

# THE QUANTUM MEASUREMENT OF TIME

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## Abstract

Traditionally, in non-relativistic Quantum Mechanics, time is considered to be a *parameter*, rather than an *observable* quantity like space. In relativistic Quantum Field Theory, space and time are treated equally by reducing space to also be a parameter. Herein, after a brief review of other measurements, we describe a third possibility, which is to treat time as a directly observable quantity.

## 1 The Measurements of Space and Momentum

Here we postulate the existence of position eigenstates, such that their corresponding eigenkets (defined by  $\hat{x}|x\rangle = x|x\rangle$ , where  $\hat{x}$  is the position operator) resolve the identity operator

$$\hat{I}_x = \int_{-\infty}^{+\infty} dx |x\rangle\langle x| \quad (1)$$

for the Hilbert space that they span,  $\mathcal{H}_x \leftrightarrow \{|x\rangle : x \in [-\infty, +\infty]\}$ . Thus, any state  $|\psi\rangle \in \mathcal{H}_x$  can be expressed as a weighted sum of position eigenstates:

$$|\psi\rangle = \int_{-\infty}^{+\infty} dx |x\rangle \psi(x) \quad (2)$$

where  $\psi(x) \equiv \langle x|\psi\rangle$  is Schrodinger's time independent wavefunction, the magnitude-square of which gives the probability of obtaining the value  $x$  in what we call the *measurement of  $\hat{x}$* .

An infinitesimal translation in physical space, denoted by  $\hat{T}(\delta x)$ , is defined by its mapping of the position eigenstates:

$$\hat{T}(\delta x)|x\rangle = |x + \delta x\rangle, \quad (3)$$

and linear canonical momentum, denoted as  $\hat{p}$ , is defined to be the generator of these translations:

$$\lim_{\delta x \rightarrow 0} \hat{T}(\delta x) = \hat{I}_x - i\hat{p}\delta x / \hbar. \quad (4)$$

It is important to note that in writing equation (3) we are assuming that space is unbounded, i.e.  $x \in [-\infty, +\infty]$ . From equations (3) and (4) we can easily prove [1] that there is a Fourier transform relation between  $\psi(x)$  and  $\varphi(p) \equiv \langle p|\psi\rangle$ , where  $\hat{p}|p\rangle = p|p\rangle$ . Rayleigh's theorem [2] then guarantees that  $|\varphi(p)|^2$  is a normalizable distribution and

hence (since it must also be positive semi-definite) it corresponds to a probability distribution function (pdf) that describes the “measurement of  $\hat{p}$ .” This can also be seen by virtue of the fact that a normalized  $|\varphi(p)|^2$  implies that the momentum eigenkets must also resolve the identity operator:

$$\int_{-\infty}^{+\infty} dp |p\rangle\langle p| = \hat{I}_x. \quad (5)$$

In the case of the measurements of space and momentum our descriptions are complete in the sense that both operators are Hermitian thereby ensuring that each has an orthogonal set of eigenkets so that the corresponding wavefunctions can collapse.

## 2 “Time-like” Measurements

Time evolution, by an infinitesimal amount,  $\delta t$ , denoted as  $\hat{U}(\delta t)$ , is defined by

$$\hat{U}(\delta t)|\psi, t\rangle = |\psi, t + \delta t\rangle, \quad (6)$$

where  $|\psi, t\rangle$  is the ket  $|\psi\rangle$  when the time *parameter* takes on the value  $t$ . This is very different than the interpretation of equation (3), i.e. we are NOT postulating that  $|\psi, t\rangle$  is an eigenket of a time operator. Thus, Schrodinger’s time-dependent wavefunction,  $\psi(x, t) \equiv \langle x|\psi, t\rangle$ , still describes the measurement of *space* (not time) as is clear from the fact that it is still the completeness of the position eigenkets that allows us to interpret  $|\psi(x, t)|^2$  as a pdf for  $x$  so that at each instant in time we have

$$\int_{-\infty}^{+\infty} dx |\psi(x, t)|^2 = 1. \quad (7)$$

We can obtain “time-like” information by performing this measurement of space (on identically prepared systems) at different values of the time parameter, but we are only *inferring* time information from the spatial measurement.

