

PARTICLE LOCALIZATION, SPINOR TWO-VALUEDNESS,  
AND FERMI QUANTIZATION OF TENSOR SYSTEMS

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**Abstract**

Recent studies of particle localization show that square-integrable positive energy bispinor fields in a Minkowski space-time cannot be physically distinguished from constrained tensor fields. In this paper we generalize this result by characterizing all classical tensor systems, which admit Fermi quantization, as those having unitary Lie-Poisson brackets. Examples include Euler's tensor equation for a rigid body and Dirac's equation in tensor form.

**1. INTRODUCTION**

It is a common misconception that fermions can only be represented in quantum field theory by bispinor fields. Recent studies of particle localization [1], [2], [3], [4] have shown that particle wave functions cannot vanish in regions of positive measure for any set of times of positive measure. This demonstrates that gedanken experiments, designed to observe the two-valuedness of bispinors, are not physically realizable since such experiments require absolute isolation of particle wave functions [4], [5], [6].

Moreover, in previous work we showed that square-integrable positive energy bispinor fields in a Minkowski space-time cannot be physically distinguished from constrained tensor fields. That is, the non-localization of particle wave functions implies that the two-valuedness of bispinors is unobservable in a Minkowski space-time. Thus, beam splitting experiments designed to observe the rotation properties of bispinors, in fact, describe the rotation properties of constrained tensor fields [4].

Furthermore, it was shown that Dirac's bispinor equation can be expressed, in an equivalent tensor form, as a constrained Yang-Mills equation in the limit of an infinitely large coupling constant. It was also shown [4], [7] that the free tensor Dirac equation is a completely integrable classical Hamiltonian system with (non-canonical) unitary Lie algebra type Poisson brackets, from which Fermi quantization can be derived directly without using bispinors.

In this paper we generalize this result by characterizing all classical tensor systems which admit Fermi quantization. As shown in Section 2, these tensor systems have Lie algebra type Poisson brackets

associated with a unitary symmetry group acting on the classical phase space. Two examples of classical tensor systems which admit Fermi quantization are Euler's equations for a rigid body and the tensor form of Dirac's equation. It is well known that Euler's equation can be Fermi (or Bose) quantized [8]. It is less well known that Dirac's equation can be written in a classical tensor form which can be directly Fermi quantized in the same manner as Euler's tensor equation.

## 2. FERMION QUANTIZATION OF CLASSICAL TENSOR SYSTEMS

In this section, Fermi quantization is derived for the Euler and Dirac tensor equations by representing their classical Lie-Poisson brackets as commutators of Heisenberg operators on a Fock space of Fermi occupation states.

We define a Fock space, that is, a Hilbert space  $\mathcal{H}$  of occupation states of a single field, which is suitable for both fermions and bosons, as follows. We suppose that there exists a denumerable set of operators  $A_p$ , where  $p = 1, 2, 3, \dots$ , such that all  $A_p$  and their adjoints  $A_p^*$  are defined on an invariant dense subspace  $\mathcal{D} \subset \mathcal{H}$ . For the fermion case, the operators  $A_p$  and  $A_p^*$  will be bounded, in which case  $\mathcal{D} = \mathcal{H}$ .

For each pair of indices  $p$  and  $q$  we define:

$$N_{pq} = A_p^* A_q \quad (1)$$

which is an operator defined on  $\mathcal{D}$ . The following can be taken as a set of axioms which are satisfied by fermions and bosons alike, when  $p = 1, 2, 3, \dots$  is interpreted as an index labelling the degrees of freedom (modes) of a single field,  $N_p = N_{pp}$  as the occupation number operator, and  $A_p^*$  and  $A_p$  as creation and annihilation operators for the mode  $p$ .

- a) There is a zero-occupancy state in  $\mathcal{D}$ , denoted by  $|0\rangle$ , such that for all modes  $p$ :

$$A_p |0\rangle = 0 \quad (2)$$

Furthermore, there are at least two distinct modes  $p$  and  $q$  such that  $A_p \neq A_q$ , and none of the operators  $A_p: \mathcal{D} \rightarrow \mathcal{D}$  are the zero operator.

