Parametric Uncertainty Modeling for Application to Robust Control

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Presentation Outline

• Parametric Uncertainty Modeling
• Multilinear Solution Framework
  - Results
  - Example
  - Extension to Rational Case
• Concluding Remarks
• Further Work
Parametric Uncertainty Modeling

Motivation:

- Robust Control Theory & Tools
  - Required Uncertainty Model Structure:
    - Separated $P\Delta$ Form:
    - Computational Efficiency Depends on Dimension of $\Delta$ Block
  - Minimal $P\Delta$ Model Desired:

- Practical Robust Control Applications
  - $P\Delta$ Model Difficult to Form for Real Parameter Variations
  - No General Systematic Approach for Minimal $P\Delta$ Modeling
  - Multidimensional Minimal Realization Problem
  - Problem to be Addressed in this Paper

Parametric Uncertainty Modeling (cont)

General Problem Definition:

Given State Space Model of Uncertain System:

\[
\begin{bmatrix}
  x \\
  u
\end{bmatrix} \rightarrow \begin{bmatrix}
  A(p) & B(p) \\
  C(p) & D(p)
\end{bmatrix} \rightarrow \begin{bmatrix}
  \dot{x} \\
  y
\end{bmatrix} \quad \begin{aligned}
  \dot{x} &= A(p)x + B(p)u \\
y &= C(p)x + D(p)u
\end{aligned}
\]

Any Element of $A(p), B(p), C(p), D(p)$ \rightarrow Explicit Function* of $p$:

Uncertain Parameters: $p = [p_1, p_2, \ldots, p_m]$

\[ p_i \leq p_i \leq p_i \quad \rightarrow \quad p_i = p_i + \delta_i = p_i + s_i \delta_i, \quad |\delta_i| \leq 1 \]

Form a $P\Delta$ Uncertainty Model:

$P$ - Constant Matrices

$\Delta(\delta)$ - Uncertain Parameters

\[ \Delta(\delta) = \text{diag}(\delta_1, \delta_2, \ldots, \delta_m) \]
Parametric Uncertainty Modeling (cont)

General Problem (cont):

Any Element of $A(p), B(p), C(p), D(p) \rightarrow \text{Explicit Function}^* \text{ of } p$

*Explicit Functional Forms:

- **Linear Function**: $a_{ij}(p) = p_1 + p_2 a_0$
- **Multilinear Function**: $a_{ij}(p) = p_1 + p_1 p_2 a_0$
- **Rational Function**: $a_{ij}(p) = \frac{p_1 + p_2 a_0 + p_3^2 p_5}{p_1 p_2 + a_1 p_3}$

** Formal Solution by Morton & McAfoos (1985 ACC & CDC)

⇒ Many Practical Problems:
- Multilinear (Rational, ...)

Objective

**Develop**: Systematic Method for Obtaining a $P-\Delta$ Model

**Given**: State-Space Model of a MIMO Uncertain System such that:

- Any Element of $A(p), B(p), C(p), D(p)$ is a Multilinear Function of $p$:
  \[ a_{ij}(p) = p_1 + p_1 p_2 a_0 \]

- The Resulting $P-\Delta$ Model is Minimal (or Near Minimal), i.e.:
  \[ \Delta(\delta) = \text{diag}(\delta_1 I, \delta_2 I, \ldots, \delta_m I_m) \]
  has Minimal Dimension for the Given State-Space Model

**Extend**: Multilinear Results to Rational Case
General Solution Framework

Block Diagram Perspective:

\[
\begin{bmatrix}
\dot{x} \\
y
\end{bmatrix} = \begin{bmatrix}
S(p) & \Delta \\
A(p) & B(p) \\
C(p) & D(p)
\end{bmatrix} \begin{bmatrix}
\dot{x} \\
y
\end{bmatrix}
\]

\[
\begin{bmatrix}
\dot{x} \\
y
\end{bmatrix} = S(p) \begin{bmatrix}
\dot{x} \\
y
\end{bmatrix} = (S(p_o) + S_o) \begin{bmatrix}
\dot{x} \\
y
\end{bmatrix}
\]

Equating Given & Desired Models:

\[
P_{21} = S(p_o)
\]

\[
\Delta = \text{diag}(\delta_1, \delta_2, \ldots, \delta_m)
\]

\[
P_{21}(I - \Delta \delta P_{12})^{-1} \Delta (\delta P_{12}) = S_o
\]

Solution of \(P_{21}, P_{12}, \text{and } P_{11}\) Matrices:

\[
P_{21}(I - \Delta (\delta) P_{12})^{-1} \Delta (\delta) P_{12} = S_o
\]

Unknown Matrix Elements

Known Matrix Elements (Function of \(\delta's\))

General Solution Requires: Direct Matrix Inversion

\[
(I - \Delta (\delta) P_{12})^{-1}
\]

Symbolic Matrix Inversion & Subsequent Solution Difficult for Many Practical Problems
Multilinear Solution Framework

$$(I - \Delta(\delta) P_{11})^{-1} = I + (\Delta(\delta) P_{11}) + (\Delta(\delta) P_{11})^2 + \ldots$$

such that: $$(\Delta(\delta) P_{11})^r = 0 \Rightarrow \text{Requires Special Structure for } P_{11}$$

where: $r$ - Determined by Maximum Crossterm Order in $A, B, C, D$

$$S_{\delta}(\delta) = \begin{bmatrix} A_{\delta}(\delta) B_{\delta}(\delta) \\ C_{\delta}(\delta) D_{\delta}(\delta) \end{bmatrix} = P_{21} \left[ I + \Delta(\delta) P_{11} + (\Delta(\delta) P_{11})^2 + \ldots \right] \Delta(\delta) P_{12}$$

Note: 1.) nth Order Terms 2.) Inverse Terms

\[ \text{Ex.: } p_i^2 = p_i p_{i+1} \quad \quad \text{Ex.: } \frac{1}{p_i} = \bar{p}_i \]

