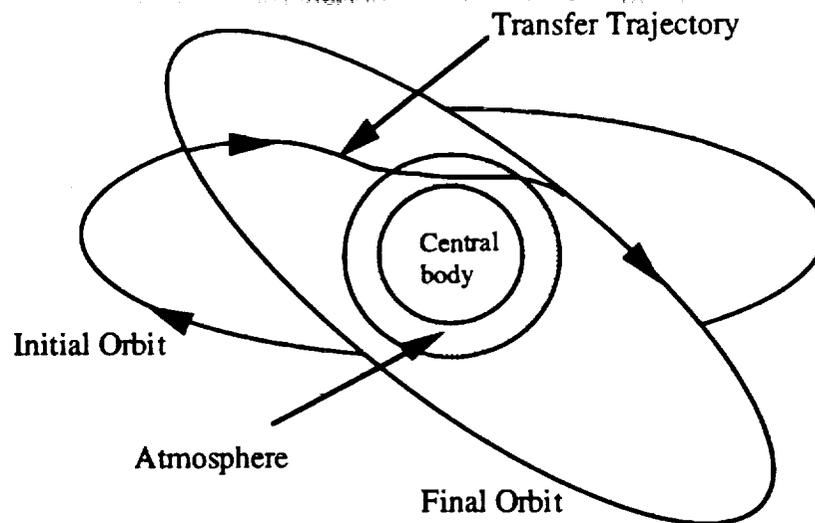


Figure 2.1. Aeroassisted Orbit Transfer

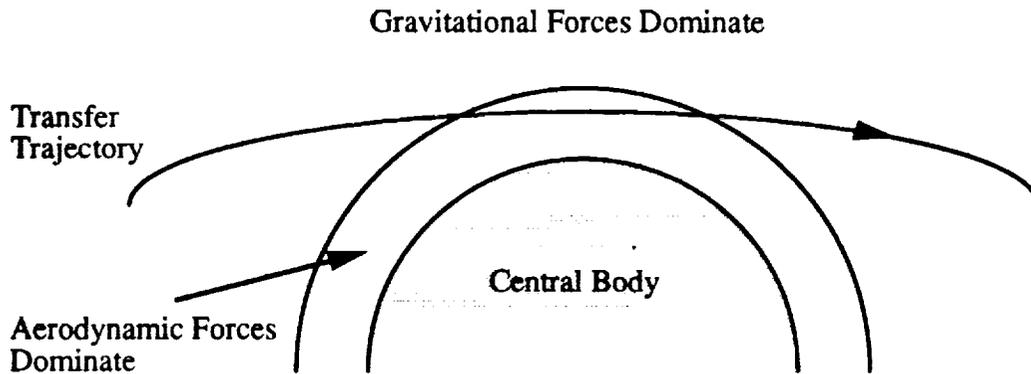


Figure 2.2. Typical Skip Trajectory

In the high altitude (outer) region, gravitational and inertial forces dominate, and the motion can be approximated as Keplerian, with the atmospheric effect considered as a perturbation. The low altitude (inner) region is dominated by aerodynamic forces, with the gravitational and the inertial effects considered as perturbations. It is crucial to note that the initial (approaching) and final (retreating) parts of the transfer trajectory are distinctly different in their trajectory parameters (when approximated as Keplerian arcs) as a consequence of what occurs in the aerodynamically dominated region.

The method of MAE can be used as a mathematical realization of the above intuitive description^{26,27}. It decomposes the total problem into two simpler subproblems (known as the inner and outer problems), which are appropriate for the separate regions of the total trajectory. The process of matching the solutions of these subproblems, and forming a composite solution, accounts for the perturbing effects as well, in a mathematically precise manner. For purposes of guidance law development, the objective is to obtain approximate analytical solutions in the outer and inner regions separately, and then to combine them to form a composite solution which is uniformly valid for the entire maneuver.

To elaborate on this idea, consider the system of equations

$$dx/dt = f(x, t, t/\varepsilon, \varepsilon) \quad (2.1)$$

The function f is assumed analytic with respect to its arguments in the region of interest, and in addition it is assumed to have the property that $\lim_{\varepsilon \rightarrow 0} f(x, t, t/\varepsilon, \varepsilon)$ exists for $t \neq 0$.

The problem is singular in that the limit is not defined at $t = 0$. Conceptually, optimal control problem formulations may appear in this form after eliminating the control using the optimality condition, so that only state and costate variables appear in the equations of motion. This results in a two point boundary value problem, with left and right boundary conditions of the form $\Psi_L(x_i, t_i) = 0$ and $\Psi_R(x_f, t_f) = 0$. Of particular interest here is the situation where $t_i < 0 < t_f$.

The solution of Eq. (2.1) is sought in the form of an asymptotic series in ϵ

$$x^\circ(t; \epsilon) = x_0^\circ(t) + \epsilon x_1^\circ(t) + \epsilon^2 x_2^\circ(t) + \dots \quad (2.2)$$

which is referred to as the "outer" expansion. The leading term in (2.2) is obtained as the solution of (2.1) for $\epsilon=0$. However, due to the singularity in f at $t = 0$, it is not a uniformly valid $O(\epsilon)$ approximation of $x(t, \epsilon)$. Note that for the situation $t_i < 0 < t_f$, two outer expansions are required, one for $t < 0$, and one for $t > 0$.

Only zeroth order solutions are used in the analysis that follows in Chapters 2 and 3, and to distinguish between the left and right zeroth order outer solutions, they are denoted by ${}^L x_0^\circ(t)$ and ${}^R x_0^\circ(t)$, where the superscript o denotes outer, the subscript 0 denotes zeroth order, and the superscripts L and R distinguish the left and right solutions. These solution segments are illustrated in Fig. 2.3, where it is important to note that in general there is a discontinuity at $t=0$.

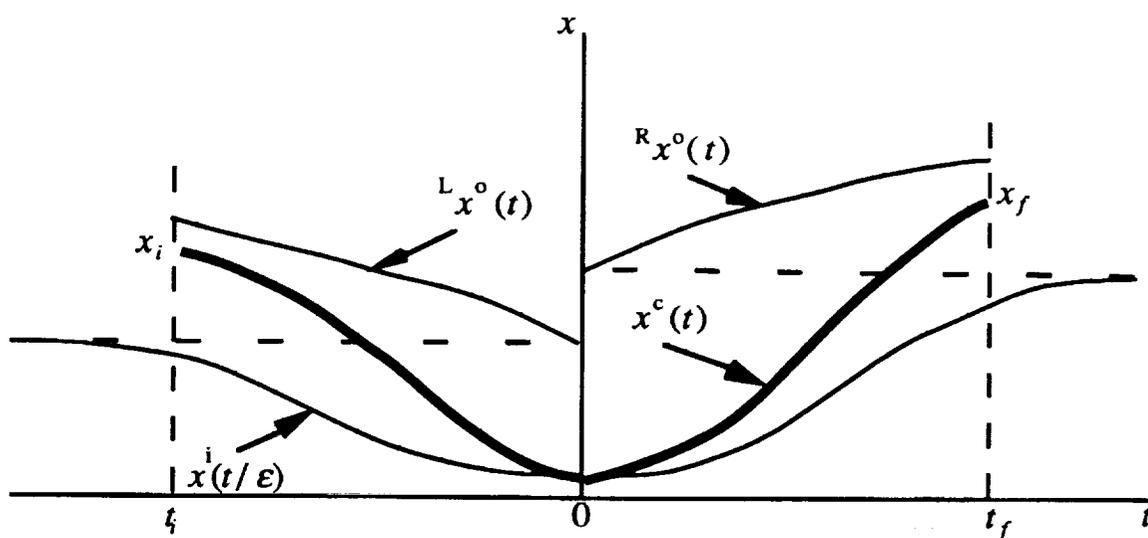


Figure 2.3. Illustration of Outer, Inner and Composite Solutions

To examine the solution in the neighborhood of $t = 0$, Eq. (2.1) is expressed in terms of a stretched independent variable $\tau = t/\varepsilon$

$$dx/d\tau = \varepsilon f(x, \varepsilon\tau, \tau, \varepsilon) \quad (2.3)$$

The solution of Eq. (2.3) is sought in the form of an asymptotic series in ε

$$x^i(\tau, \varepsilon) = x_0^i(\tau) + \varepsilon x_1^i(\tau) + \varepsilon^2 x_2^i(\tau) + \dots \quad (2.4)$$

which is referred to as the "inner" expansion. Again, only the leading term in Eq. (2.4) is considered in this Chapter and in Chapter 3.

Matching the inner solution to the outer solution is performed separately for the left and right parts of the outer solution. To zeroth order in ε , this is accomplished by enforcing the following relationships:

$$x_0^i(-\infty) = {}^L x_0^o(0), \quad x_0^i(\infty) = {}^R x_0^o(0) \quad (2.5)$$

where $x_0^i(-\infty)$, ${}^L x_0^o(0)$, $x_0^i(\infty)$ and ${}^R x_0^o(0)$ are obtained by taking appropriate limits in their respective arguments τ and t . The consequence of Eq. (2.5) on the limit behavior of $x_0^i(\tau)$ is illustrated in Fig. 2.3, where the solution is shown superimposed with $x_0^o(t)$ in terms of the original independent variable, t . The limit values in Eq. (2.5) also define the common parts of the inner and outer zeroth order solutions.

At this stage a uniformly valid composite approximation can be constructed by the method of additive composition. The additive composition is obtained by taking the sum of the outer and inner solutions and subtracting the common part. In the left part of the trajectory, the composite solution is given by:

$${}^L x_0^c(t; \varepsilon) = {}^L x_0^o(t) + x_0^i(t/\varepsilon) - {}^L x_0^o(0), \quad t < 0 \quad (2.6)$$

and in the right part:

$${}^R x_0^c(t; \varepsilon) = {}^R x_0^c(t) + x_0^i(t/\varepsilon) - {}^R x_0^c(0), \quad t > 0 \quad (2.7)$$

From Eq's. (2.6) and (2.7) it is seen that at $t=0$ the composite solution takes the form:

$${}^L x_0^c(0; \varepsilon) = {}^R x_0^c(0; \varepsilon) = x_0^i(0) \quad (2.8)$$

Thus the composite solution is continuous at $t=0$.

From the above discussion it is apparent that the inner solution (and the subsequent matching and forming of the composite solution) can be viewed as a process whereby the discontinuity between ${}^L x_0^o(t)$ and ${}^R x_0^o(t)$ at $t=0$ is taken into account. In a skip trajectory, this discontinuity is caused by the change that occurs in the trajectory parameters during the oscillating atmospheric portion of the maneuver. This point was not realized for example in Ref. 17 where a single outer solution was defined. The resulting composite solution is illustrated by the continuous bold line in Fig. 2.3.

From Fig. 2.3 it is also apparent that the composite solution must be used to satisfy the boundary conditions. In Ref.'s. 16-18 only the outer solution was used to satisfy the boundary conditions at $t=t_j$. In general, the contribution that $x_0^i(t/\varepsilon)$ makes to the boundary conditions depends on t_j and t_f . When the boundary conditions are far away from the region of influence of the inner solution, this contribution may be small. However, in an optimization problem it is well known that solutions exhibit large sensitivity to the boundary values of the costate variables. Also, in the context of developing a guidance algorithm, the initial condition should always be viewed as occurring anywhere along the trajectory. In Ref. 18 use of the outer solution alone to satisfy initial conditions led to discrepancies in the matching conditions which could not be completely resolved.

In the analysis that follows, the convention of using

$$t = h = (r - r_s)/r_s \quad (2.9)$$

is adopted, where r_s is a reference radius. While h is not monotonic, it is apparent from Fig. 2.3 that this presents no conceptual difficulty in the outer solution, since separate constants of integration are defined for entry and exit. It is however important to transform

to a monotonic independent variable when integrating the inner dynamics. For this reason the transformation used in Ref. 9 is employed. The left and right composite solutions in Eq's. (2.6) and (2.7) are then distinguished by the sign of the flight path angle.

Note that it is immediately obvious from Fig. 2.3 that $t=h=0$ corresponds to $r_s = r_{\min}$, where r_{\min} is the minimum radius over the trajectory. Thus it is essential to use the condition

$$\gamma_0^c(0; \epsilon) = \gamma_0^i(0) = 0 \quad (2.10)$$

to determine r_s , in contrast to the arbitrary selections made in Ref.'s. 17 and 18.

The approach followed in the next chapter is first to express the inner and outer solutions in terms of constants of integration. The constants of integration for the left outer solution will be viewed as the variables used to ultimately satisfy the left and right boundary conditions. The constants of integration for the inner solution will be viewed as unknowns to be determined in terms of the left outer solution constants, using the left matching condition in Eq. (2.5). The constants of integration for the right outer solution are in turn evaluated in terms of the inner solution constants, using the right matching condition in Eq. (2.5). Finally, the following relationship, which holds when the j^{th} component of x is constant in the outer solution, is to be noted

$$x_{0j}^c(t; \epsilon) = x_{0j}^i(t / \epsilon) \quad (2.11)$$

and the following relationship which holds when the j^{th} component of x is constant in the inner solution

$$x_{0j}^c(t; \epsilon) = x_{0j}^o(t) \quad (2.12)$$

which follows directly from Eq's. (2.6) and (2.7). These relationships are used several times in the analysis that follows.

2.2 A Simple Example

2.2.1 Problem Formulation

The exact dynamic is given by the following equation

$$\dot{x} = x + ue^{-t/\epsilon} / \epsilon, \quad x(t_i) \text{ given}, \quad t_f = 1, \quad |u| \leq 1 \quad (2.13)$$

where x is a scalar state variable and u is the control. Find a control u to maximize the performance index $J=x_f$. Using the maximum principle to evaluate the optimal control yields $u^{\text{opt}}=1$ and the exact solutions of the state and costate are:

$$x(t; x(t_i), t_i) = [(1 + \epsilon)x(t_i)e^{t-t_i} + e^{t-t_i-1/\epsilon} - e^{-t/\epsilon}] / (1 + \epsilon) \quad (2.14)$$

$$\lambda(t; x(t_i), t_i) = e^{1-t} \quad (2.15)$$

2.2.2 Outer Solution

The zeroth order outer dynamic is obtained by taking the limit $\epsilon \rightarrow 0$ in Eq. (2.12)

$$dx_0^\circ / dt = x_0^\circ \quad (2.16)$$

It is clear that this limit is not defined when $t=O(\epsilon)$. This is precisely what makes this problem singularly perturbed and it should not be expected that the zeroth order outer solution alone will be a zeroth order uniformly valid approximation in t in the entire domain of interest. Specifically, it is not valid in the sub-region where $t=O(\epsilon)$. We will elaborate on this more in Section 4.

The zeroth order outer Hamiltonian is defined as

$$H_0^\circ = \lambda_0^\circ x_0^\circ \quad (2.17)$$

and the zeroth order outer costate equation is given by

$$d\lambda_0^o / dt = -\partial H_0^o / \partial x_0^o = -\lambda_0^o \quad (2.18)$$

Eq's. (2.16) and (2.18) can be integrated to obtain the zeroth order outer solution:

$$x_0^o(t) = ae^t \quad (2.19)$$

$$\lambda_0^o(t) = be^{-t} \quad (2.20)$$

where a and b are integration constants.

2.2.3 Inner Solution

In the inner problem a new stretched independent variable is defined as

$$\tau = t / \varepsilon \quad (2.21)$$

and the inner exact dynamics is obtained by using Eq. (2.21) in Eq. (2.13)

$$dx^i / d\tau = \varepsilon x^i + e^{-\tau} \quad (2.22)$$

The zeroth order inner dynamics is obtained by taking the limit $\varepsilon \rightarrow 0$ in Eq. (2.22)

$$dx_0^i / d\tau = e^{-\tau} \quad (2.23)$$

It is clear that Eq. (2.23) is not a zeroth order uniformly valid approximation in τ to the exact dynamics in Eq. (2.22), because the exact dynamics is unstable, whereas the zeroth order dynamics is stable. Hence, for large τ , the zeroth order inner solution is not a valid approximation to the exact solution.

The zeroth order inner Hamiltonian is defined as

$$H_0^i = \lambda_0^i e^{-\tau} \quad (2.24)$$

and the zeroth order inner costate equation is given by

$$d\lambda_0^i / dt = -\partial H_0^i / \partial x_0^i = 0 \quad (2.25)$$

Eq's. (2.23) and (2.25) can be integrated to give the zeroth order inner solution:

$$x_0^i(\tau) = -e^{-\tau} + c \quad (2.26)$$

$$\lambda_0^i(\tau) = d \quad (2.27)$$

where c and d are integration constants.

2.2.4 Zeroth Order Matching Condition

The first step to remove the non uniformity from the zeroth order solution is to carry out the process of the matching the outer and inner solutions. To zeroth order the matching conditions entails equating the outer solution evaluated at small t (t=0) with the inner solution evaluated at large τ ($\tau \rightarrow \infty$):

$$a = x_0^o(0) = x_0^i(\infty) = c \quad (2.28)$$

$$b = \lambda_0^o(0) = \lambda_0^i(\infty) = d \quad (2.29)$$

2.2.5 Zeroth Order Composite Solution

The composite solution is constructed by the method of additive composition by taking the sum of the outer and inner solutions and subtracting the common part.

$$x_0^c(t) = x_0^o(t) + x_0^i(t/\epsilon) - x_0^o(0) \quad (2.30)$$

$$\lambda_0^c(t) = \lambda_0^o(t) + \lambda_0^i(t/\epsilon) - \lambda_0^o(0) \quad (2.31)$$

Making use of Eq's. (2.19),(2.20) and (2.26- 2.29) in Eq's. (2.30) and (2.31) results in:

$$x_0^c(t) = ae^t - e^{-t/\epsilon} \quad (2.32)$$

$$\lambda_0^c(t) = be^{-t} \quad (2.33)$$

2.2.6 Boundary Conditions

The composite solution is uniformly valid to zeroth order, hence it is used to enforce the boundary conditions. The initial condition on the state x is given in Eq. (2.13). Using it in Eq. (2.32) evaluated at t_i and solving for a yields

$$a = e^{-t_i}[x(t_i) + e^{-t_i/\epsilon}] \quad (2.34)$$

Using this in Eq. (2.32) the zeroth order composite solution for the state variable becomes

$$x_0^c(t; x(t_i), t_i) = e^{-t_i}[x(t_i) + e^{-t_i/\epsilon}]e^t - e^{-t/\epsilon} \quad (2.35)$$

The boundary condition on the costate is found from the definition of the performance index, defined as $J=x_f$, as

$$be^{-1} = \lambda_0^c(1) = \partial J / \partial x_f = 1 \quad (2.36)$$

Solving for b and using it in Eq. (2.33), the zeroth order composite solution for λ becomes

$$\lambda_0^c(t; x(t_i), t_i) = e^{1-t} \quad (2.37)$$

Note that since the zeroth order inner costate solution is constant, the composite solution is equal to the outer solution, and in this example it also equals the exact solution in Eq. (2.15).

The individual zeroth order outer and inner solutions are:

$$x_0^o(t; x(t_i), t_i) = [x(t_i) + e^{-t_i/\epsilon}]e^{t-t_i} \quad (2.38)$$

$$\lambda_0^o(t; x(t_i), t_i) = e^{t-t_i} \quad (2.39)$$

$$x_0^i(\tau; x(t_i), t_i) = -e^{-\tau} + [x(t_i) + e^{-t_i}]e^{-\tau} \quad (2.40)$$

$$\lambda_0^i(\tau; x(t_i), t_i) = e^{\tau} \quad (2.41)$$

Note that neither the inner or the outer zeroth order solutions for x , nor the inner zeroth order solution for λ , are uniformly valid zeroth order approximations to the exact solution. For the costate this is obvious since the outer solution in Eq. (2.39), which is also the composite solution, is equal to the exact solution in Eq. (2.15). The inner solution for λ in Eq. (2.41) is a zeroth order approximation to the exact solution only for the sub-region $t=O(\epsilon)$. To show that the composite solution for x is a zeroth order uniformly valid approximation to the exact solution in Eq. (2.14), ϵ is set to zero in Eq. (2.14) to get the zeroth order term in the expansion of the exact solution

$$\lim_{\epsilon \rightarrow 0} [x(t; x(t_i), t_i)] = e^{-t} [x(t_i) + e^{-t_i/\epsilon}]e^t - e^{-t/\epsilon} \quad (2.42)$$

which is precisely the zeroth order composite solution for x given in Eq. (2.35).

2.2.7 Location of the Initial Condition

The singularity in this example is located at $t=0$. Fig 2.4 gives the solution for the state x in the case $\epsilon=0.1$, $t_i=0$ and $x(t_i)=0$ that is, the initial condition is given at the singularity. Eq. (2.30) indicates that at $t=0$ the composite solution equals the inner solution. Hence the inner solution and the composite solution, but not the outer solution, satisfy the initial condition as is apparent in Fig. 2.4. It is also apparent that only the composite solution is a uniformly valid approximation to the exact solution and that the matching condition is satisfied.

