ON THE PREDICTION OF MULTIGRID EFFICIENCY THROUGH LOCAL MODE ANALYSIS

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ABSTRACT

A single grid local mode analysis is used to predict the smoothing properties of numerical schemes for solving the Navier-Stokes equations with factorization based on Stone's Strongly Implicit Method. Four difference approximations for the convection terms are considered, namely, hybrid, central, second-order upwind, and third-order upwind. Smoothing factors from the analysis are compared with practical convergence factors in a multigrid method for flow over a backward facing step and it is found that the local mode analysis correctly predicts the effects of Reynolds number and higher-order schemes.

1 INTRODUCTION

The successful use of multigrid methods to accelerate convergence rates is dependent on the ability of the numerical algorithm to dampen high frequency error components since these components cannot be resolved on coarser grids. High frequency components have short coupling ranges; therefore, their smoothing is a localized process meaning that only one isolated computational stencil need be analyzed and the effect of boundaries can be neglected. This is the approach of local mode analysis for the prediction of smoothing properties which was first introduced by Brandt [1] for various partial differential equations and numerical algorithms. Shaw and Sivaloganathan [2] extended this analysis to the SIMPLE pressure correction algorithm using alternating direction implicit (ADI) relaxation for the solution of the algebraic system of equations for varying Reynolds numbers and under-relaxation factors. Convection terms were approximated using a hybrid of first-order upwind and second-order central differencing.

The present paper uses local mode analysis to predict the smoothing properties of numerical algorithms for calculation of two-dimensional recirculating flows using higher-order difference schemes for convection terms introduced via deferred correction and Stone's Strongly Implicit Method for factorization of the resulting system of algebraic equations. Reynolds number and higher-order convection approximation effects are addressed and compared to multigrid results for laminar flow over a backward facing step.
2 THEORETICAL ANALYSIS

2.1 Governing equations

The equations governing steady, two-dimensional, incompressible flow can be written as:

\[
\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0
\]

(1)

\[
\frac{\partial}{\partial x}(\rho u^2) + \frac{\partial}{\partial y}(\rho uv) = -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)
\]

(2)

\[
\frac{\partial}{\partial x}(\rho uv) + \frac{\partial}{\partial y}(\rho v^2) = -\frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)
\]

(3)

where \( u \) and \( v \) are the velocity components in the \( x \) and \( y \) directions, respectively, \( p \) is the pressure, \( \mu \) is the absolute viscosity, and \( \rho \) is the density. Equations (1) - (3) represent the conservation of mass and momentum in the \( x \) and \( y \) directions, respectively.

The solution sequence is a predictor-corrector method which follows the SIMPLE algorithm of Patankar and Spalding [3]. Factorization of the system of equations is based on Stone’s Strongly Implicit Method. The flow geometry and boundary conditions are shown in Figure 1.

The governing equations are discretized by integrating over a set of three staggered control volumes and the locations of variables are shown in Figure 2. The central control volume is for the pressure. Equation (2) can be discretized by integrating over the left-shifted control volume for the \( u \) component of velocity. This leads to:

\[
a_p u_p = a_{EE} u_E + a_W u_W + a_N u_N + a_S u_S + a_{EE} u_{EE} + a_{WW} u_{WW} + a_{NN} u_{NN} + a_{SS} u_{SS}
\]

\[
- \frac{(p_p - p_w)}{h_x} + \left( \frac{\mu_o}{h_x h_y} \right) (v_N - v_{NW} + v_w - v_p) + \frac{\mu_o}{h_x^3} (u_E - 2 u_p + u_w)
\]

(4)

Equation (3) is discretized by integrating over the bottom-shifted control volume for the \( v \) component of velocity:

\[
a_p v_p = a_{EE} v_E + a_W v_W + a_N v_N + a_S v_S + a_{EE} v_{EE} + a_{WW} v_{WW} + a_{NN} v_{NN} + a_{SS} v_{SS}
\]

\[
- \frac{(p_p - p_s)}{h_y} + \left( \frac{\mu_o}{h_x h_y} \right) (u_E - u_{SE} + u_s - u_p) + \frac{\mu_o}{h_y^3} (v_N - 2 v_p + v_s)
\]

(5)

where the \( a_i \) coefficients contain convection and diffusion terms, the subscripts of \( u, v, \) and \( p \) refer to the location of the variables (see figure 2), \( h_x \) and \( h_y \) are the grid spacing in the \( x \) and \( y \) directions, respectively, and \( \mu_o \) is the absolute viscosity of the fluid, assumed constant.