Uncertainty Modeling Procedure

To Obtain a Minimal (or Near Minimal) $P-\Delta$ Uncertainty Model:

0. Determine $P_{22}$ and Extract $S_{\delta}(\delta)$:

\[ P_{22} = S(p_0) = \begin{bmatrix} A(p_0) & B(p_0) \\ C(p_0) & D(p_0) \end{bmatrix} \quad S_{\delta}(\delta) = \begin{bmatrix} A_{\delta}(\delta) B_{\delta}(\delta) \\ C_{\delta}(\delta) D_{\delta}(\delta) \end{bmatrix} \]

1. Define $\Delta$ Matrix: $\Delta(\delta) = \text{diag}(\delta_1, \delta_2, \ldots, \delta_n L_m)$

Repeated Parameters Only for nth Order Uncertain Parameters

2. Determine $P_{21}$ and $P_{12}$ Using Linear Terms (Morton & McAfoose):

\[ P_{21} \Delta(\delta) P_{12} = [S_{\delta}(\delta)]_b = \begin{bmatrix} A_{\delta}(\delta) B_{\delta}(\delta) \\ C_{\delta}(\delta) D_{\delta}(\delta) \end{bmatrix} \]

Known Linear Uncertain Parameter Terms Only
(No Uncertain Parameter Crossterms)
Modeling Procedure (cont)

3. Determine \( P_{11} \) Using Uncertain Parameter Cross terms:

\[
\begin{align*}
P_{21} \Delta(\delta) \Delta(\delta) P_{21} &= \left[ S_{2}(\delta) \right]_{1} = \begin{bmatrix} A_{1}(\delta)B_{1}(\delta) \\ C_{1}(\delta)D_{1}(\delta) \end{bmatrix} \quad \text{Known First-Order Cross terms} \\
P_{22} \Delta(\delta) \Delta(\delta) P_{22} &= \left[ S_{2}(\delta) \right]_{2} = \begin{bmatrix} A_{2}(\delta)B_{2}(\delta) \\ C_{2}(\delta)D_{2}(\delta) \end{bmatrix} \quad \text{Known Second-Order Cross terms} \\
& \vdots \\
P_{2n} \Delta(\delta) \Delta(\delta) P_{2n} &= \left[ S_{2}(\delta) \right]_{n} = \begin{bmatrix} A_{n}(\delta)B_{n}(\delta) \\ C_{n}(\delta)D_{n}(\delta) \end{bmatrix} \quad \text{Known nth-Order Cross terms}
\end{align*}
\]

with nilpotency condition satisfied.

If \( P_{11} \) cannot be found such that all of the above equations and condition are satisfied:

a.) Determine which parameters need to be repeated
b.) Repeat procedure from Step 1 augmenting \( A \) matrix

Once \( P_{11} \) has been determined, minimal (or near minimal) \( P-\Delta \) model has been found.

Example

*Given Uncertain System Model:*

\[
A(p) = \begin{bmatrix} -V \frac{\alpha}{L_w} & 0 & 0 \\ 0 & -V \frac{\beta}{L_w} & 0 \\ 0 & \frac{\alpha}{L_w} \sqrt{3V_a \frac{2\pi}{L_w}} & -V \frac{\beta}{L_w} \end{bmatrix}, \quad B(p) = \begin{bmatrix} \frac{V}{L_w} \sqrt{\frac{V_a}{\pi}} & 0 \\ 0 & -V \frac{(1 - \frac{1}{\sqrt{3}})}{L_w} \\ 0 & 0 \end{bmatrix},
\]

\[
C(p) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad D(p) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}
\]

\[
\Rightarrow A(p) = \begin{bmatrix} -V \frac{\alpha}{L_w} & 0 & 0 \\ 0 & -V \frac{\beta}{L_w} & 0 \\ 0 & \frac{\alpha}{L_w} \sqrt{3V_a \frac{2\pi}{L_w}} & -V \frac{\beta}{L_w} \end{bmatrix}, \quad B(p) = \begin{bmatrix} \frac{V}{L_w} \sqrt{\frac{V_a}{\pi}} & 0 \\ 0 & -V \frac{(1 - \frac{1}{\sqrt{3}})}{L_w} \\ 0 & 0 \end{bmatrix},
\]

where: \( L_v = \frac{1}{L_w}, \quad L_w = \frac{1}{L_w} \)
**Example (cont)**

**P-Δ Model Solution:**

\[
\begin{bmatrix}
\dot{x} \\
y
\end{bmatrix} = P_{22} \begin{bmatrix}
x \\
u
\end{bmatrix} + P_{21} u_\Delta
\]

\[
y_\Delta = P_{12} \begin{bmatrix}
x \\
u
\end{bmatrix} + P_{11} u_\Delta
\]

\[
u_\Delta = \Delta(\delta) y_\Delta
\]

where:

\[
P_{22} = \begin{bmatrix}
-v_a \Gamma_w & 0 & 0 & \sigma_a \Gamma_w \sqrt{V_a} & 0 \\
0 & 0 & -v_a \Gamma_w & 0 & \sigma_a \Gamma_w \sqrt{V_a} (1 - \frac{1}{f_3}) \\
0 & \sigma_a \Gamma_w \sqrt{V_a} & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