Similarly, we can perform measurements of other quantities, such as a spin component (e.g.  $\hat{S}_z$ ) at different times, but here again we would be inferring rather than directly measuring the temporal information. For example, consider a particle of spin  $s$ . We have an identity operator

$$\hat{I}_s = \sum_{m=-s}^s |m\rangle\langle m| \quad (8)$$

for the spin part of our state space,  $\mathcal{H}_s \leftrightarrow \{|m\rangle : m = -s, -s+1, \dots, s-1, s\}$ , where  $\hat{S}_z|m\rangle = m\hbar|m\rangle$ . Thus the measurement of  $\hat{S}_z$  (performed at time  $t$ ) is still described by a

probability mass function for  $m$ , so that for every instant in time we have  $\sum_{m=-s}^s P_t(m) = 1$ , where  $P_t(m) \equiv \langle m | \psi, t \rangle^2$ .

### 3 The Direct Measurement of Time

We can parallel the discussion of the spatial measurement given in section 1 under the substitutions of  $t$  for  $x$ , the Hamiltonian  $\hat{H}$  for  $\hat{p}$ , and  $\hat{U}(\delta t)$  for  $\hat{T}(\delta x)$ , thereby obtaining a *temporal wavefunction* [3],  $\psi(t)$ , that is complementary to the energy representation:  $\psi(t) \overset{\mathcal{F}}{\leftrightarrow} \psi(E)$ , i.e.

$$\psi(t) = \int_0^{\infty} dE \psi(E) e^{-iEt/\hbar} \quad (9)$$

where  $\psi(E) \equiv \langle E | \psi \rangle$  and  $\hat{H}|E\rangle = E|E\rangle$ . As it stands however, this rather obvious approach does not give a complete description of the measurement of time [4] due to the existence of a lower bound on the energy eigenspectra (i.e. a “ground state”).

For the sake of definiteness, consider a single harmonic oscillator of (“rigged”) Hilbert space  $\mathcal{H}_1 \leftrightarrow \{|n_1\rangle : n_1 = 0, 1, 2, \dots, \infty\}$ , where  $\hat{n}_1|n_1\rangle = n_1|n_1\rangle$ ,  $\hat{H}_1 = \hbar\omega \left( \hat{n}_1 + \frac{1}{2} \hat{I}_1 \right)$ , and  $\hat{I}_1 = \sum_{n_1=0}^{\infty} |n_1\rangle\langle n_1|$ . The temporal wavefunction in this case is

$$\psi(t) = e^{-i\phi/2} \sum_{n_1=0}^{\infty} \psi_{n_1} e^{-in_1\phi} = e^{-i\phi/2} \psi(\phi), \quad (10)$$

where  $\phi \equiv \omega t$ ,  $\psi_{n_1} \equiv \langle n_1 | \psi \rangle$ , and the gauge-induced topological phase,  $e^{-i\phi/2}$ , is not observable without performing interference with another system since  $|\psi(t)|^2 = |\psi(\phi)|^2$ . Clearly the lower bound on photon number ( $n_1 \geq 0$ ) prevents  $\psi(t)$  (or  $\psi(\phi)$ ) from collapsing to a delta-function. This implies that the underlying phase kets  $\{|\phi\rangle\}$  are not orthogonal,  $\langle \phi_1 | \phi_2 \rangle \neq \delta(\phi_1 - \phi_2)$  (where  $\psi(\phi) = \langle \phi | \psi \rangle$ ) and yet

$$\int_{-\pi}^{\pi} \frac{d\phi}{2\pi} |\phi\rangle\langle\phi| = \sum_{n_1=0}^{\infty} |n_1\rangle\langle n_1| = \hat{I}_1 \quad (11)$$

so that  $\frac{1}{2\pi} |\psi(\phi)|^2$  is a perfectly valid pdf (as can also be seen from Parseval’s theorem [2]). This pdf must somehow correspond to a realizable quantum measurement (as can also be seen from the formalism of *probability operator measures* or POMs [5]) and yet our description of the measurement is “incomplete” in the sense that we do not have wavefunction collapse (the  $\{|\phi\rangle\}$  is not an orthogonal set) and likewise, the non-Hermitian operator associated with this measurement does not commute with its adjoint.