2.2 Approximation of convection terms

The \( a_i \) coefficients in equations (4) and (5) are dependent on the approximation used for the convection terms.
The present analysis investigates the hybrid, central, second-order upwind (2nd OU), and third-order upwind (3rd OU) approximation schemes.

The coefficients for the hybrid scheme have the following form:

\[ a_E = \max \{0, \frac{\mu_o}{h_x^2} - \frac{|\rho u_o|}{2h_x} \} + \max \{-\frac{\rho u_o}{h_x}, 0\} \]  
\[ a_w = \max \{0, \frac{\mu_o}{h_x^2} - \frac{|\rho u_o|}{2h_x} \} + \max \{-\frac{\rho u_o}{h_x}, 0\} \]  
\[ a_N = \max \{0, \frac{\mu_o}{h_y^2} - \frac{|\rho v_o|}{2h_y} \} + \max \{-\frac{\rho v_o}{h_y}, 0\} \]  
\[ a_s = \max \{0, \frac{\mu_o}{h_y^2} - \frac{|\rho v_o|}{2h_y} \} + \max \{-\frac{\rho v_o}{h_y}, 0\} \]  
\[ a_{EE} = a_{WW} = a_{NN} = a_{SS} = 0 \]  
\[ a_p = \Sigma a_i \]  

where the sum for \( a_p \) is taken over the \( a \) coefficients, \( u_o \) and \( v_o \) are the frozen velocity components due to the linearization of equations (2) and (3), the \( \max(a,b) \) operator selects the maximum of the arguments \( a \) and \( b \), and \( \| \) represents the absolute value.

The coefficients for the higher-order schemes have the following general form:

\[ a_E = \frac{\mu_o}{h_x^2} - \frac{\rho u_o}{2h_x} + f_1 \left[ \max \left\{ \frac{\rho u_o}{h_x}, 0 \right\} + \max \left\{ -\frac{\rho u_o}{h_x}, 0 \right\} \right] + f_2 \max \left\{ -\frac{\rho u_o}{h_x}, 0 \right\} \]  
\[ a_w = \frac{\mu_o}{h_x^2} + \frac{\rho u_o}{2h_x} + f_1 \left[ \max \left\{ \frac{\rho u_o}{h_x}, 0 \right\} + \max \left\{ -\frac{\rho u_o}{h_x}, 0 \right\} \right] + f_2 \max \left\{ \frac{\rho u_o}{h_x}, 0 \right\} \]  
\[ a_N = \frac{\mu_o}{h_y^2} - \frac{\rho v_o}{2h_y} + f_1 \left[ \max \left\{ \frac{\rho v_o}{h_y}, 0 \right\} + \max \left\{ -\frac{\rho v_o}{h_y}, 0 \right\} \right] + f_2 \max \left\{ -\frac{\rho v_o}{h_y}, 0 \right\} \]  
\[ a_s = \frac{\mu_o}{h_y^2} + \frac{\rho v_o}{2h_y} + f_1 \left[ \max \left\{ \frac{\rho v_o}{h_y}, 0 \right\} + \max \left\{ -\frac{\rho v_o}{h_y}, 0 \right\} \right] + f_2 \max \left\{ \frac{\rho v_o}{h_y}, 0 \right\} \]  
\[ a_{EE} = -f_1 \max \left\{ -\frac{\rho u_o}{h_x}, 0 \right\} \]  
\[ a_{WW} = -f_1 \max \left\{ \frac{\rho u_o}{h_x}, 0 \right\} \]  
\[ a_{NN} = -f_1 \max \left\{ -\frac{\rho v_o}{h_y}, 0 \right\} \]  
\[ a_{SS} = -f_1 \max \left\{ \frac{\rho v_o}{h_y}, 0 \right\} \]  
\[ a_p = \Sigma a_i \]
The values of $f_1$ and $f_2$ depend on the higher-order method to be used. For central differencing, $f_1 = f_2 = 0$, for 2nd OU differencing, $f_1 = 1/2$ and $f_2 = 1$, and for 3rd OU differencing, $f_1 = 1/8$ and $f_2 = 1/4$.