In realistic situations the initial conditions, or the boundary conditions, do not in general occur at the singularity (see the discussion after Eq. (2.8)). Fig 2.5 gives the

solution for the state x in the case $\varepsilon=0.1$, $t_i=0.2$ and $x(t_i)=0.5$, that is the initial condition and the singularity do not coincide. It is evident that neither the inner solution nor the outer solution satisfy the initial condition. The composite solution does satisfy the initial condition and it is the only solution that uniformly approximates the exact solution, as before. Also note that the region of singularity is best approximated by the inner solution alone, as before. This situation basically represents either the left or the right side in Fig. 2.3 for aeroassisted maneuvers. It clarifies why two sets of matching condition are needed in aeroassisted maneuvers, and the role that the inner solution plays in joining the discontinuity of the outer solution to give a continuous composite solution at the singularity.

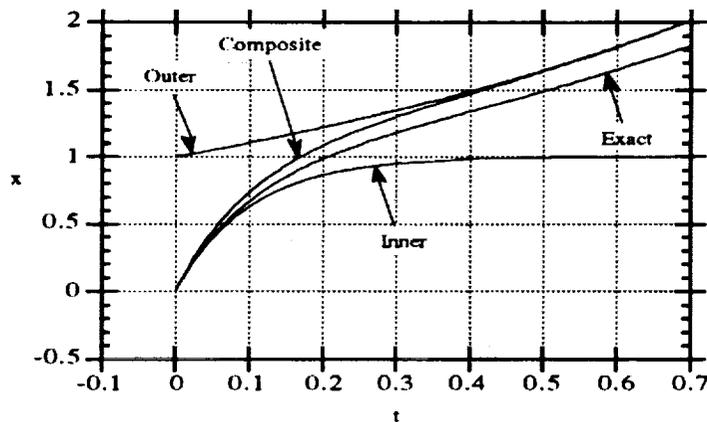


Figure 2.4 Exact, Composite, Outer and Inner State Solutions for initial Condition $x(0.0)=0.0$

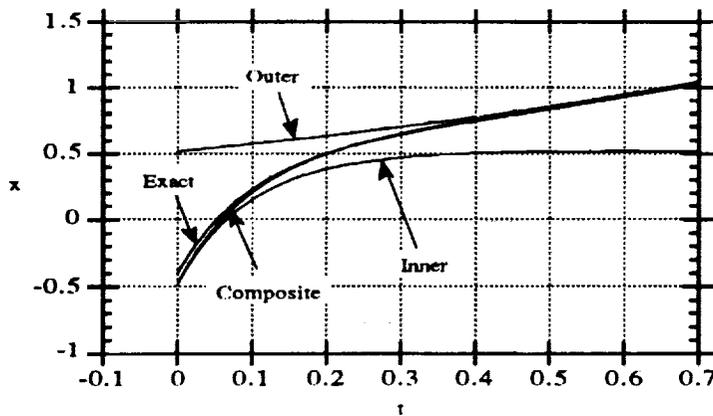


Figure 2.5 Exact, Composite, Outer and Inner State Solutions for initial Condition $x(0.2)=0.5$

Section III

Inclination Change With Minimum Energy Loss

In this Section the procedure for matching inner and outer solutions, and using the zero order composite solution to satisfy boundary conditions is rigorously followed in developing a complete algebraic solution to the problem of inclination change with minimum energy loss. Repeated solution of these algebraic equations along the trajectory, treating each current state as an initial state, constitutes a zero order feedback guidance algorithm.

3.1 Problem Formulation

The three dimensional point mass equations of motion for a lifting vehicle over a spherical non-rotating planet are given by:

$$dr/dt = V \sin \gamma \quad (3.1)$$

$$d\theta/dt = V \cos \gamma \cos \psi / r \cos \phi \quad (3.2)$$

$$d\phi/dt = V \cos \gamma \sin \psi / r \quad (3.3)$$

$$dV/dt = -\rho s C_D V^2 / 2m - g \sin \gamma \quad (3.4)$$

$$V d\gamma/dt = \rho s C_L V^2 \cos \mu / 2m - (g - V^2/r) \cos \gamma \quad (3.5)$$

$$V d\psi/dt = \rho s C_L V^2 \sin \mu / 2m \cos \gamma - V^2 \cos \gamma \cos \psi \tan \phi / r \quad (3.6)$$

where the coordinates system is defined in Fig. 3.1

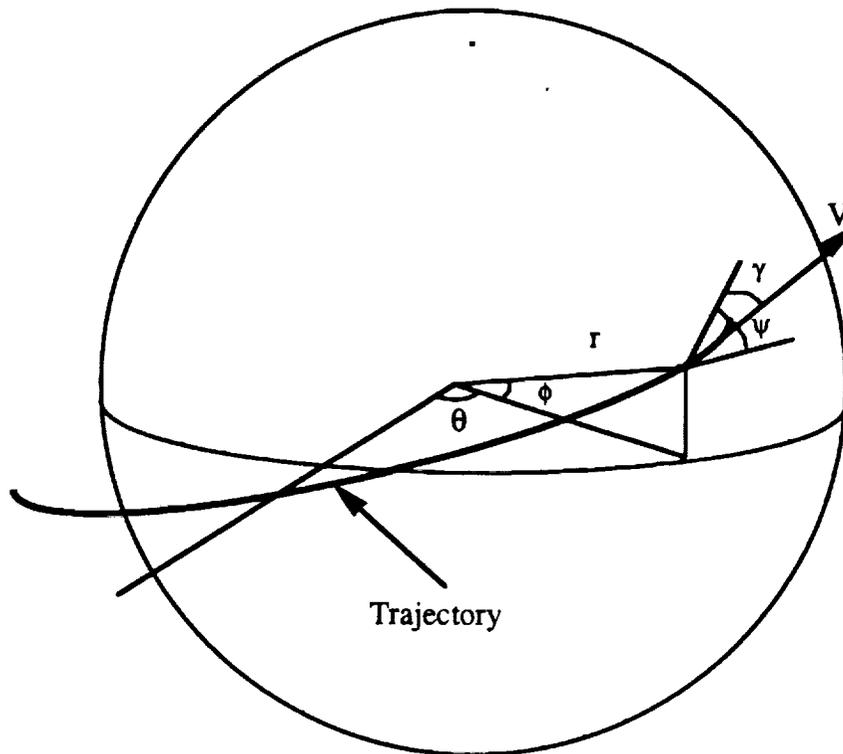


Figure 3.1. Coordinate Systems

A Newtonian gravitational field has the form

$$g(r) = g_s r_s^2 / r^2 \quad (3.7)$$

where the subscript s denotes a reference radius, defined to be the minimum trajectory radius. An exponential atmosphere model is used:

$$\rho(r) = \rho_s e^{-\beta(r-r_s)}, \quad \beta = 1/H_s \quad (3.8)$$

where H_s is the scale height. The lift and drag coefficients are assumed to be of the form

$$C_L = C_L^* \lambda \quad (3.9)$$

$$C_D = C_D^* (1 + \lambda^2) / 2 \quad (3.10)$$

where the constants C_L^* and C_D^* are the lift and drag coefficients corresponding to the maximum lift to drag ratio, and λ is the normalized lift coefficient.

Defining the following dimensionless quantities:

$$h = (r - r_s) / r_s \quad (3.11)$$

$$u = V^2 / g_s r_s \quad (3.12)$$

$$B = C_L^* \rho_s s / 2m\beta \quad (3.13)$$

$$E^* = C_L^* / C_D^* \quad (3.14)$$

$$\varepsilon = 1 / \beta r_s = H_s / r_s \quad (3.15)$$

and using h instead of t as the independent variable, the state equations can be written as follows:

$$d\theta / dh = \cos \psi \cot \gamma / (1 + h) \cos \phi \quad (3.16)$$

$$d\phi / dh = \sin \psi \cot \gamma / (1 + h) \quad (3.17)$$

$$du / dh = -Bu(1 + \lambda^2)e^{-h/\varepsilon} / \varepsilon E^* \sin \gamma - 2 / (1 + h)^2 \quad (3.18)$$

$$d\gamma / dh = B\lambda \cos \mu e^{-h/\varepsilon} / \varepsilon \sin \gamma + [1 / (1 + h) - 1 / u(1 + h)^2] \cot \gamma \quad (3.19)$$

$$d\psi / dh = B\lambda \sin \mu e^{-h/\varepsilon} / \varepsilon \sin \gamma \cos \gamma - \cos \psi \tan \phi \cot \gamma / (1 + h) \quad (3.20)$$

The objective is to achieve an inclination change with minimum fuel consumption. In Ref. 7 it is shown that for short duration maneuvers, the change in the cross range angle ϕ is small, and the change in inclination is closely approximated by the change in the heading. Moreover, fuel consumption is nearly minimized by minimizing the energy loss in the atmospheric phase of the maneuver. Furthermore, in Ref. 19 it is shown that although the actual inclination change depends on the initial inclination, the starting point of the maneuver can be timed so as to obtain the maximum inclination change that is achieved when the initial inclination is zero. A consequence of this result is that the initial plane can be taken as the plane of reference, which will be referred to as the equatorial plane. Under these assumptions, $\theta(0)$ and $\phi(0)$ can be set to zero without loss of generality, and the heading angle ψ is used to approximate the change in the inclination angle. Thus θ and ϕ become ignorable coordinates, and the equations of motion reduce to:

$$du/dh = -Bu(1 + \lambda^2)e^{-h/\epsilon} / \epsilon E^* \sin \gamma - 2/(1 + h)^2 \quad (3.21)$$

$$d\gamma/dh = B\lambda \cos \mu e^{-h/\epsilon} / \epsilon \sin \gamma + [1/(1 + h) - 1/u(1 + h)^2] \cot \gamma \quad (3.22)$$

$$d\psi/dh = B\lambda \sin \mu e^{-h/\epsilon} / \epsilon \sin \gamma \cos \gamma \quad (3.23)$$

The parameter ϵ is the ratio of the atmospheric scale height to the minimum trajectory radius. In general, r_s is nearly equal to the planet's radius, and, for the purpose of calculating ϵ to form the composite solution, it will be treated as such. For Earth, the value of ϵ is of order 10^{-3} . The controls are the normalized lift coefficient λ and the bank angle μ .

Using the definition of u given in Eq. (3.12), the objective of minimizing energy loss can be equivalently expressed as:

$$\max\{J\}, \quad J = u_f \quad (3.24)$$

The Hamiltonian function associated with the state Eq's. (3.21-3.23) has the form:

$$H = [-Bu(1 + \lambda^2)e^{-h/\epsilon} / \epsilon E^* \sin \gamma - 2/(1 + h)^2]P_u$$

$$\begin{aligned}
& + \{B\lambda \cos \mu e^{-h/\epsilon} / \epsilon \sin \gamma + [1/(1+h) - 1/u(1+h)^2] \cot \gamma\} P_\gamma \\
& + [B\lambda \sin \mu e^{-h/\epsilon} / \epsilon \sin \gamma \cos \gamma] P_\psi
\end{aligned} \tag{3.25}$$

where p_u , p_γ and p_ψ are the associated costate variables. Assuming the controls λ and μ are not beyond their limits, their optimal values are obtained as a function of the state and costate variables by solving the optimality conditions $H_\lambda=0$ and $H_\mu=0$. The resulting expressions are:

$$\lambda = E^* (P_\gamma \cos \mu + P_\psi \sin \mu / \cos \gamma) / 2uP_u \tag{3.26}$$

$$\tan \mu = P_\psi / P_\gamma \cos \gamma \tag{3.27}$$

3.2 Zero Order Outer Solution¹⁷

The zero order equations for the outer problem can be obtained by simply taking the limit as ϵ approaches zero on the right hand side of Eq's. (3.21-3.23):

$$du_0^o / dh = -2 / (1+h)^2 \tag{3.28}$$

$$d\gamma_0^o / dh = [1/(1+h) - 1/u_0^o(1+h)^2] \cot \gamma_0^o \tag{3.29}$$

$$d\psi_0^o / dh = 0 \tag{3.30}$$

The general solution for the outer system to zero order in ϵ is given by:

$$u_0^o(h) = 2[c_1 + 1/(1+h)] \tag{3.31}$$

$$\cos \gamma_0^o(h) = c_2 / (1+h) \sqrt{u_0^o(h)} \tag{3.32}$$

$$\psi_0^o(h) = c_3 \tag{3.33}$$

where c_1 , c_2 and c_3 are constants of integration. The adjoint equations are given by:

$$dP_x / dh = -H_x \quad (3.34)$$

where x is any of the state variables. In the outer region these equations to zero order in ϵ are:

$$dP_{u_0}^o / dh = -P_{\gamma_0}^o \cot \gamma_0^o / [u_0^o(1+h)]^2 \quad (3.35)$$

$$dP_{\gamma_0}^o / dh = P_{\gamma_0}^o [1/(1+h) - 1/u_0^o(1+h)^2] / \sin^2 \gamma_0^o \quad (3.36)$$

$$dP_{\psi_0}^o / dh = 0 \quad (3.37)$$

The solution to this system using Eq's. (3.31-3.33) is:

$$P_{u_0}^o(h) = -a_2 / 2u_0^o + a_1 \quad (3.38)$$

$$P_{\gamma_0}^o(h) = a_2 \tan \gamma_0^o \quad (3.39)$$

$$P_{\psi_0}^o(h) = a_3 \quad (3.40)$$

where a_1 , a_2 , and a_3 are constants of integration. Eq's. (3.31-3.33) provide the exact solution in the outer region and are the integrals of Keplerian motion that express conservation of energy and conservation of angular momentum.

3.3 Zero Order Inner Solution⁹

In the inner region, where the aerodynamic force is dominant, a new stretched altitude is defined as:

$$\eta = h/\varepsilon \quad (3.41)$$

However, in order to integrate the inner layer equations the transformations used in Ref. 9 are also adopted. These have the additional feature of transforming the independent variable from η to a monotonic independent variable, ψ . Analytic solution also requires a small flight path angle approximation.

$$\cos \gamma \approx 1, \quad \sin \gamma \approx \gamma \quad (3.42)$$

The following transformations are defined²⁰:

$$w = Be^{-\eta} \quad (3.43)$$

$$v = E^* \ln(1/g_s r_s u) \quad (3.44)$$

The controls are also transformed to vertical and horizontal components:

$$\delta = \lambda \cos \mu \quad (3.45)$$

$$\sigma = \lambda \sin \mu \quad (3.46)$$

Invoking the above transformations, Eq's. (3.21-3.23) become:

$$d\gamma/d\psi = [\delta + \varepsilon \bar{M}(\varepsilon)]/\sigma \quad (3.47)$$

$$dw/d\psi = -\gamma/\sigma \quad (3.48)$$

$$dv/d\psi = [1 + \delta^2 + \sigma^2 + \varepsilon G(\varepsilon)]/\sigma \quad (3.49)$$

where $\varepsilon\bar{M}(\varepsilon)$ is Loh's term which accounts for the gravitational and inertial effects on the motion.

$$\bar{M}(\varepsilon) = \{1 - e^{v/E^*} g_r r_s / [1 + \varepsilon \ln(B/w)]\} / w[1 + \varepsilon \ln(B/w)] \quad (3.50)$$

and

$$G(\varepsilon) = e^{v/E^*} g_r r_s E^* \sin 2\gamma / w[1 + \varepsilon \ln(B/w)]^2 \quad (3.51)$$

The approximations used in Ref. 9 to integrate the state and costate equations were that $\varepsilon\bar{M}(\varepsilon)$ is constant over the trajectory, and that the term $\varepsilon G(\varepsilon)$ is negligible. We note here that setting $\varepsilon = 0$ in Eq's. (3.47) and (3.49), to obtain the zero order inner solution, corresponds to the approximations in Ref. 9 with $\varepsilon\bar{M}(\varepsilon) = 0$. Actually, $\varepsilon\bar{M}$ is nearly zero during entry but undergoes a sharp variation during the exit phase. This is the main shortcoming to the approximation in Ref. 9. The interesting feature in this analysis is that the zero order inner solution corresponds to the $\bar{M} = 0$ solution in Ref. 9, but that \bar{M} variations are accounted for in the outer solution. It should also be pointed out that while the integrated solution bears a close resemblance to that in Ref. 9, it is totally different in the method of evaluating the constants of integration.

The transformed zero order state equations in the inner region will be obtained by letting $\varepsilon=0$ in Eq's. (3.47) and (3.49). The zero order state equations for γ_0^i and v_0^i become:

$$d\gamma_0^i / d\psi_0^i = \delta / \sigma \quad (3.52)$$

$$dv_0^i / d\psi_0^i = [1 + \delta^2 + \sigma^2] / \sigma \quad (3.53)$$

Using Eq. (3.44), the performance index Eq. (3.24), expressed in the transformed variables, takes the form:

$$\max\{J\}, \quad J = e^{-v_0/E^*} / g_r r_s \quad (3.54)$$

Note that u_f in Eq. (3.24) and v_f in Eq. (3.54) represent the composite solution of the transformed final velocity to zero order, and not the inner or outer solutions alone. The zero order Hamiltonian function in the inner region is given by:

$$H_0^i = P_{0\gamma}^i \delta / \sigma - P_{0w}^i \gamma^i / \sigma + P_{0v}^i (1 + \delta^2 + \sigma^2) / \sigma \quad (3.55)$$

and is constant since it does not depend on the independent variable ψ_0^i . The control functions are obtained from the optimality conditions $H_{0\delta}^i = 0$ and $H_{0\sigma}^i = 0$. They are given by:

$$\delta = -P_{0\gamma}^i / 2P_{0v}^i \quad (3.56)$$

$$\sigma^2 = (-P_{0w}^i \gamma^i - P_{0\gamma}^{i2} / 4P_{0v}^i) / P_{0v}^i + 1 \quad (3.57)$$

The corresponding transformed zero order costate equations in the inner region are obtained using Eq. (3.34):

$$dP_{0\gamma}^i / d\psi_0^i = P_{0w}^i / \sigma \quad (3.58)$$

$$dP_{0w}^i / d\psi_0^i = 0 \quad (3.59)$$

$$dP_{0v}^i / d\psi_0^i = 0 \quad (3.60)$$

In Appendix A it is shown that $H_0^i = 2\sigma P_{0v}^i$, and a consequence of this result and Eq. (3.60) is that σ is constant too.

At this stage the state Eq's. (3.48,3.52,3.53) and the costate Eq's. (3.58-3.60) can be integrated with respect to ψ_0^i to result in:

$$\gamma_0^i(\psi_0^i) = -k_1 \psi_0^{i2} / 2 + k_2 \psi_0^i + k_3 \quad (3.61)$$

$$w_0^i(\psi_0^i) = [k_1\psi_0^{i3} / 6 - k_2\psi_0^{i2} / 2 - k_3\psi_0^i + k_4] / \sigma \quad (3.62)$$

$$v_0^i(\psi_0^i) = (\sigma + 1/\sigma)\psi_0^i + \sigma[(\psi_0^i k_1 - k_2)^3] / 3k_1 + k_5 \quad (3.63)$$

and:

$$P_{0\gamma}^i(\psi_0^i) = (P_{0w}^i / \sigma)\psi_0^i + c \quad (3.64)$$

$$P_{0w}^i = \text{constant} \quad (3.65)$$

$$P_{0v}^i = \text{constant} \quad (3.66)$$

where k_3 , k_4 , k_5 and c , P_{0w}^i , P_{0v}^i are the constants of integration of the state and costate equations respectively. The constants k_1 and k_2 are defined in terms of these constants as:

$$k_1 = P_{0w}^i / 2\sigma^2 P_{0v}^i, \quad k_2 = -c / 2\sigma P_{0v}^i \quad (3.67)$$

3.4 Matching Conditions

The method of additive composition (as described in Section 2), is used to combine the zero order inner and outer solutions into a single, uniformly valid approximation. The additive composition is obtained by taking the sum of the solutions in the different regions and subtracting the common part. Matching implies agreement between the outer solution for small values of h ($h \rightarrow 0$), and the inner solution for large values of η ($\eta \rightarrow \infty$). Since the inner solution lies between two discontinuous outer solutions, matching is done separately in the left and right parts of the transfer trajectory, and two sets of matching conditions result. Each set of conditions involves matching of both the state and costate variables.

3.4.1 State Matching Conditions

The solution for the states in the outer region is given by Eq's. (3.31-3.33). Taking the limit $h \rightarrow 0$ yields:

$${}^L u_0^o(0) = 2(c_1^L + 1) \quad (3.68)$$

$$\cos[{}^L \gamma_0^o(0)] = c_2^L / [2(c_1^L + 1)]^{1/2} \quad (3.69)$$

$${}^L \psi_0^o(0) = c_3^L \quad (3.70)$$

where superscript L denotes the constants of integration for the left outer solution.