## 4 The Complete Description of The Direct Measurement of Time

In order to get a complete description of the direct measurement of time we *must* deal with a Hilbert space that is larger than  $\mathcal{H}_1$ . For example, we can use the product space for *two* oscillators,  $\mathcal{H}_{1\otimes 2} \equiv \mathcal{H}_1 \otimes \mathcal{H}_2$ . If we are willing to restrict our states to a subset of  $\mathcal{H}_{1\otimes 2}$  where each value of the energy difference ( $m \equiv n_1 - n_2$ ) occurs with a *unique* value of the energy sum ( $j \equiv n_1 + n_2$ ) then there are an infinite number of Hermitian time operators since there is an infinite number of such subsets (one such example, where we restrict to  $\mathcal{H}' \subset \{\mathcal{H}_{1\otimes 2} : n_1 n_2 = 0\}$ , is discussed in [6], [7], and [8]). If we do **not** wish to restrict our states, then there are two physically reasonable alternatives. We perform a *relative time measurement* that treats the different  $j$  states as either: (1) distinguishable; or (2) indistinguishable. These two procedures *also* correspond to performing: (1) a ‘‘marginal measurement’’ in which we average over all values of absolute time; and (2) a ‘‘conditional measurement’’ in which we determine the relative time distribution at an instant in absolute time, as we now demonstrate.

Complementarity suggests that for an arbitrary two-mode excitation, with number representation  $\psi_{n_1, n_2} \equiv {}_1\langle n_1 | {}_2\langle n_2 | \psi \rangle_{1\otimes 2}$ , we take a two-dimensional Fourier transform

$$\Psi(\phi_1, \phi_2) \equiv {}_1\langle \phi_1 | {}_2\langle \phi_2 | \psi \rangle_{1\otimes 2} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \psi_{n_1, n_2} e^{-in_1\phi_1} e^{-in_2\phi_2}. \quad (12)$$

Rewriting this in terms of  $\phi_{\Sigma} \equiv (\phi_1 + \phi_2)/2$  and  $\phi_{\Delta} \equiv (\phi_1 - \phi_2)/2$ , we have

$$\psi(\phi_{\Delta}, \phi_{\Sigma}) \equiv \langle \phi_{\Delta}, \phi_{\Sigma} | \psi \rangle_{1\otimes 2} = \sum_{j=0}^{\infty} \sum_{m=-j}^{+j} \psi_{j, m} e^{-im\phi_{\Delta}} e^{-ij\phi_{\Sigma}} \quad (13)$$

where

$$|\phi_{\Delta}, \phi_{\Sigma}\rangle \equiv \sum_{j=0}^{\infty} \sum_{m=-j}^{+j} |j, m\rangle e^{im\phi_{\Delta}} e^{ij\phi_{\Sigma}} \quad \text{and} \quad |j, m\rangle \equiv |n_1\rangle_1 |n_2\rangle_2 \Big|_{j=n_1+n_2, m=n_1-n_2}. \quad (14)$$

We see that the  $\phi_{\Sigma}$  part of  $\psi(\phi_{\Delta}, \phi_{\Sigma})$  cannot collapse due to the boundedness of its complement, the energy sum ( $j \geq 0$ ).

We can eliminate  $\phi_{\Sigma}$  to obtain a complete description of the measurement of  $\phi_{\Delta}$  on  $\mathcal{H}_{1\otimes 2}$  by applying an ‘‘absolute time average’’ to  $|\phi_{\Delta}, \phi_{\Sigma}\rangle \langle \phi_{\Delta}, \phi_{\Sigma}|$  resulting in

$$(2\pi) d\hat{\Pi}_1(\phi_{\Delta}) \equiv \int_{-\pi}^{+\pi} \frac{d\phi_{\Sigma}}{2\pi} |\phi_{\Delta}, \phi_{\Sigma}\rangle \langle \phi_{\Delta}, \phi_{\Sigma}| = \sum_{j=0}^{\infty} \left[ \left( \sum_{m=-j}^{+j} |j, m\rangle e^{im\phi_{\Delta}} \right) \left( \sum_{m'=-j}^{+j} \langle j, m' | e^{-im'\phi_{\Delta}} \right) \right]. \quad (15)$$