- b) There are no states in  $\mathcal{H}$  (except 0) which are orthogonal to all the occupation states:

$$|n_1, n_2, \dots\rangle = (A_1^*)^{n_1} (A_2^*)^{n_2} \dots |0\rangle \quad (3)$$

where  $n_p = 0, 1, 2, \dots$  is the occupation number for the mode  $p$ , and all but a finite number of  $n_p$  are zero.

- c) For all modes  $p, q$ , and  $r$ , the operators  $N_{pq}$  and  $A_r$  satisfy the following commutation relation on  $\mathcal{D}$ :

$$[N_{pq}, A_r] = -A_q \delta_{pr} \quad (4)$$

and its adjoint on  $\mathcal{D}$

$$[N_{pq}, A_r^*] = A_p^* \delta_{rq} \quad (5)$$

where  $\delta_{pq}$  equals one if  $p = q$  and equals zero otherwise.

Note that axiom (c) is required for a quantum field theory in which field operators (Fermi or Bose) satisfy Heisenberg's equation [9]. Note also that axiom (c) does not involve any anti-commutation relations, and hence is applicable to tensor systems. We can prove that any Fock space satisfying (a), (b), and (c) is either Fermi or Bose, and for a given set of modes, the Fermi and Bose Fock spaces satisfying axioms (a), (b), and (c) are unique up to isomorphism [10].

Axioms (a) and (c), or equivalent axioms, are assumed in more general Fock spaces which have been used to derive parastatistics [10]. However, the parastatistical Fock spaces do not assume axiom (b). That is, in a parastatistical Fock space, there is more than one state for each set of occupation numbers. Since from the current state of knowledge, one can assume that the occupation numbers  $n_1, n_2, \dots$  determine a unique state  $|n_1, n_2, \dots\rangle$ , we do not consider Fock spaces with alternatives to axiom (b).

For a Fermi Fock space we add:

- d) For each mode  $p$ , the occupation number  $n_p = 0, 1$ .

As a consequence of axiom (d),  $A_p$  and  $A_p^*$  are bounded operators defined on  $\mathcal{H}$  (i.e.,  $\mathcal{D} = \mathcal{H}$ ).

Fermi quantization of tensor fields is derived from the following theorem:

#### THEOREM 1:

Given any denumerable set of modes  $p, q, r, s, \dots$ , there exists a Fermi Fock space  $\mathcal{H}$  (unique up to isomorphism) satisfying (a), (b), (c), and (d). Moreover, let  $\hat{H}$  be a self-adjoint operator whose domain is a dense subspace of  $\mathcal{H}$ . Then there exist operators  $\hat{a}_{pq}(t)$  defined on  $\mathcal{H}$ , indexed by each pair of modes  $p$  and  $q$ , and depending on time  $t \in \mathbb{R}$  satisfying:

- i) The adjoint relation:

$$\hat{a}_{pq}^* = \hat{a}_{qp} \quad (6)$$

- ii) The (equal time) commutation relation:

$$[\hat{a}_{pq}, \hat{a}_{rs}] = \hat{a}_{ps} \delta_{rq} - \hat{a}_{rq} \delta_{ps} \quad (7)$$

- iii) The Heisenberg equation:

$$\frac{d\hat{a}_{pq}}{dt} = -i [\hat{a}_{pq}, \hat{H}] \quad (8)$$

PROOF:

The (unique) Fermi Fock space exists by explicit construction [10]. Since  $\hat{H}$  is self-adjoint, we may define the following operators on  $\mathcal{H}$ :

$$\hat{a}_{pq}(t) = e^{i\hat{H}t} N_{pq} e^{-i\hat{H}t} \quad (9)$$

Formulas (6) and (7) follow from formulas (1), (4), and (5). Formula (8) follows by differentiating formula (9) with respect to time  $t$ . Q.E.D.