The solutions for the transformed inner variables are expressed with ψ_0^i as the independent variable. Conceptually, it is possible to perform an inverse transformation to express the inner solution in the original variables with η as the independent variable. However, the inverse transformation can be bypassed by first recognizing from Eq's. (3.43, 3.44) that

$${}^L w_0^i(\infty) = 0 \quad (3.71)$$

$${}^L v_0^i(\infty) = E^* \ln[1 / g_s r_s {}^L u_0^i(\infty)] = E^* \ln[1 / 2g_s r_s (c_1^L + 1)] \quad (3.72)$$

where the second relationship in Eq. (3.72) follows from Eq. (3.68) and enforcement of the matching condition ${}^L u_0^o(0) = {}^L u_0^i(\infty)$. Applying the matching condition to γ_0^i and ψ_0^i gives:

$${}^L \gamma_0^i(\infty) = {}^L \gamma_0^o(0) = \cos^{-1}\{c_2^L / [2(c_1^L + 1)]^{1/2}\} \quad (3.73)$$

$${}^L \psi_0^i(\infty) = {}^L \psi_0^o(0) = c_3^L \quad (3.74)$$

Using Eq's. (3.71-3.74) to evaluate Eq's. (3.61-3.63) at $\eta = \infty$ yields:

$$\cos^{-1}\{c_2^l / [2(c_1^l + 1)]^{1/2}\} = -k_1 c_3^{l2} / 2 + k_2 c_3^l + k_3 \quad (3.75)$$

$$0 = k_1 c_3^{l3} / 6 - k_2 c_3^{l2} / 2 - k_3 c_3^l + k_4 \quad (3.76)$$

$$E^* \ln[1/2 g_s r_s (c_1^l + 1)] = (\sigma + 1/\sigma) c_3^l + \sigma[(c_3^l k_1 - k_2)^3] / 3k_1 + k_5 \quad (3.77)$$

Eq's. (3.75-3.77) can be used to evaluate k_3 , k_4 and k_5 in terms of the constants of integration for the state variables in the left side outer solution. Recall that k_1 and k_2 have been defined in Eq. (3.67).

An exactly symmetric set of equations results from the matching conditions for the state variables on the right side:

$$\cos^{-1}\{c_2^r / [2(c_1^r + 1)]^{1/2}\} = -k_1 c_3^{r2} / 2 + k_2 c_3^r + k_3 \quad (3.78)$$

$$0 = k_1 c_3^{r3} / 6 - k_2 c_3^{r2} / 2 - k_3 c_3^r + k_4 \quad (3.79)$$

$$E^* \ln[1/2 g_s r_s (c_1^r + 1)] = (\sigma + 1/\sigma) c_3^r + \sigma[(c_3^r k_1 - k_2)^3] / 3k_1 + k_5 \quad (3.80)$$

Eq's. (3.78-3.80) relate c_1^r , c_2^r and c_3^r to the constants of integration for the state variables in the inner solution.

3.4.2 Costate Matching Conditions

The left and right matching conditions for the costate variables are defined below. The inner solution for the costate variables, corresponding to the transformed state variables, is given in Eq's. (3.64-3.66) in terms of the constants of integration c , P_{0w}^i and P_{0v}^i . To perform the matching with the outer costate solutions given in Eq's. (3.38-3.40), it is necessary to first find the corresponding expressions for P_{0u}^i and $P_{0\psi}^i$ for the inner solution. In effect, this amounts to transforming the inner solution back to the original problem variables.

The value of any costate variable at time t can be interpreted as the sensitivity of J to perturbations in the corresponding state variable at time t (Ref. 17). Thus it can be written:

$$P_u = \partial J / \partial u ; \quad P_v = \partial J / \partial v \quad (3.81)$$

From Eq. (3.44) it follows that:

$$\partial v / \partial u = -E^* / u \quad (3.82)$$

Combining Eq's. (3.81) and (3.82) it becomes apparent that

$$P_{0u}^i = P_{0v}^i (\partial v / \partial u) \Big|_0 = -E^* P_{0v}^i / u_0^i \quad (3.83)$$

To determine $P_{0\psi}^i$, it is noted that ψ_0^i is the independent variable in the transformed inner problem. Therefore, making use of the Hamilton-Jacobi-Bellman equation¹⁷ results in:

$$P_{0\psi}^i = \partial J / \partial \psi_0^i = -H_0^i \quad (3.84)$$

In Appendix A it is shown that $H_0^i = 2\sigma P_{0v}^i$, and thus:

$$P_{0\psi}^i = -2\sigma P_{0v}^i \quad (3.85)$$

Using Eq's. (3.83) and (3.85), the costate matching process is ready to be carried out. Taking the limit $h \rightarrow 0$ in Eq's. (3.38-3.40) and using Eq's. (3.68-3.70) the left side matching conditions results:

$${}^L P_{0u}^o(0) = -a_2^t / 4(c_1^t + 1) + a_1^t = {}^L P_{0u}^i(\infty) \quad (3.86)$$

$${}^L P_{0\gamma}^o(0) = a_2^t \tan\{\cos^{-1}\{c_2^t / [2(c_1^t + 1)]^{1/2}\}\} = {}^L P_{0\gamma}^i(\infty) \quad (3.87)$$

$${}^L P_{0\psi}^o(0) = a_3^L = {}^L P_{0\psi}^i(\infty) \quad (3.88)$$

Taking the limit $\eta \rightarrow \infty$ in Eq's. (3.64, 3.83, 3.85) and making use of Eq's. (3.68, 3.70, 3.74) and of the state matching condition ${}^L u_0^o(0) = {}^L u_0^i(\infty)$ leads to:

$${}^L P_{0u}^i(\infty) = -E^* P_{0v}^i / 2(c_1^L + 1) \quad (3.89)$$

$${}^L P_{0\gamma}^i(\infty) = (P_{0w}^i / \sigma) c_3^L + c \quad (3.90)$$

$${}^L P_{0\psi}^i(\infty) = -2\sigma P_{0v}^i \quad (3.91)$$

Substituting Eq's. (3.89-3.91) in Eq's. (3.86-3.88) yields:

$$-a_2^L / 4(c_1^L + 1) + a_1^L = -E^* P_{0v}^i / 2(c_1^L + 1) \quad (3.92)$$

$$a_2^L \tan\{\cos^{-1}\{c_2^L / [2(c_1^L + 1)]^{1/2}\}\} = (P_{0w}^i / \sigma) c_3^L + c \quad (3.93)$$

$$a_3^L = -2\sigma P_{0v}^i \quad (3.94)$$

Finally, in Appendix A it is shown that σ can be written as:

$$\sigma = 1 / (1 + k_2^2 + 2k_1 k_3)^{1/2} \quad (3.95)$$

Eq's. (3.92-3.94) relate c , P_{0w}^i , and P_{0v}^i to the constants of integration for the left outer costate solution.

A precisely symmetrical set of equations results from the matching conditions for the costate variables on the right side:

$$-a_2^R / 4(c_1^R + 1) + a_1^R = -E^* P_{0v}^i / 2(c_1^R + 1) \quad (3.96)$$

$$a_2^R \tan\{\cos^{-1}\{c_2^R / [2(c_1^R + 1)]^{1/2}\}\} = (P_{0w}^i / \sigma)c_3^R + c \quad (3.97)$$

$$a_3^R = -2\sigma P_{0v}^i \quad (3.98)$$

Eq's. (3.96-3.98) relate a_1^R , a_2^R , and a_3^R to the constants of integration for the inner costate solution.

3.5 Zeroth Order Composite Solution

The composite solution is constructed by the method of additive composition as outlined in Section 2. The form of the composite solution for the state and the costate variables is:

$$x_0^c(h, \varepsilon) = x_0^o(h) + x_0^i(h/\varepsilon) - x_0^o(0) \quad (3.99)$$

where now $\varepsilon = H_s/\bar{r}_s$, \bar{r}_s is the planet radius (see the comment following Eq. (3.23)) and x is any of the state or costate variables. The solution for the outer state variables is given in Eq's. (3.31-3.33) and for the inner state variables in Eq's. (3.61-3.63). The outer left solution, evaluated at $h=0$, is given in Eq's. (3.68-3.70). Using Eq. (3.44), the left zeroth order state composite solution is given by:

$${}^L u_0^c(h, \varepsilon) = 2/(1+h) + e^{-v_0^i(h/\varepsilon)/E^*} / g_s r_s - 2 \quad (3.100)$$

$$\begin{aligned} {}^L \gamma_0^c(h, \varepsilon) &= \cos^{-1}\{c_2^L / (1+h)[2(c_1^L + 1/(1+h))]^{1/2}\} \\ &\quad + \{-k_1[\psi_0^i(h/\varepsilon)]^2 / 2 + k_2\psi_0^i(h/\varepsilon) + k_3\} \\ &\quad - \cos^{-1}\{c_2^L / [2(c_1^L + 1)]^{1/2}\} \end{aligned} \quad (3.101)$$

$${}^L \psi_0^c(h, \varepsilon) = \psi_0^i(h/\varepsilon) \quad (3.102)$$

Note that the composite solution for ψ is simply the inner solution since the outer solution is constant (see the general comment at the end of Section 2). Since no outer constants of

integration appear explicitly in the left composite solutions of u and of ψ , the left and right composite solutions are identical. The right zeroth order composite solution for γ is given by:

$$\begin{aligned} {}^R\gamma_0^c(h, \varepsilon) &= \cos^{-1}\{c_2^R / (1 + h)[2(c_1^R + 1/(1 + h))]^{1/2}\} \\ &+ \{-k_1[\psi_0^i(h/\varepsilon)]^2 / 2 + k_2\psi_0^i(h/\varepsilon) + k_3\} \\ &- \cos^{-1}\{c_2^R / [2(c_1^R + 1)]^{1/2}\} \end{aligned} \quad (3.103)$$

By Eq. (3.43), w_0^i as a function of h/ε is given as:

$$w_0^i(h/\varepsilon) = Be^{-h/\varepsilon} \quad (3.104)$$

and Eq. (3.62) is used to solve for $\psi_0^i(h/\varepsilon)$ in terms of the altitude h and the constants of integration. Eq. (3.63) is then used to evaluate $v_0^i(h/\varepsilon)$. Thus Eq's. (3.100-3.103) together with Eq's. (3.62, 3.63, 3.104), provide the composite solution for the state variables as a function of h .

The form of the zeroth order composite solution of the costate variables is given in Eq. (3.99) and it is constructed the same way the composite solution for the states was obtained. The solution for the outer costate variables is given in Eq's. (3.38-3.40) and for the inner costate variables in Eq's. (3.64-3.66). The outer left solution evaluated at $h=0$ is given in Eq's. (3.86-3.88). Using Eq's. (3.44) and (3.83, 3.85), the left composite solution is given by:

$$\begin{aligned} {}^L P_{0u}^c(h, \varepsilon) &= -a_2^L / 4[c_1^L + 1/(1 + h)] \\ &- P_{0v}^i E^* / [e^{-v_0^i(h/\varepsilon)/E^*} / g_s r_s] + a_2^L / 4(c_1^L + 1) \end{aligned} \quad (3.105)$$

$$\begin{aligned} {}^L P_{0\gamma}^c(h, \varepsilon) &= a_2^L \tan\{\cos^{-1}\{c_2^L / (1 + h)2^{1/2}[c_1^L + 1/(1 + h)]^{1/2}\}\} \\ &+ (P_{0w}^i / \sigma)\psi_0^i(h/\varepsilon) + c \\ &- a_2^L \tan\{\cos^{-1}\{c_2^L / 2^{1/2}[c_1^L + 1]^{1/2}\}\} \end{aligned} \quad (3.106)$$

$${}^L P_{0\psi}^c(h, \varepsilon) = a_3^L \quad (3.107)$$

The values of $v_0^i(h/\varepsilon)$ and $\psi_0^i(h/\varepsilon)$ are found from Eq's. (3.62, 3.63) and (3.104). Eq's. (3.40) and (3.85) for $P_{0\psi}$ show that it is constant in the outer and inner regions. Hence the composite solution for $P_{0\psi}$ is also constant and is simply the outer costate constant of integration. On the right side, the composite solution takes the form:

$$\begin{aligned} {}^*P_{0u}^c(h, \varepsilon) &= -a_2^* / 4[c_1^* + 1/(1+h)] \\ &\quad - P_{0v}^i E^* / [e^{-v_0^i(h/\varepsilon)/E^*} / g_s r_s] + a_2^* / 4(c_1^* + 1) \end{aligned} \quad (3.108)$$

$$\begin{aligned} {}^*P_{0\gamma}^c(h, \varepsilon) &= a_2^* \tan\{\cos^{-1}\{c_2^* / (1+h) 2^{1/2} [c_1^* + 1/(1+h)]^{1/2}\}\} \\ &\quad + (P_{0w}^i / \sigma) \psi_0^i(h/\varepsilon) + c \\ &\quad - a_2^* \tan\{\cos^{-1}\{c_2^* / 2^{1/2} [c_1^* + 1]^{1/2}\}\} \end{aligned} \quad (3.109)$$

$${}^*P_{0\psi}^c(h, \varepsilon) = a_3^* \quad (3.110)$$

From Eq's. (3.94) and (3.98) it is seen that $a_3^i = a_3^*$, thus the zeroth order composite solution for $P_{0\psi}$ is constant all along the transfer trajectory. Eq's. (3.105-3.110) provide the zeroth order composite solution for the costate variables as a function of h .

The reference radius, r_s , is also treated as an unknown parameter in the problem. As discussed in Section 2, it is the distance to the lowest point of the transfer trajectory at which $h=0$. Using the fact that at this point the composite solution for γ is zero, provides the relationship needed to evaluate r_s . The composite solution for γ is given in Eq's. (3.101) and (3.103). At $h=0$ it becomes the inner solution evaluated at $h=0$,

$${}^i\gamma_0^c(0, \varepsilon) = {}^* \gamma_0^c(0, \varepsilon) = \gamma_0^i(0) = 0 \quad (3.111)$$

The solution for γ_0^i is given in Eq. (3.61). Equating it to 0 gives a relationship for the value of ψ_0^i corresponding to $h=0$,

$$0 = -k_1 \psi_0^i(0)^2 / 2 + k_2 \psi_0^i(0) + k_3 \quad (3.112)$$

and the solution for $\psi_0^i(0)$ is:

$$\psi_0^i(0) = k_2/k_1[1 \pm (1 + 2k_1k_3/k_2)^{1/2}] \quad (3.113)$$

This value of $\psi_0^i(0)$ is used in Eq. (3.62) to evaluate w_0^i at $h=0$

$$w_0^i(0) = [k_1\psi_0^i(0)^3/6 - k_2\psi_0^i(0)^2/2 - k_3\psi_0^i(0) + k_4]/\sigma \quad (3.114)$$

Finally, use of Eq's. (3.114) and (3.13) in Eq. (3.104) for $h=0$ yields:

$$w_0^i(0) = C_1^* \rho_s s / 2m\beta \quad (3.115)$$

from which ρ_s (and thus r_s) can be found in terms of the constants of integration of the problem.

3.6 Enforcing Boundary Conditions

To complete the solution of the problem, the initial conditions and transversality conditions at the end point must be satisfied. As discussed in Section 2, this is performed by using the composite solution, and can be thought of as the process by which the constants of integration for the left outer solution are evaluated.

Assume that the initial conditions u_i , γ_i and ψ_i are given, along with the initial radius r_i . For the purpose of exposition, assume that γ_f , ψ_f and the final radius r_f are given. The corresponding transversality condition is:

$$P_u^c(h_f) = 1 \quad (3.116)$$

where

$$h_i = (r_i - r_s) / r_s, \quad h_f = (r_f - r_s) / r_s \quad (3.117)$$

Using u_i , γ_i , ψ_i , ψ_f , γ_f and Eq. (3.116), the following equations result from enforcing the boundary conditions:

$$u_i = 2 / (1 + h_i) + e^{-v_0^i(h_i/\epsilon)/E^*} / g_s r_s - 2 \quad (3.118)$$

$$\begin{aligned} \gamma_i &= \cos^{-1} \{ c_2^i / (1 + h_i) [2(c_1^i + 1 / (1 + h_i))]^{1/2} \} \\ &+ \{-k_1 [\psi_0^i(h_i / \epsilon)]^2 / 2 + k_2 \psi_0^i(h_i / \epsilon) + k_3\} \\ &- \cos^{-1} \{ c_2^i / [2(c_1^i + 1)]^{1/2} \} \end{aligned} \quad (3.119)$$

$$\psi_i = \psi_0^i(h_i / \epsilon) \quad (3.120)$$

$$\psi_f = \psi_0^i(h_f / \epsilon) \quad (3.121)$$

$$\begin{aligned} 1 &= -a_2^s / 4 [c_1^s + 1 / (1 + h_f)] \\ &- P_{0v}^i E^* / [e^{-v_0^i(h_f/\epsilon)/E^*} / g_s r_s] + a_2^s / 4 (c_1^s + 1) \end{aligned} \quad (3.122)$$

$$\begin{aligned} \gamma_f &= \cos^{-1} \{ c_2^s / (1 + h_f) [2(c_1^s + 1 / (1 + h_f))]^{1/2} \} \\ &+ \{-k_1 [\psi_0^i(h_f / \epsilon)]^2 / 2 + k_2 \psi_0^i(h_f / \epsilon) + k_3\} \\ &- \cos^{-1} \{ c_2^s / [2(c_1^s + 1)]^{1/2} \} \end{aligned} \quad (3.123)$$

where h_i and h_f are defined in Eq. (3.117). To summarize, Eq's. (3.75-3.80, 3.92-3.98, 3.115, 3.118-3.123) constitute a set of 20 equations for the 18 unknown constants of integration c_1^i , c_2^i , c_3^i , k_3 , k_4 , k_5 , c_1^s , c_2^s , c_3^s , a_1^i , a_2^i , a_3^i , c , P_{0w}^i , P_{0v}^i , a_1^s , a_2^s , a_3^s , and for the parameters σ and r_s .

3.7 Zeroth Order MAE Numerical Solution

3.7.1 Solution Procedure

In this section, by exploiting the structure of the MAE solution procedure, it is shown how to simplify the original problem by further reducing it to a set of 6 implicit equations in 6 unknowns. The unknowns are the common parts of the inner and outer solution (which are equal to one another).

The iteration procedure involves repeated solution of the inner and outer problems using the common parts as artificial boundary conditions. The common parts are adjusted in the iteration process until the actual boundary conditions are satisfied by the composite solution. The matching conditions are enforced at each iteration by simply equating the inner and outer common parts. We also exploit the fact that heading is constant in the outer solution, which permits enforcement of the actual boundary conditions on heading using the inner solution alone. This allows enforcement of two of the six boundary conditions at each iteration, and further reduces the problem to four equations in four unknowns, for which a Newton method is used to obtain a solution. Representative zeroth order MAE approximate solutions were calculated to exhibit the resulting inner, outer and composite solutions.

The vehicle used in this study is the Maneuverable Research Re-entry Vehicle (MRRV). Its aerodynamic and mass characteristics are presented in Ref. 24. The maximum lift to drag ratio is 2.362, the lift coefficient corresponding to $(L/D)_{\max}$ is $C_L^* = 0.1512$, the reference area is $s = 11.69 \text{ m}^2$ and the mass is $m = 4898.7 \text{ kg}$. The constants needed for the exponential atmospheric density function are obtained by fitting to the standard atmosphere densities at 30 and 60 km. The resulting value of the scale height is $H_S = 1/\beta = 7625.4 \text{ m}$.

A Newton method was first tried to solve for the 20 unknowns. This approach was not successful due to the complexity of the relationships. An alternative approach was then derived wherein the number of coupled equations was reduced to 6 by defining the unknowns to be the common parts of the separate inner and outer solutions. The equations that determine these unknowns are the original boundary conditions enforced on the composite solutions. The basic idea is to first use the common parts as artificial boundary conditions to evaluate the constants of integration in Eq's. (3.31-3.33, 3.38-3.40) and (3.61-3.66), starting with an initial guess. Then a Newton method is used to iterate on the

common parts until the original boundary conditions are satisfied by the composite solution.

This procedure was slightly modified to capitalize on the structure of the MAE solution for this problem. Since ψ_0° is constant in the outer solution, it follows from Eq. (3.102) and the matching conditions that ψ_0° (the zeroth order composite solution for ψ) is simply the zeroth order inner solution (ψ_0^i). This allows the boundary conditions on ψ to be enforced using only the inner solution at each stage of the iteration process. Nested Newton iteration algorithms were implemented to determine the 6 unknowns using the boundary conditions for the composite solution. The inner iteration procedure corresponds to solving the inner problem with the boundary conditions on ψ enforced. The outer iteration process is used to enforce the four remaining boundary conditions on the composite solution.

The first step in the procedure is to calculate the 6 inner solution integration constants $k_3, k_4, k_5, c, P_{0w}^i$, and P_{0v}^i in Eq's (3.61-3.66) using a guess for the common part values of γ_0^i, v_0^i on the left and for the common part values of γ_0^i, P_{0v}^i on the right (the constants k_1 and k_2 are given in terms of the other constants in Eq. (3.67)). Note that an initial guess for the common parts of w_0^i is not needed since the common part corresponds to $\eta \rightarrow \infty$. Hence it follows from Eq. (3.43) that the common part values for w_0^i are zero on both the left and right sides. This particular arrangement concerning which of the common part values are treated as unknowns was chosen to agree with the actual boundary conditions for the original problem, which greatly simplifies the problem of forming an initial guess. The procedure of forming an initial guess and the equations for evaluating the constants of integration are given in Appendix B.