2.3 Local mode analysis

The Strongly Implicit Method (SIP) by Stone [4] solves the algebraic equations shown in equations (4) and (5) in a fully implicit manner. All variables are treated as unknowns, as opposed to line-relaxation methods which consider only lines of constant $x$ or $y$ as unknowns while sweeping through the computational domain. In the local mode analysis outlined below, the variables that are updated at the end of a relaxation sweep will be denoted by a dot over the variable, such as $u_p$, while those from the beginning of the relaxation sweep or those unchanged by the current relaxation sweep will be written as above, such as $u_p$. Higher-order approximations for the convection terms are introduced via deferred correction (see Khosla and Rubin [5]). In this procedure, the $a_i$ coefficients are calculated initially using equations (6) - (11) for the hybrid scheme. As the solution proceeds, the higher-order scheme is slowly introduced via corrections to the source terms. At the end when the solution is fully converged, the coefficients are effectively those of the higher-order scheme outlined in equations (12) - (20). The base hybrid coefficients will be denoted by an additional subscript $h$, such as $a_{Eh}$, while the higher-order coefficients will not have an additional subscript and will be written as above, such as $a_E$.

The relaxation of equation (4) (the discretized x momentum equation), with under-relaxation and deferred correction can be written as:

$$a_p h u_p = a_p h u_p + r_m (a_E h u_E + a_w h u_w + a_N h u_N + a_s h u_s - a_p h u_p)$$

$$+ r_m \left\{ \frac{-2 v_w}{s} + \frac{\mu_o}{h_s h_y} (v_{NW} + v_w - v_p) + \frac{\mu_o}{h_s^2} (u_E - 2 u_p + u_W) \right\}$$

$$+ r_1 r_m [(a_E - a_E h) (u_E - u_p) + (a_w - a_w h) (u_w - u_p) + (a_N - a_N h) (u_N - u_p) + (a_s - a_s h) (u_s - u_p)]$$

$$+ a_{EE} (u_{EE} - u_p) + a_{WW} (u_{WW} - u_p) + a_{NN} (u_{NN} - u_p) + a_{SS} (u_{SS} - u_p) \right\}$$

where $r_m$ is the under-relaxation factor for the $x$ and $y$ momentum equations, $r_j$ is the relaxation factor for introducing higher-order coefficients and is set to unity for the analysis. The exact solution for $U$, $V$, and $P$ also satisfies equation (21). If an equation written with the exact solution for $U$, $V$, and $P$ is subtracted from equation (21), which is written in terms of the approximate solution, $u$, $v$, and $p$, the error in the solution can be introduced. The error has components defined as; $\varepsilon^u = U - u$, $\varepsilon^v = V - v$, and $\varepsilon^p = P - p$.

Equation (21) written in terms of the error becomes:

$$a_p h \varepsilon^u = a_p h \varepsilon^u + r_m (a_E h \varepsilon^u + a_w h \varepsilon^u + a_N h \varepsilon^u + a_s h \varepsilon^u - a_p h \varepsilon^u)$$

$$+ r_m \left\{ \frac{-2 \varepsilon^w}{s} + \frac{\mu_o}{h_s h_y} \left( \varepsilon^u_N - \varepsilon^u_{NW} + \varepsilon^u_w - \varepsilon^u_p \right) + \frac{\mu_o}{h_s^2} \left( \varepsilon^u_E - 2 \varepsilon^u_p + \varepsilon^u_W \right) \right\}$$

$$+ r_1 r_m [(a_E - a_E h) \varepsilon^u + (a_w - a_w h) \varepsilon^u + (a_N - a_N h) \varepsilon^u + (a_s - a_s h) \varepsilon^u]$$

$$+ a_{EE} (\varepsilon^u_{EE} - \varepsilon^u) + a_{WW} (\varepsilon^u_{WW} - \varepsilon^u) + a_{NN} (\varepsilon^u_{NN} - \varepsilon^u) + a_{SS} (\varepsilon^u_{SS} - \varepsilon^u) \right\}$$

