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Effects of Noise Variance Model on Optimal Feedback Design and Actuator Placement

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ABSTRACT

In optimal placement of actuators for stochastic systems, it is commonly assumed that the actuator noise variances are not related to the feedback matrix and the actuator locations. In this paper, we will discuss the limitation of that assumption and develop a more practical noise variance model. Various properties associated with optimal actuator placement under the assumption of this noise variance model are discovered through the analytical study of a second order system.

Introduction

Refs. [1–4] are typical works in the literature for actuator placement of stochastic systems. In all of these works, it is assumed that the noise strengths of the actuators are given and not related to the feedback and actuator locations — an assumption made in the LQG theory. However, as shown in Ref. [5], this noise variance model is not always true in practice. For example, the noise strength of an actuator may depend on its capacity (the largest signal it can produce) and the magnitude of its producing signal. Clearly, if a person (actuator) is required to push an object with 1 lbf (small signal), the error of the produced force will be probably several ounces (small variance). However, if he is required to push the object with 100 lbf (large signal), the error of the produced force will be several or ten pounds (large variance). Also, the noise of a reaction wheel (actuator) may be caused by the bearing and eccentricity of the wheel, etc. If a reaction wheel is required to produce a larger signal (larger capacity), it is usually required to increase the inertial of the wheel and/or the maximum spinning speed, then the noise caused by the bearing and eccentricity will be greater. These facts show that the actuator noise strength usually depends on the capacity and the signal magnitude of the actuator. Since the signal magnitude and the required capacity of the actuator depend on the feedback matrix and the actuator location, the noise strength of the actuator implicitly depends on the feedback matrix and the actuator location. Clearly, when an actuator is placed at a carefully selected location, the control force and the required capacity of the actuator will be smaller. Also, the signal and capacity will depend on the feedback matrix because a smaller feedback (slower system) usually requires a smaller control force. Since the ordinary LQG theory neglects these facts, it cannot be used to reduce the noise variance of the actuator through the selection of a feedback matrix and actuator locations, and thus results in unnecessarily noisy systems.

A New Noise Variance Model

In most practical applications, the actuator noise variance increases with its capacity and signal magnitude. Since the required capacity in steady state is related to the signal variance of the actuator, we can reasonably use the signal variance to represent the actuator capacity in the new noise variance model. To take into account the effects of the signal magnitude on the noise variance, we may use the signal square in the noise variance model. However, this method will result in time-dependent noise variance and make analysis very complicated. In order to simplify the analysis, we can use the time average method, then signal square again becomes signal variance. According to the discussion above, we can develop a realistic noise variance model of an actuator as

$$W = \alpha' \sigma_u^2 + \alpha'' \sigma_u^2 + \beta \triangleq \alpha \sigma_u^2 + \beta \qquad (1)$$

where σ_u^2 is the variance of the actuator signal in steady state. α', α'' and β are non-negative constants which depend on manufacturing processes. The term $\alpha' \sigma_u^2$ reflects the contribution of the actuator signal magnitude, and the term $\alpha'' \sigma_u^2$ reflects the contribution of the actuator's capacity. An advantage of this noise variance model is that the noise is still white, Gaussian with constant variance, and thus analysis can be simplified. The only difference from the ordinary model is that the noise variance in the new model will depend on the capacity and signal magnitude of the actuator, and will thus implicitly depend on the feedback and actuator locations.