**Example (cont)**

**P-Δ Model Solution (cont):**

\[
P_{21} = \begin{bmatrix}
-1 & \sigma_a \Gamma_w & 0 & 0 & 0 & \Gamma_{w_a} & 0 \\
0 & 0 & -v_a & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & \Gamma_{w_a} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
P_{12} = \begin{bmatrix}
s_{L_v} V_a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2s_{L_v} \sqrt{V_a} & 0 & s_{L_v} & 0 \\
0 & s_{L_v} & 0 & 0 & 0 & 0 & s_{L_v} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & s_{L_v} \\
0 & 2s_{L_v} \sigma_a \Gamma_w \sqrt{V_a} & -s_{L_v} V_a & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2s_{L_v} \sigma_a \Gamma_w \sqrt{V_a} & 0 & 0 & 0 \\
0 & s_{\alpha} \sqrt{V_a} & 0 & 0 & 0 & s_{\alpha} \sqrt{V_a} & 0 & 0 \\
0 & 0 & s_{\alpha} \sqrt{V_a} & 0 & 0 & 0 & s_{\alpha} \sqrt{V_a} & 0
\end{bmatrix}
\]
Example (cont)

P-Δ Model Solution (cont):

\[
P_{11} = \begin{bmatrix}
0 & \frac{S_n}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{S_n}{4} & \frac{S_n}{2} & 0 & 0 \\
0 & 0 & 0 & \frac{S_n}{2} & \frac{S_n}{2} & 0 \\
0 & 0 & 0 & 0 & \frac{S_n}{2} & \frac{S_n}{2} \\
0 & 0 & 0 & 0 & 0 & \frac{S_n}{2} \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\Delta(\delta) = \text{diag} \left[ \delta_{L_r} I_2, \delta_{L_t} I_3, \delta_{s_{\alpha}}, \delta_{o_{\alpha}} \right]
\]

Extension to Rational Case

\[
\begin{bmatrix} x \\ u \end{bmatrix} \rightarrow S_{D_{11}}^{-1} S_N \rightarrow \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ u \end{bmatrix} \rightarrow S_D S_N \rightarrow \begin{bmatrix} x \\ y \end{bmatrix}
\]

\[
\begin{bmatrix} x \\ u \end{bmatrix} \rightarrow S_{D_{11}} S_{N_{11}} \rightarrow \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ u \end{bmatrix} \rightarrow S_{D_{11}} S_{N_{11}} \rightarrow \begin{bmatrix} x \\ y \end{bmatrix}
\]

\[
[S_{N_{11}}]_{S_{D_{11}}} \begin{bmatrix} \Delta \\
\delta \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \end{bmatrix}
\Rightarrow \begin{bmatrix} S_{N_{11}} \\
S_{D_{11}} \end{bmatrix} = \begin{bmatrix} P_{11} & (I - \Delta P_{11})^{-1} \Delta P_{12} \\
0 & 0 \end{bmatrix}
\]

(Multilinear Problem)
Extension to Rational Case (cont.)

System Equations:

\[
\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = S_{n_1}S_{n_2}^{-1}\begin{bmatrix} \dot{x} \\ u \end{bmatrix} + \begin{bmatrix} P_{12n_1} \\ P_{22n_2} \end{bmatrix} - S_{n_1}S_{n_2}^{-1}\begin{bmatrix} P_{12n_2} \\ P_{22n_1} \end{bmatrix} u_h
\]

\[y_h = [P_{12} P_{12}]S_{n_2}^{-1}\begin{bmatrix} \dot{x} \\ u \end{bmatrix} + \begin{bmatrix} P_{11} -[P_{12} P_{12}]S_{n_2}^{-1}\begin{bmatrix} P_{21n_1} \\ P_{31n_1} \end{bmatrix} u_h
\]

\[u_h = \Delta y_h\]

where:

\[S_{n_1}S_{n_2}^{-1} = S_n = \begin{bmatrix} A_n B_n \\ C_n D_n \end{bmatrix} = P_{22}\]

\[S_{n_1} = \begin{bmatrix} A_{n_1} B_{n_1} \\ C_{n_1} D_{n_1} \end{bmatrix}, \quad S_{n_2}^{-1} = \begin{bmatrix} A_{n_2} B_{n_2}^{-1} \\ C_{n_2} D_{n_2} \end{bmatrix}\]

Concluding Remarks

- **Multilinear Solution Framework**
  - Solves Multilinear Parameter Case
    - Accomodates nth Order and Inverse Terms
  - Eliminates Symbolic Matrix Inversion In Computation of P_{11}
    - Computationally Tractable for Symbolic Solution
      (Symbolic Algebra Tool Required)
  - Can be Extended to Rational Parameter Case
    - Preliminary Results

- **Systematic Procedure for (Near) Minimal P-Δ Modeling**
  - Minimality is Relative to Given State Space Realization
    - A Lower Dimension P-Δ Model May Exist for Different Realization
  - (Near) Minimality by Construction
    - Minimality may not Always be Assured
Further Work

- **Evaluate/Refine/Generalize Procedure**
  - Wider Class of Problems
  - Multidimensional System Theory

- **Automate Modeling Procedure**
  - Mathematica/Maple
  - Output Files to Matlab

- **Apply to HSCT Problems**
  - Configuration Evaluation
  - Control System Analysis & Design
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Abstract

Advanced robust control system analysis and design is based on the availability of an uncertainty description which separates the uncertain system elements from the nominal system. Although this modeling structure is relatively straightforward to obtain for multiple unstructured uncertainties modeled throughout the system, it is difficult to formulate for many problems involving real parameter variations. Furthermore, it is difficult to ensure that the uncertainty model is formulated such that the dimension of the resulting model is minimal. This paper presents a procedure for obtaining an uncertainty model for real uncertain parameter problems in which the uncertain parameters can be represented in a minimal form. Furthermore, the procedure is formulated such that the resulting uncertainty model is minimal (or near minimal) relative to a given state space realization of the system. The approach is demonstrated for a multivariable third-order example problem having four uncertain parameters.

