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FINITE ELEMENT APPROXIMATION OF AN OPTIMAL CONTROL PROBLEM FOR THE VON KARMAN EQUATIONS

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ABSTRACT

This paper is concerned with optimal control problems for the von Kármán equations with distributed controls. We first show that optimal solutions exist. We then show that Lagrange multipliers may be used to enforce the constraints and derive an optimality system from which optimal states and controls may be deduced. Finally we define finite element approximations of solutions for the optimality system and derive error estimates for the approximations.

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1. Introduction

The von Kármán equations for a clamped plate are given by (see, e.g., [7] or [9])

$$\Delta^2\psi_1 + \frac{1}{2}[\psi_2, \psi_2] = 0 \quad \text{in } \Omega$$

and

$$\Delta^2\psi_2 - [\psi_1, \psi_2] = \lambda g \quad \text{in } \Omega$$

where

$$[\psi, \phi] = \frac{\partial^2\psi}{\partial x_1^2}\frac{\partial^2\phi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_2^2}\frac{\partial^2\phi}{\partial x_1^2} - 2\frac{\partial^2\psi}{\partial x_1 x_2}\frac{\partial^2\phi}{\partial x_1 x_2}\,.$$

Here, Ω is a bounded, convex polygonal domain in $\mathbb{R}^2$, ψ_1 denotes the Airy stress function, ψ_2 denotes the deflection of the plate in the direction normal to the plate, and λg is an external load normal to the plate which depends on the loading parameter λ.

The boundary conditions on $\Gamma = \partial\Omega$ are given as follow:

$$\psi_1 = \frac{\partial\psi_1}{\partial n} = \psi_2 = \frac{\partial\psi_2}{\partial n} = 0 \quad \text{on } \Gamma\,,$$

where $\partial(\cdot)/\partial n$ denotes the normal derivative in the direction of the outer normal to Γ.

For reasons to be explained later we introduce appropriate rescalings, i.e., by replacing ψ_1 by $\lambda\psi_1$, ψ_2 by $\lambda\psi_2$, and g by λg. Then we can rewrite the Kármán equations as follows:

$$\Delta^2\psi_1 + \frac{\lambda}{2}[\psi_2, \psi_2] = 0 \quad \text{in } \Omega\,, \tag{1.1}$$

$$\Delta^2\psi_2 - \lambda\,[\psi_1, \psi_2] = \lambda\, g \quad \text{in } \Omega\,, \tag{1.2}$$

and

$$\psi_1 = \frac{\partial\psi_1}{\partial n} = \psi_2 = \frac{\partial\psi_2}{\partial n} = 0 \quad \text{on } \Gamma\,. \tag{1.3}$$

We introduce the standard Sobolev spaces (see, e.g., [1])

$$H_0^2(\Omega) = \left\{\psi \in H^2(\Omega) \;\middle|\; \psi = 0\,,\ \frac{\partial\psi}{\partial n} = 0 \text{ on } \Gamma\right\}\,,$$

$$H_0^2(\Omega) = \left(H_0^2(\Omega)\right)^2\,, \quad H^{-2}(\Omega) = \left(H_0^2(\Omega)\right)^*\,, \quad \text{and} \quad H^{-2}(\Omega) = \left(H_0^2(\Omega)\right)^*\,,$$

the bilinear form

$$a(\psi, \phi) = \int_\Omega \Delta\psi\,\Delta\phi\, d\Omega \quad \forall\, \psi, \phi \in H^2(\Omega)$$

and the $L^2(\Omega)$-inner product

$$(\psi, \phi) = \int_\Omega \psi\,\phi\, d\Omega \quad \forall\, \psi, \phi \in L^2(\Omega)\,.$$

Then we may define the following weak formulation of the von Kármán equations (1.1)-(1.3): find $\boldsymbol{\psi} = (\psi_1, \psi_2)^T \in H_0^2(\Omega)$ such that

$$a(\psi_1, \phi_1) + \frac{\lambda}{2}\big([\psi_2, \psi_2], \phi_1\big) = 0 \quad \forall\, \phi_1 \in H_0^2(\Omega) \tag{1.4}$$

and

$$a(\psi_2, \phi_2) - \lambda\big([\psi_1, \psi_2], \phi_2\big) = \lambda\langle g, \phi_2\rangle \quad \forall\, \phi_2 \in H_0^2(\Omega)\,. \tag{1.5}$$

Here the notation $(\cdot,\cdot)$ stands for the $L^2(\Omega)$-inner product and $\langle\cdot,\cdot\rangle$ the duality pairing. Using the identity

$$\big([\psi, \phi], \zeta\big) = \big([\psi, \zeta], \phi\big) \quad \forall\, \psi, \phi, \zeta \in H_0^2(\Omega)\,, \tag{1.6}$$

one can show that for each $g \in H^{-2}(\Omega)$, (1.4)-(1.5) possesses at least one solution $\boldsymbol{\psi} = (\psi_1, \psi_2)^T \in H_0^2(\Omega)$ and that all solutions of (1.4)-(1.5) satisfy the a priori estimate

$$\|\psi_1\|_2 + \|\psi_2\|_2 \le C\|g\|_{-2}\,; \tag{1.7}$$

see, e.g., [9], for details. In the sequel a solution to (1.1)-(1.3) will be understood in the sense of (1.4)-(1.5).

Given a desired state $\boldsymbol{\psi}_0 = \big(\psi_{10}, \psi_{20}\big)^T \in L^2(\Omega)$, we define the functional

$$\begin{aligned} \mathcal{J}(\boldsymbol{\psi}, g) &= \mathcal{J}(\psi_1, \psi_2, g) \\ &= \frac{\lambda}{2}\int_\Omega \Big((\psi_1 - \psi_{10})^2 + (\psi_2 - \psi_{20})^2\Big)\, d\Omega + \frac{\lambda}{2}\int_\Omega g^2\, d\Omega \end{aligned} \tag{1.8}$$

for all $\boldsymbol{\psi} = (\psi_1, \psi_2)^T \in H_0^2(\Omega)$ and $g \in L^2(\Omega)$. We then consider the following optimal control problem associated with the von Kármán plate equations:

$$min\,\{\,\mathcal{J}(\boldsymbol{\psi}, g) \mid \boldsymbol{\psi} \in H_0^2(\Omega)\,,\ g \in \Theta\,\} \quad \text{subject to} \quad (1.4)\text{-}(1.5)\,, \tag{1.9}$$

where Θ is a convex, closed subset of $L^2(\Omega)$. The physical interpretation of this optimal control problem is that we wish to match a desired Airy stress distribution and a deflection distribution on the entire plate by choosing an (optimal) external load g from an admissible set Θ.

Our plan is as follows. In §2, we show that an optimal solution exists. In §3, we use Lagrange multiplier techniques to derive an optimality system from which optimal states and controls can be deduced. We then specialize to the case $\Theta = L^2(\Omega)$ and derive an optimality system of equations. In §4, we define conforming finite element approximations of the optimality system of equations and derive error estimates.

2. Existence of an optimal solution

We will prove in this section that an optimal solution exsits for the minimization problem (1.9).

THEOREM 2.1. *There exists a* $(\boldsymbol{\phi}, g) \in H_0^2(\Omega) \times \Theta$ *such that* (1.8) *is minimized subject to* (1.4)-(1.5).

Proof. We choose a $g^{(0)} \in \Theta$ and let $\boldsymbol{\psi}^{(0)} = \big(\psi_1^{(0)}, \psi_2^{(0)}\big)^T \in H_0^2(\Omega)$ be a solution of

$$a(\psi_1^{(0)}, \phi_1) + \frac{\lambda}{2}\big([\psi_2^{(0)}, \psi_2^{(0)}], \phi_1\big) = 0 \quad \forall\, \phi_1 \in H_0^2(\Omega)$$

and

$$a(\psi_2^{(0)}, \phi_2) - \lambda\big([\psi_1^{(0)}, \psi_2^{(0)}], \phi_2\big) = \lambda\langle g^{(0)}, \phi_2\rangle \quad \forall\, \phi_2 \in H_0^2(\Omega)\,.$$

The existence of such a $\boldsymbol{\psi}^{(0)}$ was established in [9]. We see that $(\boldsymbol{\psi}^{(0)}, g^{(0)}) \in H_0^2(\Omega)\times\Theta$ satisfies the constraints (1.4)-(1.5) and $\mathcal{J}(\boldsymbol{\psi}^{(0)}, g^{(0)}) < \infty$. We also note that $\mathcal{J}(\boldsymbol{\psi}, g) \geq 0$ for all $(\boldsymbol{\psi}, g) \in X\times\Theta$. Thus we may choose a minimizing sequence $\{(\boldsymbol{\psi}^{(n)}, g^{(n)})\} \subset H_0^2(\Omega)\times\Theta$ such that for some constant $M > 0$,

$$\mathcal{J}(\boldsymbol{\psi}^{(n)}, g^{(n)}) \leq M\,, \tag{2.1}$$

$$a(\psi_1^{(n)}, \phi_1) + \frac{\lambda}{2}([\psi_2^{(n)}, \psi_2^{(n)}], \phi_1) = 0 \quad \forall\, \phi_1 \in H_0^2(\Omega) \tag{2.2}$$

and

$$a(\psi_2^{(n)}, \phi_2) - \lambda([\psi_1^{(n)}, \psi_2^{(n)}], \phi_2) = \lambda\langle g^{(n)}, \phi_2\rangle \quad \forall\, \phi_2 \in H_0^2(\Omega)\,. \tag{2.3}$$