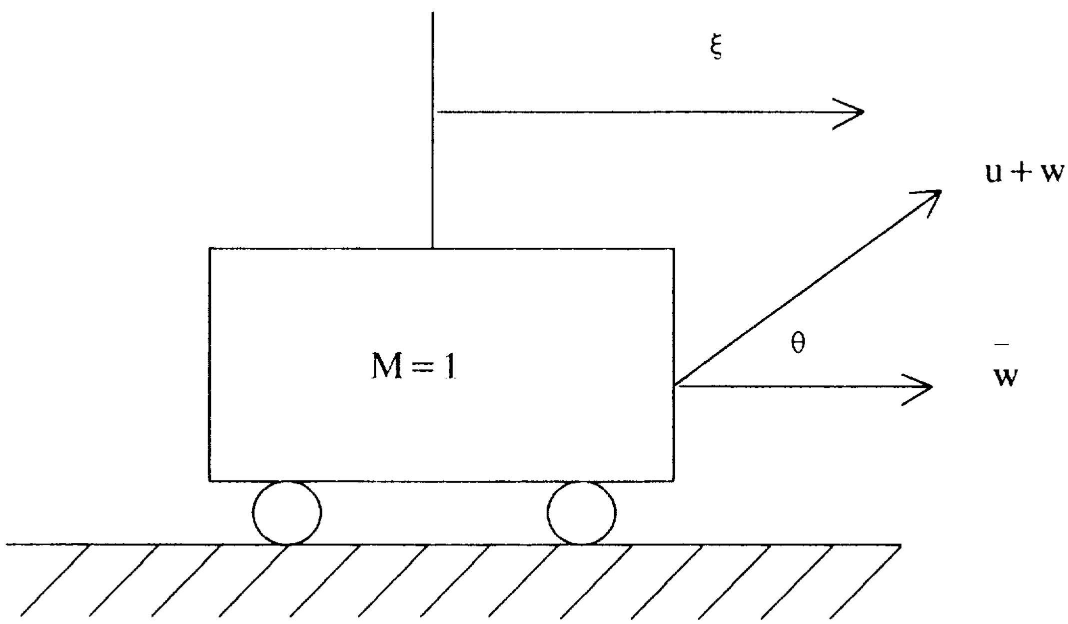

Figure 1. A second order system.

A Physical System

The new noise variance model, Eq. (1), will be applied to a second order system shown in figure 1. In the system, $\bar{w}$ is the plant disturbance with given strength $\bar{W}$, but w is actuator noise whose strength is governed by Eq. (1). The actuator orientation (location) is specified by angle, θ. Obviously, the actuator is most efficient when $\theta = 0$, and is most desirable for a deterministic system. However, as shown in Ref. [1], the selection of $\theta = 0$ may not give optimal performance for stochastic systems, especially when the ratio of plant disturbance to actuator noise is small.

The state equation is given by

$$\dot{x} = Ax + B(u+w) + G\bar{w} \tag{2}$$

with

$$x = \begin{bmatrix} \xi \\ \dot{\xi} \end{bmatrix} ; \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} ; \quad B = \begin{bmatrix} 0 \\ b \end{bmatrix} ; \quad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{3}$$

where $b = \cos\theta$. Without losing generality, we assume $0 \leq b \leq 1$. Clearly, matrix G is fixed but matrix B changes with actuator location. The feedback control law is given by

$$u = -Fx \tag{4}$$

The objective of the problem is to find the optimal feedback F, and the optimal actuator orientation b, so that the following cost function is minimized:

$$J = E_{\infty}[\, q\, x^T x + r\, u^2\,] \tag{5}$$

where q and r are given weights, E_{∞} is the mean operator when the time period approaches infinity.

Since the noise is still white and Gaussian with constant variance, we can use stochastic control theory to find the variance of the state, P_x:

$$P_x (A - BF)^T + (A - BF) P_x + BWB^T + G\bar{W}G^T = 0 \tag{6}$$

The solution of Eq. (6) is given by

$$P_x = \text{diag}\,[P_{x1}, P_{x2}] \tag{7}$$

with

$$P_{x_1} = \frac{b^2 \beta + \bar{W}}{2b^2 f_1 f_2 - \alpha b^2 f_1^2 - \alpha b^3 f_1 f_2^2} \tag{8}$$

$$P_{x_2} = b f_1 P_{x_1} \tag{9}$$

The cost function (5) can be rewritten as

$$J = (q + r f_1^2) P_{x_1} + (q + r f_2^2) P_{x_2} \tag{10}$$

where f_1 and f_2 are elements of the feedback matrix. When the weights, noise parameters and the actuator orientation are all given, the cost will be a function of f_1 and f_2. The optimal feedback can be obtained by our equating the partial derivatives of the cost with respect to f_1 and f_2 to zero. After substantial mathematical manipulation, the equations for optimal feedback become