(22)
Since the continuous governing equations (1) - (3) have been linearized during the discretization, a single Fourier component of the error can be considered as:

$$
e^{i \Phi} = \alpha^{i \Phi} e^{i \theta} e^{i \frac{x}{h_x} + \frac{y}{h_y}}$$

$$
e_{w}^{i \Phi} = \alpha_{w}^{i \Phi} e^{i \theta} e^{i \frac{y}{h_y}}$$

$$
e_{s}^{i \Phi} = \alpha_{s}^{i \Phi} e^{i \theta} e^{i \frac{x}{h_x} + \frac{y}{h_y}}$$

where $i = \sqrt{-1}$, $\theta_1$ and $\theta_2$ are the components of the phase angle vector, $\alpha_0$, which is the error amplitude of the single Fourier mode $\theta_1$, $\theta_2$. Similar expressions exist for other grid points to the east and north and for the variables $v$ and $p$. Substituting the single Fourier modes into equation (22) and dividing through by $e^{i \theta_1} e^{i \theta_2} e^{i \theta_3}$, equation (22) becomes:

$$\alpha_{0}^{i \Phi} \left[ a_{p}^{i \Phi} r_{m} \left( e^{i \theta_1} + e^{i \theta_2} + e^{i \theta_3} - \frac{4 \mu_s^2}{h_x^2} \right) \right] =$$

$$\alpha_{w}^{i \Phi} \left[ a_{w}^{i \Phi} \left( e^{i \theta_1} + e^{i \theta_2} + e^{i \theta_3} - \frac{4 \mu_s^2}{h_x^2} \right) \right]$$

Equation (24) can be written in a more compact form, if the following variables are defined:

$$s_1 = \sin \left( \theta_1 / 2 \right)$$
$$s_2 = \sin \left( \theta_2 / 2 \right)$$

Equation (24) can then be written as:

$$\dot{\alpha}_{0}^{i \Phi} = \frac{1}{\eta^{i \Phi}} \left( \nu \alpha^{i \Phi} - \varphi^{i \Phi} - \zeta^{i \Phi} \alpha^{i \Phi} \right)$$

Following the same procedure for equation (5) (the discretized y momentum equation) yields:
where:

\[
\eta^v = a_p l_a - r_m \left( a_{e} l_{e} e^{i \theta_1} + a_{w} l_{w} e^{-i \theta_1} + a_{n} l_{n} e^{i \theta_1} + a_{s} l_{s} e^{-i \theta_1} - \frac{4 \mu_{e}^{2}}{h_{y}} \right)
\]

\[
\zeta^v = \frac{2 r_m s_2}{h_{y}}
\]

Equations (25) and (26) can be combined to give the amplification matrix \( A_f \) for the complete operation. This yields:

\[
\begin{bmatrix}
\alpha_u^0 \\
\alpha_v^0 \\
\alpha_\theta^0 \\
\alpha_p^0
\end{bmatrix} =
\begin{bmatrix}
\frac{\nu}{\eta^u} & -\frac{\varphi_1}{\eta^u} & -\frac{\varphi_2}{\eta^u} & \frac{\varphi_3}{\eta^u} \\
-\frac{\varphi_1}{\eta^v} & \frac{\psi_1}{\eta^v} & \frac{\psi_2}{\eta^v} & \frac{\psi_3}{\eta^v} \\
-\frac{\varphi_2}{\eta^v} & -\frac{\psi_2}{\eta^v} & \frac{\psi_3}{\eta^v} & \psi \\
0 & 0 & \frac{\varphi_3}{\eta^v} & \frac{\psi_3}{\eta^v}
\end{bmatrix}
\begin{bmatrix}
\alpha_u^f \\
\alpha_v^f \\
\alpha_\theta^f \\
\alpha_p^f
\end{bmatrix}
\]

In compact form:

\[
\dot{\alpha}_0 = A_f \alpha_0
\]  

Equation (27) yields the amplification matrix defining how the amplitude of the Fourier mode, with phase angles \( \theta_1 \) and \( \theta_2 \), is amplified during relaxation of the \( x \) and \( y \) momentum equations.