Next, an inner loop Newton search is performed on the left and right common part values of ψ_0^i (using only the inner solution) so that the boundary conditions $\psi_0^i(w_i) = \psi_i$ and $\psi_0^i(w_f) = \psi_f$ are satisfied, where w_i and w_f correspond to $r = r_i$ and $r = r_f$ in Eq's. (3.43) and (3.117). This is done by solving the cubic equation in Eq. (3.62) while taking into account that $w_0^i(\psi)$ for the range of ψ of interest must be positive for the solution to have physical meaning. The inner solution procedure also determines r_s and σ (see Appendix B).

Fig. 4.1 presents an example of a converged solution of this equation for $\psi_i=0^\circ$, $\psi_f=20^\circ$. The resulting left and right common parts of ψ_0^i are -1.10° and 20.75° respectively, which correspond to $w_0^i = 0$ ($\eta \rightarrow \infty$) in this figure. The values of w_0^i at $\psi = 0^\circ$ and $\psi = 20^\circ$ map to $z_i = z_f = 60$ km altitude above sea level via Eq. (3.43). These

altitudes where specified as part of the boundary conditions. Thus the boundary conditions $\psi_i(r_i) = 0^\circ$ and $\psi_f(r_f) = 20^\circ$ have been met.

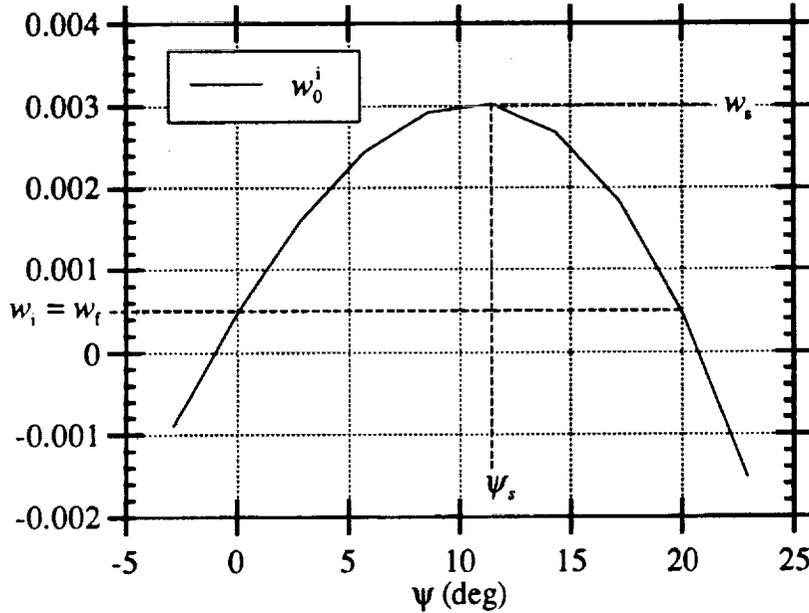


Figure 3.2. Example of Converged Inner Newton Iteration Process for w as a Function of ψ

The left and right outer solution integration constants in Eq's. (3.31-3.33) and (3.38-3.40) are obtained by enforcing the matching conditions in Eq's. (3.75-3.80) and (3.92-3.94, 3.96-3.98). This amounts to equating the common parts for the outer solution (Eq's. (3.31-3.33, 3.38-3.40) evaluated at $h=0$) with the common parts from the inner solution (Eq's. (3.61-3.66) evaluated at the left and right common part values of ψ_0^i corresponding to $w_0^i=0$ in Fig. 3.2). Since some of the outer solution variables are different from the inner solution variables, it is necessary to employ the transformations in Eq's. (3.43, 3.44) and (3.83). Note that c_3 and a_3 are not needed since the outer solutions for ψ_0^i and $P_{0\psi}^i$ are constant. Hence the zeroth order composite solution for these variables is the inner solution alone. The zeroth order composite solution is then evaluated to determine if the boundary conditions are satisfied:

$$u_0^c(h_i) - u_i = 0, \quad \gamma_0^c(h_i) - \gamma_i = 0,$$

$$P_{0u}^c(h_f) - 1 = 0, \quad \gamma_0^c(h_f) - \gamma_f = 0 \quad (3.124)$$

The error in these equations is used in an outer loop Newton iteration to iterate on the remaining four unknown common parts. These are the left common part values for u and γ , and the right common part values for P_u and γ . A summary of this procedure is given in Fig 3.3

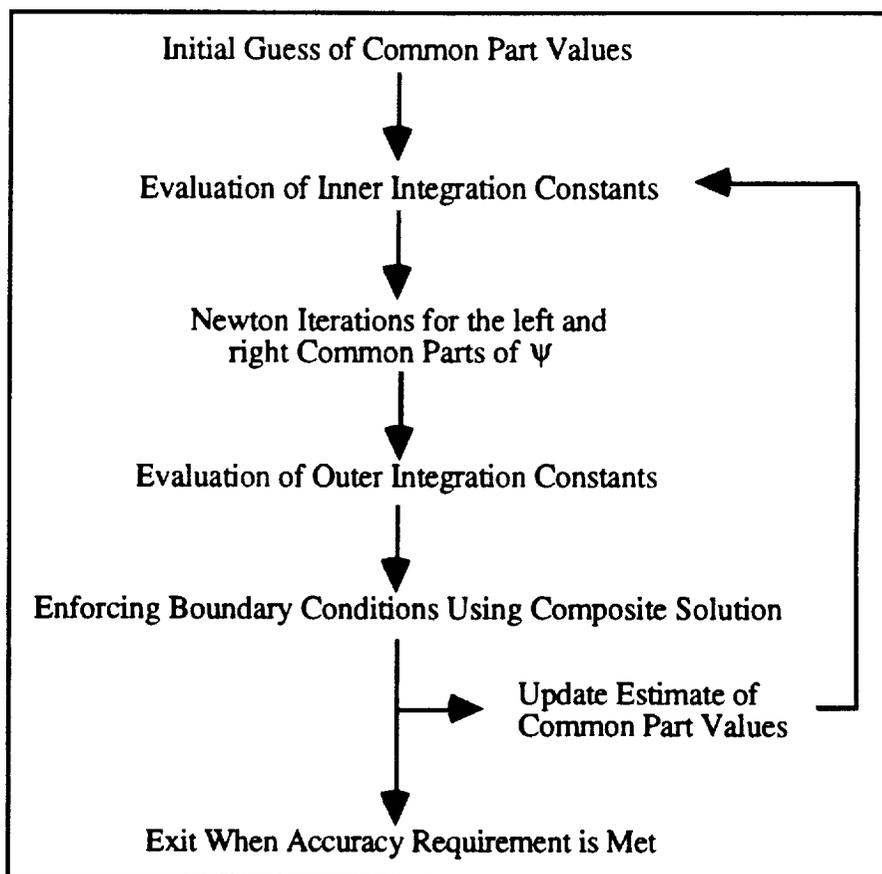


Figure 3.3. Summary of the Numerical Solution Procedure

3.7.2 Open Loop Numerical Results

Fig's. 3.4-3.7 present the zeroth order MAE converged solution for the boundary conditions: $z_i = z_f = 60$ km (initial and final altitudes above sea level), $V_i = 7851$ m/sec, $\gamma_i = 1.35^\circ$, $\gamma_f = 1.0^\circ$ and $\Delta\psi = 20^\circ$. The corresponding control time histories are given in Fig's. 3.8 and 3.9 and the altitude history in Fig. 3.10. The value of the expansion parameter used here is $\epsilon = 1/\beta r_s = 11.87e-4$. In Fig's. 3.4-3.7 the inner solution is shown for the complete range of ψ between its left and right common part values, which correspond to $\eta \rightarrow \infty$ ($w'_0 = 0$ in Fig. 3.2). This was done to illustrate the fact that the matching conditions in Eq's. (3.75-3.80) and (3.92-3.94, 3.96-3.98) are satisfied on the left and right portions of the solution (see also Fig. 2.3)

These results clearly indicate that the composite solutions for γ and p_γ are significantly different from the inner solution. In particular, the major variation in p_γ is due to the outer solution, which in effect amounts to a correction for the large variation in Loh's term during the exit phase. We note that the outer solution plays an important role in forming a uniformly valid zeroth order approximation to the exact solution. The normalized lift control λ is always near 1.0, corresponding to flight at near maximum lift to drag ratio. The bank angle μ is always near 90° indicating that most of the aerodynamic force is utilized in performing the turn.

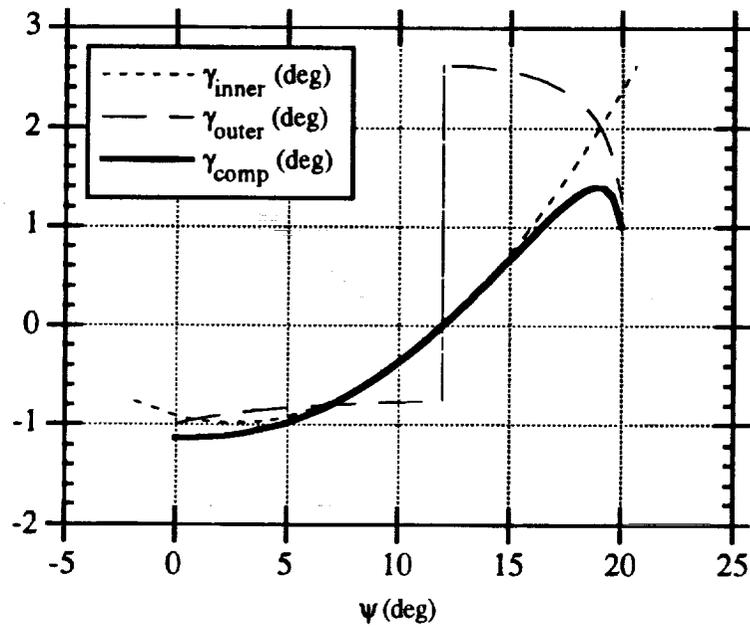


Figure 3.4. Zero Order MAE Inner, Outer and Composite γ Solution

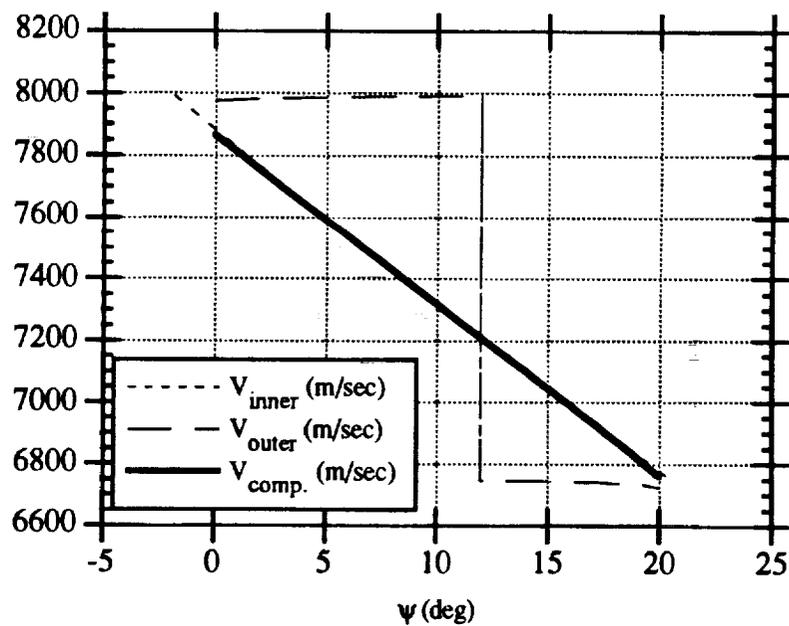


Figure 3.5. Zero Order MAE Inner, Outer and Composite Velocity Solution

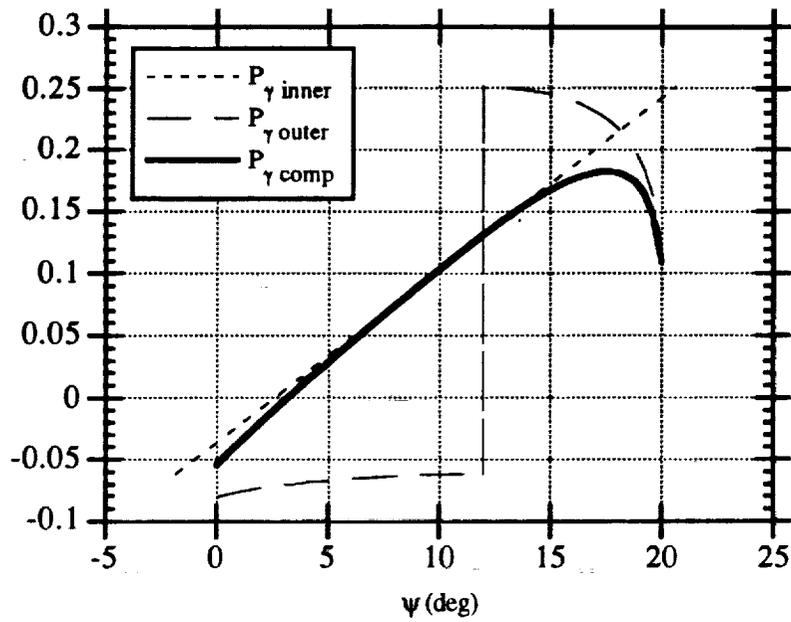


Figure 3.6. Zero Order Inner, Outer and Composite P_γ Solution

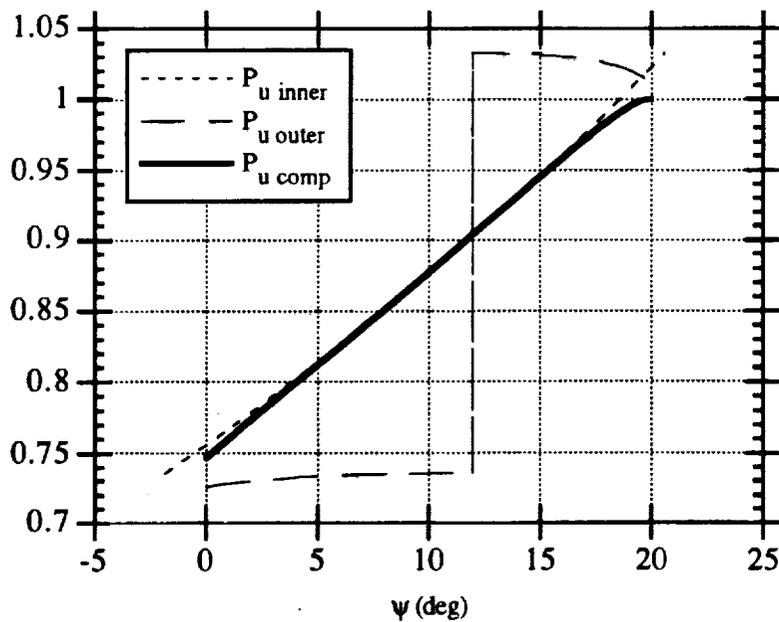


Figure 3.7. Zero Order MAE Inner, Outer and Composite P_u Solution

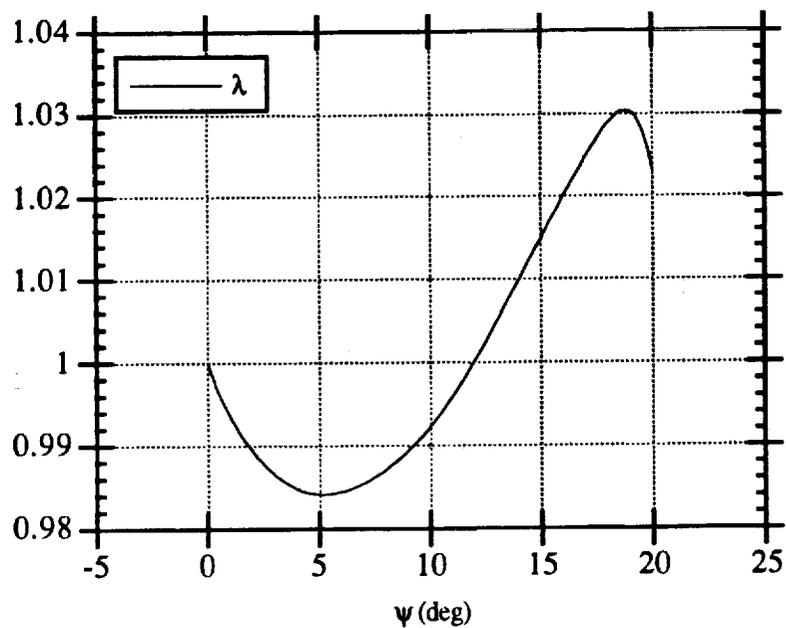


Figure 3.8. Zeroth Order MAE Normalized Lift Control

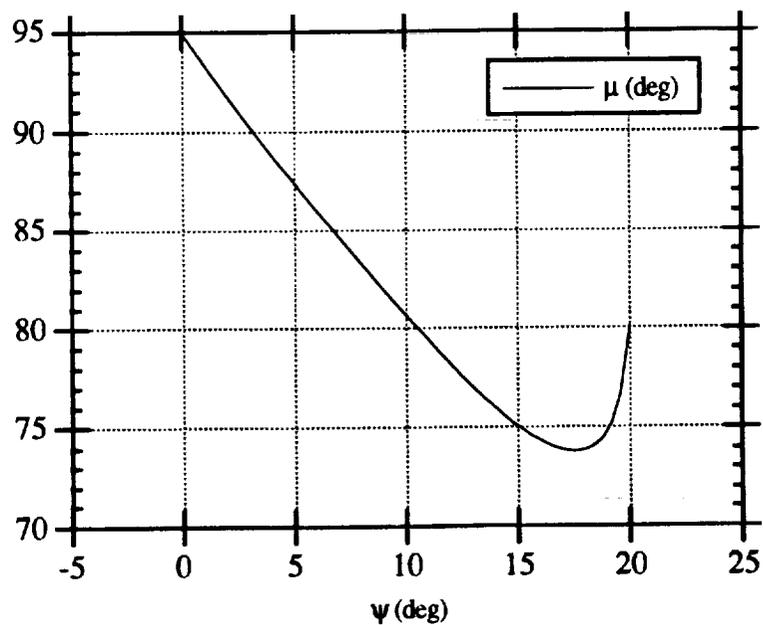


Figure 3.9. Zeroth Order Bank Angle Control

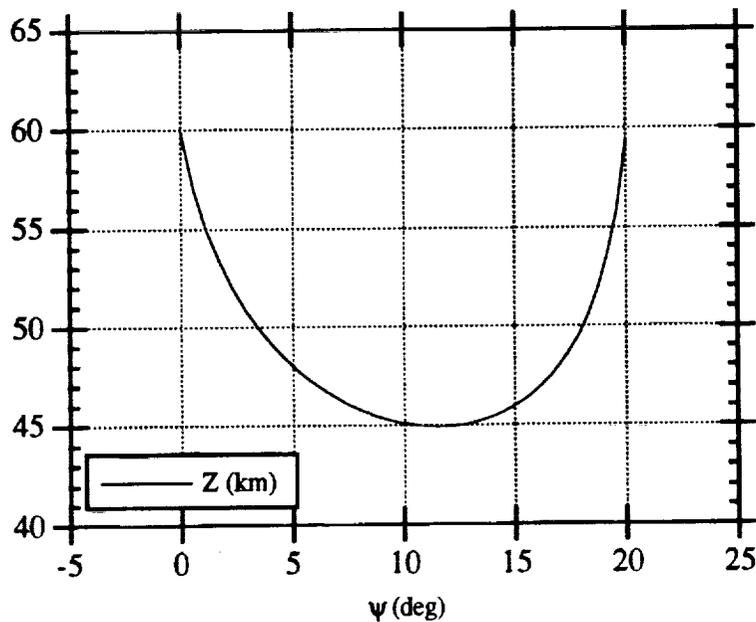


Figure 3.10. Zero Order MAE Altitude History

3.8 Zeroth Order Guided Solution

3.8.1 Solution Procedure

Zeroth order closed loop guided solutions are obtained by using the optimal control expressions given in Eq's (3.26) and (3.27). These expressions involve both the states and the costates, thus knowledge of both states and costates is needed to evaluate the controls. Assuming that the states are available for feedback, only estimates of the costates are required at each control computation along the trajectory. Feedback implementation entails treating the current state (from the simulation) at each control update as a new initial state, and calculating the costate values corresponding to the same time instant. The estimate for these costates (to zeroth order) are obtained by repetitively solving the zero order MAE problem.

In the first step, an initial guess and boundary conditions are supplied to initiate the procedure of obtaining a zero order MAE converged solution (Appendix B). Next, the costate expressions in Eq's. (3.38-3.40) and (3.64-3.66) can be evaluated as a function of

the corresponding independent variables and used in Eq's. (3.105-3.107) to construct the composite costate expressions. These are in turn used in Eq's (3.26) and (3.27) to compute the controls. When a predetermined time increment has been reached, the current states are used as initial conditions for the next MAE calculation. It follows that the initial guess is available in every step of the zero order MAE calculation after the first.