Note that since both of the inner sums use the same value of  $j$ , interference among the different  $j$  states is excluded and we have (for pure states) the pdf

$$P_1(\phi_\Delta) = \text{Tr}(\hat{\rho} d\hat{\Pi}_1(\phi_\Delta)) = \frac{1}{2\pi} \sum_{j=0}^{\infty} \left| \psi^{(j)}(\phi_\Delta) \right|^2 \quad (16)$$

where:  $\text{Tr}()$  denotes trace;  $\hat{\rho}$  is the density matrix; and

$$\psi^{(j)}(\phi_\Delta) \equiv \sum_{m=-j}^{+j} \psi_{j,m} e^{-im\phi_\Delta} . \quad (17)$$

This procedure treats the  $j$  states as distinguishable (**adding** the non-interfering **probabilities** that each contributes to the measurement of  $\phi_\Delta$ ).

We can also eliminate  $\phi_\Sigma$  by conditioning  $|\phi_\Delta, \phi_\Sigma\rangle\langle\phi_\Delta, \phi_\Sigma|$  to  $\phi_\Sigma = 0$  resulting in

$$(2\pi) d\hat{\Pi}_2(\phi_\Delta) \equiv \frac{1}{P(\phi_\Sigma = 0)} |\phi_\Delta, \phi_\Sigma = 0\rangle\langle\phi_\Delta, \phi_\Sigma = 0| = \frac{1}{P(\phi_\Sigma = 0)} \left( \sum_{j=0}^{\infty} \sum_{m=-j}^{+j} |j, m\rangle e^{im\phi_\Delta} \right) \left( \sum_{j'=0}^{\infty} \sum_{m'=-j'}^{+j'} \langle j', m'| e^{-im'\phi_\Delta} \right) \quad (18)$$

where the renormalization constant is

$$P(\phi_\Sigma = 0) = \sum_{m=-\infty}^{\infty} \left\{ \left| \left( \sum_{j=0}^{\infty} \psi_{j,m} \right) \right|^2 \right\} . \quad (19)$$

Herein we are taking a “snapshot” in absolute time so that the inner sums use different values of  $j$  thereby permitting interference among the different  $j$  states so that (for pure states) we have the pdf

$$P_2(\phi_\Delta) = \text{Tr}(\hat{\rho} d\hat{\Pi}_2(\phi_\Delta)) = \frac{1}{2\pi P(\phi_\Sigma = 0)} \left| \sum_{j=0}^{\infty} \psi^{(j)}(\phi_\Delta) \right|^2 . \quad (20)$$

This measurement treats the  $j$  states as indistinguishable (**adding** the interfering **amplitudes** that each contributes).

## 5 Discussion

It may be of interest to note that in the “marginal measurement” defined by equation (15) (which reduces to equation (16) for pure states) we are **directly** measuring the relative phase angle between our two “clock arms.” Thus, two uniformly (randomly) distributed clocks result in a uniform (random) distribution in  $\phi_\Delta$ . This is different than what one would obtain from the *marginal pdf* calculated from the joint distribution of our two clock arms. Rather than directly measuring a phase difference, the marginal pdf

would describe the procedure of *first* measuring  $\phi_1$  and  $\phi_2$ , and *then* subtracting the results of these two measurements (resulting in a non-uniform distribution for the case of two random clocks, due to the mod  $2\pi$  range of  $\phi_1$  and  $\phi_2$ ).

It may also be of interest to note that physical intuition regarding the connection between the issue of *distinguishability* and “absolute time average” versus “snapshot” can be reinforced by contrasting electromagnetic field moments with the angular measurement (which is equivalent to the measurement of  $\phi_\Delta$  when the two oscillators are the right and left circularly polarized modes of an electromagnetic wave [3]). Furthermore, in the case of the “snapshot” (equations (18) and (19)) we determine the angular distribution of the field vectors (and their quantum fluctuations) at a point in absolute time, whereas the “absolute time average” (equations (15) and (16)) traces out the quantum version of the polarization ellipse.

## 6 References

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