Using (2.1) and the definition of the functional $\mathcal{J}$ we deduce that the sequence $\{\|g^{(n)}\|_0\}$ is bounded so that by the a priori estimate (1.7), the sequence $\{\|\boldsymbol{\psi}^{(n)}\|_2\}$ is bounded as well. Hence by choosing a subsequence, we have that

$$g^{(n)} \rightharpoonup g \quad \text{in } L^2(\Omega)$$

and

$$\boldsymbol{\psi}^{(n)} \rightharpoonup \boldsymbol{\psi} \quad \text{in } H_0^2(\Omega)$$

for some $g \in L^2(\Omega)$ and $\boldsymbol{\psi} \in H_0^2(\Omega)$. Using the convexity and closedness of Θ, we deduce that $g \in \Theta$. By the compact imbedding $H_0^2(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$, we obtain $\boldsymbol{\psi}^{(n)} \to \boldsymbol{\psi}$ in $L^2(\Omega)$ so that for each $\boldsymbol{\phi} = (\phi_1, \phi_2)^T \in H_0^2(\Omega)$, we have that (also using the continuous imbedding $H_0^2(\Omega) \hookrightarrow L^\infty(\Omega)$)

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{2}\left([\psi_2^{(n)}, \psi_2^{(n)}], \phi_1\right) &= \lim_{n\to\infty} \frac{1}{2}\left([\psi_2^{(n)}, \phi_1], \psi_2^{(n)}\right)\\ &= \frac{1}{2}([\psi_2, \phi_1], \psi_2) = \frac{1}{2}([\psi_2, \psi_2], \phi_1)\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty} -\left([\psi_1^{(n)}, \psi_2^{(n)}], \phi_2\right) &= \lim_{n\to\infty} -\left([\psi_1^{(n)}, \phi_2], \psi_2^{(n)}\right)\\ &= -([\psi_1, \phi_2], \psi_2) = -([\psi_1, \psi_2], \phi_2)\,.\end{aligned}$$

Hence we may pass to the limit in (2.2)-(2.3) to show that $(\boldsymbol{\psi}, g)$ satisfies (1.4)-(1.5).

Finally, we use the sequential lower semi-continuity of the functional $\mathcal{J}$ to obtain that

$$\mathcal{J}(\boldsymbol{\psi}, g) \leq \lim_{n\to\infty} \mathcal{J}(\boldsymbol{\psi}^{(n)}, g^{(n)})\,.$$

Hence, we have proved that a solution $(\boldsymbol{\psi}, g) \in H_0^2(\Omega)\times\Theta$ exists that minimizes (1.8) subject to (1.4)-(1.5). □

3. An optimality system of equations

In this section we assume that $(\boldsymbol{\psi}, g) \in H_0^2(\Omega)\times\Theta$ is an optimal solution for the minimization problem (1.9) and we attemp to characterize the optimal solution as the solution for a system of partial differential equations. To be precise, we use Lagrange multiplier rules to derive an optimality system of equations.

We define the Lagrangian for the constrained minimization problem (1.9) as follows:

$$\mathcal{L}(\boldsymbol{\psi}, g, \boldsymbol{\eta}) = \mathcal{J}(\boldsymbol{\psi}, g) - \Big\{ a(\psi_1, \eta_1) + \frac{\lambda}{2}([\psi_2, \psi_2], \eta_1) + a(\psi_2, \eta_2) - \lambda([\psi_1, \psi_2], \eta_2) - \lambda(g, \eta_2) \Big\}$$

for all $(\boldsymbol{\psi}, g, \boldsymbol{\eta}) \in H_0^2(\Omega) \times L^2(\Omega) \times H_0^2(\Omega)$.

By formally taking variations in the Lagrangian with respect to $\boldsymbol{\psi}$ and g, we obtain:

$$a(\zeta_1, \eta_1) - \lambda([\psi_2, \eta_2], \zeta_1) = \lambda(\psi_1 - \psi_{10}, \zeta_1) \quad \forall\, \zeta_1 \in H_0^2(\Omega), \tag{3.1}$$

$$\begin{aligned} a(\zeta_2, \eta_2) + \lambda([\psi_2, \eta_1], \zeta_2) - \lambda([\psi_1, \eta_2], \zeta_2) \\ = \lambda(\psi_2 - \psi_{20}, \zeta_2) \quad \forall\, \zeta_2 \in H_0^2(\Omega) \end{aligned} \tag{3.2}$$

and

$$\frac{\lambda}{2}(z, z) + \lambda(z, \eta_2) - \frac{\lambda}{2}(g, g) - \lambda(g, \eta_2) \geq 0 \quad \forall\, z \in \Theta. \tag{3.3}$$

For each $\epsilon \in (0, 1)$ and each $t \in \Theta$, we set $z = \epsilon t + (1 - \epsilon) g \in \Theta$ in (3.3) to obtain

$$\frac{\epsilon^2}{2}(t - g, t - g) + \epsilon(t - g, g) + \epsilon(t - g, \eta_2) \geq 0 \quad \forall\, t \in \Theta$$

so that, after dividing by $\epsilon > 0$ and then letting $\epsilon \to 0^+$, we obtain

$$(t - g, g + \eta_2) \geq 0 \quad \forall\, t \in \Theta. \tag{3.4}$$

Now we show that there does exist an $\boldsymbol{\eta} \in H_0^2(\Omega)$ satisfying (3.1)-(3.3), or equivalently, (3.1)-(3.2) and (3.4), so that we are justified to compute a triplet $(\boldsymbol{\psi}, g, \boldsymbol{\eta})$ from (1.4)-(1.5), (3.1)-(3.2) and (3.4). In this paper we will not address the uniqueness of solutions for the system (1.4)-(1.5), (3.1)-(3.2) and (3.4).

We first quote the following abstract Lagrange multiplier rule whose proof can be found in [10].

THEOREM 3.1. *Let X_1 and X_2 be two Banach spaces and Θ an arbitrary set. Suppose $\mathcal{J}$ is a functional on $X_1 \times \Theta$ and M a mapping from $X_1 \times \Theta$ to X_2. Assume that (u, g) is a solution to the following constrained minimization problem:*

$$\begin{aligned} &M(u, g) = 0 \text{ and there exists an } \epsilon > 0 \text{ such that } \mathcal{J}(u, g) \leq \mathcal{J}(v, z) \\ &\text{for all } (v, z) \text{ such that } \|u - v\|_{X_1} \leq \epsilon \text{ and } M(v, z) = 0. \end{aligned} \tag{3.5}$$

Let U be an open neighborhood of u in X_1. Assume further that the following conditions are satisfied:

(A) *for each $z \in \Theta$, $v \mapsto \mathcal{J}(v, z)$ and $v \mapsto M(v, z)$ are Fréchet-differentiable at $v = u$;*

(B) *for any $v \in U$, $z_1, z_2 \in \Theta$, and $\gamma \in [0, 1]$, there exists a $z_\gamma = z_\gamma(v, z_1, z_2)$ such that*

$$M(v, z_\gamma) = \gamma M(v, z_1) + (1 - \gamma) M(v, z_2)$$

and

$$\mathcal{J}(v, z_\gamma) \leq \gamma \mathcal{J}(v, z_1) + (1-\gamma)\mathcal{J}(v, z_2)\,;$$

(C) *Range*$(M_u(u,g))$ *is closed with a finite codimension*;

and

(D) *the algeraic sum* $M_u(u,g)X_1 + M(u,\Theta)$ *contains* $0 \in X_2$ *as an interior point, where* $M_u(u,g)$ *denotes the Fréchet derivative of* M *with respect to* u. *Then, there exists a a* $\eta \in X_2^*$ *such that*

$$\langle \eta, M_u(u,g)v\rangle - \langle \mathcal{J}_u(u,g), v\rangle = 0 \quad \forall\, v \in X_1$$

(*or equivelently,* $[M_u(u,g)]^*\eta - \mathcal{J}_u(u,g) = 0$) *and*

$$\min_{z\in\Theta} \mathcal{L}(u,z,\eta) = \mathcal{L}(u,g,\eta)\,,$$

where $\mathcal{L}(u,g,\eta) = \mathcal{J}(u,g) - \langle \eta, M(u,g)\rangle$ *is the Lagrangian for the constrained minimization problem* (2.5) *and where* $\mathcal{J}_u(u,g)$ *denotes the Fréchet derivative of* $\mathcal{J}$ *with respect to* u. □

We now define some spaces and operators in order to rewrite the constraint equations (1.4)-(1.5) into a form that will facilitate our verification of the assumptions in Theorem 3.1.

We define the spaces $X = H_0^2(\Omega)$, $Y = H^{-2}(\Omega)$, $G = L^2(\Omega)$, and $Z = L^1(\Omega)$. By compact imbedding results, $Z \hookrightarrow\hookrightarrow Y$. The control set Θ is a closed, convex subset of $G = L^2(\Omega)$. We also note that $Y = X^*$.