For control computation between MAE solution updates, the integration constants from the last update are used. The transformations defined in Eq's. (3.11, 3.12, 3.43) and (3.44) are used to transform the simulated dimensional variables into the inner and outer dependent and independent variables. The transformed altitude h is used in Eq's. (3.31, 3.32, 3.38) and (3.39) to compute the left and right outer costates. The heading ψ is used in Eq. (3.64) to evaluate P_{0v}^i , and Eq's. (3.105-107) are used to calculate the composite costates. The composite costates and the current γ and u (from Eq. (3.12)) are used in Eq's. (3.26) and (3.27) to evaluate the optimal controls between the time instants where the MAE solution is updated.

During the exit phase, the left outer solution is discarded, and matching is required only between the right outer solution and the inner solution. In this case, the constants of integration for the inner solution are viewed as free parameters used to satisfy the boundary conditions.

3.8.2 Numerical Results

Fig's 3.11 through 3.14 present a comparison between the optimal solution (obtained using a multiple shooting method²⁵) and the zero order guided solution for $\Delta\psi=20^\circ$. The corresponding control time histories are given in Fig's. 3.15 and 3.16. Loh's term (corresponding to the optimal solution) is given in Fig. 3.17 and the altitude in Fig. 3.18. The time increment between guided solution updates is 5 seconds, with the control updated at every integration step following the procedure described above. These results indicate that the guided solutions and the optimal solutions are in a very good agreement throughout the trajectory. The error that does exist in some of the guided solution variables, for example the γ and the altitude (z) solutions, indicate the need for a first order correction. A general procedure for MAE expansion of the HJB equation to first order is developed in Section 4, along with a description of its potential application to aero-assisted orbit transfer.

Fig's 3.13 and 3.15 clearly indicate how the variation in Loh's term near the end of the trajectory is partially accounted for in the zero order guided solution. Specifically, the normalized lift coefficient λ given in Fig. 3.15 does not saturate in the exit phase, but reduces to near zero. This is due to the fact that the correction in P_y from the outer solution is too large (see also Fig. 3.6). Fig. 3.19 compares the velocity histories near the end of the trajectory. The optimal and guided solution values for terminal velocities are 6751 m/sec and 6736 m/sec respectively. The difference of the guided value from the optimal one is 15 m/sec.

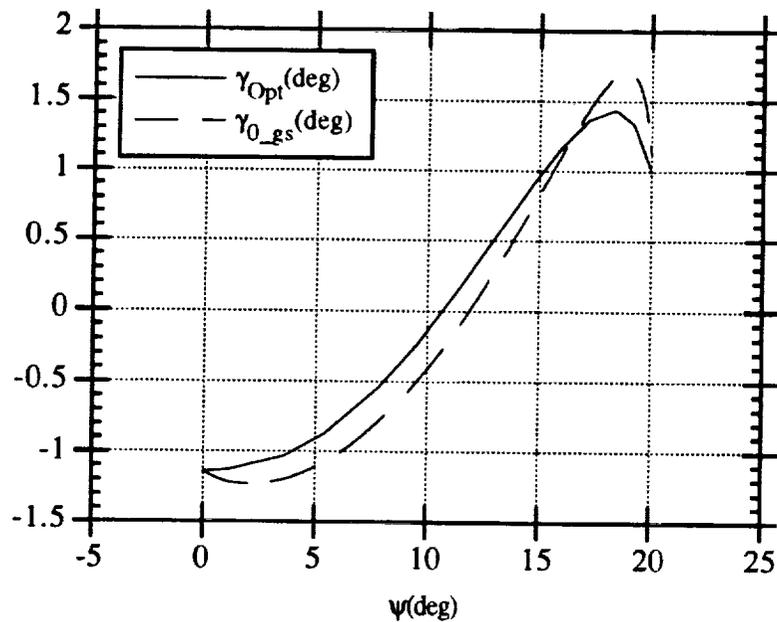


Figure 3.11. Optimal and Guided γ Solutions

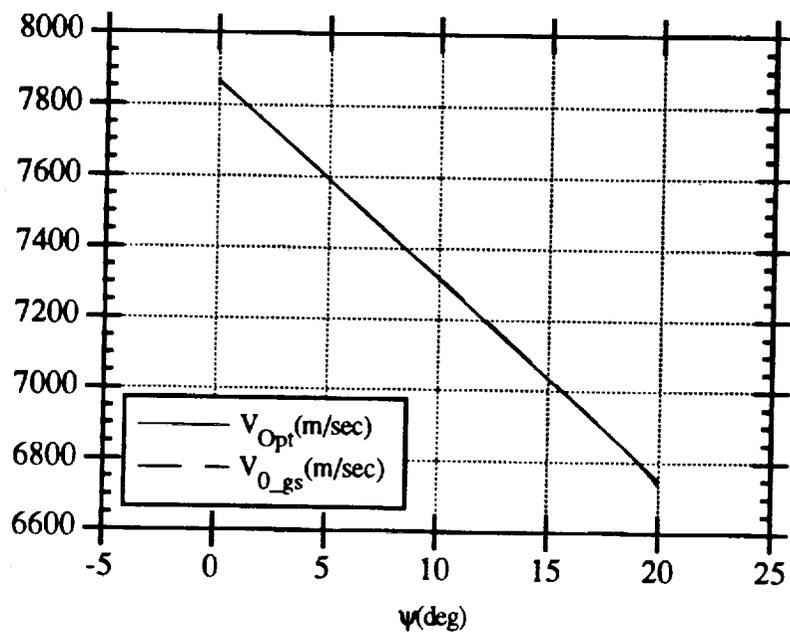


Figure 3.12. Optimal and Guided Velocity Solutions

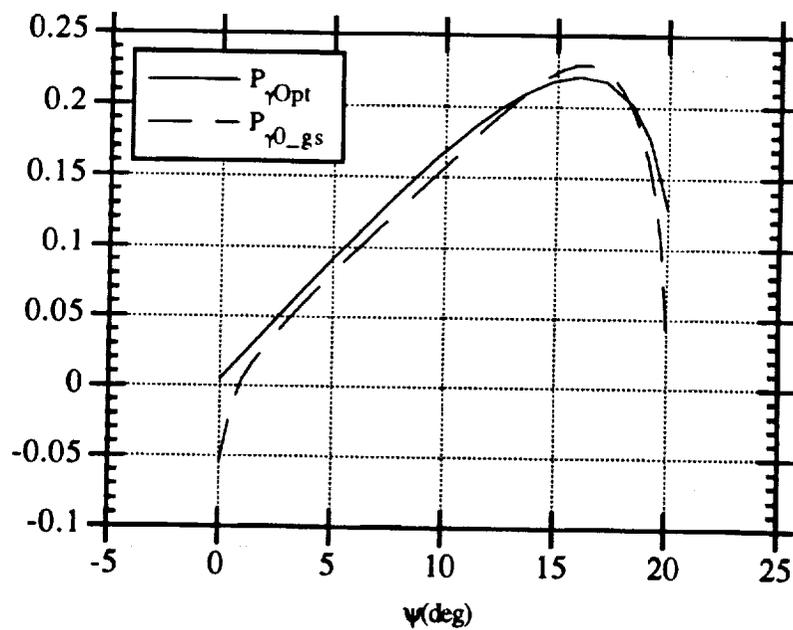


Figure 3.13. Optimal and Guided P_{γ} Solutions

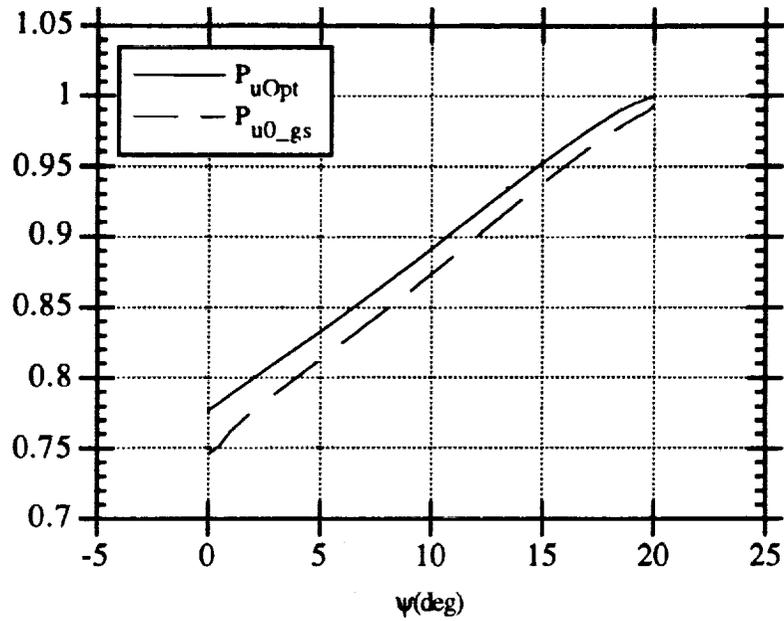
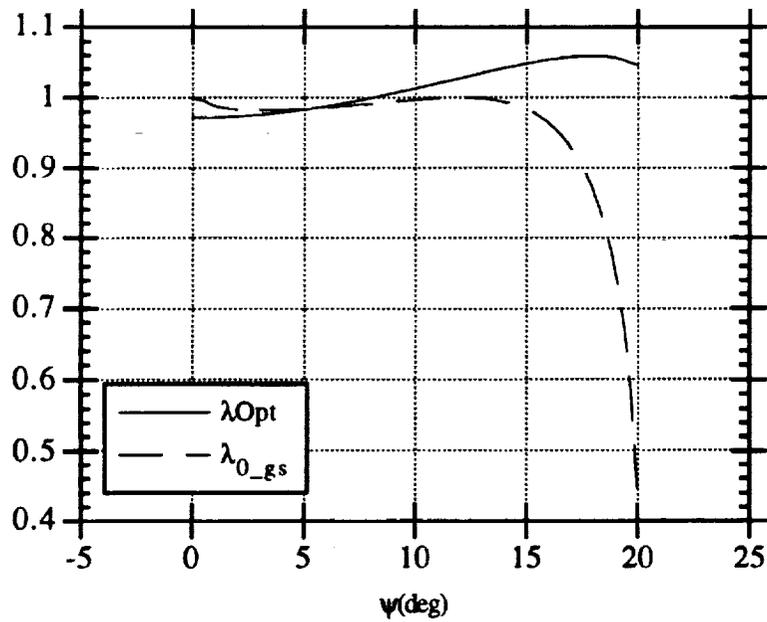
Figure 3.14. Optimal and Guided P_u Solutions

Figure 3.15. Optimal and guided Normalized Lift Control

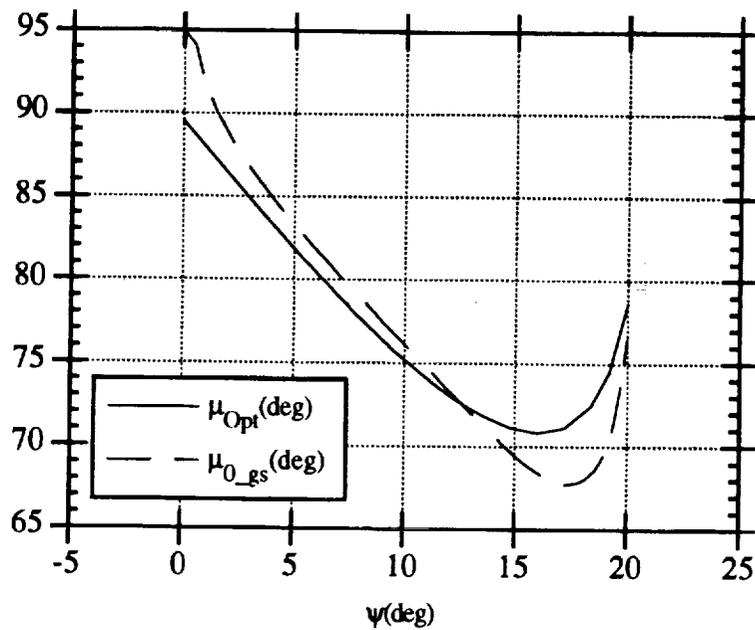


Figure 3.16. Optimal and Guided Bank Angle Control

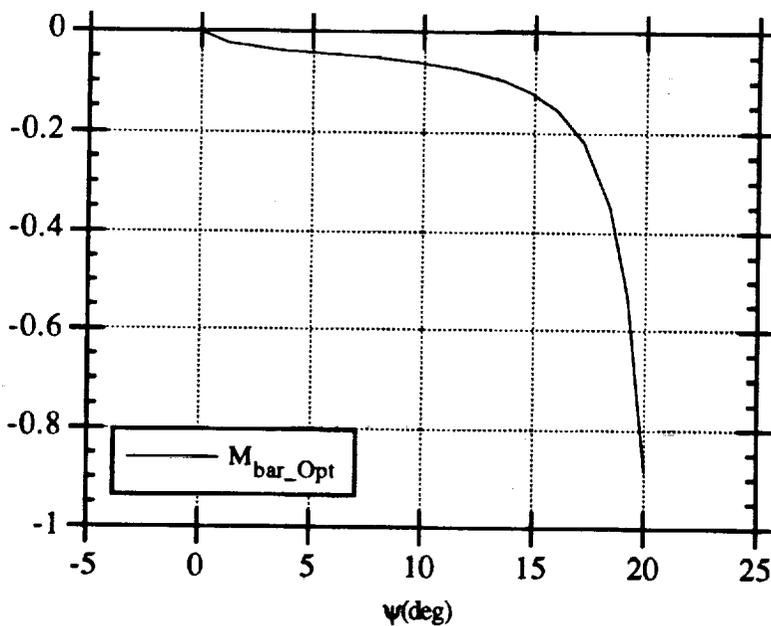


Figure 3.17. Optimal Loh's Term

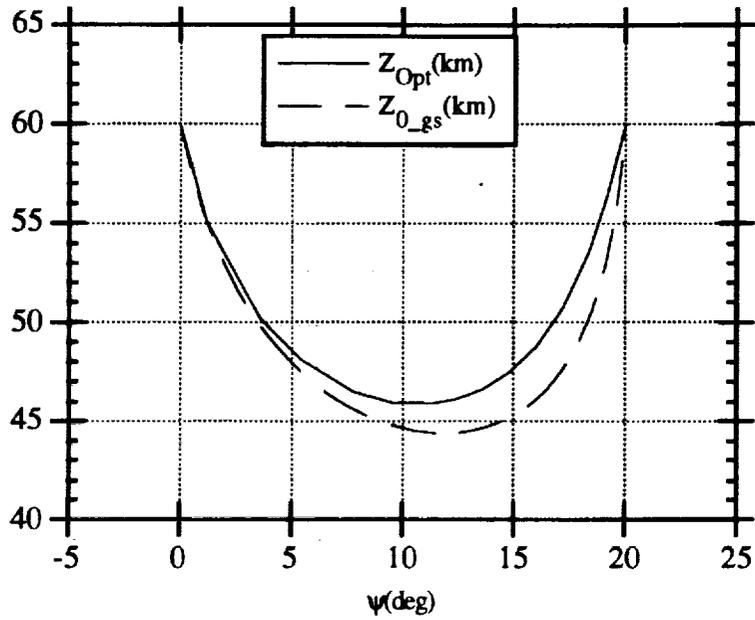


Figure 3.18. Optimal and Guided Altitude Solution

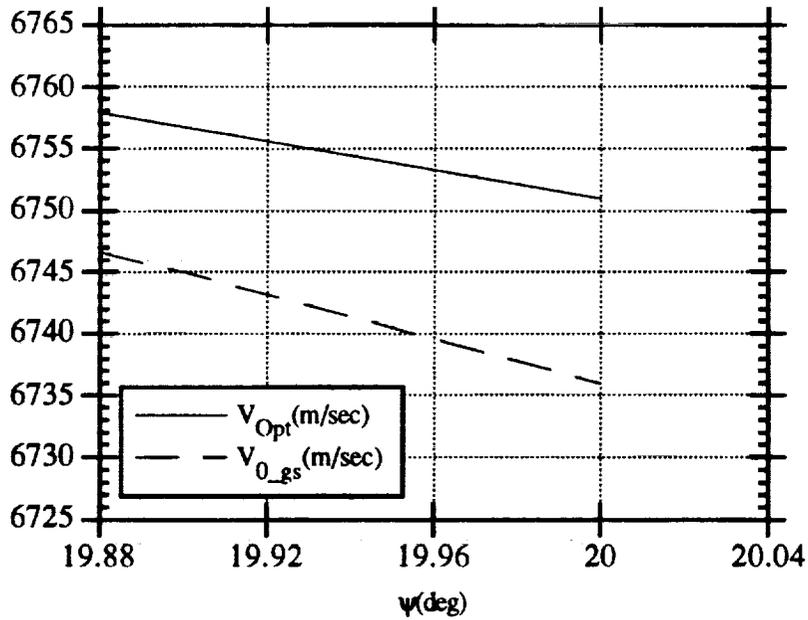


Figure 3.19. Optimal and Guided Performance

Section IV

Matched Asymptotic Expansion of the Hamilton-Jacobi-Bellman Equation

In this Section a general procedure for constructing a matched asymptotic expansion of the Hamilton-Jacobi-Bellman equation, based on the method of characteristics, is developed. The development is valid for a class of perturbation problems whose solution exhibits two-time-scale behavior. A regular expansion for problems of this type is shown to be inappropriate since it is not valid over a narrow range of the independent variable. That is, it is not uniformly valid. Of particular interest here is the manner in which matching and boundary conditions are enforced when the expansion is carried out to first order. Two cases are distinguished - one where the left boundary condition coincides with or lies to the right of the singular region and another one where the left boundary condition lies to the left of the singular region.

4.1 Singularly Perturbed Hamilton-Jacobi-Bellman Equation

Ref.'s. 10 and 11 have formulated the aeroassisted plane change maneuver as a regular perturbation problem and employed a regular expansion of the optimal return function (P) to first order to obtain an approximate guidance law for that problem. However, as shown in Section 3, the optimal aeroassisted plane change maneuver is actually a singularly perturbed problem. For the solution of such a problem to be uniformly valid, inner and outer expansions of P and a matching procedure have to be performed so that a uniformly valid composite solution for the optimal return function can be constructed. This Section treats the expansion to first order for a class of singularly perturbed problems that are characterized by the presence of t and t/ϵ in the right hand sides of the differential equations.

We assume that the dynamic system in Eq. (2.1) has the following form

$$dx/dt = f(x,u,t) + g(x,u,t/\epsilon)/\epsilon, \quad x(t_i) \text{ given} \quad (4.1)$$

where $x \in R^n$ is the state vector, $u \in R^m$ is the control vector, ϵ is a small parameter and t is the independent variable. The functions f and g are assumed analytic with respect to their arguments in the region of interest and, in addition, g is assumed to have the property that

$$\lim_{\epsilon \rightarrow 0} g(x,u,t/\epsilon)/\epsilon^i = 0, \quad t \neq 0, \quad i=1, \dots \quad (4.2)$$

The singular characteristic of this type of problem arises at $t=0$, where the above property is not satisfied. To investigate the behavior in the region of singularity near $t=0$, a stretched independent variable τ is defined as $\tau=t/\epsilon$ and Eq. (4.1) in terms of τ is given by

$$dx/d\tau = \epsilon f(x,u,\epsilon\tau) + g(x,u,\tau), \quad x(\tau_i) \text{ given} \quad (4.3)$$

where the function f is assumed to have the property that

$$\lim_{\epsilon \rightarrow 0} \epsilon f(x,u,\epsilon\tau) = 0 \quad (4.4)$$

Note that both systems in Eq's. (4.1) and (4.3) represent the exact dynamics. Eq. (4.1) is called the outer system and all the variables associated with it will be denoted by the superscript o . Eq. (4.3) is called the inner system, and all the variables associated with it will be denoted by the superscript i .

The assumption in Eq. (4.2) regarding the form of g was chosen because the aeroassisted plane change problem satisfies this property. It is typically satisfied when g in Eq. (4.1) has the form $g = e^{-t/\epsilon} h(x,u)$. Also, the division by ϵ in Eq. (4.1) insures that the zeroth order inner dynamics in Eq. (4.3) are not zero. As will become evident, this specialization is not essential to the development in this section, and the main results are not restricted to it. Generalizations (and specializations) to other assumed forms for both f and

g are straightforward extensions of the main ideas to be presented below. The only requirement is that the zeroth order subproblems remain well defined.

Note that when the transformation to Eq. (4.3) is performed, it appears to result in a regular perturbation problem, when in fact the original dynamics in Eq. (4.1) are singular. This is precisely what was done in Ref.'s. 10 and 11. It will be shown later, in the application section, that transformation of our equations that begin in the form of Eq (4.1) to the variables used in Ref. 11 results in a set of equations in the form of Eq. (4.3). This means that the inner expansion alone was used in these earlier studies to satisfy the boundary conditions, and that the expansions are valid only within the region of singularity. In an MAE expansion, the solution is dominated by the inner expansion within this region, because the outer expansion nearly cancels with the common part of the solution. Therefore, the differences will be most apparent outside this region, and for problems involving skipping trajectories through the atmosphere this will be most evident where Loh's term undergoes as large variation. That is, whenever the orbital forces dominate. These forces are ignored in the zeroth order regular expansion solution, and are accounted for in the zeroth order MAE solution. This point was illustrated in the numerical results in Section 3.