The SIMPLE pressure correction of Patankar and Spalding [3] follows the relaxation of the \( x \) and \( y \) momentum equations. A single dot over a variable will denote a value at the completion of the relaxation of the \( x \) and \( y \) momentum equations. A double dot over a variable will denote a value at the completion of the pressure correction. The variables \( u \), \( v \), and \( p \) are corrected following Shaw and Sivaloganathan [2]:

\[
\bar{u}_p = \bar{u}_p - \frac{r_{uv}}{a_p h_x} (\delta p_p - \delta p_w)
\]

\[
\bar{v}_p = \bar{v}_p - \frac{r_{uv}}{a_p h_y} (\delta p_p - \delta p_s)
\]

\[
\bar{p}_p = \bar{p}_p + r_p \delta p_p
\]

where \( r_{uv} \) is the relaxation factor for correcting \( u \) and \( v \) velocities and \( r_p \) is the relaxation factor for updating pressure. The value \( \delta p \) is a pressure increment such that the velocity field \( \bar{u} \) and \( \bar{v} \) will satisfy conservation of mass. It is obtained by discretizing equation (3) and substituting for the velocities and corrections given in equations (28) - (30). This yields an equation for the pressure correction:

\[
\delta^p p + \delta p_p = \delta^p s \delta p_s + \delta^p g \delta p_E + \delta^p w \delta p_w - \frac{1}{h_x} (\bar{u}_E - \bar{u}_p) - \frac{1}{h_y} (\bar{v}_N - \bar{v}_p)
\]  

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where:
\[
d^2_n = \frac{1}{d^2_p h^2_x} = d^2_s
\]
\[
d^2_E = \frac{1}{d^2_p h^2_y} = d^2_w
\]
\[
d^2_p = d^2_n + d^2_s + d^2_E + d^2_w
\]

Equations (28) - (31) can be written in compact form as:
\[
\begin{align*}
\ddot{u}_p &= \dot{u}_p - \frac{\tau_{uv}}{\sigma_p} \delta^x \delta_p h \\
\ddot{v}_p &= \dot{v}_p - \frac{\tau_{uv}}{\sigma_p} \delta^y \delta_p h \\
\ddot{\rho}_h &= \dot{\rho}_h + r_p \delta_p h \\
P_h \delta_p h &= \delta^x h \dot{u}_h + \delta^y h \dot{v}_h
\end{align*}
\]

If equation (35) is solved for \( \delta_p h \) and used in equation (32)-(34):
\[
\begin{align*}
\ddot{u}_h &= \dot{u}_h - \frac{\tau_{uv}}{\sigma_p} \delta^x h P^{-1} h (\delta^x h \dot{u}_h + \delta^y h \dot{v}_h) \\
\ddot{v}_h &= \dot{v}_h - \frac{\tau_{uv}}{\sigma_p} \delta^y h P^{-1} h (\delta^x h \dot{u}_h + \delta^y h \dot{v}_h) \\
\ddot{\rho}_h &= \dot{\rho}_h + r_p P^{-1} h (\delta^x h \dot{u}_h + \delta^y h \dot{v}_h)
\end{align*}
\]

This assumes that the pressure correction equation has been solved exactly. As before, the error is introduced by writing equations (36)-(38) using the exact solution and then subtracting the result from equations (36)-(38) respectively. The errors become:
\[
\begin{align*}
\dot{\varepsilon}^u_h &= \dot{\varepsilon}^u_h - \frac{\tau_{uv}}{\sigma_p} \delta^x h P^{-1} h (\delta^x h \varepsilon^u_h + \delta^y h \varepsilon^y_h) \\
\dot{\varepsilon}^v_h &= \dot{\varepsilon}^v_h - \frac{\tau_{uv}}{\sigma_p} \delta^y h P^{-1} h (\delta^x h \varepsilon^u_h + \delta^y h \varepsilon^y_h) \\
\dot{\varepsilon}^\rho_h &= \dot{\varepsilon}^\rho_h + r_p P^{-1} h (\delta^x h \varepsilon^u_h + \delta^y h \varepsilon^y_h)
\end{align*}
\]