1. Introduction

Advanced robust control system analysis and design is based on the availability of an uncertainty description which separates the uncertain system elements from the nominal system. More specifically, the uncertain system components are contained in a block-diagonal $\Delta$ matrix, which is connected to the nominal system, $P(s)$, such that the closed-loop uncertain system is described by a linear fractional transformation (LFT). The idea of separating the uncertain part of a system from its nominal part in this manner, for use in robust control system analysis and design, was first posed by John Doyle (see [3] and [4]), and the robust control theory associated with this structured description of uncertainty continues to be an important area of research. A block diagram of this modeling structure can be depicted as follows in Figure 1:

![Figure 1. Block Diagram of General Uncertain System](image)

where $u$ contains all external inputs to the system (e.g., disturbances, control inputs, etc.), $y$ contains all outputs from the system (e.g., controlled outputs, measured outputs, etc.) and $\Delta$ and $\Delta_y$ connect the uncertainties represented by $\Delta$ to the nominal system, $P(s)$. Although this modeling structure is relatively straightforward to obtain for multiple unstructured uncertainties which occur throughout the system, it is difficult to formulate for many problems involving real parameter variations. Furthermore, it is difficult to ensure that the uncertainty model is formulated such that the dimension of the resulting model is minimal (i.e., the number of repeated parameters in $\Delta$ is minimized). Although formulating an uncertainty model is a requirement for utilizing the recently developed robust control analysis and design techniques mentioned above, very little research has been reported in the literature which addresses this problem, particularly for the real parameter uncertainty case. Results to date primarily apply to multiple uncertain parameters which enter the system model in a linear functional form, although some work involving nonlinear special cases have been worked [10]. The results for linear uncertain parameters were first presented in [8] (Morton & McAfoos, 1985) and [9] (Morton, 1985). A later paper [10] (Steinbuch, et al., 1991) summarizes the general uncertainty modeling problem and the results to date, and presents two simple scalar nonlinear uncertain parameter examples. However, no solution to the general minimal uncertainty modeling problem has been found. The objective of this paper is to present an important extension to these uncertainty modeling results. Specifically, a procedure is presented for obtaining a minimal (or near minimal) uncertainty model (having the form of Figure 1) given the state space realization of an uncertain system with multiple parametric uncertainties entering the model in a multilinear functional form. It should be noted that minimality here is relative to the given state space realization. As discussed in [1] and [2] (Belcastro, et al., 1989 and 1991), the dimension of the uncertainty model (i.e., the dimension of the $\Delta$ matrix) is dependent on the state space realization of the system. Thus, one can consider the minimality of an uncertainty model for a particular state space realization, or one can consider the achievable minimality of the uncertainty model irrespective of the state space realization. In this paper, we present a method of obtaining a minimal (or near minimal) uncertainty model relative to the given state space model of the uncertain system for multiple uncertainties entering the model in a multilinear functional form. The multilinear framework significantly reduces the computational complexity involved in obtaining a solution, as compared to solving the problem directly for the rational parameter case. Moreover, it can be shown that the multilinear solution framework can actually be used to solve the rational parameter case, as well. Thus, it provides a means of determining an uncertainty model for many difficult problems of practical interest.

The paper is organized in the following manner. Section 2 presents a formal problem definition for the general uncertain parameter case, briefly summarizes results for the special case of linear parametric uncertainty, and defines the problem to be addressed in this paper. Section 3 summarizes our results for this defined problem, and Section 4 presents an example problem which demonstrates these results. Section 5 briefly discusses the application of the multilinear solution framework to solve the rational uncertain parameter problem, and concluding remarks are given in Section 6.

2. Parametric Uncertainty Modeling: Problem Definition

2.1 General Problem Definition

Consider the state space model of an uncertain system:

\[
\begin{align*}
\dot{x} &= A(p)x + B(p)u, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m \\
y &= C(p)x + D(p)u, \quad y \in \mathbb{R}^n
\end{align*}
\]

(1)

where $p$ represents a vector of real uncertain parameters:

\[
p = [p_1, p_2, \ldots, p_m] \in \mathbb{R}^m
\]

(2)

It is assumed that each entry of the model presented in equation (1) is a function of the parameters $p$. For the general rational case considered in this paper, the uncertain parameters can appear in a rational multivariate functional form within each element of the system model. For example, as given in [10] (Steinbuch et al., 1991), the $(ij)_{th}$ entry of the $A$ matrix could have the form:
\[ A_0(\delta) = \frac{P_1 + P_2 a_2 + P_2^2 P_3}{P_1 P_3 + s_1 P_4} \]  

where \( a_2 \) and \( a_1 \) are constants. It should be noted that the \( n \)th-order terms are included here because they can be handled within a multilinear framework by defining \( n-1 \) additional uncertain parameters which are equal to the parameter being raised to the \( n \)th power. For this example, a new uncertain parameter, \( P_2^2 = P_2 \), could be defined and \( P_2^2 \) would then be replaced by \( p_2^2 \).

The uncertainty modeling problem consists of three components: scaling of the uncertain parameters, extraction of the uncertainties from the nominal system, and formulation of a linear fractional transformation (LFT) (see [5], Doyle, et. al., 1991 for a review of LFT's). These components are reviewed below.

**Uncertainty Scaling:**

Each uncertain parameter \( p_i \) in \( p \) can be bounded by an upper bound, \( p_{\text{max}} \), and a lower bound, \( p_{\text{min}} \), as follows:

\[ p_{\text{min}} \leq p_i \leq p_{\text{max}} \]  

Then the parameter can be written in terms of some nominal value within this range of uncertainty. One way to do this is shown below:

\[ p_i = p_{\text{nom}} + \delta_i = p_{\text{nom}} + s_i \delta_i \]  

\[ p_{\text{nom}} = \frac{p_{\text{min}} + p_{\text{max}}}{2} \]  

\[ s_i = \frac{p_{\text{max}} - p_{\text{min}}}{2} \]  

\[ |\delta_i| \leq 1 \]  

Equations (4) - (7) can also be written in vector form by stacking each associated parameter quantity into vectors. The \( \delta_i \) terms as defined in equations (5) and (8) are the uncertain terms that will be separated into the \( \Delta \) matrix of Figure 1.