The optimization problem is to find $u(x,t)$ that minimizes $J = \phi(x_f)$, subject to the dynamic constraints in Eq. (4.1) and the terminal constraint $\psi(x_f) = 0$, where $x_f = x(t_f)$ and t_f is the final time. The outer HJB equation is

$$P_t^o = -\min_{u \in U} H = -P_x^o (f^{opt} + g^{opt} / \epsilon) \quad (4.5)$$

where U is a class of continuous bounded controls, $f^{opt} \equiv f(x, u^{opt}, t)$, $g^{opt} \equiv g(x, t / \epsilon, u^{opt}(x, t))$, and $u^{opt}(x, P_x, t)$ is given by the optimality condition $H_u = 0$, assuming that H_{uu} is positive definite. $P^o(x^o(t_i), t_i)$ is the optimal return function defined as the optimal value of the performance index for an optimal path starting at $x(t_i)$ and t_i and satisfying the terminal constraint. In terms of the stretched independent variable τ the inner HJB equation is given by

$$P_\tau^i = -\min_{u \in U} H = -P_x^i (\epsilon f^{opt} + g^{opt}) \quad (4.6)$$

where $P^i(x^i(\tau_i), \tau_i)$ is the same optimal return function as defined for Eq. (4.5), but expressed in terms of the inner variables. It is important to note that in both formulations the exact solutions for the return functions are identical and depend on the same initial condition $x^o(t_i) \equiv x^i(\tau_i) \equiv x(t_i)$, where $\tau_i = t_i/\epsilon$. This is not true for the corresponding inner and outer expansions of the solution, since the boundary conditions is satisfied with the composite solution. Therefore, only the composite solution should be compared with the exact solution. However, the inner and outer expansion solution terms may be viewed as being dependent on $x(t_i)$ and t_i through the matching conditions and the boundary conditions enforced on the composite solution. This dependence will be made explicit in the following development.

Consider a power series expansion in ϵ of the return functions for both the outer or inner formulations:

$$P^o(x(t_i), t_i, \epsilon) = \sum_{j=0}^{\infty} P_j^o(x(t_i), t_i) \epsilon^j \quad (4.7)$$

$$P^i(x(t_i), t_i, \epsilon) = \sum_{j=0}^{\infty} P_j^i(x(t_i), t_i) \epsilon^j \quad (4.8)$$

By substituting these expansions into the outer and inner expressions of the optimal control (derived from the optimality condition) and expanding in a power series in ϵ , the outer and inner expansions of the optimal control in a feedback form are obtained as:

$$u^{o\text{opt}}(x(t_i), P_x^o, t_i) = \sum_{j=0}^{\infty} u_j^o(x(t_i), t_i) \epsilon^j = \underline{u}^{o\text{opt}}(x(t_i), t_i, \epsilon) \quad (4.9)$$

$$u^{i\text{opt}}(x(t_i), P_x^i, t_i) = \sum_{j=0}^{\infty} u_j^i(x(t_i), t_i) \epsilon^j = \underline{u}^{i\text{opt}}(x(t_i), t_i, \epsilon) \quad (4.10)$$

The details regarding this expansion for a regular perturbation problem are given in Ref. 11, and they apply directly for the separate expansions in Eq's. (4.9) and (4.10). The zeroth order terms u_0^o and u_0^i are the optimal controls for the zeroth order outer and inner problems that are obtained by setting $\epsilon=0$ in Eq's. (4.1) and (4.3). If analytic solutions are

available for the zeroth order problem, then the higher order solutions are determined by the expansions of the outer and inner HJB equations:

$$P_t^o = \sum_{j=0}^{\infty} P_{jt}^o \varepsilon^j = - \left(\sum_{j=0}^{\infty} P_{jx}^o \varepsilon^j \right) \left(\sum_{j=0}^{\infty} f_j^o(x, t) \varepsilon^j \right) \quad (4.11)$$

$$P_\tau^i = \sum_{j=0}^{\infty} P_{j\tau}^i \varepsilon^j = - \left(\sum_{j=0}^{\infty} P_{jx}^i \varepsilon^j \right) \left(\varepsilon \sum_{j=0}^{\infty} f_j^i(x, \varepsilon \tau) \varepsilon^j + \sum_{j=0}^{\infty} g_j^i(x, \tau) \varepsilon^j \right) \quad (4.12)$$

where the expansions for f and g come from substituting the series in Eq's (4.9) and (4.10) for u in Eq's. (4.1) and (4.3). The reader is again referred to Ref. 11 for the details, while noting the exception that the dependence of f in Eq. (4.3) on ε enters both through u and $\varepsilon \tau$. Equating like powers of ε in Eq's. (4.11) and (4.12) leads to outer and inner first order linear partial differential equations for P_j^o and P_j^i

$$P_{jt}^o + P_{jx}^o f_0^o = R_j^o(x(t_i), t_i, P_{j-1}^o, \dots, P_0^o), \quad j=1, \dots \quad (4.13)$$

$$P_{j\tau}^i + P_{jx}^i g_0^i = R_j^i(x(t_i), t_i, P_{j-1}^i, \dots, P_0^i), \quad j=1, \dots \quad (4.14)$$

where the forcing terms R_j^o and R_j^i are functions of the lower terms in the solution. This procedure was carried out in Ref. 11 for a regular expansion, and is identical in form for the inner and outer expansions defined here with obvious accounting for differences in notation, and the exception noted above regarding the dependence of f on $\varepsilon \tau$ in the inner expansion. In particular, the $j=1$ expressions are:

$$R_1^o = 0, \quad R_1^i = -P_{0,x}^i f_0^i(x, 0) \quad (4.15)$$

after taking into account the optimality condition for the zeroth order problem. Note that R_1^o is zero as a consequence of the assumption in Eq. (4.2), which also accounts for the fact that an expansion of g is not required in Eq. (4.11). In the sequel it is not assumed that $R_1^o = 0$ so as to allow for more general assumptions regarding the form for g .

Singular perturbation expansions differ from regular expansions in that solutions to any order of the outer and inner problems are not uniformly valid approximations of the exact solution. Uniformly valid approximations are constructed from the outer and inner solutions by the MAE method. This process involves matching of the inner and outer solutions and enforcing the boundary conditions on an additive composite solution made up of the outer solution plus the inner solution minus the common part. See Sections 2 and 3 for details.

The partial differential equations in Eq's. (4.13) and (4.14) may be solved by the method of characteristics (Ref. 18). The characteristic curves for any order term of P^o and P^i are defined by the zeroth order differential equations:

$$\dot{x}_0^o = f^o(x_0^o, u_0^o, t) \quad (4.16)$$

$$dx_0^i / d\tau = g^i(x_0^i, \tau, u_0^i) \quad (4.17)$$

whose solutions are denoted by $x_0^o(t; x(t_i), t_i)$ and $x_0^i(\tau; x(t_i), t_i)$ respectively. It is shown in Appendix C that the boundary conditions for Eq's. (4.16) and (4.17) are also defined by the MAE method. Therefore the characteristic curves for the inner and outer expansions are the inner and outer solution components of a zeroth order MAE solution. Note that to any order, the boundary conditions are satisfied by the composite solution and not individually by the inner and outer solution components. The same is true for the characteristic curves which satisfy Eq's. (4.16) and (4.17). However, both the inner and outer characteristic curves depend on $x(t_i)$ and t_i through the matching conditions and the boundary conditions. See Sections 2 and 3 for details. Note that the expansions P_j^o and P_j^i in Eq's. (4.13) and (4.14) are useful only at the initial time $t_0 = \varepsilon\tau_0$, because the characteristic curves are computed for a specified $x(t_i)$ and t_i .

4.1.1 First Order Solution for the Case $t_i \geq 0$

In order to express the matching condition, it is necessary to express the outer solution of Eq. (4.13) for the interval $0 \leq t \leq t_i$, and the inner solution of Eq. (4.14) for the interval $0 \leq \tau \leq \infty$. We also assume in this section that the initial time satisfies $t_i \geq 0$, and

will subsequently generalize the result for arbitrary t_i . The solutions for P_1^o and P_1^i are expressed as:

$$P_1^o(t; x(t_i), t_i) = \int_0^t R_1^o(\tau; x(t_i), t_i) d\tau + P_1^o(0; x(t_i), t_i) \quad (4.18)$$

$$P_1^i(\tau; x(t_i), t_i) = \int_0^\tau R_1^i(\tau; x(t_i), t_i) d\tau + P_1^i(0; x(t_i), t_i) \quad (4.19)$$

A necessary condition in MAE analysis is that the zeroth order solution at $t=0$ serves as a stable equilibrium point for the zeroth order inner solution. This means that the characteristic curve for the inner expansion must approach a well defined limit as $\tau \rightarrow \infty$. Hence it is assumed that $R_1^i(\tau; x(t_i), t_i)$ reaches a well defined limit as $\tau \rightarrow \infty$ which will be denoted by $R_1^{im}(x(t_i), t_i)$. Thus, in order to express P_1^i for large values of τ , it is necessary to rewrite Eq. (4.19) in the following form:

$$P_1^i(\tau; x(t_i), t_i) = \int_0^\tau [R_1^i(\tau; x(t_i), t_i) - R_1^{im}] d\tau + \tau R_1^{im} + P_1^i(0; x(t_i), t_i) \quad (4.20)$$

We are now ready to perform the matching procedure between the outer and the inner solutions of the return function by enforcing the following rule:

$$P^o(\text{small } t) = P^i(\text{large } \tau) \Big|_{\tau=t/\epsilon} \quad (4.21)$$

where the dependency on $x(t_i), t_i$ is omitted to save notation. Retaining terms to first order in t and ϵ yields

$$P_0^o(0) + P_{0,t}^o(0)t + \epsilon P_1^o(0) = P_0^i(\infty) + \epsilon [P_1^i(0) + \int_0^\infty [R_1^i(\tau) - R_1^{im}] d\tau + R_1^{im} \tau] \quad (4.22)$$

where Eq. (4.20) is used to express P^i for large τ . Equating like powers in t and ϵ , the following matching conditions are obtained:

$$\epsilon^0: \quad P_0^o(0) = P_0^i(\infty) \quad (4.23)$$

$$\epsilon^1: \quad P_1^o(0) = P_1^i(0) + \int_0^{\infty} [R_1^i(\tau) - R_1^{i\infty}] d\tau \quad (4.24)$$

$$t^1: \quad R_1^{i\infty} = P_{0,t}^o(0) = -P_{0,x}^o f^o(0) \quad (4.25)$$

Next the boundary conditions are applied to the composite solution expressed to first order. The composite solution is constructed by adding the outer and inner solutions (expressed to first order) and subtracting their common part. The common part is the left hand side of Eq (4.22). Evaluation of the resulting first order composite solution at the final time yields the following boundary condition

$$P_1^c(t_f) = P_0^c(t_f) + \epsilon [P_1^o(t_f) + P_1^i(\tau_f)] - [P_{0,t}^o(0)t_f + \epsilon P_1^o(0)] = \phi(x_f) \quad (4.26)$$

where the zeroth order composite return function is given by

$$P_0^c(t_f) = P_0^o(t_f) + P_0^i(\tau_f) - P_0^o(0) \quad (4.27)$$

From Eq's. (4.18) and (4.20) the first order terms evaluated at t_f and τ_f are related to their respective values at $t = \tau = 0$ by:

$$P_1^o(t_f) = \int_0^{t_f} R_1^o(t) dt + P_1^o(0) \quad (4.28)$$

$$P_1^i(\tau_f) = \int_0^{\tau_f} [R_1^i(\tau) - R_1^{i\infty}] d\tau + R_1^{i\infty} \tau_f + P_1^i(0) \quad (4.29)$$

Using Eq's. (4.25) and (4.27-4.29) in Eq. (4.26), and recalling that $\tau_f = t_f / \varepsilon$ yields:

$$\varepsilon^0: \quad P_0^c(t_f) = \phi(x_f) \quad (4.30)$$

$$\varepsilon^1: \quad P_1^i(0) = -\int_0^{t_f} R_1^o(t) dt - \int_0^{\tau_f} [R_1^i(\tau) - R_1^{im}] d\tau \quad (4.31)$$

Eq's. (4.31) and (4.24) are used to evaluate the constants $P_1^o(0)$ and $P_1^i(0)$ in Eq's. (4.18) and (4.20). Finally, using these results in Eq. (4.26) evaluated for an arbitrary initial time t_i such that $0 \leq t_i \leq t_f$ results in

$$P_1^c(x(t_i), t_i, \varepsilon) = P_0^c(x(t_i), t_i) + \varepsilon \left\{ -\int_{t_i}^{t_f} R_1^o(t; x(t_i), t_i) dt \right. \\ \left. - \int_{\tau_i}^{\tau_f} [R_1^i(\tau; x(t_i), t_i) - R_1^{im}] d\tau \right\} \quad (4.32)$$

Partial differentiation of this expression with respect to $x(t_i)$ provides an approximation for the costate variables to first order, which, when used in the optimality condition, results in an approximation for the feedback control to first order.¹¹

$$P_{1_{x(t_i)}}^c(x(t_i), t_i, \varepsilon) = P_{0_{x(t_i)}}^c(x(t_i), t_i) + \varepsilon \left\{ -\int_{t_i}^{t_f} R_{1_{x(t_i)}}^o(t; x(t_i), t_i) dt \right. \\ \left. + R_1^o(t_f; x(t_i), t_i) \partial t_f / \partial x(t_i) - \int_{\tau_i}^{\tau_f} [R_{1_{x(t_i)}}^i(\tau; x(t_i), t_i) - R_{1_{x(t_i)}}^{im}] d\tau \right. \\ \left. + [R_1^i(\tau_f; x(t_i), t_i) - R_1^{im}] \partial \tau_f / \partial x(t_i) \right\} \quad (4.33)$$

4.1.2 A Simple Example

Here, the simple example that was presented in Section 2.2 is followed up. The problem formulation is

$$\dot{x} = x + ue^{-t/\varepsilon} / \varepsilon, \quad x(t_i) \text{ given}, \quad t_f = 1, \quad |u| \leq 1 \quad (4.34)$$

where x is a scalar state variable and u is the control. Find a control u to maximize the performance index $J=x_f$. To repeat the main results, the exact solutions of the state and costate are:

$$x(t; x(t_i), t_i) = [(1 + \varepsilon)x(t_i)e^{t-t_i} + e^{t-t_i-1/\varepsilon} - e^{-t/\varepsilon}] / (1 + \varepsilon) \quad (4.35)$$

$$\lambda(t; x(t_i), t_i) = e^{1-t} \quad (4.36)$$

and the zeroth order outer and inner solutions are:

$$x_0^o(t; x(t_i), t_i) = [x(t_i) + e^{-t_i/\varepsilon}]e^{t-t_i} \quad (4.37)$$

$$\lambda_0^o(t; x(t_i), t_i) = e^{1-t} \quad (4.38)$$

$$x_0^i(\tau; x(t_i), t_i) = -e^{-\tau} + [x(t_i) + e^{-t_i/\varepsilon}]e^{-\tau} \quad (4.39)$$

$$\lambda_0^i(\tau; x(t_i), t_i) = e^1 \quad (4.40)$$

In this case the zeroth order composite solution for λ equals the outer solution since the inner solution is constant. Comparison of Eq's. (4.36) and (4.38) shows that it also equals to the exact solution. Hence, no correction terms for λ to first and higher orders are expected.

Using Eq. (4.15) it is found that

$$R_1^o(t) = 0 \quad (4.41)$$

$$R_1^i(\tau) = -P_{0x}^i(\tau)x_0^i(\tau) = -e^1[-e^{-\tau} + e^{-t_i/\varepsilon}(x(t_i) + e^{-t_i/\varepsilon})] \quad (4.42)$$

Evaluating $R_1^i(\tau)$ in Eq. (4.42) at $\tau \rightarrow \infty$ yields

$$R_1^{im} = -e^1 [e^{-t_i} (x(t_i) + e^{-t_i/\epsilon})] \quad (4.43)$$

Note from Eq's. (4.37,4.38) and (4.43) that the matching condition in Eq. (4.25) is automatically satisfied. This will always be the case when the outer and inner zeroth order MAE solutions are used to define the characteristic curves (see Appendix C). Using Eq's. (4.41-4.43) in Eq. (4.32) gives

$$\begin{aligned} P_1^c(x(t_i), t_i, \epsilon) &= P_0^c(t_i) - \epsilon \int_{t_i/\epsilon}^{1/\epsilon} [R_1^i(\tau) - R_1^{im}] d\tau \\ &= P_0^c(t_i) - \epsilon e^1 \int_{t_i/\epsilon}^{1/\epsilon} e^{-\tau} d\tau = P_0^c(t_i) + \epsilon e^1 [e^{-1/\epsilon} - e^{-t_i/\epsilon}] \end{aligned} \quad (4.44)$$

Note that the first order term in Eq. (4.44) does not depend on $x(t_i)$, hence there is no correction to first order for the costate function as expected. The exact return function is the final value of x from Eq. (4.35), with t replaced by $t_f = 1$.

$$P(x(t_i), t_i, \epsilon) = [(1 + \epsilon)x(t_i)e^{1-t_i} + e^{1-t_i-t_i/\epsilon} - e^{-1/\epsilon}] / (1 + \epsilon) \quad (4.45)$$

To show that Eq. (4.44) is a uniformly valid approximation to $O(\epsilon)$, the exact solution in Eq. (4.45) is expanded to first order

$$P(x(t_i), t_i, \epsilon) = P_0^c(x(t_i), t_i) + \epsilon [e^{-1/\epsilon} - e^{1-t_i-t_i/\epsilon}] \quad (4.46)$$

The difference between the first order terms in Eq's. (4.44) and (4.46) is

$$\Delta(\epsilon, t_i) = \epsilon e^1 (1 - e^{-t_i}) e^{-t_i/\epsilon} + \epsilon e^{-1/\epsilon} (1 - e^1) = \epsilon E(\epsilon, t_i) \quad (4.47)$$

The range of interest for t_i is $0 \leq t_i \leq 1$. The region of singularity corresponds to $t_i = O(\epsilon)$, or to the range $0 \leq t_i \leq k\epsilon$, where k is some constant. Outside this region $t_i = O(1)$. We now investigate the size of $E(\epsilon, t_i)$ both inside and outside the region of singularity:

$$\text{for } t_i = O(\varepsilon): \quad \lim_{\varepsilon \rightarrow 0} E(\varepsilon, t_i) / \varepsilon = ke^{1-k} < \infty \quad (4.48)$$

$$\text{for } t_i = O(1): \quad \lim_{\varepsilon \rightarrow 0} E(\varepsilon, t_i) / \varepsilon = 0 \quad (4.49)$$

Thus, $E(\varepsilon, t_i) = O(\varepsilon)$ when t_i is $O(\varepsilon)$ close to zero and $o(\varepsilon)$ when t_i is $O(1)$. Hence, $\Delta(\varepsilon, t_i)$ in Eq. (4.47) is $O(\varepsilon^2)$ uniformly in t_i . It follows that the solution in Eq. (4.44) is a uniformly valid first order approximation to the exact solution in Eq. (4.46). When the regular expansion of Ref. 11 is applied to this problem with $\tau = t/\varepsilon$ as the independent variable (this transforms the dynamics to the form of Eq. (4.3)), then the first order solution is not valid in the region outside the neighborhood of $t=0$.

4.1.3 First Order Solution for the Case $t_i \leq 0$

In this case the region of singularity near $t=0$ occurs between the initial time $t_i < 0$ and the final time $t_f > 0$. As detailed in Sections 2 and 3, two outer expansions are required for this situation, one for the interval $t < 0$ and the other corresponding to $t > 0$. The outer expansions are in general discontinuous at $t = 0$, but the composite solution is always continuous¹¹. The inner expansion must be considered for $-\infty < \tau < \infty$ in order that matching may be performed on both sides. This is illustrated in Fig. 4.1.