We define the continuous linear operator $A \in \mathcal{L}(X;Y)$ as follows: for $\boldsymbol{\psi} \in X = H_0^2(\Omega)$, $A\boldsymbol{\psi} = f \in Y = H^{-2}(\Omega)$ if and only if

$$a(\psi_1, \phi_1) = \langle f_1, \phi_1\rangle \quad \forall\, \phi_1 \in H_0^2(\Omega)$$

and

$$a(\psi_2, \phi_2) = \langle f_2, \phi_2\rangle \quad \forall\, \phi_2 \in H_0^2(\Omega)\,.$$

It can be easily verified that A is self-adjoint, invertible, and $A^{-1} \in \mathcal{L}(Y;X)$.

We define the (differentiable) nonlinear mapping $N : X \to Y$ by

$$N(\boldsymbol{\psi}) = \begin{pmatrix} \frac{1}{2}[\psi_2, \psi_2] \\ -[\psi_1, \psi_2] \end{pmatrix} \quad \forall\, \boldsymbol{\psi} \in X$$

or equivalently

$$\langle N(\boldsymbol{\psi}), \boldsymbol{\phi}\rangle = \frac{1}{2}([\psi_2, \psi_2], \phi_1) - ([\psi_1, \psi_2], \phi_2) \quad \forall\, \boldsymbol{\phi} = (\phi_1, \phi_2)^T \in X$$

and define $K : g \in L^2(\Omega) \to Y$ by

$$Kg = -\begin{pmatrix} 0 \\ g \end{pmatrix},$$

or equivalently,

$$\langle Kg, \boldsymbol{\phi}\rangle = -\langle g, \phi_2\rangle \quad \forall\, \boldsymbol{\phi} = (\phi_1, \phi_2) \in X\,.$$

Clearly, the constraint equations (1.4)-(1.5) can be expressed as

$$A\boldsymbol{\psi} + \lambda\, N(\boldsymbol{\psi}) + \lambda\, Kg = 0\,.$$

We are now in a position to prove the existence of a Lagrange multiplier $\boldsymbol{\eta} \in H_0^2(\Omega)$ that satisfies (3.1)-(3.2) and (3.4).

THEOREM 3.2. *Assume* $(\boldsymbol{\psi}, g) \in H_0^2(\Omega) \times \Theta$ *is an optimal solution that minimizes* (1.8) *subject to* (1.4)-(1.5). *Then, for almost all* $\lambda \in \Lambda$, *there exists a Lagrange multiplier* $\boldsymbol{\eta} \in H_0^2(\Omega)$ *satisfying the Euler equations* (3.1)-(3.2) *and* (3.4).

Proof. We define a mapping $M : H_0^2(\Omega) \times \Theta \to H^{-2}(\Omega)$ (i.e., $X \times \Theta \to Y$) by

$$M(\boldsymbol{\phi}, z) \equiv A\boldsymbol{\phi} + \lambda\, N(\boldsymbol{\phi}) + \lambda\, Kz \qquad \forall\, (\boldsymbol{\phi}, z) \in H_0^2(\Omega) \times \Theta\,.$$

We see that minimizing (1.8) subject to (1.4)-(1.5) can be stated as minimizing $\mathcal{J}(\boldsymbol{\psi}, g)$ subject to $M(\boldsymbol{\psi}, g) = 0$. We first verify that the hypotheses (A)-(C) of Theorem 3.1 hold with $X_1 = X$ and $X_2 = Y$.

Obviously, for each $z \in \Theta$, the mappings $\boldsymbol{\phi} \mapsto \mathcal{J}(\boldsymbol{\phi}, z)$ and $\boldsymbol{\phi} \mapsto M(\boldsymbol{\phi}, z)$ are both Fréchet differentiable, i.e., (A) holds.

Since Θ is convex and the mapping K is linear, we have that for any $\boldsymbol{\phi} \in H_0^2(\Omega)$, $z_1 \in \Theta$, $z_2 \in \Theta$ and $\gamma \in (0,1)$, the element $z_\gamma \equiv \gamma z_1 + (1-\gamma) z_2$ belongs to Θ and

$$\begin{aligned} M(\boldsymbol{\phi}, z_\gamma) &= A\boldsymbol{\phi} + \lambda\, N(\boldsymbol{\phi}) + \lambda\, Kz_\gamma \\ &= \gamma\big(A\boldsymbol{\phi} + \lambda\, N(\boldsymbol{\phi})\big) + (1-\gamma)\big(A\boldsymbol{\phi} + \lambda\, N(\boldsymbol{\phi})\big) \\ &\quad + \lambda\big(\gamma K z_1 + (1-\gamma) K z_2\big) \\ &= \gamma M(\boldsymbol{\phi}, z_1) + (1-\gamma) M(\boldsymbol{\phi}, z_2)\,. \end{aligned}$$

Moreover, the mapping $g \mapsto \|g\|_0^2$ is obviously convex from Θ into $\mathbb{R}$ so that

$$\begin{aligned} \mathcal{J}(\boldsymbol{\phi}, z_\gamma) &= \frac{\lambda}{2}\, \|\boldsymbol{\phi} - \boldsymbol{\psi}_0\|_0^2 + \frac{\lambda}{2}\, \|\gamma z_1 + (1-\gamma) z_2\|_0^2 \\ &\le \frac{\lambda}{2}\, \big(\gamma\, \|\boldsymbol{\phi} - \boldsymbol{\psi}_0\|_0^2 + (1-\gamma)\, \|\boldsymbol{\phi} - \boldsymbol{\psi}_0\|_0^2\big) + \frac{\lambda}{2}\, \big(\gamma\, \|z_1\|_0^2 + (1-\gamma)\, \|z_2\|_0^2\big) \\ &= \gamma\, \mathcal{J}(\boldsymbol{\phi}, z_1) + (1-\gamma)\, \mathcal{J}(\boldsymbol{\phi}, z_2)\,. \end{aligned}$$

Thus, (B) holds.

The operator $M_{\boldsymbol{\psi}}(\boldsymbol{\psi}, g)$ from X to Y is defined by

$$M_{\boldsymbol{\psi}}(\boldsymbol{\psi}, g) \cdot \boldsymbol{\phi} = A\boldsymbol{\phi} + \lambda N'(\boldsymbol{\psi}) \cdot \boldsymbol{\phi} \qquad \forall\, \boldsymbol{\phi} \in X = H_0^2(\Omega)\,,$$

or simply,

$$M_{\boldsymbol{\psi}}(\boldsymbol{\psi}, g) = A + \lambda N'(\boldsymbol{\psi})\,,$$

where for any $\boldsymbol{\psi} \in X$, the operator $N'(\boldsymbol{\psi}) : X \to Y$ is given by

$$N'(\boldsymbol{\psi}) \cdot \boldsymbol{\phi} = \begin{pmatrix} [\psi_2, \phi_2] \\ -[\psi_1, \phi_2] - [\psi_2, \phi_1] \end{pmatrix} \quad \forall\, \boldsymbol{\phi} = (\phi_1, \phi_2)^T \in X\,.$$

Thus, using the definition of $[\cdot,\cdot]$, we obtain that $N'(\psi)\cdot\phi \in L^1(\Omega) = Z$. Using the compact imbedding $Z \hookrightarrow\hookrightarrow Y$, we see that $N'(\psi)$ is a compact operator from X to X. As a result, $M_\psi(\psi,g) = A + \lambda N'(\psi) = A\big(I + \lambda A^{-1}N'(\psi)\big)$ is a Fredholm operator so that it has a closed range with a finite codimension, i.e., (C) holds.

We now verify that the hypothesis (D) holds for almost all λ. In fact, if $\lambda \neq 0$ and $(1/\lambda) \notin \sigma\big(-A^{-1}N'(\psi)\big)$, where $\sigma\big(-A^{-1}N'(\psi)\big)$ denotes the spectrum of the operator $\big(-A^{-1}N'(\psi)\big)$, then it follows that

$$Y = Range\big(A(I + \lambda A^{-1}N'(\psi))\big) = Range\big(M_\psi(\psi,g)\big)$$

so that $Range\big(M_\psi(\psi,g)\big)$ contains $0 \in Y$ as an interior point, i.e., (D) holds. Since the spectrum of a compact operator is at most countable, we conclude that for almost all λ, (D) holds.