**Uncertainty Extraction:**

Using equation (5), the state space model of the uncertain system given in (1) can be rewritten in compact form as follows:

\[ \begin{bmatrix} \dot{x} \\ y \end{bmatrix} = S(p) \begin{bmatrix} x \\ u \end{bmatrix} = S(p_{\text{nom}}) \begin{bmatrix} x \\ u \end{bmatrix} + S_A(\delta) \begin{bmatrix} x \\ u \end{bmatrix} \]  

where:

\[ \delta = [\delta_1, \delta_2, \ldots, \delta_m] \in \mathbb{R}^m \]  

\[ S(p) = \begin{bmatrix} A(p) & B(p) \\ C(p) & D(p) \end{bmatrix} = S(p_{\text{nom}}) + S_A(\delta) \]  

\[ S(p_{\text{nom}}) = \begin{bmatrix} A(p_{\text{nom}}) & B(p_{\text{nom}}) \\ C(p_{\text{nom}}) & D(p_{\text{nom}}) \end{bmatrix} \]  

\[ S_A(\delta) = \begin{bmatrix} A_0(\delta) & B_0(\delta) \\ C_0(\delta) & D_0(\delta) \end{bmatrix} \]  

Separation of \( S(p) \) into nominal and uncertain parts, \( S(p_{\text{nom}}) \) and \( S_A(\delta) \), respectively, results in the extraction of the uncertainties from the nominal system.

**Formulation of a Linear Fractional Transformation (LFT):**

Equation (9) can be rewritten in the form of an upper (time domain) LFT by defining an input vector, \( u_A \), and an output vector, \( y_A \), associated with the uncertain part of the system as follows:

\[ y_A = P_{11} u_A + P_{12} \begin{bmatrix} x \\ u \end{bmatrix} \]  

\[ \begin{bmatrix} \dot{x} \\ y \end{bmatrix} = P_{21} u_A + P_{22} \begin{bmatrix} x \\ u \end{bmatrix} \]  

\[ u_A = \Delta(\delta) y_A \]  

\[ \Delta(\delta) = \text{diag}(\delta_1, \delta_2, \ldots, \delta_m) \]  

\[ \Delta(\delta) \in \mathbb{R}^{n_u \times n_y} \]  

\[ n_A = \sum_{i=1}^m r_i, r_i = \text{dim}(\delta_i) \]  

where \( P_{11}, P_{12}, P_{21}, \) and \( P_{22} \) are constant matrices with \( P_{22} = S(p_{\text{nom}}) \), and the matrices \( P_{11}, P_{12}, \) and \( P_{21} \) are related to \( S_A(\delta) \). The \( I_i \) terms in equation (16a) represent the identity matrix with dimension equal to the repeatedness of parameter \( \delta_i \). For example, the squared uncertain parameter of equation (3), i.e. \( p_2^2 \) results (after scaling) in the term \( \delta_2^2 \). Thus, this example would require that both \( \delta_2 \) and \( \delta_2 = \delta_2 \) (associated with the uncertain parameter \( p_2 \) discussed above) appear in \( \Delta \), which means that \( I_2 \) in equation (16a) would be a 2-dimensional identity matrix.

The objective of the uncertainty modeling problem is to find the matrices \( P_{11}, P_{12}, P_{21} \), and \( P_{22} \) such that the system of equations represented by (13) - (16) is equivalent to the system represented by equation (9). To do this, equations (13) - (15) are combined such that \( u_A \) and \( y_A \) are eliminated, as follows:

\[ \begin{bmatrix} \dot{x} \\ y \end{bmatrix} = [P_{22} + P_{21}(I - \Delta(\delta)P_{11})^{-1}\Delta(\delta)P_{12}] \begin{bmatrix} x \\ u \end{bmatrix} \]  

Thus, the uncertainty modeling problem can be thought of as a multi-dimensional (minimal) realization problem defined by the following equation:

\[ S_A(\delta) = P_{21}(I - \Delta(\delta)P_{11})^{-1}\Delta(\delta)P_{12} \]  

where \( \delta \) represents the uncertain parameter vector defined in equation (10).

**2.2 Summary of Results for Linear Parametric Uncertainties**

As indicated previously in this paper, uncertainty modeling results have primarily focused on the special uncertainty case involving multiple uncertain parameters that enter the system model linearly. Results for this case were first presented by [8] (Morton & McAfoos, 1985), and involve solving equation (19) with \( P_{11} = 0 \). For this case, \( P_{21} \) and \( P_{12} \) can easily be found by expanding \( S_A(\delta) \) as a linear combination of the \( \delta_i \) terms, and decomposing the resulting coefficient matrices. If any of the coefficient matrices has rank greater than one, then the associated \( \delta_i \) term must be repeated in \( \Delta \) a corresponding number of times in order to perform the decomposition. For example, if the coefficient matrix for \( \delta_i \) is rank 2, then \( \delta_i \) must appear twice in the \( \Delta \) matrix. This is also discussed in [9] (Morton, 1985).

**2.3 Specific Problem Definition for this Paper: Multilinear Parametric Uncertainties**

In this paper, we consider the case of multiple uncertain parameters which enter any element of the system described in equation (1) in a multilinear manner. It should be noted that rational multivariate elements involving only one denominator term can be represented in a multilinear form directly. For example,
\[ A_0(p) = \frac{P_1 + P_2 a_o + P^2 P_3}{P_1 P_3} \]

where:
\[ \tilde{p}_1 = \frac{1}{p_1}, \tilde{p}_2 = \frac{1}{P_3} \]

The general multivariate rational uncertainty case containing multiple uncertain terms in the denominator (defined in Section 2.1) could be redefined. For example, an uncertain model element represented by equation (3) could be approximated in a multilinear form as follows:

\[ A_u(p) = \frac{P_1 + P_2 a_o + P^2 P_3}{P_1 P_3} \]

where:
\[ \tilde{p}_4 = \frac{1}{P_1 P_3 + a_1 P_4} \]

Thus, in this formulation the fourth uncertain parameter, \( \tilde{p}_4 \), is dependent on the uncertain parameters \( p_1, P_2, \) and \( p_4 \). This approach therefore poses a slight restriction to the general case. However, a brief discussion of a technique for formulating the rational problem in such a way that the multilinear solution framework can be used is presented in Section 5.