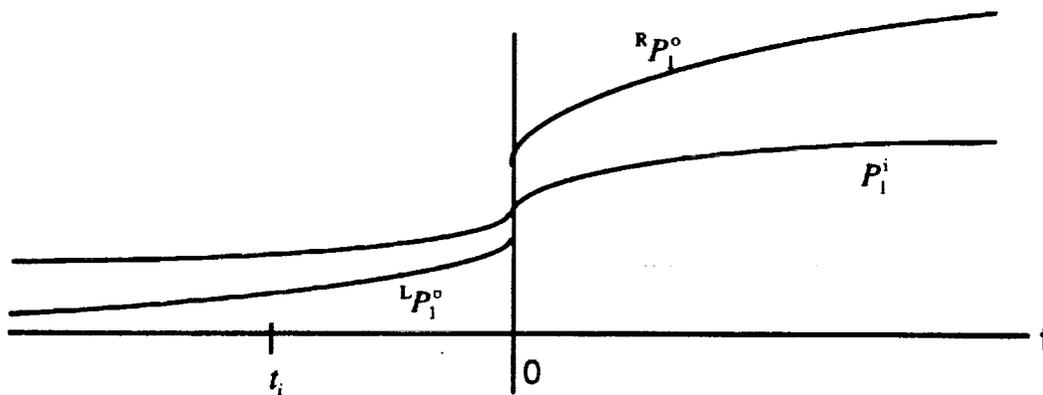


Figure 4.1. Inner and Outer Expansions of P in the Left and Right Regions

The equivalent expressions to Eq's. (4.18) and (4.20) on the left side for $t < 0$ are:

$$P_1^0(0; x(t_i), t_i) = \int_t^0 R_1^0(\tau; x(\tau), t_i) d\tau + P_1^0(t; x(t), t_i) \quad (4.50)$$

$$P_1^i(0; x(t_i), t_i) = \int_t^0 [R_1^i(\tau; x(\tau), t_i) - {}^L R_1^{im}] d\tau - \tau {}^L R_1^{im} + P_1^i(\tau; x(t), t_i) \quad (4.51)$$

Note that, in general, the inner forcing function R_1^{im} has different limit values on the left and right sides as denoted by the L and R superscripts. Matching on the left side is done in a fashion similar to that on the right side and the equivalent expression to Eq. (4.21) is

$$P^0(\text{small negative } t) = P^i(\text{large negative } \tau) \Big|_{\tau=t/\epsilon} \quad (4.52)$$

Retaining terms to first order in t and ϵ gives

$$\begin{aligned} {}^L P_0^0(0) + {}^L P_{0t}^0(0)t + \epsilon {}^L P_1^0(0) = \\ P_0^i(-\infty) + \epsilon [P_1^i(0) - \int_{-\infty}^0 [R_1^i(\tau) - {}^L R_1^{im}] d\tau + {}^L R_1^{im} \tau] \end{aligned} \quad (4.53)$$

Equating like powers in t and ϵ yields:

$$\epsilon^0: \quad {}^L P_0^0(0) = P_0^i(-\infty) \quad (4.54)$$

$$\epsilon^1: \quad {}^L P_1^0(0) = P_1^i(0) - \int_{-\infty}^0 [R_1^i(\tau) - {}^L R_1^{im}] d\tau \quad (4.55)$$

$$t^1: \quad {}^L R_1^{im} = {}^L P_{0t}^0(0) = -{}^L P_{0x}^0(0) f^0(0) \quad (4.56)$$

Since the inner characteristic curve is continuous at $t=0$, so is $P_1^i(0)$, and therefore it satisfies the right side expression in Eq. (4.31). Using Eq. (4.31) in Eq's. (4.51) and

(4.55), and then Eq. (4.55) in Eq. (4.50) gives the inner and outer solutions of the return function on the left side. The composite solution on the left side is constructed the same way Eq. (4.26) was constructed and, after some manipulation, it takes the form

$$P_1^c(t_i) = P_0^c(t_i) + \varepsilon \left\{ -\int_{t_i}^0 R_1^o(t) dt + P_1^i(0) - \int_{\tau_i}^0 [R_1^i(\tau) - {}^L R_1^{i\infty}] d\tau \right\} \quad (4.57)$$

Finally, using Eq. (4.31) in Eq. (4.57) yields

$$P_1^c(x(t_i), t_i) = P_0^c(x(t_i), t_i) + \varepsilon \left\{ -\int_{t_i}^{t_f} R_1^o(t; x(t_i), t_i) dt - \int_{\tau_i}^0 [R_1^i(\tau; x(t_i), t_i) - {}^L R_1^{i\infty}] d\tau - \int_0^{\tau_f} [R_1^i(\tau; x(t_i), t_i) - {}^R R_1^{i\infty}] d\tau \right\} \quad (4.58)$$

where ${}^R R_1^{i\infty}$ satisfies the right side matching condition in Eq. (4.25). Differentiating Eq. (4.58) with respect to $x(t_i)$ yields

$$\begin{aligned} P_{1_{x(t_i)}}^c(x(t_i), t_i) &= P_{0_{x(t_i)}}^c(x(t_i), t_i) + \varepsilon \left\{ -\int_{t_i}^{t_f} R_{1_{x(t_i)}}^o(t; x(t_i), t_i) dt \right. \\ &\quad - R_1^o(t_f; x(t_i), t_i) \partial t_f / \partial x(t_i) - \int_{\tau_i}^0 [R_{1_{x(t_i)}}^i(\tau; x(t_i), t_i) - {}^L R_{1_{x(t_i)}}^{i\infty}] d\tau \\ &\quad - \int_0^{\tau_f} [R_{1_{x(t_i)}}^i(\tau; x(t_i), t_i) - {}^R R_{1_{x(t_i)}}^{i\infty}] d\tau \\ &\quad \left. - [R_1^i(\tau_f; x(t_i), t_i) - {}^R R_1^{i\infty}] \partial \tau_f / \partial x(t_i) \right\} \quad (4.59) \end{aligned}$$

which, when used in the optimality condition, results in an approximation for the feedback control to first order.

As noted earlier, the first order corrections in Eq's. (4.33) and (4.59) are only useful at t_0 , and therefore the quadratures must be repeated at each control update.

4.2 Application to Aeroassisted Plane-Change

In Section 3 analytic zeroth order solutions were obtained for the aeroassisted plane-change problem. In this problem, the singularity occurs at the lowest altitude point, hence the result in Eq. (4.59) must be used for initial conditions when they correspond to the entry phase of the maneuver. As shown in Section 3, the reduced three dimensional point mass equations of motion (3.21-3.23) for a lifting vehicle over a spherical non-rotating planet have been written in the form of Eq. (4.1). Note that these equations satisfy the condition in Eq. (4.2) and that h plays the role of the independent variable, which is identified as t in Eq. (4.1). It is a monotonic decreasing variable to the left of the singularity (entry phase of the trajectory) and a monotonic increasing variable to the right of the singularity (exit phase of the trajectory). Therefore τ in Eq. (4.3) is h/ϵ , which is denoted by $\eta=h/\epsilon$ in Eq. (3.41).

The control expressions in Eq's. (3.26) and (3.27) depend on P_ψ , P_γ and P_u . These can be obtained to first order using Eq. (4.59). From the first of Eq. (4.15) $R_1^0 = 0$, hence only the inner quadrature is needed. From the second of Eq. (4.15) and Eq's. (3.21)-(3.23)

$$R_1^i = P_{0u}^i(-2) + P_{0\gamma}^i[(1 - 1/u_0^i) / \tan \gamma_0^i] \quad (4.60)$$

$R_1^{i\infty}$ in Eq. (4.59) can be obtained by using the matching conditions $P_{0u}^i(\infty) = P_{0u}^0(0)$, $P_{0\gamma}^i(\infty) = P_{0\gamma}^0(0)$, $u_0^i(\infty) = u_0^0(0)$ and the expressions in Eq's. (3.31), (3.38) and (3.39). The result is simply

$${}^R R_1^{i\infty}(\infty) = 2^R a_1 - {}^R a_2, \quad {}^L R_1^{i\infty}(\infty) = 2^L a_1 - {}^L a_2 \quad (4.61)$$

Since $\tan \gamma_0^i$ appears in the denominator of Eq. (4.60) and $\gamma_0^i(0) = 0$ [see Eq. (3.111)], a singularity occurs in the integrand of Eq. (4.59) for this problem at $\eta=0$. However, the singularity is removable by simply transforming the variable of integration (η) to the independent variable for the inner problem (ψ). Using Eq's. (3.23), (3.41) and (3.43) it follows that

$$d\eta = \sin \gamma \cos \gamma d\psi / \sigma w \quad (4.62)$$

which removes the singularity in Eq. (4.59) when the variable of integration is changed from η to ψ .

Let ψ_s denote $\psi_0^i(\eta=0)$. This quantity is known from the zeroth order solution. Then, the specialization of Eq. (4.59) to the AOTV problem becomes:

$$P_{1\psi_i}^C(t_i) = P_{0\psi_i}^C(t_i) + \varepsilon \left\{ - \int_0^{\psi_s} \left[\frac{\gamma_0^i}{\sigma \omega_0^i} [R_1^i(\psi) - {}^L R_1^{i\infty}] \right]_{\psi_i} d\psi \right. \\ \left. - \int_{\psi_s}^{\psi_f} \left[\frac{\gamma_0^i}{\sigma \omega_0^i} [R_1^i(\psi) - {}^R R_1^{i\infty}] \right]_{\psi_i} d\psi - \left[\frac{\gamma_0^i}{\sigma \omega_0^i} [R_1^i(\psi) - {}^R R_1^{i\infty}] \right]_{\psi_f} \partial \psi_f / \partial \psi_i \right\} \quad (4.63)$$

$$P_{1\gamma_i}^C(t_i) = P_{0\gamma_i}^C(t_i) + \varepsilon \left\{ - \int_0^{\psi_s} \left[\frac{\gamma_0^i}{\sigma \omega_0^i} [R_1^i(\psi) - {}^L R_1^{i\infty}] \right]_{\gamma_i} d\psi \right. \\ \left. - \int_{\psi_s}^{\psi_f} \left[\frac{\gamma_0^i}{\sigma \omega_0^i} [R_1^i(\psi) - {}^R R_1^{i\infty}] \right]_{\gamma_i} d\psi - \left[\frac{\gamma_0^i}{\sigma \omega_0^i} [R_1^i(\psi) - {}^R R_1^{i\infty}] \right]_{\psi_f} \partial \psi_f / \partial \gamma_i \right\} \quad (4.64)$$

$$P_{1u_i}^C(t_i) = P_{0u_i}^C(t_i) + \varepsilon \left\{ - \int_0^{\psi_s} \left[\frac{\gamma_0^i}{\sigma \omega_0^i} [R_1^i(\psi) - {}^L R_1^{i\infty}] \right]_{u_i} d\psi \right. \\ \left. - \int_{\psi_s}^{\psi_f} \left[\frac{\gamma_0^i}{\sigma \omega_0^i} [R_1^i(\psi) - {}^R R_1^{i\infty}] \right]_{u_i} d\psi - \left[\frac{\gamma_0^i}{\sigma \omega_0^i} [R_1^i(\psi) - {}^R R_1^{i\infty}] \right]_{\psi_f} \partial \psi_f / \partial u_i \right\} \quad (4.65)$$

where a small flight path angle approximation $\gamma = \sin \gamma \cos \gamma$ is employed for simplicity of notation.

The zeroth order composite terms in Eq's. (4.63-4.65) are known from the zeroth order solution as developed in Section 3. The expression for $R_1^i(\psi)$ follows from Eq. (4.60), Eq's. (3.44) and (3.83) that relate u_0^i to v_0^i and P_{0u}^i to P_{0v}^i , and Eq's. (3.61), (3.63), (3.64) and (3.67) that relate $\gamma_0^i(\psi)$, $v_0^i(\psi)$, $P_{0\gamma}^i(\psi)$ and P_{0v}^i to ψ and the constants of integration. $R_1^i(\psi)$ is evaluated along the inner characteristic curve, which corresponds to the zeroth order inner solution in Section 3. Hence, differentiation of R_1^i with respect to $x(t_i)$ implies that $\partial x_0^i / \partial x(t_i)$ must be computed. The dependency of the zeroth order inner solution on the initial conditions is expressed in Section 3 through a set of 20 equations for 20 unknown parameters that define the inner and outer zeroth order solutions. The procedure to obtain the required derivatives involves use of the chain rule,

where first the partials of $R_1^j(\psi)$ with respect to the 20 parameters is taken, and then multiplied by the partials of the parameters with respect to the initial conditions. The details are omitted, but the procedure is similar to that developed for a regular expansion in Ref. 11.

4.3 First Order Guided Solution

Closed loop guided solutions are obtained by using the optimal control expressions given in Eq's (3.26) and (3.27). Feedback implementation entails treating the current state (from the simulation) at each control update as a new initial state, and calculating the costate values corresponding to the same time instant. The estimate for these costates to first order is obtained by repetitively following two steps. First the zeroth order MAE problem is solved, providing the zeroth order Euler solution which defines the characteristic curves for the first order expansion of the HJB equation. The procedure for this step has been detailed in Section 3. Next the quadratures in Eq's. (4.63-4.65) are performed to correct the costates to first order. When a specified time increment has been reached, the current states are used as initial conditions for the next MAE calculation followed by the first order correction step. During the exit phase, the left outer solution is discarded. This situation represents the case where $t_0 \geq 0$ in the analysis of Section 4.1.

Figure 4.2 compares the exact solution, the zeroth order guided solution and the first order calculation for the left side (the heading in the entry phase of the maneuver is between 0 and 11.5°). The exact solution is computed at heading increments of 0.25° where the initial conditions are determined by the values of the zeroth order guided simulation at the end of the last control update segment. The first order correction is calculated but not included in the guided solution. It is clear that initially the first order calculation overcorrects the zeroth order solution. Close to and in the inner region though, the first order calculation does provide an excellent correction to the zeroth order solution. These partial results indicate the need for further investigation of this behavior. The cause may be a numerical ill conditioning in the evaluation of the quadratures in Eq's. (4.63-4.65), and perhaps further development of the theory may be required.

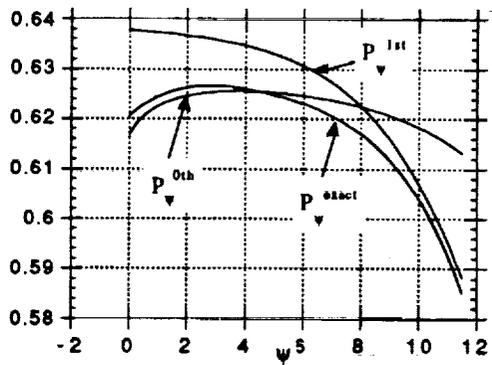
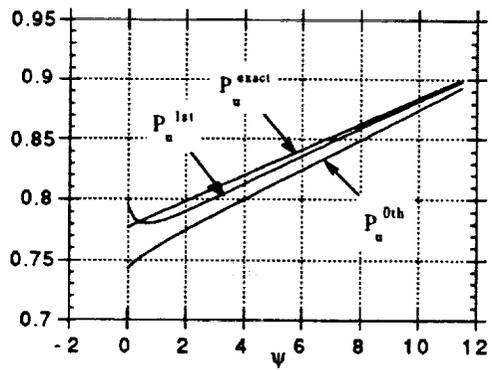
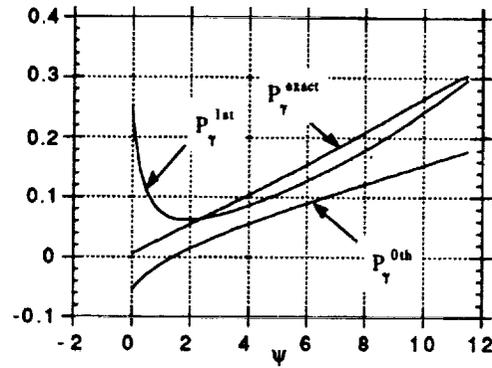


Figure 4.2 Exact and zeroth Order Guided Solutions of the Costates and first Order Calculation

Section V

Conclusions and Recommendations

5.1 Conclusions

In this report it has been demonstrated that the dynamics associated with skip trajectories are singularly perturbed, with the perturbation parameter defined as the ratio of the atmospheric scale height to a reference radius close to the planet radius. Earlier studies have either employed regular perturbation analysis of the inner dynamics, or have incorrectly attempted a matched asymptotic expansion (MAE) analysis. We have clearly demonstrated that the transformations employed in the earlier regular expansion studies in effect transformed the original problem to the inner dynamics, and therefore only have the appearance of being the type that occur in regularly perturbed problems. The resulting solutions are therefore not uniformly valid. For skip trajectories, this results in a poor approximation of the optimal solution near the end of the trajectory, where there is little control authority available.

With regard to MAE analysis, all of the issues improperly dealt with in earlier analyses of this type that have been attempted in the past for skip trajectories, have been corrected. The first issue deals with the fact that both the inner and outer expansions are crucially involved in satisfying the boundary conditions. The use of the outer expansion alone to satisfy initial conditions leads to discrepancies in the matching conditions. A second issue is the need for separate left and right outer expansions, and the role that the inner expansion plays in joining the discontinuities that occur between the outer expansions through the matching conditions. The true optimal solution is, by its nature, continuous; therefore, in order for the composite solution to serve as a uniformly valid approximation to the exact solution, it also must be continuous. In a skip trajectory, a discontinuity between the left and right zeroth order outer solutions is caused by the change that occurs in the trajectory parameters during the osculating atmospheric maneuver. We have demonstrated

that the zeroth order inner solution can be viewed as a process whereby the discontinuity between the left and right zeroth order outer solutions is taken into account.

A third issue concerns the proper selection of the reference altitude, used in defining the independent variable of integration in the outer solution, which has significant implications on the solution process. An arbitrary choice of this altitude leads to a situation where the outer solution for the flight path angle cannot be evaluated, thereby preventing a zeroth order composite solution from being formed. A systematic approach to obtain a relationship to determine the correct reference altitude is to use the condition that the composite zeroth order flight path angle solution is zero at the lowest point of the trajectory.

Application of these ideas to the problem of inclination change with minimum energy loss has resulted in a zero order solution in the form of a set of 20 algebraic equations. By exploiting the structure inherent in the matching procedure it is possible to reduce the problem to 6 equations in 6 unknowns. A further simplification was employed which permits satisfaction of two of the boundary conditions by partially separating two of the equations from the remaining four equations. Numerical experience shows that the zeroth order solution is close to the optimal solution, and that the outer solution plays a critical role in accounting for the variations in Loh's term near the exit phase of the maneuver. This feature is what differentiates a MAE analysis from a regular perturbation analysis of skip trajectories. However, the deficiency that remains in several of the critical variables indicates the need for a first order correction.

Expansion of the solution to first order required further development of the MAE expansion procedure. We have developed a general procedure for constructing a matched asymptotic expansion of the Hamilton-Jacobi-Bellman (HJB) equation, based on the method of characteristics. Of particular interest here is the manner in which matching and boundary conditions are enforced when the expansion is carried out to first order. We have recognized the need to distinguish between two cases pertaining to the location of the singular region with respect to the boundary conditions. The first is where the left boundary condition coincides with, or lies to the right of, the singular region, and the second is where the singular region lies between the boundary conditions. It is shown that the boundary conditions for the characteristic curves of the HJB equations are also defined by the MAE method. The characteristic curves for the inner and outer expansions of the HJB equation are the inner and outer solution components of the zeroth order MAE solution. Another consequence of the analysis is that whenever the outer and inner zeroth order MAE solutions are used to define the characteristic curves, the first order matching

conditions are automatically satisfied. A simple example is used to illustrate the procedure, where the obtained solution is uniformly valid to $O(\epsilon^2)$. The procedure's potential application to aeroassisted plane change was also evaluated.

5.2 Recommendations

Based on the results developed in this report, the following recommendations are made for further research in this area.

5.2.1 Completion of the First Order Correction Analysis of the Aeroassisted Plane Change Problem

In Section 3 the zeroth order guided solution was obtained and compared with the exact solution. The error that exists in some of the zeroth order guided solution variables with respect to the exact solution indicated the need for a first order correction. Section 4 developed a general procedure to obtain a first order correction to the zeroth order problem and its potential application to the aeroassisted plane change problem. The numerical results indicate the need for further evaluation of the approach, and perhaps further development of the theory. Simple examples of increasing complexity proved useful in understanding and developing the theory. It is recommended that the first order correction for the aeroassisted plane change problem be further analyzed to determine if it is numerically ill conditioned, or if an alternative approach or further development of the theory is needed. One alternative approach for first order analysis is to expand the Euler system equations.²⁸

5.2.2 Aerodynamic Heating Requirements

Aerodynamic heating is an important aspect of the aeroassisted maneuver that was not considered in this report. To make the guidance law useful in realistic applications, it is necessary to include aerodynamic heating requirements in the problem formulation in a form such that the zeroth order problem remains tractable. Minimization of the time integral of the flight path angle squared may have desirable features, as the numerical work

in Ref. 2 indicates. The numerical solution to this problem results in nearly grazing trajectories that are considered an useful engineering compromise between energy requirements and aerodynamic heating requirements. Hence, a choice of performance index that includes a flight path angle squared term is recommended.

A second approach for formulations that render the zeroth order problem intractable (such as enforcing a hard constraint on the heating rate or the heating load) would be to investigate combining an analytically tractable portion of the solution with a numerical method, such as collocation, to develop a numerically efficient zeroth order solution. The approach was developed and illustrated for a launch vehicle application in Ref. 28.