Hence, by Theorem 3.1, we obtain that for almost all λ, there exists a $\eta \in Y^* = H_0^2(\Omega)$ such that

$$A^*\eta + \lambda\,[N'(\psi)]^*\cdot\eta - \mathcal{J}_\psi(\psi,g) = 0 \tag{3.6}$$

and

$$\mathcal{L}(\psi,g,\eta) \le \mathcal{L}(\psi,z,\eta) \quad \forall\, z \in \Theta\,. \tag{3.7}$$

Recalling the definition of the operators A and $N'(\psi)$ as well as the fact that A is self-adjoint, i.e., $A^* = A$, we see that (3.6) is equivalent to (3.1)-(3.2). Using the definition of the Lagrangian, we easily see that (3.7) is simply a restatement of (3.3), the latter being equivalent to (3.4). Thus we have proved that a Lagrange multiplier $\eta \in H_0^2(\Omega)$ exists such that (3.1)-(3.2) and (3.4) hold. □

Remark. Now we can explain the rescalings that were introduced in §1. With these rescalings, the parameter λ appears in the form of an eigen value in the operator $M_\psi(\psi,g) \equiv A + \lambda N'(\psi)$ so that spectrum theories for compact operators can be readily employed in the proof of Theorem 3.2. □

So far Θ has only been assumed to be a closed, convex subset of $L^2(\Omega)$. In the sequel we specialize to the case $\Theta = L^2(\Omega)$. Then we can obtain two improvements in Theorem 3.2, the first being that the results of Theorem 3.2 now hold for all λ, the second being that (3.4) becomes an equality.

THEOREM 3.3. *Assume $\Theta = L^2(\Omega)$ and $(\psi,g) \in H_0^2(\Omega)\times L^2(\Omega)$ is an optimal solution that minimizes* (1.8) *subject to* (1.4)-(1.5). *Then, for all $\lambda \in \Lambda$, there exists a Lagrange multiplier $\eta \in H_0^2(\Omega)$ satisfying the Euler equations* (3.1)-(3.2) *and*

$$(t, g + \eta_2) = 0 \quad \forall\, t \in L^2(\Omega)\,. \tag{3.8}$$

Proof. In view of Theorem 3.2 and its proof, we only need to verify (D) when $(1/\lambda) \in \sigma\big(-A^{-1}N'(u)\big)$, i.e., we need to show that for each $\tilde f \in H^{-2}(\Omega)$, there exists a $\tilde g \in L^2(\Omega)$ and a $\tilde\psi \in H_0^2(\Omega)$ such that

$$A\tilde\psi + \lambda N'(\psi)\cdot\tilde\psi + \lambda K\tilde g = \tilde f\,,$$

or equivalently,

$$a(\tilde\psi_1,\phi_1) + \lambda\big([\psi_2,\tilde\psi_2],\phi_1\big) = \langle\tilde f_1,\phi_1\rangle \quad \forall\, \phi_1 \in H_0^2(\Omega) \tag{3.9}$$

and

$$a(\tilde{\psi}_2,\phi_2)-\lambda([\tilde{\psi}_1,\psi_2],\phi_2)-\lambda([\psi_1,\tilde{\psi}_2],\phi_2)-\lambda(\tilde{g},\phi_2) = \langle \tilde{f}_2,\phi_2\rangle \quad \forall\, \phi_2 \in H_0^2(\Omega)\,. \tag{3.10}$$

To show this, we first let $\tilde{\boldsymbol{\psi}} \in H_0^2(\Omega)$ be a solution of

$$a(\tilde{\psi}_1,\phi_1)+\lambda([\psi_2,\tilde{\psi}_2],\phi_1) = \langle \tilde{f}_1,\phi_1\rangle \quad \forall\, \phi_1 \in H_0^2(\Omega)$$

and

$$a(\tilde{\psi}_2,\phi_2)-\lambda([\tilde{\psi}_1,\psi_2],\phi_2) = \langle \tilde{f}_2,\phi_2\rangle \quad \forall\, \phi_2 \in H_0^2(\Omega)\,.$$

The existence of such a $\tilde{\boldsymbol{\psi}}$ can be shown in a manner similar to that for showing the existence of a solution to the von Kármán equation; the key step is that by adding the two equations with the test function $\boldsymbol{\phi}$ replaced by $\tilde{\boldsymbol{\psi}}$, we have the a priori estimate

$$a(\tilde{\psi}_1,\tilde{\psi}_1)+a(\tilde{\psi}_2,\tilde{\psi}_2) = \langle \tilde{f}_1,\tilde{\psi}_1\rangle + \langle \tilde{f}_2,\tilde{\psi}_2\rangle\,.$$

Now, having chosen such a $\tilde{\boldsymbol{\psi}}$, we simply set $\tilde{g} = -[\psi_1,\tilde{\psi}_2]$. Note that regularity results for the biharmonic equation applied to (1.4)-(1.5) yield $\boldsymbol{\psi} \in H^4(\Omega)$ (see [3]). Hence, using imbedding theorems we deduce that $\tilde{g} \in L^2(\Omega)$. It is obvious that $\tilde{g}$ and $\tilde{\boldsymbol{\psi}}$ satisfy (3.9)-(3.10). Thus, we have verified (D) so that by Theorem 3.1, there exists a $\boldsymbol{\eta} \in X$ such that (3.1)-(3.2) and (3.4) hold.

Finally, (3.4) trivially reduces to (3.8) in the present case of $\Theta = L^2(\Omega)$. □

Combining the results of §2 and this section we see that we have proved the existence of a triplet $(\boldsymbol{\psi}, g, \boldsymbol{\eta}) \in H_0^2(\Omega) \times L^2(\Omega) \times H_0^2(\Omega)$ such that the system (1.4)-(1.5), (3.1)-(3.2) and (3.8) is satisfied. For convenience we collect these equations here to form *an Optimality System of Equations*:

$$a(\psi_1,\phi_1)+\frac{\lambda}{2}([\psi_2,\psi_2],\phi_1) = 0 \quad \forall\, \phi_1 \in H_0^2(\Omega)\,, \tag{3.11}$$

$$a(\psi_2,\phi_2)-\lambda([\psi_1,\psi_2],\phi_2) = \lambda\langle g,\phi_2\rangle \quad \forall\, \phi_2 \in H_0^2(\Omega)\,, \tag{3.12}$$

$$a(\zeta_1,\eta_1)-\lambda([\psi_2,\eta_2],\zeta_1) = \lambda(\psi_1-\psi_{10},\zeta_1) \quad \forall\, \zeta_1 \in H_0^2(\Omega)\,, \tag{3.13}$$

$$a(\zeta_2,\eta_2)+\lambda([\psi_2,\eta_1],\zeta_2)-\lambda([\psi_1,\eta_2],\zeta_2) = \lambda(\psi_2-\psi_{20},\zeta_2) \quad \forall\, \zeta_2 \in H_0^2(\Omega) \tag{3.14}$$

and

$$(t, g+\eta_2) = 0 \quad \forall\, t \in L^2(\Omega)\,. \tag{3.15}$$

4. Finite element approximations and error estimates

4.1. Definition of finite element approximations.

A finite element discretization of the optimality system (3.11)-(3.15) is defined in the usual manner. For simplicity, we will only study conforming finite element approximations in this paper. However, the error estimation techniques used in this paper are equally applicable to mixed finite element approximations based on the Hellan-Hermann-Johnson scheme for biharmonic equations.

See [4] (also [5]) for the definition and discussions of the mixed Hellan-Hermann-Johnson scheme for biharmonic equations.

We first choose families of finite dimensional subspaces $X^h \subset H_0^2(\Omega)$ and $G^h \subset L^2(\Omega)$ parameterized by a parameter h that tends to zero and satisfying the following approximation properties: there exists a constant C and an integer r such that

$$\inf_{\boldsymbol{\phi}^h \in X^h} \|\boldsymbol{\phi} - \boldsymbol{\phi}^h\|_2 \leq Ch^m \|\boldsymbol{\phi}\|_{m+2}\,, \quad \forall\, \boldsymbol{\phi} \in H^{m+2}(\Omega)\,,\ 1 \leq m \leq r \tag{4.1}$$

and

$$\inf_{z^h \in G^h} \|z - z^h\|_0 \leq Ch^m \|z\|_m\,, \quad \forall\, z \in H^m(\Omega)\,,\ 1 \leq m \leq r\,. \tag{4.2}$$

One may consult, e.g., [2] and [6] for some finite element spaces satisfying (4.1) and (4.2). For example, one may choose $X^h = V^h \times V^h$ where V^h is the piecewise quintic-$C^1(\overline{\Omega})$ finite element space constrained to satisfy the given boundary conditions and defined with respect to a family of triangulations of Ω. In this case, h is a measure of the grid size.

Once the approximating spaces have been chosen, we may formulate the approximate problem for the optimality system (3.11)-(3.15): seek $\boldsymbol{\psi}^h \in X^h$, $g^h \in G^h$, and $\boldsymbol{\eta}^h \in X^h$ such that

$$a(\psi_1^h, \phi_1^h) + \frac{\lambda}{2}\big([\psi_2^h, \psi_2^h], \phi_1^h\big) = 0 \quad \forall\, \phi_1^h \in V^h\,, \tag{4.3}$$

$$a(\psi_2^h, \phi_2^h) - \lambda\big([\psi_1^h, \psi_2^h], \phi_2^h\big) = \lambda(g^h, \phi_2^h) \quad \forall\, \phi_2^h \in V^h\,, \tag{4.4}$$

$$a(\zeta_1^h, \eta_1^h) - \lambda\big([\psi_2^h, \eta_2^h], \zeta_1^h\big) = \lambda(\psi_1^h - \psi_{10}, \zeta_1^h) \quad \forall\, \zeta_1^h \in V^h\,, \tag{4.5}$$

$$a(\zeta_2^h, \eta_2^h) + \lambda\big([\psi_2^h, \eta_1^h], \zeta_2^h\big) - \lambda\big([\psi_1^h, \eta_2^h], \zeta_2^h\big) = \lambda(\psi_2^h - \psi_{20}, \zeta_2^h) \quad \forall\, \zeta_2^h \in V^h \tag{4.6}$$

and

$$(z^h, g^h + \eta_2^h) = 0 \quad \forall\, z^h \in G^h\,. \tag{4.7}$$

Note that in the last equation, if $G^h = V^h$, then we have $g^h = -\eta_2^h$ so that the variable g^h can be eliminated to simplify the approximate problem. But in general we have to deal with the entire system (4.3)-(4.7).