$$b\, f_1^2 - b\, f_2^2 + 2\, f_1 = 0 \tag{11}$$

$$f_1^2 = \frac{q}{r}(1 - \alpha b\, f_2) \tag{12}$$

These equations give optimal feedback when the actuator orientation is fixed. We can see that the feedback matrix does not depend on the plant disturbance and β since they correspond to the ordinary noise variance in LQG theory. Eqs. (11–12) are a parabolic equation and a hyperbolic equation. Those equations can be plotted in the f_1–f_2 plane (Fig. 2), and may give up to 4 intersection points. By inspection, only one point out of the 4 corresponds to a stable system. It should be noted that for the new noise variance model the solution obtained from ordinary LQG method is no longer optimal. The solution of the feedback corresponding to ordinary noise variance model (LQG) can be obtained by our equating α to zero, and is also plotted in the figure. The trends of the new and ordinary solutions and their difference can be seen clearly from the figure when q/r, b, or α is changed. It shows that the optimal f_1 is between 0 and $1/(\alpha b)$, and the optimal f_2 is between 0 and sqrt(q/r). When α becomes larger or q/r becomes larger, the difference between the optimal solutión and the ordinary LQG solution becomes more significant. Both elements of the optimal feedback matrix are smaller than those obtained by LQG method. Clearly, smaller feedback elements help to reduce the actuator noise.

To find optimal actuator orientation, we differentiate the cost with respect to b by considering the feedback elements as functions of b. By equating the derivative to zero, we obtain a really complicated equation for optimal actuator location. After much mathematical manipulation the equation becomes:

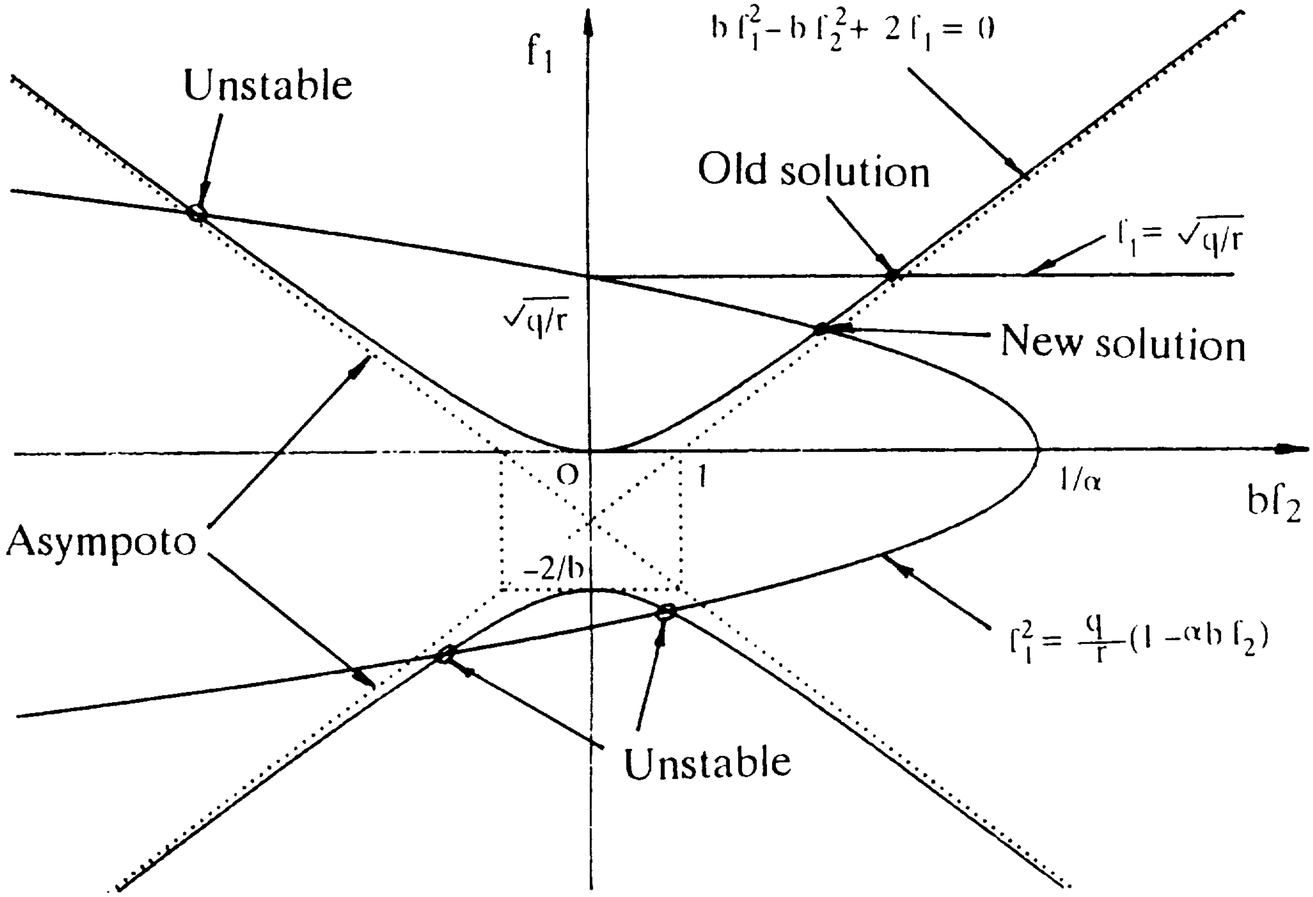

Figure 2. The plot of feedback for ordinary and new solutions.

$$b^2(1+bf_1)\left[\frac{q}{r}\alpha^2(1+bf_1)^2+4f_2^2-4\alpha f_1 f_2\right]-(3+bf_1)\left[2f_2-3\alpha f_1-b\alpha f_1^2\right]^2\frac{\bar{W}}{\beta}=0 \tag{13}$$

It can be seen that $\bar{W}/\beta$ does not affect the feedback but it does affect the optimal actuator location. The optimal actuator location differs from that obtained by the ordinary noise variance model in Ref. [1], and the latter can also be obtained by our equating α to zero in Eq. (13).

Eqs. (11–13) can be solved simultaneously to obtain the optimal feedback and the optimal actuator location. Some properties of Eqs. (11–13) can simplify the computation of the optimal feedback and optimal actuator location. For example, there is one and only one solution of f_1 between 0 and sqrt(q/r) if α is not zero, and the left side of Eq. (13) is a monotonous function of b.

It is particularly interesting to investigate Eq. (13) when $b = 1$. In this case, for different values of α, we can plot $\bar{W}/\beta$ as a function of q/r, as shown in Fig. 3. In the $\bar{W}/\beta$–q/r plane, for a specific α, the optimal b in the area above the corresponding curve is larger than 1, and the optimal b in the area below the corresponding curve is smaller than 1. Since b ($=\cos\theta$) can not be greater than 1, we must use $b = 1$ in the area above the corresponding curve.

Fig. 4 shows the optimal b as a function of α and q/r when $\bar{W}/\beta = 1$. Clearly, optimal b decreases with α and q/r and could be significantly less than 1. Computation also shows that when $\bar{W}/\beta$ decreases the optimal b will also decrease; when $\bar{W}/\beta$ increases the optimal b will also increase.

Conclusion

In many applications, a more practical noise variance model of an actuator than the one in LQG theory is that its noise variance increases with its signal variance. In this paper, we investigated the optimal control and optimal actuator placement when the actuator noise variance increases linearly with its signal variance. In this case the feedback and actuator location obtained by ordinary LQG theory are no longer optimal.