2.4 Formal Problem Statement

A formal problem statement based on the above discussion can be summarized as follows:

**Given:** An uncertain system in state space form as in equation (1), i.e.:
\[ \dot{x} = A(p)x + B(p)u, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m \]
\[ y = C(p)x + D(p)u, \quad y \in \mathbb{R}^m \]
which can be rewritten as in equation (9), i.e.:
\[ \begin{bmatrix} \dot{x} \\ y \end{bmatrix} = S(p) \begin{bmatrix} x \\ u \end{bmatrix} = S(p)_{(A\beta)} \begin{bmatrix} x \\ u \end{bmatrix} + S_d(\delta) \begin{bmatrix} x \\ u \end{bmatrix} \]

**Find:** The matrices \( P_{21}, P_{12}, \) and \( P_{11} \) such that the above system can be expressed as in equations (13-16), i.e.:
\[ y_\delta = P_{11} u_\delta + P_{12} \begin{bmatrix} x \\ u \end{bmatrix} \]
\[ \begin{bmatrix} \dot{x} \\ y \end{bmatrix} = P_{21} u_\delta + P_{22} \begin{bmatrix} x \\ u \end{bmatrix} \]
\[ u_\delta = A(\delta) y_\delta \]
\[ \Delta(d) = \text{diag}(\delta_{11}, \delta_{22}, \ldots, \delta_{m \times m}) \]

A detailed discussion of a solution to this problem for uncertainties which are represented within a multilinear framework, as discussed above, will be presented in the next section.

3. Parametric Uncertainty Modeling: A Multilinear Problem Solution

3.1 Multilinear Solution Framework

As indicated in Section 2, the solution to the uncertainty modeling problem posed above involves finding the matrices \( P_{21}, P_{12}, \) and \( P_{11} \) such that the \( S_d(\delta) \) matrices given by (12) and (19) are equal, i.e.:

\[ S_d(\delta) = \begin{bmatrix} A_d(\delta) & B_d(\delta) \\ C_d(\delta) & D_d(\delta) \end{bmatrix} \]

where the \( A_d(\delta), B_d(\delta), C_d(\delta), \) and \( D_d(\delta) \) terms in equation (22) are formed by scaling the uncertain parameters \( p \) and extracting the uncertain \( \delta \) terms from the nominal system, as discussed in Section 2, and \( P_{21} \) and \( P_{12} \) are partitioned appropriately. Thus, the matrices \( A_d(\delta), B_d(\delta), C_d(\delta), \) and \( D_d(\delta) \) are known matrix functions of the \( \delta \) parameters, and the matrices \( P_{21}, P_{12}, \) and \( P_{11} \) are the unknown matrix variables for which equation (22) is solved. This section presents the main result of the paper — namely a solution to the above problem for uncertainties that are represented within the multilinear framework described in Section 2.3.

As stated above, the solution to this problem involves solving equation (22) for \( P_{21}, P_{12}, \) and \( P_{11} \). However, the inversion of the quantity \( (I - A(\delta) P_{11}) \) in equation (22) for multiple parameter problems can become very cumbersome because \( P_{11} \) is of the same dimension as \( A(\delta) \), and the inversion has to be performed symbolically. Moreover, each element of \( P_{11} \) must be determined such that equation (22) is satisfied. Within the multilinear framework, however, this quantity can be replaced by a finite series. To see this, consider the matrix equation:

\[ I - A^{n+1} = (I - A)^n + (I - A)^{n-1} A + (I - A)^{n-2} A^2 + \ldots + A^{n-1} A^n \]

which can be rewritten for any matrix \( A \). Assuming that the matrix \( (I - A) \) is invertible, this equation can be rewritten as:

\[ (I - A)^{-1} (I - A^{n+1}) = I + A + A^2 + A^3 + \ldots + A^n \]

If matrix \( A \) is structured such that \( A^{n+1} = 0 \) (i.e., \( A \) is nilpotent), then:

\[ (I - A)^{-1} = I + A + A^2 + A^3 + \ldots + A^n \]

This development is similar to the Neumann series expansion developed in [6] (Halmos, 1974) for a matrix \( A \) such that \( ||A|| < 1 \). For our problem, however, \( A = A(\delta) P_{11} \), where \( A(\delta) \) is a diagonal matrix and \( P_{11} \) is unknown. Although \( A(\delta) \) is norm-bounded by unity, \( P_{11} \) is not norm-bounded. However, since \( P_{11} \) is to be determined, requiring \( P_{11} \) to be structured such that:

\[ (A(\delta) P_{11})^{n+1} = 0 \]

yields:

\[ (I - A(\delta) P_{11})^{-1} = I + (A(\delta) P_{11})^2 + (A(\delta) P_{11})^3 + \ldots + (A(\delta) P_{11})^n \]

Substituting this into equation (22) results in:

\[ S_d(\delta) = \begin{bmatrix} A_d(\delta) & B_d(\delta) \\ C_d(\delta) & D_d(\delta) \end{bmatrix} = P_{21} \begin{bmatrix} I + A(\delta) P_{11} (A(\delta) P_{11})^2 + \ldots + (A(\delta) P_{11})^n \end{bmatrix} \]

which can be rewritten as:

\[ S_d(\delta) = P_{21} A(\delta) P_{11} + P_{21} (A(\delta) P_{11})^2 + \ldots (A(\delta) P_{11})^n A(\delta) P_{12} \]