4.2. Quotation of results concerning the approximation of a class of nonlinear problems.

The error estimate to be derived in Section 4.3 makes use of results of [4] and [8] concerning the approximation of a class of nonlinear problems. These results imply that, under certain hypotheses, the error of approximation of solutions of certain nonlinear problems is basically the same as the error of approximation of solutions of related linear problems. Here, for the sake of completeness, we will state the relevant results, specialized to our needs.

The nonlinear problems considered in [4] and [8] are of the following type. For given $\lambda \in \Lambda$, we seek $\psi \in \mathcal{X}$ such that

$$\mathcal{H}(\lambda, \psi) \equiv \psi + \mathcal{T}\mathcal{G}(\lambda, \psi) = 0\,, \tag{4.8}$$

where $\mathcal{T} \in \mathcal{L}(\mathcal{Y}; \mathcal{X})$, $\mathcal{G}$ is a C^2 mapping from $\Lambda \times \mathcal{X}$ into $\mathcal{Y}$, $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, and Λ is a compact interval of $\mathbb{R}$. We say that $\{(\lambda, \psi(\lambda)) : \lambda \in \Lambda\}$ is a *branch of solutions* of (4.8) if

$\lambda \to \psi(\lambda)$ is a continuous function from Λ into $\mathcal{X}$ such that $\mathcal{H}(\lambda, \psi(\lambda)) = 0$. The branch is called a *regular branch* if we also have that $\mathcal{H}_\psi(\lambda, \psi(\lambda))$ is an isomorphism from $\mathcal{X}$ into $\mathcal{X}$ for all $\lambda \in \Lambda$. Here, $\mathcal{H}_\psi(\cdot,\cdot)$ denotes the Fréchet derivative of $\mathcal{H}(\cdot,\cdot)$ with respect to the second argument. We assume that there exists another Banach space $\mathcal{Z}$, contained in $\mathcal{Y}$, with continuous imbedding, such that

$$\mathcal{G}_\psi(\lambda, \psi) \in \mathcal{L}(\mathcal{X}; \mathcal{Z}) \quad \forall\, \lambda \in \Lambda \text{ and } \psi \in \mathcal{X}\,, \tag{4.9}$$

where $\mathcal{G}_\psi(\cdot,\cdot)$ denotes the Fréchet derivative of $\mathcal{G}(\cdot,\cdot)$ with respect to the second argument.

Approximations are defined by introducing a subspace $\mathcal{X}^h \subset \mathcal{X}$ and an approximating operator $\mathcal{T}^h \in \mathcal{L}(\mathcal{Y}; \mathcal{X}^h)$. Then, given $\lambda \in \Lambda$, we seek $\psi^h \in \mathcal{X}^h$ such that

$$\mathcal{H}^h(\lambda, \psi^h) \equiv \psi^h + \mathcal{T}^h \mathcal{G}(\lambda, \psi^h) = 0\,. \tag{4.10}$$

Concerning the operator $\mathcal{T}^h$, we assume the approximation properties

$$\lim_{h\to 0} \|(\mathcal{T}^h - \mathcal{T})\omega\|_{\mathcal{X}} = 0 \quad \forall\, \omega \in \mathcal{Y} \tag{4.11}$$

and

$$\lim_{h\to 0} \|(\mathcal{T}^h - \mathcal{T})\|_{\mathcal{L}(\mathcal{Z};\mathcal{X})} = 0\,. \tag{4.12}$$

Note that whenever the imbedding $\mathcal{Z} \subset \mathcal{Y}$ is compact, (4.12) follows from (4.11) and, moreover, (4.9) implies that the operator $\mathcal{T}\mathcal{G}_\psi(\lambda, \psi) \in \mathcal{L}(\mathcal{X}; \mathcal{X})$ is compact.

We can now state the result of [4] or [8] that will be used in the sequel. In the statement of the theorem, $D^2\mathcal{G}$ represents any and all second Fréchet derivatives of $\mathcal{G}$.

THEOREM 4.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and Λ a compact subset of $\mathbb{R}$. Assume that $\mathcal{G}$ is a C^2 mapping from $\Lambda \times \mathcal{X}$ into $\mathcal{Y}$ and that $D^2\mathcal{G}$ is bounded on all bounded sets of $\Lambda \times \mathcal{X}$. Assume that* (4.9), (4.11), *and* (4.12) *hold and that $\{\psi(\lambda); \lambda \in \Lambda\}$ is a branch of regular solutions of* (4.8). *Then, there exists a neighborhood $\mathcal{O}$ of the origin in $\mathcal{X}$ and, for $h \le h_0$ small enough, a unique C^2 function $\lambda \to \psi^h(\lambda) \in \mathcal{X}^h$ such that $\{\psi^h(\lambda); \lambda \in \Lambda\}$ is a branch of regular solutions of* (4.10) *and $\psi^h(\lambda) - \psi(\lambda) \in \mathcal{O}$ for all $\lambda \in \Lambda$. Moreover, there exists a constant $C > 0$, independent of h and λ, such that*

$$\|\psi^h(\lambda) - \psi(\lambda)\|_{\mathcal{X}} \le C\|(\mathcal{T}^h - \mathcal{T})\mathcal{G}(\lambda, \psi(\lambda))\|_{\mathcal{X}} \quad \forall\, \lambda \in \Lambda\,. \quad \square \tag{4.13}$$

4.3. Error estimates.

In this section we will derive error estimates for the finite element approximations of solutions of the optimality system.

We begin by fitting the optimality system and its finite element approximations into the abstract framework in §4.2.

We set $\mathcal{X} = X \times G \times Y^*$, $\mathcal{Y} = Y \times X^*$, $\mathcal{Z} = Z \times Z$, and $\mathcal{X}^h = X^h \times G^h \times (Y^*)^h$. Then we have that $\mathcal{Z}$ is compactly imbedded into $\mathcal{Y}$. (We recall that the spaces $X = H_0^2(\Omega) = Y^*$, $Y = H^{-2}(\Omega) = X^*$, $Z = L^1(\Omega)$ and $G = L^2(\Omega)$ were all introduced in §3.) We define $\mathcal{T} \in \mathcal{L}(\mathcal{Y}; \mathcal{X})$ as follows: $\mathcal{T}(\tilde{r}, \tilde{\boldsymbol{r}}) = (\tilde{\boldsymbol{\psi}}, \tilde{g}, \tilde{\boldsymbol{\eta}})$ for $(\tilde{r}, \tilde{\boldsymbol{r}}) \in \mathcal{Y}$ and $(\tilde{\boldsymbol{\psi}}, \tilde{g}, \tilde{\boldsymbol{\eta}}) \in \mathcal{X}$ if and only if

$$a(\tilde{\boldsymbol{\psi}}, \boldsymbol{\phi}) = (\tilde{r}, \boldsymbol{\phi}) \quad \forall\, \boldsymbol{\phi} \in H_0^2(\Omega)\,, \tag{4.14}$$

$$a(\tilde{\boldsymbol{\eta}},\boldsymbol{\omega}) = (\tilde{\boldsymbol{\tau}},\boldsymbol{\omega}) \quad \forall\, \boldsymbol{\omega} \in H_0^2(\Omega) \tag{4.15}$$

and

$$(t, \tilde{g} + \tilde{\eta}_2) = 0 \quad \forall\, t \in L^2(\Omega)\,. \tag{4.16}$$

Similarly, the operator $\mathcal{T}^h \in \mathcal{L}(\mathcal{Y};\mathcal{X}^h)$ is defined as follows: $\mathcal{T}^h(\tilde{r},\tilde{\boldsymbol{\tau}}) = (\tilde{\boldsymbol{\psi}}^h, \tilde{g}^h, \tilde{\boldsymbol{\eta}}^h)$ for $(\tilde{r},\tilde{\boldsymbol{\tau}}) \in \mathcal{Y}$ and $(\tilde{\boldsymbol{\psi}}^h, \tilde{g}^h, \tilde{\boldsymbol{\eta}}^h) \in \mathcal{X}^h$ if and only if

$$a(\tilde{\boldsymbol{\psi}}^h, \boldsymbol{\phi}^h) = (\tilde{r}, \boldsymbol{\phi}^h) \quad \forall\, \boldsymbol{\phi}^h \in X^h\,, \tag{4.17}$$

$$a(\tilde{\boldsymbol{\eta}}^h, \boldsymbol{\omega}^h) = (\tilde{\boldsymbol{\tau}}, \boldsymbol{\omega}^h) \quad \forall\, \boldsymbol{\omega}^h \in X^h \tag{4.18}$$

and

$$(t^h, \tilde{g}^h + \tilde{\eta}_2^h) = 0 \quad \forall\, t^h \in G^h\,. \tag{4.19}$$