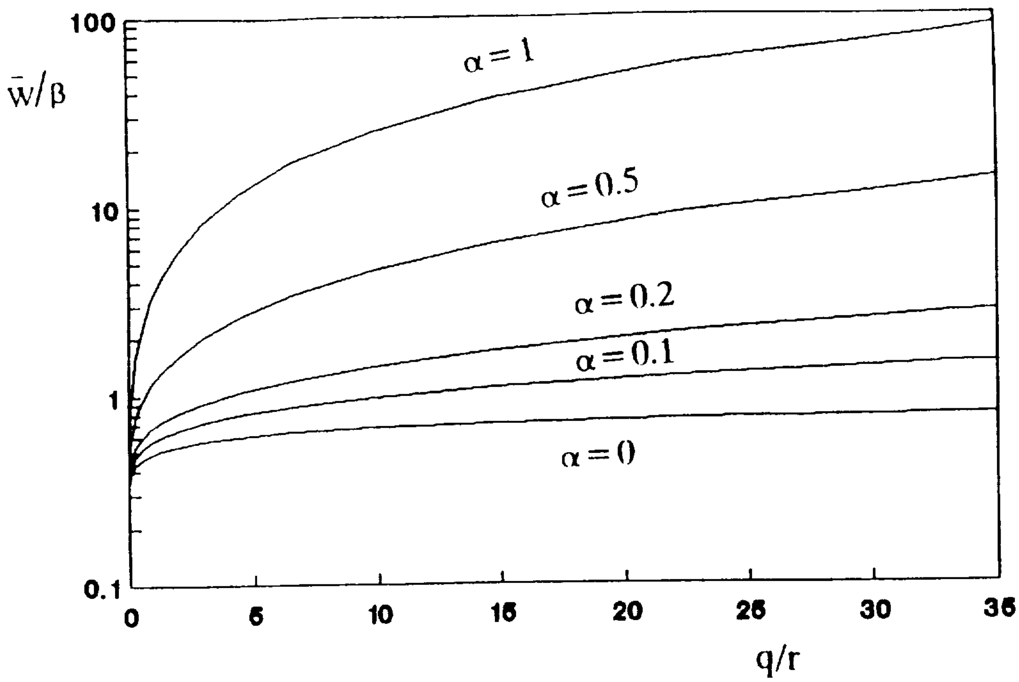

Figure 3. The areas of $b < 0$ and $b = 1$ for several values of α.

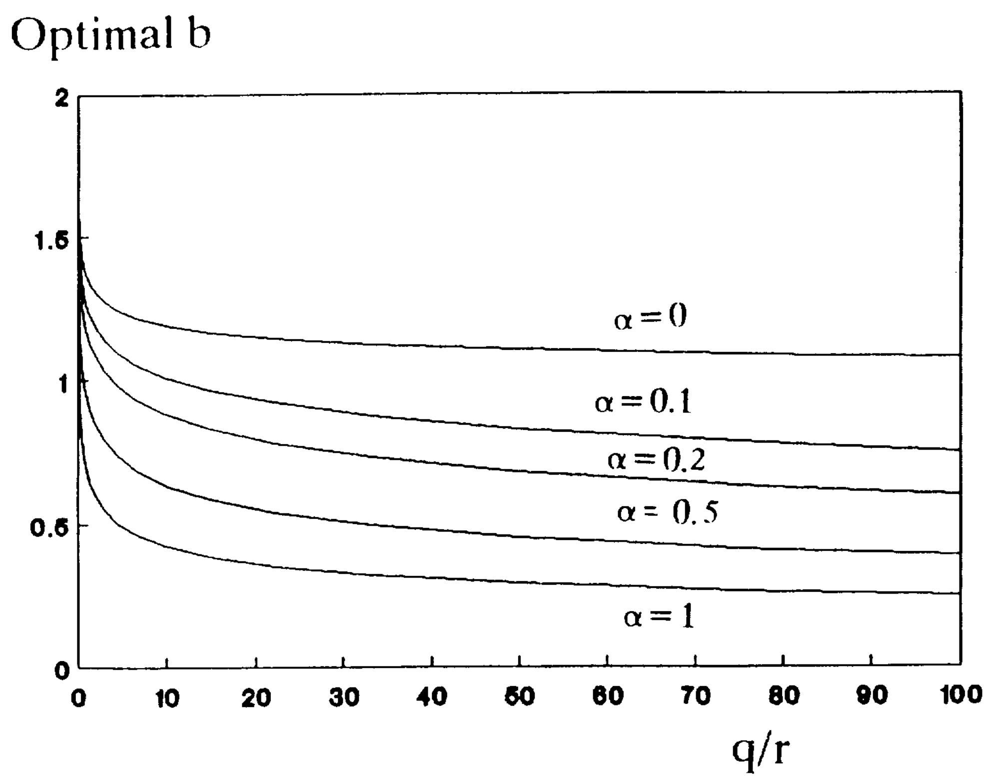

Figure 4. The optimal actuator location b as a function of q/r and α, when $\bar{W}/\beta = 1$.

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