Note that by formulas (6) and (7), the operators  $\hat{a}_{pq}(t)$  generate a unitary Lie algebra [11]. For example, in the case of two modes, linear combinations of  $\hat{a}_{pq}(t)$  for  $p, q = 1, 2$  satisfy the commutation relations of angular momentum operators [12], which allows Euler's tensor equation to be Fermi quantized [8]. The following corollary of Theorem 1 characterizes all classical systems that can be Fermi quantized.

COROLLARY:

Classical systems that can be Fermi quantized are described by bimodal complex amplitudes<sup>1</sup>  $a_{pq}(t)$  satisfying:

$$a_{pq}(t) = \overline{a_{qp}(t)} \quad (10)$$

(where the bar denotes ordinary complex conjugation), and a Hamiltonian function  $H = H(a_{pq})$  which depends on the amplitudes  $a_{pq}(t)$ , such that the Hamiltonian equation is given by:

$$\frac{da_{pq}}{dt} = \{a_{pq}, H\} \quad (11)$$

where the Lie-Poisson brackets  $\{ , \}$  are defined by:

$$\{a_{pq}, a_{rs}\} = -i (a_{ps} \delta_{rq} - a_{rq} \delta_{ps}) \quad (12)$$

Furthermore, Fermi quantization of such classical systems is unique up to isomorphism.

The chief application of Theorem 1 is the Fermi quantization of Dirac's equation in its tensor form. As previously shown [4], there is a double covering map which takes a bispinor field  $\psi$  to a constrained set of  $SL(2, \mathbb{C}) \times U(1)$  gauge potentials  $A_\alpha^K$  and a complex scalar field  $\rho$ , where Lorentz indices are denoted by  $\alpha, \beta, \gamma = 0, 1, 2, 3$  and gauge indices by  $J, K, L = 0, 1, 2, 3$ . Repeated indices will be contracted using Minkowski metrics  $g_{\alpha\beta}$  and  $g_{JK}$ . Since the Lie algebra of  $SL(2, \mathbb{C})$  is regarded as the complexification of the Lie algebra of  $SU(2)$ , the gauge potentials  $A_\alpha^j$  for  $j = 1, 2, 3$  are complex, while the  $U(1)$  gauge potential  $A_\alpha^0$  is real.  $A_\alpha^K$  and  $\rho$  satisfy the following constraint:

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<sup>1</sup> Note that the set of observables  $\{a_{pq}\}$  defines a complex phase space  $P_c$  for the classical system, which is a Lie algebra under the Lie-Poisson brackets (12). In general, the "physical" phase space  $P$  need not coincide with  $P_c$ . All that is needed is that  $P$  have Lie-Poisson brackets, and there is a homomorphism of  $P_c$  onto  $P$ . All observables of  $P$  thus become observables of  $P_c$ . For a more complete discussion of Lie-Poisson brackets see reference [13].

$$A_{\alpha}^K A_{K\beta} = -|\rho|^2 g_{\alpha\beta} \quad (13)$$

With this constraint, Dirac's equation is obtained from the following Yang-Mills Lagrangian  $L_g$  in the limit of an infinitely large coupling constant  $g$ :

$$L_g = -\frac{1}{4} \text{Re} [A_K^{\alpha\beta} A_{\alpha\beta}^K] + (\overline{D_{\alpha}\rho}) (D^{\alpha}\rho) - \frac{g^2}{2} |\rho+2m|^4 \quad (14)$$

where  $mg$  is the mass, and the covariant derivative  $D_{\alpha}$  and curvature tensor  $A_{\alpha\beta}^K$  are given by:

$$\begin{aligned} D_{\alpha}\rho &= \nabla_{\alpha}\rho + ig A_{\alpha}^0\rho \\ A_{\alpha\beta}^0 &= \nabla_{\alpha}A_{\beta}^0 - \nabla_{\beta}A_{\alpha}^0 \\ \vec{A}_{\alpha\beta} &= \nabla_{\alpha}\vec{A}_{\beta} - \nabla_{\beta}\vec{A}_{\alpha} - g\vec{A}_{\alpha} \times \vec{A}_{\beta} \end{aligned} \quad (15)$$