The Fourier components of the error can be substituted as before to give the \( A_2 \) amplification matrix which governs the amplification of errors during the pressure correction phase:
\[
\left[
\begin{array}{c}
\dot{a}^u_h \\
\dot{a}^v_h \\
\dot{a}^\rho_h
\end{array}
\right] = \left[
\begin{array}{ccc}
1 + \frac{\tau_{uv}^2}{\sigma_p \rho h_x} & 0 & \frac{\tau_{uv}^2 \rho h_y}{\sigma_p} \\
\frac{\tau_{uv}^2 \rho h_y}{\sigma_p} & 1 + \frac{\tau_{uv}^2}{\sigma_p \rho h_y} & 0 \\
2 \rho \frac{\tau_{uv}^2 \rho h_y}{\rho h_x} & 2 \rho \frac{\tau_{uv}^2 \rho h_y}{\rho h_y}
\end{array}
\right]
\left[
\begin{array}{c}
\dot{a}^u_h \\
\dot{a}^v_h \\
\dot{a}^\rho_h
\end{array}
\right]
\]
In compact form:

$$\ddot{\alpha}_\theta = A_2 \dot{\alpha}_\theta$$

where:

$$\dot{P}_h = a_0 e^{i\theta_1} + a_1 e^{-i\theta_1} + a_2 e^{i\theta_1} + a_3 e^{-i\theta_1} - d_p$$

Equation (42) gives the amplification matrix defining how the amplitude of the Fourier mode with phase angles $\theta_1$ and $\theta_2$ is amplified during the pressure correction phase of the algorithm. The amplification matrix for relaxation of the $x$ and $y$ momentum equations and the pressure correction is obtained by combining equation (27) and (42) as:

$$\ddot{\alpha}_\theta = A_2 A_1 \alpha_\theta = A \alpha_\theta$$

(43)

2.4 Smoothing factor

The smoothing factor is a measure of the worst reduction of the high frequency error components for one complete relaxation sweep. It is calculated as the largest eigenvalue of the amplification matrix $A$ given by equation (43) for the Fourier modes $\theta_1$ and $\theta_2$ in the high frequency range defined as:

$$\pi/2 \leq |\Theta_i| \leq \pi$$ and $$\pi/2 \leq |\Theta_j| \leq \pi.$$ 

3 RESULTS

To test the predictive capability of the LMA presented in Section 2, flow over a backward facing step (BFS) was computed using a multigrid code based on the FAS-FMG (full approximation storage - full multi-grid) algorithm proposed by Brandt [1]. Higher-order schemes were introduced through deferred correction only on the finest of three grids with constant grid spacing in the $x$ and $y$ directions. The grid sizes from coarsest to finest grid are, $nx \times ny = 66 \times 18, 130 \times 34, \text{and} 258 \times 66,$ where $nx$ is the number of grid points in the $x$ direction and $ny$ is the number of grid points in the $y$ direction. Smoothing properties on the two coarser grids are identical for the four schemes since hybrid coefficients were used on these grids. Local mode analysis was used to estimate the smoothing factor on the finest grid and this result was compared to the number of work units to reach convergence for the multigrid result. The work units (WU) and convergence factor (CF) are indicators of the smoothing properties of the algorithm and numerical scheme. The work units for a two-dimensional problem with grid refinement in the $x$ and $y$ directions are defined as:

$$WU = \sum_{i=1}^{N} \tau_i 2^{2(i-N)}$$

(44)

where $\tau_i$ is the number of iterations on the $i^{th}$ grid at convergence, $i = 1$ for the coarsest grid, and $i = N$ for the finest grid. The convergence factor is defined as:

$$CF = (r_f/r_i)^{1/AWU}$$

(45)

where $r_i$ is the initial norm of the residuals of the $x$ momentum, $y$ momentum, and pressure correction equation on the fine grid, $r_f$ is the norm of the residuals at convergence on the fine grid, and $AWU$ is the
change in work units on the fine grid.