5.2.3 Atmospheric and Parametric Uncertainties

The aerodynamic force used to modify the vehicle's trajectory during the aeroassisted maneuver is uncertain due to uncertainties associated with estimates of the vehicle state vector, atmospheric density and the vehicle's aerodynamic coefficients. The effect of these uncertainties along with uncertainties in the entry conditions may result in significant trajectory deviations from the nominal trajectory and in large errors in the final conditions.²⁹ An important determinate of the guidance system performance during maneuvers, such as aerocapture or landing on the surface of Mars, is the accuracy of the information on which the guidance law is based.³⁰ A further study to improve onboard estimation and parameter identification for aeroassisted applications is recommended.

A second viewpoint is to design the guidance law so that performance is maintained in the presence of uncertainty (robust performance). Such a design can be achieved by treating the uncertainty as an opponent in a differential game formulation³¹. In the case of linear quadratic games, such a formulation is intimately connected to the design of a controller that minimizes the infinity norm of the transfer function from disturbances to performance outputs. It may also be beneficial to investigate the use of thrust within the atmosphere, such as an Aerocruise maneuver to achieve plane change. It has been shown in Ref. 32 that Aerocruise is less sensitive to atmospheric uncertainties than aeroglide.

5.2.4 Expansion of the Euler Equations

An alternative approach to obtaining higher order corrections is to consider an expansion of the Euler system of equations. The higher order equations for the states and costates are coupled, linear and inhomogeneous, and the contributions of higher order outer terms to the composite solution are not zero (as they are in the HJB expansion). The solution of these equations requires calculation of a state transition matrix, which can be derived analytically using the analytical zeroth order solution, and performing a quadrature on both state and costate equations. It has been shown that in the case of a regular perturbation, the first order correction obtained by expansion of the HJB equation is equal to the correction obtained by expanding the Euler system of equations to first order.³³ However, this method has an important potential advantage over the HJB expansion in that it may be possible to fix the zeroth order solution, and precompute and store the quadratures along the zeroth order solution as a function of a monotonic variable (such as total energy). Then, at each control update, the state perturbations from the zeroth order solution are accounted for in the first order correction by treating them as initial conditions for the first order solution.²⁸ This is not possible in an HJB expansion, since the first order correction is valid only for the current values of the state and the independent variable, and the quadrature must be repeated at each control update.

5.2.5 Other Problem Formulations

Extensions of the existing problem formulation would include accounting for the effects of the planet's rotation and cross range angle, which are ignored in the present analysis. It is easy to show that these effects are not present in the zeroth order dynamics, which means that they appear only in the first order corrections that are computed by quadrature. Therefore, conceptually it should be rather straightforward to include these effects in the analysis. Also, there are a range of other problems that are of practical interest, such as the aerocapture problem and reentry problems. In the aerocapture problem the total energy of a vehicle on a hyperbolic trajectory is to be reduced so that it is captured by the gravitational field of a planet. Aerodynamic heating limits are an important consideration in this problem formulation as well.

5.2.6 Parametric Studies

In the context of parametric studies it is necessary to determine the range of problem parameters for which approximate solutions are obtainable, and to conduct a comparison with purely exoatmospheric maneuvers in terms of energy consumption. Typical parameters to be considered are the initial and final orbital parameters, and the maximum lift to drag ratio. Therefore, it is of interest to conduct evaluations of a more extensive range of parameter values than in the limited study conducted here.

Appendix A

Relationship for the Zeroth Order Control σ

Expressed in the transformed variables in the inner region, the Hamiltonian function has the form given in Eq. (3.55)

$$H_0^i = P_{0\gamma}^i \delta / \sigma - P_{0w}^i \gamma^i / \sigma + P_{0v}^i (1 + \delta^2 + \sigma^2) / \sigma \quad (\text{A1})$$

The control functions are obtained from the optimality conditions $H_{0\delta}^i = 0$ and $H_{0\sigma}^i = 0$. For the above expression of H_0^i these conditions, after some manipulation, are:

$$H_{0\delta}^i = (P_{0\gamma}^i + 2P_{0v}^i \delta) / \sigma = 0 \quad (\text{A2})$$

$$H_{0\sigma}^i = P_{0v}^i - (-P_{0\gamma}^i / 4P_{0v}^i - P_{0w}^i \gamma_0^i + P_{0v}^i) / \sigma^2 = 0 \quad (\text{A3})$$

and the control functions are obtained as:

$$\delta = -P_{0\gamma}^i / 2P_{0v}^i \quad (\text{A4})$$

$$\sigma^2 = (-P_{0w}^i \gamma_0^i - P_{0\gamma}^i / 4P_{0v}^i) / P_{0v}^i + 1 \quad (\text{A5})$$

Substituting these functions back into the Hamiltonian in Eq. (A1), after some manipulation it takes the form:

$$H_0^i = (2P_{0v}^i / \sigma)[(-P_{0w}^i \gamma_0^i - P_{0\gamma}^{i2} / 4P_{0v}^i) / P_{0v}^i + 1] = 2\sigma P_{0v}^i \quad (\text{A6})$$

where the expression in the brackets is recognized to be σ^2 . Since H_0^i and P_{0v}^i are constant in this problem, so is σ . Thus it should be possible to express σ in term of constant parameters only, which is not immediately recognized in Eq. (A5) since it contains the functions $P_{0\gamma}^i$ and γ_0^i . To show this, use of Eq. (3.64) in Eq. (A5) yields

$$\sigma^2 = \{-(P_{0w}^i / \sigma)\psi_0^i + c\} / 2P_{0v}^i - P_{0w}^i \gamma_0^i / P_{0v}^i + 1 \quad (\text{A7})$$

From Eq. (3.67) P_{0w}^i and c can be written as:

$$P_{0w}^i = 2\sigma^2 P_{0v}^i k_1 \quad (\text{A8})$$

$$c = -2\sigma P_{0v}^i k_2 \quad (\text{A9})$$

Using these equations and Eq. (3.61) in Eq. (A7), σ becomes:

$$\sigma = 1 / (1 + k_2^2 + 2k_1 k_3)^{1/2} \quad (\text{A10})$$

Appendix B

Initial Guess and Evaluation of the Inner Integration Constants

B.1 Initial Guess Procedure

To calculate the 6 inner integration constants k_3 , k_4 , k_5 , c , P_{0w}^i , and P_{0v}^i in Eq's. (3.61-3.66), 6 boundary values must be supplied. These are the left common part values of γ , v and ψ and the right common part values of γ , P_v and ψ which will be denoted by the subscripts "L" and "R" respectively. At the very first step of the procedure, the actual left boundary values of γ , v and ψ and the right boundary values of γ and ψ are used as an initial guess for γ_L , v_L , ψ_L , γ_R and ψ_R respectively. The value of P_{0w} is estimated using the boundary condition in Eq. (3.116) that $P_{0w}^C(h_f) = 1.0$. Since h_f is far from the region of singularity ($h=0$), this is closely approximated by the outer solution, so ${}^R P_{0w}^O(h_f) = 1.0$. Since $h \ll 1$, the value of ${}^R u_0^O(h)$ does not change substantially in Eq. (3.31), so ${}^R u_0^O(0) = {}^R u_0^O(h_f)$ and from Eq. (3.38) ${}^R P_{0u}^O(0) = {}^R P_{0u}^O(h_f) = 1.0$. These properties can be verified by examining the converged solution for $P_{0u}^O(\psi)$ in Fig. 3.7. Next the transformation in Eq. (3.83) is used to relate ${}^R P_{0u}^O(0)$ to ${}^R P_{0v}^O(0)$, where ${}^R u_0^O(0) = {}^R u_0^O(h_f)$ and from Eq. (3.44)

$${}^R u_0^O(h_f) = e^{-R v_0^O(h_f)/E^*} / g_s r_s \quad (B1)$$

Now ${}^R v_0^O(h_f) = v_0^i(\psi_f) = v_0^C(\psi_f)$ and $v_0^i(\psi_f)$ is calculated by approximating the dynamics in Eq. (3.49) using $dv_0^i/d\psi = 2.0$. All of the above properties may be verified from Fig. 3.5. Finally, the matching condition in Eq. (3.86) $P_{0v}^O(0) = P_{0v}^i(\infty) = P_{vR}$ is used to provide an estimate for P_{vR} .

B.2 Inner Integration Constants

Using Eq's. (3.61) and (3.62) at the left and right common part boundary conditions, k_1 , k_2 and k_3 become:

$$k_1 = -12\sigma\Delta w / \Delta\psi^3 - 6(\gamma_R + \gamma_L) / \Delta\psi^2 \quad (\text{B2})$$

$$k_2 = \Delta\gamma / \Delta\psi + k_1(\psi_R + \psi_L) / 2 \quad (\text{B3})$$

$$k_3 = -k_1\psi_R\psi_L / 2 + (\gamma_L\psi_R - \gamma_R\psi_L) / \Delta\psi \quad (\text{B4})$$

where $\Delta(\cdot) = (\cdot)_R - (\cdot)_L$ and σ is found as a function of the boundary conditions by using Eq's. (B2-B4) in Eq. (A10). The expression for σ becomes

$$\sigma = \sqrt{\Delta\psi^2 / [\Delta\psi^2 + 3(\gamma_R + \gamma_L)^2 + \Delta\gamma^2]} \quad (\text{B5})$$

From Eq. (3.43) at $\eta \rightarrow \infty$, the common part values of w are zero, thus $w_L = w_R = 0$ when the initial conditions are in the left side (entry phase). In this case $\Delta w = 0$ in Eq. (B.2). When the initial conditions are given in the right side (exit Phase), the left matching condition is dropped and w_L is equal to the current w . Use of Eq's. (B2) and (B3) and the estimate of P_v^i in Eq. (3.67) provides the values of the constants P_{0w}^i and c .

Next, k_4 and k_5 can be found using Eq's. (3.62) and (3.63) at the left boundary conditions:

$$k_4 = \sigma w_L - k_1\psi_L^3 / 6 + k_2\psi_L^2 / 2 + k_3\psi_L \quad (\text{B6})$$

$$k_5 = v_L - (\sigma + 1/\sigma)\psi_L - \sigma[(\psi_L k_1 - k_2)^3] / 3k_1 \quad (\text{B7})$$

At this stage all the inner states and costates can be evaluated at any ψ between ψ_L and ψ_R as a function of the boundary conditions using Eq's. (3.61-3.66). In particular, the initial and final inner solutions are needed to construct the composite solution, which is

used to enforce the boundary conditions. Also, at the lowest point in the trajectory ($h=0$, $r=r_s$) the composite solution is equal to the inner solution and the flight path angle $\gamma_0^c(0, \varepsilon)$ is zero. Thus, the value of ψ_s that corresponds to this point can be found from Eq. (3.61) by equating it with zero, and then using the result in Eq. (3.62) to calculate the corresponding value of w_s . This point is illustrated in Fig. 3.2. Finally, ρ_s and hence, the reference radius r_s , can be obtained by using w_s and the relationships defined in Eq. (3.43).

Appendix C

Relationship Between the Zeroth and the First Order Matching Conditions

The quantity $R_1^i(\tau)$ needed for the quadratures in Eq. (4.59) is given by Eq. (4.15)

$$R_1^i = -P_{0x}^i(\tau)f_0^i(x(\tau),0) \quad (C1)$$

where $x(\tau)$ satisfies Eq. (4.17). Therefore

$$R_1^{i\infty} = -P_{0x}^i(\infty)f_0^i(x_0^i(\infty),0) \quad (C2)$$

where $x_0^i(\tau)$ and $P_{0x}^i(\tau)$ are the state and costate solutions along the inner characteristic curve. The right side matching condition in Eq. (4.25) imposes a constraint that must be satisfied by the inner and outer characteristic curves. That is

$$P_{0x}^i(\infty)f_0^i(x_0^i(\infty),0) = {}^R P_{0x}^o(0)f_0^o({}^R x_0^o(0),0) \quad (C3)$$

where the superscript R is used in Eq. (C3) in recognition of the fact that the outer characteristic curves are discontinuous at $t=0$. Noting that f_0^i and f_0^o are the same functions and use of Eq. (C3) together with the matching conditions in Eq. (4.23) imply that:

$$P_{0x}^i(\infty) = {}^R P_{0x}^o(0) , \quad x_0^i(\infty) = {}^R x_0^o(0) \quad (C4)$$

A similar consideration of the left side matching conditions in Eq's. (4.54) and (4.56) leads to the conditions:

$$P_{0x}^i(-\infty) = {}^iP_{0x}^o(0) , \quad x_0^i(-\infty) = {}^i x_0^o(0) \quad (C5)$$

It can be seen that the inner and outer characteristic curves must also satisfy the MAE matching conditions of Eq. (2.5). Thus it follows that the Euler solutions from a zeroth order MAE analysis, as outlined in Section 2, serve as the characteristic curves along which the quadratures in Eq. (4.59) are performed.

Bibliography

1. Mease, K.D., "Optimization of Aeroassisted Orbital Transfer: Current Status," *The Journal of Astronautical Sciences*, Vol. 36, Nos. 1/2, January-June 1988.
2. Miele, A., Basapur, V.K., Mease, K.D., "Nearly-Grazing Optimal Trajectories for Aeroassisted Orbital transfer," *Journal of the Astronautical sciences*, Vol. 34, No. 1, 1986, pp. 3-18.
3. Miele, A., Lee, W.Y., Mease, K.D., "Nearly-Grazing Optimal Trajectories for Aeroassisted Orbital transfer," *Journal of the Astronautical sciences*, Vol. 36, Nos. 1/2, Jan.-June 1988, pp. 139-157.
4. Miele, A., Lee, W.Y., "Optimal Trajectories for Hypersonic Flight," 1989 American Control Conference, pp. 2017-2023.
5. Mease, K.D., Vinh, N.X., Kou, S.H., "Optimal Plane Change during Constant Altitude Hypersonic Flight," AIAA Atmospheric Flight Mechanics conference, Aug. 15-17 1988.
6. Lee, J.Y., Hull, D.G., "Maximum Orbit Plane Change with Heat Transfer Considerations," *Journal of Guidance, Control and Dynamics*, Vol. 13, No. 3, May-June 1990, pp. 492-497.
7. Hull, D.G., Giltner, J.M., Speyer, J.L., Mapar, J., "Minimum Energy-Loss Guidance for Aero-Assisted Orbital Plane Change," *AIAA Journal of Guidance, Control, and Dynamics* Vol. 8, No. 4, July-August 1985.

8. Hull, D.G., McClendon, J.R., Speyer, J.L., "Improved Aero-Assisted Plane Change Using Successive Approximation," *The Journal of Astronautical Sciences*, Vol. 36, Nos. 1/2, January-June 1988.
9. Hull, D.G., "New Analytical Results for AOTV Guidance," 1985 AIAA Guidance, Navigation, and Control Conf., Snowmass, CO, August 1985.
10. Mishne, D., Speyer, J.L., "Optimal Control of Aeroassisted Plane-Change Maneuver Using Feedback Expansions," *Proceedings of the AIAA Atmospheric Flight Mechanics Conference*, AIAA, New York, 1986, pp. 253-258.
11. Speyer, J.L., Crues, E.Z., "Approximate Atmospheric Guidance Law for Aeroassisted Plane Change Maneuvers," *AIAA Journal of Guidance, Control, and Dynamics*, Vol. 13, No. 5, September-October 1990.
12. Shi, Y.Y., "Matched Asymptotic Solutions for Optimum Lift Controlled Atmospheric Entry," *AIAA Journal*, Vol. 9, No. 11, 1971.
13. Shi, Y.Y., Pottsepp, L., "A Matched Asymptotic Solution for Skipping Entry Into Planetary Atmosphere," *AIAA Journal*, Vol. 9, No. 4, 1971.
14. Calise, A.J., Bae, G.H., "Optimal Heading Change with Minimum Energy Loss for a Hypersonic Gliding Vehicle," *AIAA Journal of Guidance, Control, and Dynamics*, Vol. 13, No. 4, July-August 1990.
15. Calise, A.J., Bae, G.H., "A Near Optimal Guidance Algorithm for Aeroassisted Orbit Transfer," *AIAA Guidance, Navigation and Control Conf.*, Minneapolis, MN, August 1988.
16. Naidu, D.S., "Three-Dimensional Atmospheric Entry Problem Using Method of Matched Asymptotic Expansions," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. 25, No. 5, September 1989.

17. Frostic, F., Vinh, N.H., "Optimal Aerodynamic Control by Matched Asymptotic Expansions," *Acta Astronautica*, Vol. 3, 1976.
18. Mease, K.D., McCreary, F.A., "Atmospheric Guidance Law for Planar Skip Trajectories," *AIAA Atmospheric Flight Mechanics Conference*, Snowmass, CO, August 1985.
19. Mease, K.D., Lee, J.Y., Vinh, N.X., "Orbital Change During Hypersonic Aerocruise," *The Journal of Astronautical Sciences*, Vol. 36, Nos. 1/2, January-June 1988.
20. Vinh, X.N., "Optimal Trajectories in Atmospheric Flight," Elsevier Scientific Publishing Company, Amsterdam-Oxford-New York, 1981.
21. Bryson, A.E., Jr., Ho, Y.C., "Applied Optimal Control," Halstead Press, New York, 1975.
22. Breakwell, J.V., Shinar, J., and Visser, H.G., "Uniformly Valid Feedback Expansions for Optimal Control of singularly Perturbed Dynamic Systems," *J. of Optimization Theory and Applications*, Vol. 46, No. 4, 1985, pp. 441-453.
23. Courant, R., and Hilbert, D., "Methods of Mathematical Physics," Vol. 2, Interscience, New York, 1962.
24. Hull, D.G., Speyer, J.L., "Optimal Reentry and Plane Change Trajectories," *Journal of the Astronautical sciences*, Vol. xxx, April-June 1982, pp. 117-130.
25. Oberle H.J., W. Grimm, "BNDSCO - A Program for the Numerical Solution of Optimal Control Problems," English Translation of DFVLR-Mitt. 85-05, German Aerospace Research Establishment DLR, Oberpfaffenhofen, Germany.
26. Nayfeh, H., "Perturbations Methods," John Wiley & Sons, New York, New York, 1973

27. Ardema, D.M., "Singular Perturbations in Flight Mechanics," NASA Technical Memorandum, NASA TM X-62,380.
28. Leung, M.K.S., "A Real-Time Near-Optimal Guidance Approach for Launch Vehicle," Ph.D. Thesis, Georgia Institute of Technology, Atlanta GA, March 1992, pp. 36-42.
29. Tigges, M., Ling, L., "A predictive Guidance Algorithm for Mars Entry," 27th Aerospace Sciences Meeting, January 9-12, 1989, Reno, Nevada, Paper AIAA 89-0632.
30. Mease, K.D., Chattrjee, K.A., Purvis, R.C., McCreary, A.F., "State Estimation and Parameter Identification During Aerocapture at Mars," AAS/AIAA Astrodynamics Specialist Conference, Kalispell, Montana, August 10-13, 1987, Paper AAS 87-505.
31. Ilgen, M.R., Speyer, J.L., "Robust Approximate Optimal Plane Change Guidance Using Differential Game Theoretical Methods," AIAA Guidance, Navigation and Control Conf., Hilton Head, SC, Aug. 1992, AIAA Paper 92-4453.
32. Hougi, S., Mishne, D., "An Approximate guidance Law for atmospheric Plane Change Maneuver of an Aerocruise Space Vehicle,"
33. Feeley, T. S., "Approximate Optimal Guidance for the Advanced Launch System," Ph.D. Dissertation, University of Texas, Austin Tx, May 1992.
34. Calise, A.J., Melamed, N., "Optimal Guidance of Aeroassisted Transfer Vehicles based on Matched Asymptotic Expansions," AIAA Guidance, Navigation and Control Conf., New Orleans, LA, Aug. 1991, pp. 1048-1058. (Submitted for publication in the Journal of Guidance, Control and Dynamics)
35. Melamed, N, Calise, A.J., "Evaluation of an Optimal Guidance algorithm for Aeroassisted Orbit Transfer," AIAA Guidance, Navigation and Control Conf., Hilton Head, SC, Aug. 1992, pp. 632-640. (Submitted for publication in the Journal of Guidance, Control and Dynamics)

36. Calise, A.J., Melamed, N., "Matched Asymptotic Expansion of the Hamilton-Jacobi-Bellman Equation for Aeroassisted Plane-Change Maneuvers," Submitted for presentation in the AIAA Guidance, Navigation and Control Conf., Monterey, CA, Aug. 9-11, 1993.
37. Calise, A.J., Melamed, N., "Analytical Guidance Law Development for Aerocapture at Mars," Progress Report to NASA Langley Research Center for the period 1 May 1990 - 31 October 1990.
38. Calise, A.J., Melamed, N., "Analytical Guidance Law Development for Aerocapture at Mars," Progress Report to NASA Langley Research Center for the period 1 December 1990 - 30 April 1991.
39. Calise, A.J., Melamed, N., "Analytical Guidance Law Development for Aerocapture at Mars," Progress Report to NASA Langley Research Center for the period 1 May 1991 - 31 October 1991.
40. Calise, A.J., Melamed, N., "Analytical Guidance Law Development for Aerocapture at Mars," Progress Report to NASA Langley Research Center for the period 1 November 1991 - 31 July 1992.