where we denote  $A_{\alpha}^K = (A_{\alpha}^0, \vec{A}_{\alpha})$  with  $\vec{A}_{\alpha} = (A_{\alpha}^1, A_{\alpha}^2, A_{\alpha}^3)$ , and  $\nabla_{\alpha}$  denotes the space-time partial derivatives. The tensor form of Dirac's equation  $L$  is given by:

$$L = \lim_{g \rightarrow \infty} g^{-1} L_g \quad (16)$$

By standard Yang-Mills formulas, we obtain the energy-momentum tensor  $T_g^{\alpha\beta}$ , the spin-polarization tensor  $S_g^{\alpha\beta\gamma}$ , and the electric current  $J_g^{\alpha}$  derived from  $L_g$ . The corresponding bispinor observables, denoted by  $T^{\alpha\beta}$ ,  $S^{\alpha\beta\gamma}$ , and  $J^{\alpha}$ , are given by (see formula (16)):

$$T^{\alpha\beta} = \lim_{g \rightarrow \infty} g^{-1} T_g^{\alpha\beta} \quad (17)$$

and similarly for  $S^{\alpha\beta\gamma}$  and  $J^{\alpha}$ .

For quantum theory in a Minkowski space-time, it suffices [4] to consider tensor fields  $(A_{\alpha}^K, \rho)$  that: (i) are enclosed in an arbitrarily large cube  $K \subset \mathbb{R}^3$  and (ii) satisfy periodic boundary conditions for all times  $t \in \mathbb{R}$ . We quantize  $(A_{\alpha}^K, \rho)$  by considering the classical fields to be defined on  $K \times \mathbb{R}$ , and by defining the classical Hamiltonian  $H$  to be:

$$H = \int_K T^{00}(\vec{x}, t) d\vec{x} \quad (18)$$

where  $T^{\alpha\beta}$  is the energy-momentum tensor (17), and where points in  $K \subset \mathbb{R}^3$  are denoted by  $\vec{x} = (x^1, x^2, x^3)$  and  $d\vec{x} = dx^1 dx^2 dx^3$  denotes the volume measure of  $K$ . Note that by conservation of energy, the Hamiltonian  $H$  is independent of time  $t$ .

By the map from bispinors to tensors [4],  $T^{\alpha\beta}$ ,  $S^{\alpha\beta\gamma}$ , and  $J^{\alpha}$  have expansions of the form:

$$T^{\alpha\beta}(\vec{x}, t) = \sum_p \sum_q T_{pq}^{\alpha\beta}(\vec{x}) a_{pq}(t) \quad (19)$$

(and similarly for  $S^{\alpha\beta\gamma}$  and  $J^\alpha$ ) where the sum is over all pairs of fermion modes  $p$  and  $q$ , and the amplitudes  $a_{pq}(t)$  are complex functions of time satisfying the complex conjugate relation (10). The coefficients of the amplitudes  $a_{pq}(t)$ , denoted by  $T_{pq}^{\alpha\beta}(\vec{x})$ , are fixed complex functions of  $\vec{x} \in K$ . Thus, at any time  $t$ , the amplitudes  $a_{pq}(t)$  suffice to specify  $T^{\alpha\beta}$  (and similarly,  $S^{\alpha\beta\gamma}$  and  $J^\alpha$ ), and hence can be considered as classical phase space variables. Substituting (19) into formula (18), we get:

$$H = \sum_p \omega_p a_{pp} \quad (20)$$

where  $\omega_p$  is the frequency of the mode  $p$ . The classical Hamiltonian equations (which are equivalent to the Euler-Lagrange equation for the tensor Dirac Lagrangian (16)) are given by formulas (10), (11), (12), and (20) which, by the corollary to Theorem 1, can be (uniquely) Fermi quantized.

This then reproduces the existing second quantized theory for fermions. This also shows that bispinors are not more fundamental than the tensor Hamiltonian equations (10), (11), (12), and (20), which we derived from the tensor Dirac Lagrangian (16).

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## **SECTION 5**

### **UNCERTAINTY RELATIONS**