The nonlinear mapping $\mathcal{G} : \Lambda \times \mathcal{X} \to \mathcal{Y}$ is defined as follows: $\mathcal{G}\big(\lambda, (\tilde{\boldsymbol{\psi}}, \tilde{g}, \tilde{\boldsymbol{\eta}})\big) = (\tilde{r}, \tilde{\boldsymbol{\tau}})$ for $\lambda \in \Lambda$, $(\tilde{\boldsymbol{\psi}}, \tilde{g}, \tilde{\boldsymbol{\eta}}) \in \mathcal{X}$, and $(\tilde{r}, \tilde{\boldsymbol{\tau}}) \in \mathcal{Y}$ if and only if

$$\tilde{r} = \begin{pmatrix} -\frac{\lambda}{2}[\tilde{\psi}_2, \tilde{\psi}_2] \\ \lambda[\tilde{\psi}_1, \tilde{\psi}_2] + \lambda\tilde{g} \end{pmatrix}$$

and

$$\tilde{\boldsymbol{\tau}} = \begin{pmatrix} \lambda[\tilde{\psi}_2, \tilde{\eta}_2] + \lambda(\tilde{\psi}_1 - \psi_{10}) \\ -\lambda[\tilde{\psi}_2, \tilde{\eta}_1] + \lambda[\tilde{\psi}_1, \tilde{\eta}_2] + \lambda(\tilde{\psi}_2 - \psi_{20}) \end{pmatrix},$$

or equivalently,

$$\begin{aligned} &\Big\langle \mathcal{G}\big(\lambda, (\tilde{\boldsymbol{\psi}}, \tilde{g}, \tilde{\boldsymbol{\eta}})\big), (\boldsymbol{\phi}, \boldsymbol{\zeta}) \Big\rangle \\ &= -\frac{\lambda}{2}\big([\tilde{\psi}_2, \tilde{\psi}_2], \phi_1\big) + \lambda\big([\tilde{\psi}_1, \tilde{\psi}_2], \phi_2\big) + \lambda\langle \tilde{g}, \phi_2\rangle \\ &\quad + \lambda\big([\tilde{\psi}_2, \tilde{\eta}_2], \zeta_1\big) + \lambda(\tilde{\psi}_1 - \psi_{10}, \zeta_1) - \lambda\big([\tilde{\psi}_2, \tilde{\eta}_1], \zeta_2\big) \\ &\quad + \lambda\big([\tilde{\psi}_1, \tilde{\eta}_2], \zeta_2\big) + \lambda(\tilde{\psi}_2 - \psi_{20}, \zeta_2) \quad \forall\, (\boldsymbol{\phi}, \boldsymbol{\zeta}) \in H_0^2(\Omega) \times H_0^2(\Omega)\,. \end{aligned}$$

Using the operators N and K introduced in §3, we can also write the defining equations for $\mathcal{G}\big(\lambda, (\tilde{\boldsymbol{\psi}}, \tilde{g}, \tilde{\boldsymbol{\eta}})\big)$ as follows:

$$\tilde{r} = \lambda\, N(\tilde{\boldsymbol{\psi}}) + \lambda\, K\tilde{g} \tag{4.20}$$

and

$$\tilde{\boldsymbol{\tau}} = \lambda\, [N'(\tilde{\boldsymbol{\psi}})]^* \tilde{\boldsymbol{\eta}} - \lambda\, (\tilde{\boldsymbol{\psi}} - \boldsymbol{\psi}_0)\,. \tag{4.21}$$

It is easy to see that the optimality system (3.11)-(3.15) and its finite dimensional counterpart (4.3)-(4.7) can be written as

$$(\boldsymbol{\psi}, g, \boldsymbol{\eta}) + \mathcal{T}\mathcal{G}\big(\lambda, (\boldsymbol{\psi}, g, \boldsymbol{\eta})\big) = 0$$

and

$$(\boldsymbol{\psi}^h, g^h, \boldsymbol{\eta}^h) + \mathcal{T}^h\mathcal{G}\big(\lambda, (\boldsymbol{\psi}^h, g^h, \boldsymbol{\eta}^h)\big) = 0\,,$$

respectively, i.e., with $\psi = (\boldsymbol{\psi}, g, \boldsymbol{\eta})$ and $\psi^h = (\boldsymbol{\psi}^h, g^h, \boldsymbol{\eta}^h)$, in the form of (4.8) and (4.10), respectively.

We now turn to the verification of the assumptions in Theorem 4.1. We first examine the approximation results for the linear operators $\mathcal{T}$ and $\mathcal{T}^h$.

LEMMA 4.2. *For every* $(\tilde{r}, \tilde{\boldsymbol{r}}) \in \mathcal{Y}$,

$$\lim_{h\to 0} \|(\mathcal{T} - \mathcal{T}^h)(\tilde{r}, \tilde{\boldsymbol{r}})\|_{\mathcal{X}} = 0\,.$$

Furthermore, if $\mathcal{T}(\tilde{r}, \tilde{\boldsymbol{r}}) = (\tilde{\boldsymbol{\psi}}, \tilde{g}, \tilde{\boldsymbol{\eta}}) \in H^{m+2}(\Omega) \times H^m(\Omega) \times H^{m+2}(\Omega)$, *then*

$$\|(\mathcal{T} - \mathcal{T}^h)(\tilde{r}, \tilde{\boldsymbol{r}})\|_{\mathcal{X}} \le Ch^m(\|\tilde{\boldsymbol{\psi}}\|_{m+2} + \|\tilde{g}\|_m + \|\tilde{\boldsymbol{\eta}}\|_{m+2})\,.$$

Proof. For each given $(\tilde{r}, \tilde{\boldsymbol{r}}) \in \mathcal{Y}$, let $(\tilde{\boldsymbol{\psi}}, \tilde{g}, \tilde{\boldsymbol{\eta}}) = \mathcal{T}(\tilde{r}, \tilde{\boldsymbol{r}})$ and $(\tilde{\boldsymbol{\psi}}^h, \tilde{g}^h, \tilde{\boldsymbol{\eta}}^h) = \mathcal{T}^h(\tilde{r}, \tilde{\boldsymbol{r}})$; i.e., $(\tilde{\boldsymbol{\psi}}, \tilde{g}, \tilde{\boldsymbol{\eta}}) \in \mathcal{X}$ and $(\tilde{\boldsymbol{\psi}}^h, \tilde{g}^h, \tilde{\boldsymbol{\eta}}^h) \in \mathcal{X}^h$ satisfy (4.14)-(4.16) and (4.17)-(4.19), respectively. From the well-known results concerning the approximation of biharmonic equations (see, e.g., [2] and [6]), we obtain that

$$\lim_{h\to 0} \|\tilde{\boldsymbol{\psi}} - \tilde{\boldsymbol{\psi}}^h\|_2 = 0$$

and

$$\lim_{h\to 0} \|\tilde{\boldsymbol{\eta}} - \tilde{\boldsymbol{\eta}}^h\|_2 = 0\,. \tag{4.22}$$

Furthermore, if $\tilde{\boldsymbol{\psi}} \in H^{m+2}(\Omega)$ and $\tilde{\boldsymbol{\eta}} \in H^{m+2}(\Omega)$, then

$$\|\tilde{\boldsymbol{\psi}} - \tilde{\boldsymbol{\psi}}^h\|_2 \le Ch^m\|\tilde{\boldsymbol{\psi}}\|_{m+2}$$

and

$$\|\tilde{\boldsymbol{\eta}} - \tilde{\boldsymbol{\eta}}^h\|_2 \le Ch^m\|\tilde{\boldsymbol{\eta}}\|_{m+2}\,.$$

(4.16) and (4.19) yields $(g - g^h, z^h) = -(\tilde{\eta}_2 - \tilde{\eta}_2^h, z^h)$ for all $z^h \in G^h$. We then have that

$$\begin{aligned}
\|\tilde{g} - \tilde{g}^h\|_0^2 &= (\tilde{g} - \tilde{g}^h, \tilde{g} - \tilde{g}^h) \\
&= (\tilde{g} - \tilde{g}^h, \tilde{g} - z^h) + (\tilde{g} - \tilde{g}^h, z^h - \tilde{g}^h) \\
&= (\tilde{g} - \tilde{g}^h, \tilde{g} - z^h) - (\tilde{\eta}_2 - \tilde{\eta}_2^h, z^h - \tilde{g}^h) \\
&\le \|\tilde{g} - z^h\|_0^2 + \frac{1}{4}\|\tilde{g} - \tilde{g}^h\|_0^2 + \frac{1}{2}\|\tilde{\eta}_2 - \tilde{\eta}_2^h\|_0^2 \\
&\qquad + \frac{1}{2}\|z^h - \tilde{g}\|_0^2 + \|\tilde{\eta}_2 - \tilde{\eta}_2^h\|_0^2 + \frac{1}{4}\|\tilde{g} - \tilde{g}^h\|_0^2
\end{aligned}$$

so that

$$\|\tilde{g} - \tilde{g}^h\|_0 \le C\left(\|\tilde{g} - z^h\|_0 + \|\tilde{\eta}_2 - \tilde{\eta}_2^h\|_0\right).$$