The BFS flow was solved for Reynolds numbers (based on the upstream channel height) of 100, 250, and 400 using the hybrid, central, 2nd OU, and 3rd OU schemes. The work units for convergence and convergence factors are shown in Table 1. The smoothing factors were calculated for conditions identical to the finest grid of the multigrid results (the same grid spacing, relaxation factors, density, etc.). The only parameter varied was the frozen velocity components needed to calculate the coefficients in equations (6) - (20) of the various schemes. The maximum velocity components provide an upper bound for the smoothing factor which will dominate the smoothing properties since it was found that as the cell-Reynolds number (Reynolds number with the length scale based on the grid spacing) approaches zero, corresponding to regions where the velocity approaches zero, the smoothing factor also decreases. An estimate of the maximum velocity components for the BFS flow is $u_o = 1.5$ at $y = 0.25$ at the inlet, and $v_o = 0.15$ near recirculation regions. Three principal flow directions are considered with velocity components given by: $u_o, v_o = (0, 0.15), (1.5, 0.15)$, and $(1.5, 0)$. The SIP method exhibits symmetry about the $x$ and $y$ axis so that other flow direction results can be obtained from the three principal flow direction results. For example, the smoothing factors for $u_o, v_o = (-1.5, 0.15), (-1.5, -0.15)$, and $(1.5, -0.15)$ are equal to the smoothing factor for $u_o, v_o = (1.5, 0.15)$. The smoothing factor is then defined as the largest eigenvalue of the amplification matrix $A$, defined by equation (43), for the three flow directions while restricting the phase angles to the high frequency range. Results from the three principal flow directions show that the flow direction $u_o, v_o = (1.5, 0.15)$ produced the largest eigenvalue for all Reynolds numbers, and thus the smoothing factor was based on this flow direction. The computed smoothing factors are shown in Table 2.

The results of Table 2 show that as the cell-Reynolds number in the LMA is increased, the smoothing factor also increases for the four schemes. More work units will be required to smooth the high frequency error components. The results in Table 1 show that the work units increase and the convergence factor deteriorates as the Reynolds number increases. For the $Re = 100$ results, the LMA predicts that the smoothing properties of the hybrid, central, and 3rd OU will be virtually identical while that of the 2nd OU will be slightly worse. The multigrid results confirm this prediction. For the $Re = 250$ and 400 results, the LMA predicts that the hybrid difference scheme will have the best smoothing properties while the central difference scheme will have the worst, and the 2nd OU and 3rd OU difference schemes should be similar with the 3rd OU difference scheme slightly better. The multigrid results confirm these predictions with the exception being that the 2nd OU difference scheme results converged in slightly less number of work units when compared to the 3rd OU difference scheme. Their convergence factors are similar.

Table I: Work Units/Convergence Factors of Multigrid Results

<table>
<thead>
<tr>
<th>Difference Scheme</th>
<th>$Re = 100$</th>
<th>$Re = 250$</th>
<th>$Re = 400$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hybrid</td>
<td>59/0.868</td>
<td>137/0.930</td>
<td>321/0.981</td>
</tr>
<tr>
<td>Central</td>
<td>58/0.873</td>
<td>166/0.964</td>
<td>569/0.993</td>
</tr>
<tr>
<td>2nd OU</td>
<td>63/0.891</td>
<td>148/0.952</td>
<td>421/0.988</td>
</tr>
<tr>
<td>3rd OU</td>
<td>58/0.875</td>
<td>152/0.956</td>
<td>428/0.988</td>
</tr>
</tbody>
</table>
Table II: Smoothing Factors from Local Mode Analysis

<table>
<thead>
<tr>
<th>Difference Scheme</th>
<th>$Re = 100$</th>
<th>$Re = 250$</th>
<th>$Re = 400$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hybrid</td>
<td>0.902</td>
<td>0.924</td>
<td>0.931</td>
</tr>
<tr>
<td>Central</td>
<td>0.910</td>
<td>0.950</td>
<td>0.968</td>
</tr>
<tr>
<td>2nd OU</td>
<td>0.931</td>
<td>0.946</td>
<td>0.950</td>
</tr>
<tr>
<td>3rd OU</td>
<td>0.899</td>
<td>0.926</td>
<td>0.935</td>
</tr>
<tr>
<td>$Re_{Ax}/Re_{Ay}$</td>
<td>11.72/0.47</td>
<td>29.30/1.18</td>
<td>46.88/1.88</td>
</tr>
</tbody>
</table>

**CONCLUSION**

Local mode analysis was performed using four schemes for the approximation of convection terms: hybrid, central, second-order upwind, and third-order upwind, over a range of cell-Reynolds numbers. The smoothing factors from this analysis were compared with actual multigrid results for flow over a backward facing step to test the predictive capability of local mode analysis. It was found that this analysis is useful in predicting the smoothing properties of the four schemes along with the effect of flow Reynolds number. This analysis could be extended to predict optimum relaxation factors, grid aspect ratios, and other solution algorithms.

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**REFERENCES**


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Figure 1. Geometry and boundary conditions for flow over a backward facing step.

Figure 2. Location of variables for staggered grid.