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The first term on the right side of equation (26) represents the linear uncertain components of $S_2(8)$, and the second term adds in the nonlinear terms. Furthermore, since the nonlinear terms of $S_2(8)$ consist of cross terms and $n^2$-order terms (which can be represented as cross terms), the order $r$ of the highest term in the series of equation (26) is defined by the highest cross term order required to realize $S_2(8)$. Thus, $r$ is defined by the order of the highest cross-term occurring in $A_2(8)$, $B_2(8)$, $C_2(8)$, and $D_2(8)$, i.e.,

$$ r = \max \{ O_A, O_B, O_C, O_D \} $$

(27a)

and $O_A$, $O_B$, $O_C$, and $O_D$ represent the order of the highest-order cross-product term in $A_2(8)$, $B_2(8)$, $C_2(8)$, and $D_2(8)$, respectively. That is, for a general uncertain matrix $M$:

$$ O_M = \max \{ \text{order} \{ m_{ij} \} : \text{for all } i = 1, 2, \ldots, n_A \text{ and } j = 1, 2, \ldots, n_A \} $$

(27b)

where the order of each $m_{ij}$ is the order of its highest-order cross-product term, and cross-product term order is defined as:

$$ \text{order} \{ \delta_1 \delta_2 \delta_3 \ldots \delta_r \} = 1 \cdot \underbrace{1 \times 1 \times 1 \ldots 1}_{r \text{ times}} $$

(27c)

Thus, the maximum value of $r$ is $\max \{ n_A \}$, where $n_A$ is the dimension of the $A$ matrix and is given by equation (17). The nilpotent uncertainty equation of equation (23) for $(\Delta \delta) P_{11}$ can be satisfied if the elements of $P_{11}$, $P_{12}$, satisfy the following structure:

1.) $p_{ii} = 0$; $i = 1, 2, \ldots, n_A$

2.) If $p_{jj} \neq 0$, then for $i = 1, 2, \ldots, n_A$ and $j = 1, 2, \ldots, n_A$:

   a.) $p_{ii} = 0$;

   b.) $p_{i,j+1} = 0$ or $p_{i,j-1} = 0$ or

   ... or $p_{i,j,n_A-1} = 0$

(28)

where the symbol "\(\oplus\)" represents "modulo $n_A$" addition [7] (Horowitz and Sahn, 1978) over the set $\{1, 2, \ldots, n_A\}$; i.e.,

$$ a \oplus b = \begin{cases} a + b & \text{if } a + b \leq n_A \\ a + b - n_A & \text{if } a + b > n_A \end{cases} $$

(29)

and $n_A$ is the dimension of $A$ (and, hence, $P_{11}$) as defined in equation (17). It should be noted that requiring $P_{11}$ to satisfy the conditions of (28) does not impose a restriction in solving the uncertainty modeling problem, but rather it is a means of removing unnecessary freedom in determining $P_{11}$ based on the uncertain system being modeled. Thus, (28) assists in the process of solving for $P_{11}$.

Using this nonlinear uncertainty equation of $S_m(8)$, $P_{21}$ and $P_{12}$ can be found using the linear uncertain terms of $S_2(8)$, and $P_{11}$ can be found using the nonlinear terms of $S_m(8)$ such that the conditions of (28) are satisfied. Thus, the procedure presented in [8] (Morton & McAfoos, 1985) (and briefly described in Section 2.2) for obtaining an uncertainty model for multiple linear uncertain parameters can be used to obtain $P_{21}$ and $P_{12}$, and these matrices can be used in the second right-hand term of equation (26) so that $P_{11}$ can be determined directly using equations (26) and (28). Details of the procedure for doing this are presented in [1] and [2] (Belcastro, et al., 1989 and 1991), and an example problem is presented in Section 4 which demonstrates these results.

3.2 Uncertainty Modeling Procedure

Obviously, in order to reduce computational complexity in robust control system analysis and design, it is desired to obtain an uncertainty model of minimal dimension. As discussed in [1] and [2] (Belcastro et. al., 1989 and 1991), the dimension of the uncertainty model is dependent on the system state space realization. These papers address the problem of obtaining a state space realization of an uncertain single-input single-output (SISO) system (given its transfer function) such that an uncertainty model of minimal dimension can be determined. For practical multivariable applications, however, it is usually desired to retain physical relevance to the problem being considered in assigning the states of the system, so that a particular state space realization may be preferred. Therefore, given a desired state space model of an uncertain system, one would like to be able to determine a minimal uncertainty model for this particular realization - which may or may not be an overall minimal uncertainty model for the system. A procedure to obtain a minimal (or near minimal) uncertainty model relative to a particular state space realization (based on the multilinear framework presented in Section 3.2) is therefore given in this section.

Given a state space realization of an uncertain system whose matrix elements are multilinear functions of the uncertainty parameters of the system, it is desired to obtain an uncertainty model of the form of Figure 1, which has a minimal (or near minimal) number of repeated parameters in $\Delta$. This can be done using the following approach:

1. Define a $\Delta$ matrix of the form of equation (16) which has only those repeated uncertain parameters necessary to realize the nth-order uncertain terms in the model, as discussed in Section 2.1.

2. Follow the procedure given in [8] (Morton & McAfoos, 1985) and [9] (Morton, 1985) for the linear uncertain parameter case to obtain $P_{21}$ and $P_{12}$ using equations (22) and (26). If problems with rank occur in defining $P_{21}$ and $P_{12}$, go back to step 1 and add a repeated parameter to $\Delta$, as described in Section 2.2.