It follows from (4.2) and (4.22) that $lim_{h\to 0}\|\tilde{g} - \tilde{g}^h\|_0 = 0$. Hence,

$$\lim_{h\to 0} \|(\mathcal{T} - \mathcal{T}^h)(\tilde{r}, \tilde{\boldsymbol{r}})\|_{\mathcal{X}} = \lim_{h\to 0}\left(\|\tilde{\boldsymbol{\psi}} - \tilde{\boldsymbol{\psi}}^h\|_2 + \|\tilde{g} - \tilde{g}^h\|_0 + \|\tilde{\boldsymbol{\eta}} - \tilde{\boldsymbol{\eta}}^h\|_2\right) = 0\,.$$

Furthermore, if $(\tilde{\boldsymbol{\psi}}, \tilde{g}, \tilde{\boldsymbol{\eta}}) \in H^{m+2}(\Omega) \times H^m(\Omega) \times H^{m+2}(\Omega)$, then

$$\|\tilde{g} - \tilde{g}^h\|_0 \le Ch^m(\|\tilde{g}\|_m + \|\tilde{\boldsymbol{\eta}}\|_{m+2})$$

so that

$$\|(\mathcal{T}-\mathcal{T}^h)(\tilde{r},\tilde{\tau})\|_{\mathcal{X}} = \|\tilde{\boldsymbol{\psi}}-\tilde{\boldsymbol{\psi}}^h\|_2 + \|\tilde{g}-\tilde{g}^h\|_0 + \|\tilde{\boldsymbol{\eta}}-\tilde{\boldsymbol{\eta}}^h\|_2$$
$$\leq Ch^m(\|\tilde{\boldsymbol{\psi}}\|_{m+2} + \|\tilde{g}\|_m + \|\tilde{\boldsymbol{\eta}}\|_{m+2}). \quad \square$$

Next, we examine the derivatives of the mapping $\mathcal{G}$.

LEMMA 4.3. *The mapping $\mathcal{G}$ is twice continuously Fréchet differentiable; the second order derivative is bounded on all bounded subsets of $\Lambda \times \mathcal{X}$; and the first order derivative maps $\mathcal{X}$ into $\mathcal{Z}$ continuously, i.e., $\mathcal{G}_{(\boldsymbol{\psi},g,\boldsymbol{\eta})}(\lambda,(\boldsymbol{\psi},g,\boldsymbol{\eta})) \in \mathcal{L}(\mathcal{X};\mathcal{Z})$.*

Proof. Using imbedding theorems, Hölder inequalities and the fact that $\mathcal{G}$ consists of polynomial maps, we deduce that the mapping $\mathcal{G}$ is twice continuously Fréchet differentiable and the second order derivative is bounded on all bounded subsets of $\Lambda \times \mathcal{X}$.

A simple calculation shows that $\mathcal{G}_{(\boldsymbol{\psi},g,\boldsymbol{\eta})}(\lambda,(\boldsymbol{\psi},g,\boldsymbol{\eta})) \in \mathcal{L}(\mathcal{X};\mathcal{Y})$ is given by

$$\mathcal{G}_{(\boldsymbol{\psi},g,\boldsymbol{\eta})}(\lambda,(\boldsymbol{\psi},g,\boldsymbol{\eta})) \cdot (\tilde{\boldsymbol{\psi}},\tilde{g},\tilde{\boldsymbol{\eta}}) = \lambda \begin{pmatrix} N'(\boldsymbol{\psi})\cdot\tilde{\boldsymbol{\psi}} + K\tilde{g} \\ [N''(\boldsymbol{\psi})\cdot\tilde{\boldsymbol{\psi}}]^*\cdot\boldsymbol{\eta} + [N'(\boldsymbol{\psi})]^*\cdot\tilde{\boldsymbol{\eta}} - \tilde{\boldsymbol{\psi}} \end{pmatrix}$$

where the operators N and K along with their derivatives were introduced in §3. Sobolev imbedding theorems imply that

$$N'(\boldsymbol{\psi})\cdot\tilde{\boldsymbol{\psi}} = \begin{pmatrix} [\psi_2,\tilde{\psi}_2] \\ -[\psi_1,\tilde{\psi}_2] - [\psi_2,\tilde{\psi}_1] \end{pmatrix} \in L^1(\Omega),$$

$$[N'(\boldsymbol{\psi})]^*\cdot\tilde{\boldsymbol{\eta}} = \begin{pmatrix} -[\psi_2,\tilde{\eta}_2] \\ [\psi_2,\tilde{\eta}_1] - [\psi_1,\tilde{\eta}_2] \end{pmatrix} \in L^1(\Omega)$$

and

$$([N''(\boldsymbol{\psi})]^*\cdot\tilde{\boldsymbol{\psi}})\cdot\boldsymbol{\eta} = \begin{pmatrix} -[\tilde{\psi}_2,\eta_2] \\ [\tilde{\psi}_2,\eta_1] - [\tilde{\psi}_1,\eta_2] \end{pmatrix} \in L^1(\Omega).$$

Of course, $\tilde{\boldsymbol{\psi}} \in L^1(\Omega)$. From the definition of the operator K we see that K maps $L^2(\Omega)$ into $\{0\} \times L^2(\Omega) \subset L^1(\Omega)$. Hence the operator $\mathcal{G}_{(\boldsymbol{\psi},g,\boldsymbol{\eta})}(\lambda,(\boldsymbol{\psi},g,\boldsymbol{\eta}))$ maps $\mathcal{X}$ into $\mathcal{Z}$. Moreover, using Holder inequalities and Sobolev imbedding theorems it is easy to see that

$$\|\mathcal{G}_{(\boldsymbol{\psi},g,\boldsymbol{\eta})}(\lambda,(\boldsymbol{\psi},g,\boldsymbol{\eta}))\cdot(\tilde{\boldsymbol{\psi}},\tilde{g},\tilde{\boldsymbol{\eta}})\|_{L^1(\Omega)}$$
$$\leq C\left(\|\boldsymbol{\psi}\|_2 + \|\boldsymbol{\eta}\|_2 + 1\right)\left(\|\tilde{\boldsymbol{\psi}}\|_2 + \|\tilde{g}\|_0 + \|\tilde{\boldsymbol{\eta}}\|_2\right)$$

so that

$$\mathcal{G}_{(\boldsymbol{\psi},g,\boldsymbol{\eta})}(\lambda,(\boldsymbol{\psi},g,\boldsymbol{\eta})) \in \mathcal{L}(\mathcal{X};\mathcal{Z}). \quad \square$$

Next, we recall the notion of regular solutions. A solution $(\boldsymbol{\psi}(\lambda),g(\lambda),\boldsymbol{\eta}(\lambda))$ of the optimality system (3.11)-(3.15) is called *regular* if

$$\Big(I + \mathcal{T}\mathcal{G}(\boldsymbol{\psi},g,\boldsymbol{\eta})(\lambda,(\boldsymbol{\psi},g,\boldsymbol{\eta}))\Big)\cdot(\tilde{\boldsymbol{\psi}},\tilde{g},\tilde{\boldsymbol{\eta}}) = (\tilde{\boldsymbol{\xi}},\tilde{h},\tilde{\boldsymbol{\zeta}})$$

has a unique solution $(\tilde{\boldsymbol{\psi}},\tilde{g},\tilde{\boldsymbol{\eta}}) \in \mathcal{X}$ for any $(\tilde{\boldsymbol{\xi}},\tilde{h},\tilde{\boldsymbol{\zeta}}) \in \mathcal{X}$. As $\mathcal{T}$ is invertible, $(\tilde{\boldsymbol{\xi}},\tilde{h},\tilde{\boldsymbol{\zeta}}) = \mathcal{T}(\tilde{r},\tilde{\boldsymbol{\tau}})$ with $(\tilde{r},\tilde{\boldsymbol{\tau}}) \in \mathcal{Y} = H^{-2}(\Omega) \times H^{-2}(\Omega)$. Thus, $(\boldsymbol{\psi}(\lambda),g(\lambda),\boldsymbol{\eta}(\lambda))$ is a regular solution of the optimality system (3.11)-(3.15) if

$$\Big(I + \mathcal{T}\mathcal{G}(\boldsymbol{\psi},g,\boldsymbol{\eta})(\lambda,(\boldsymbol{\psi},g,\boldsymbol{\eta}))\Big)\cdot(\tilde{\boldsymbol{\psi}},\tilde{g},\tilde{\boldsymbol{\eta}}) = \mathcal{T}(\tilde{r},\tilde{\boldsymbol{\tau}})$$

has a unique solution $(\tilde{\boldsymbol{\psi}}, \tilde{g}, \tilde{\boldsymbol{\eta}}) \in \mathcal{X}$ for any $(\tilde{r}, \tilde{\boldsymbol{r}}) \in \mathcal{Y}$, or equivalently, the following system (for the unknowns $(\tilde{\boldsymbol{\psi}}, \tilde{g}, \tilde{\boldsymbol{\eta}})$) is uniquely solvable for any $(\tilde{r}, \tilde{\boldsymbol{r}}) \in \mathcal{Y}$:

$$a(\tilde{\psi}_1, \phi_1) + \big([\psi_2, \tilde{\psi}_2], \phi_1\big) = \langle \tilde{r}_1, \phi_1 \rangle \quad \forall\, \phi_1 \in H_0^2(\Omega)\,, \tag{4.23}$$

$$\begin{aligned} a(\tilde{\psi}_2, \phi_2) - \lambda\big([\psi_1, \tilde{\psi}_2], \phi_2\big) - \lambda\big([\tilde{\psi}_1, \psi_2], \phi_2\big) - \lambda\langle \tilde{g}, \phi_2 \rangle \\ = \langle \tilde{r}_1, \phi_1 \rangle \quad \forall\, \phi_2 \in H_0^2(\Omega)\,, \end{aligned} \tag{4.24}$$

$$\begin{aligned} a(\zeta_1, \tilde{\eta}_1) - \lambda\big([\psi_2, \tilde{\eta}_2], \zeta_1\big) - \lambda\big([\tilde{\psi}_2, \eta_2], \zeta_1\big) - \lambda(\tilde{\psi}_1, \zeta_1) \\ = \langle \tilde{r}_1, \zeta_1 \rangle \quad \forall\, \zeta_1 \in H_0^2(\Omega)\,, \end{aligned} \tag{4.25}$$

$$\begin{aligned} a(\zeta_2, \tilde{\eta}_2) + \lambda\big([\psi_2, \tilde{\eta}_1], \zeta_2\big) + \lambda\big([\tilde{\psi}_2, \eta_1], \zeta_2\big) - \lambda\big([\psi_1, \tilde{\eta}_2], \zeta_2\big) \\ - \lambda\big([\tilde{\psi}_1, \eta_2], \zeta_2\big) - \lambda(\tilde{\psi}_2, \zeta_2) = \langle \tilde{r}_2, \zeta_2 \rangle \quad \forall\, \zeta_2 \in H_0^2(\Omega) \end{aligned} \tag{4.26}$$

and

$$(t, \tilde{g} + \tilde{\eta}_2) = 0 \quad \forall\, t \in L^2(\Omega)\,. \tag{4.27}$$

Note that the linear operator appearing on the left hand side of (4.23)-(4.27) is obtained by linearizing the optimality system (3.11)-(3.15) about $(\boldsymbol{\psi}, g, \boldsymbol{\eta})$.

LEMMA 4.4. *For almost all λ, solutions $\big(\boldsymbol{\psi}(\lambda), g(\lambda), \boldsymbol{\eta}(\lambda)\big)$ of the optimality system* (3.11)-(3.15) *are regular.*

Proof. The system (4.23)-(4.27) can be rewritten as

$$\Big(I + \lambda\, \mathcal{T}\mathcal{S}(\boldsymbol{\psi}, g, \boldsymbol{\eta})\Big)(\tilde{\boldsymbol{\psi}}, \tilde{g}, \tilde{\boldsymbol{\eta}}) = \mathcal{T}(\tilde{r}, \tilde{\boldsymbol{r}})\,, \tag{4.28}$$

where the linear operator $\mathcal{S}(\boldsymbol{\psi}, g, \boldsymbol{\eta}) : \mathcal{X} \to \mathcal{Y}$ is defined by

$$\begin{aligned} \mathcal{S}(\boldsymbol{\psi}, g, \boldsymbol{\eta}) \cdot (\tilde{\boldsymbol{\psi}}, \tilde{g}, \tilde{\boldsymbol{\eta}}) &\equiv \frac{1}{\lambda}\, \mathcal{G}_{(\boldsymbol{\psi}, g, \boldsymbol{\eta})}\big(\lambda, (\boldsymbol{\psi}, g, \boldsymbol{\eta})\big) \cdot (\tilde{\boldsymbol{\psi}}, \tilde{g}, \tilde{\boldsymbol{\eta}}) \\ &= \begin{pmatrix} N'(\boldsymbol{\psi}) \cdot \tilde{\boldsymbol{\psi}} + K\tilde{g} \\ [N''(\boldsymbol{\psi}) \cdot \tilde{\boldsymbol{\psi}}]^* \cdot \boldsymbol{\eta} + [N'(\boldsymbol{\psi})]^* \cdot \tilde{\boldsymbol{\eta}} - \tilde{\boldsymbol{\psi}} \end{pmatrix}. \end{aligned}$$

It was established in Lemma 4.3 that the mapping $\mathcal{G}_{(\boldsymbol{\psi}, g, \boldsymbol{\eta})}\big(\lambda, (\boldsymbol{\psi}, g, \boldsymbol{\eta})\big)$ is compact from $\mathcal{X}$ into $\mathcal{Y}$. Now, $\mathcal{T} \in \mathcal{L}(\mathcal{Y}; \mathcal{X})$, so that $\big(I + \lambda\, \mathcal{T}\mathcal{S}(\boldsymbol{\psi}, g, \boldsymbol{\eta})\big)$ is a compact perturbation of the identity operator from $\mathcal{X}$ to $\mathcal{X}$. Thus, for almost all λ, (4.28), or equivalently (4.23)-(4.27), is uniquely solvable, i.e., for almost all λ, the solution $\big(\boldsymbol{\psi}(\lambda), g(\lambda), \boldsymbol{\eta}(\lambda)\big)$ of the optimality system (3.11)-(3.15) is regular. □

We are now in a position to derive error estimates. In the following theorem, we will assume that the solution $\big(\boldsymbol{\psi}(\lambda), g(\lambda), \boldsymbol{\eta}(\lambda)\big)$ of the optimality system (3.11)-(3.15) is regular. Lemma 4.4 guarantees that this is indeed the case for almost all λ. Lemmas 4.2 and 4.3 verified all the assumptions in Theorem 4.1. Thus we are led to the following error estimates.

THEOREM 4.5. *Assume that Λ is a compact interval of $\mathbb{R}_+$ and that there exists a branch $\{\big(\boldsymbol{\psi}(\lambda), g(\lambda), \boldsymbol{\eta}(\lambda)\big) : \lambda \in \Lambda\}$ of regular solutions of the optimality system* (3.11)-(3.15). *Assume that the finite element spaces X^h and G^h satisfy the hypotheses* (4.1)-(4.2). *Then, there exists a*

$\delta > 0$ and an $h_0 > 0$ such that for $h \leq h_0$, the discrete optimality system (4.3)-(4.7) *has a unique branch of regular solutions* $\{(\boldsymbol{\psi}^h(\lambda), g^h(\lambda), \boldsymbol{\eta}^h(\lambda)) : \lambda \in \Lambda\}$ *satisfying*

$$\{\|\boldsymbol{\psi}^h(\lambda) - \boldsymbol{\psi}(\lambda)\|_2 + \|g^h(\lambda) - g(\lambda)\|_0 + \|\boldsymbol{\eta}^h(\lambda) - \boldsymbol{\eta}(\lambda)\|_2\} < \delta \quad \textit{for all } \lambda \in \Lambda.$$

Moreover,

$$\lim_{h \to 0} \{\|\boldsymbol{\psi}^h(\lambda) - \boldsymbol{\psi}(\lambda)\|_2 + \|g^h(\lambda) - g(\lambda)\|_0 + \|\boldsymbol{\eta}^h(\lambda) - \boldsymbol{\eta}(\lambda)\|_2\} = 0\,,$$

uniformly in $\lambda \in \Lambda$.

If, in addition, the solution of the optimality system satisfies $(\boldsymbol{\psi}(\lambda), g(\lambda), \boldsymbol{\eta}(\lambda)) \in H^{m+2}(\Omega) \times H^m(\Omega) \times H^{m+2}(\Omega)$ for $\lambda \in \Lambda$, then there exists a constant C, independent of h, such that

$$\begin{aligned} &\|\boldsymbol{\psi}(\lambda) - \boldsymbol{\psi}^h(\lambda)\|_2 + \|g(\lambda) - g^h(\lambda)\|_0 + \|\boldsymbol{\eta}(\lambda) - \boldsymbol{\eta}^h(\lambda)\|_2 \\ &\quad \leq Ch^m \left(\|\boldsymbol{\psi}(\lambda)\|_{m+2} + \|g(\lambda)\|_m + \|\boldsymbol{\eta}(\lambda)\|_{m+2}\right), \end{aligned}$$

uniformly in $\lambda \in \Lambda$. □

5. Conclusions

We studied an optimal control problem for the von Karman equations in this paper. We first gave the mathematical statement of the problem and proved the existence of an optimal solution. We then applied the Lagrange multiplier rules to derive an optimality system of equations that the optimal state must satisfy (the use of Lagrange multiplier rules was justifed). We finally defined finite element approximations for the optimality system and derived optimal error estimates. The functional we minimized was a tracking functional (i.e., the tracking of the Airy stress function and the normal deflection). The control we used was a distributed control (i.e., the external load). However, the methods used in this paper apply equally well to optimal control problems with other objectives (e.g., minimizing the stress functional in some areas) and/or other types of controls (e.g, boundary controls).

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