3. Once $P_{21}$ and $P_{12}$ have been obtained, use the nonlinear uncertain terms in equations (22) and (26) to obtain $P_{11}$ such that the conditions of (28) and, hence, equation (23) are satisfied. If $P_{11}$ cannot be determined such that all of these equations and conditions are satisfied, the dimension of $\Delta$ is not large enough. If this occurs, it must be determined which parameter must be repeated (based on the specific problem encountered in trying to satisfy the above equations), and the process begins again at step 1 with the repeated parameter being added to the $\Delta$ matrix. Once $P_{11}$ has been successfully determined such that all equations and conditions are satisfied, the minimal (or near minimal) uncertainty model for the given state space realization of the system has been determined, and equations (13) - (16) can be used to model the uncertain system as depicted in Figure 1.

It should be noted that the above procedure yields a minimal (or near minimal) uncertainty model by construction, since the initial $\Delta$ matrix defined in step 1 is of the smallest possible dimension required to model the given system, and additional parameters are added to this $\Delta$ matrix in steps 2 and 3 only if required. An example problem illustrating the above procedure is presented in Section 4.

4. Example

Consider the third-order multivariable system described in state space form as in equation (1) by the following realization:

$$
A(p) = \begin{bmatrix}
-V_A & 0 & 0 \\
0 & -V_A & 0 \\
0 & \sigma_A & -V_A/L_w
\end{bmatrix}
$$

(29a)
where the uncertain parameters $L_w$, $\sigma_n$, and $\sigma_w$ vary over the following ranges:

\[105.7 \leq L_w \leq 841.1\] (30a)

\[10.4 \leq L_w \leq 795.5\] (30b)

\[5.74 \leq \sigma_n \leq 9.69\] (30c)

\[3.95 \leq \sigma_w \leq 13.4\] (30d)

The elements of equation (29) can be expressed as multilinear functions of the uncertain parameters as follows:

\[A(p) = \begin{bmatrix}
-V \bar{l} \\
0 \\
0
\end{bmatrix}
\]

\[B(p) = \begin{bmatrix}
\sigma \bar{l} \bar{n} \bar{r} \sqrt{\frac{V_n}{2\pi}} \\
0
\end{bmatrix}
\]

\[C(p) = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[D(p) = \begin{bmatrix}
0 & 0
\end{bmatrix}
\]

The first step is to extract the uncertain $\delta$ terms from the nominal system by scaling the uncertain parameters as in equation (5), as follows:

\[\bar{\Gamma}_w = \Gamma_w + \sigma \bar{\delta}_w = \Gamma_w + \tilde{\delta}_w \]

\[\bar{\sigma}_n = \sigma_n + \epsilon \bar{\delta}_n = \sigma_n + \tilde{\delta}_n \]

\[\bar{\sigma}_w = \sigma_w + \epsilon \bar{\delta}_w = \sigma_w + \tilde{\delta}_w \]

so that, as in equation (12):

\[\Delta = \text{diag}(\tilde{\delta}_n, \tilde{\delta}_o, \tilde{\delta}_w) \]

\[P_{21} = \begin{bmatrix}
-\sigma \bar{l} \bar{n} \bar{r} \sqrt{\frac{V_n}{2\pi}} \\
0
\end{bmatrix}
\]

As can be seen by the last term in equation (37) (for either $a_{11}$, $b_{12}$, or $b_{22}$), $r = 2$ for this example problem (as defined by equation (27)). Since $S_0(\delta)$ contains 2nd-order terms associated with $L_w$ and $\sigma_n$, the $\delta$ terms associated with these variables will have to appear twice in $\Delta$. Thus, the dimension of $\Delta$ going into Step 1 of Section 3.2 is six. For a six-dimensional $\Delta$, the matrices $P_{21}$ and $P_{12}$ can be determined, as described in Step 2 of Section 3.2. However, it is impossible to obtain a $P_{11}$ matrix which satisfies all of the equations discussed in Step 3 of Section 3.2. Moreover, it is determined in that step that the $\delta$ term associated with $L_w$ must be repeated a third time. Therefore, when steps 1 - 3 of Section 3.2 are repeated, the resulting uncertainty model can be expressed as in equations (13) - (16) and (22), where:

\[\Delta = \text{diag}(\tilde{\delta}_n, \tilde{\delta}_o, \tilde{\delta}_w) \]

\[P_{21} = \begin{bmatrix}
-\sigma \bar{l} \bar{n} \bar{r} \sqrt{\frac{V_n}{2\pi}} \\
0
\end{bmatrix}
\]
uncertain system is not unique. It should also be noted that in determining an uncertainty model for a given uncertain system, the scaling terms $s_i$ were incorporated into the model at the end so as to reduce the number of symbolic terms involved in the determination of the $P_{21}$, $P_{12}$, and $P_{11}$ matrices.

5. Extension to Rational Case

The above procedure for solving the multilinear uncertainty modeling problem can in fact also be used to solve the more general rational uncertainty modeling problem. This is done by obtaining a matrix fraction description of the uncertain system, and representing the denominator matrix in a feedback loop so as to remove the inverse. The numerator and denominator matrices are then multivariate polynomial matrices which can be concatenated together and modeled using the multilinear techniques discussed above. Details of this approach will be presented in a subsequent paper.

6. Conclusions

This paper has summarized previous results in parametric uncertainty modeling, and has presented and demonstrated an important extension to these results. The extension consists of a framework for modeling multiple parametric uncertainties which can be represented in a multilinear functional form, and includes a procedure for obtaining a minimal (or near minimal) uncertainty model relative to a given state space realization of the uncertain system. As discussed in the paper, the multilinear framework can also be used to solve the more general rational uncertain parameter case, and provides a mechanism for significantly simplifying the computational complexity involved in determining an uncertainty model for a given uncertain system. Thus, many practical problems of interest can be solved within this framework.

Further work being addressed in this area includes evaluating/refining/generalizing this modeling procedure for a wider class of problems, automating the generalized modeling procedure, and applying the procedure to practical application problems.

References