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EXACT, E = 0, CLASSICAL AND QUANTUM SOLUTIONS FOR GENERAL POWER-LAW OSCILLATORS

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Abstract

For zero energy, $E = 0$, we derive exact, classical and quantum solutions for *all* power-law oscillators with potentials $V(r) = -\gamma/r^{\nu}$, $\gamma > 0$ and $-\infty < \nu < \infty$. When the angular momentum is non-zero, these solutions lead to the classical orbits $\rho(t) = [\cos\mu(\varphi(t) - \varphi_0(t))]^{1/\mu}$, with $\mu = \nu/2 - 1 \neq 0$. For $\nu > 2$, the orbits are bound and go through the origin. We calculate the periods and precessions of these bound orbits, and graph a number of specific examples. The unbound orbits are also discussed in detail. Quantum mechanically, this system is also exactly solvable. We find that when $\nu > 2$ the solutions are normalizable (bound), as in the classical case. Further, there are normalizable discrete, yet *unbound*, states. They correspond to unbound classical particles which reach infinity in a finite time. Finally, the number of space dimensions of the system can determine whether or not an $E = 0$ state is bound. These and other interesting comparisons to the classical system will be discussed.

1 Introduction

This all really started in Moscow, in 1992. That is where I met my colleague, Jamil Daboul, at the Second International Workshop on Squeezed States and Uncertainty Relations. It was held at the Conference Center-Hotel that the Russian Academy of Sciences uses. Late at night Jamil and I would get into sessions on life, physics, women, politics – you know, the usual stuff – while we drank his scotch.

The physics came around to musings about why certain problems can be solved exactly while others cannot, and the symmetries associated with such problems. There is a "folk-theorem" I often think of, which certainly is not exact but also certainly is intriguing. This theorem declares that if you can solve (or not solve) something classically the same is true quantum mechanically, and *visa versa*.

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Things stood there until Jamil visited me last year, and we took things up again. While wondering about the role of the Runge-Lenz vector in potential systems, a number of small observations started us down the line: a) the classical orbit for the attractive $-\gamma/r^4$ potential with centripetal potential barrier can be solved exactly; b) this type of quantum system is usually only discussed for $E \neq 0$; c) for $E = 0$, both the classical and quantum equations are simpler. Eventually we found out that *all* potentials of the form:

$$V(r) = -\frac{\gamma}{r^\nu} = -\frac{\gamma}{r^{2\mu+2}}, \qquad \gamma > 0, \qquad -\infty < \nu < \infty, \tag{1}$$

can be solved exactly, both classically and quantum mechanically, for zero binding energy, $E = 0$. – In what follows, it will be useful to switch back and forth between the variables ν and μ related by

$$\mu = (\nu - 2)/2, \qquad \nu = 2(\mu + 1). \tag{2}$$

Therefore, contrary to the usual scenario of solving a particular potential for all energies, one could solve an infinite system of potentials for a particular energy. The physics that came out was most amusing. In this piece I will report on this work. For further details you can consult a letter on the new results for the quantum system [1], as well as longer articles on the classical and quantum physics involved [2, 3].

In Section 2 I will demonstrate the solution to the classical problem. Section 3 contains amusing specific examples of some of the classical trajectories. The quantum solution will be given in Section 4. Section 5 contains interesting aspects of the quantum solutions, and then I give a brief closing comment.

Before continuing, I wish to further parametrize the power-law potentials as

$$V(r) \equiv -\frac{\gamma}{r^\nu} \equiv -\mathcal{E}_0\frac{g^2}{\rho^\nu} = -\frac{L_0^2}{2ma^2}\frac{g^2}{\rho^\nu}, \qquad \rho \equiv r/a. \tag{3}$$

The dimensional coupling constant, γ, is more useful in classical physics. The dimensionless coupling constant, g^2, is more useful in quantum physics. Note, in particular, that the constant L_0 becomes $\hbar$ in quantum physics. Finally, the "effective potential," including the angular-momentum barrier, is

$$U(L,r) = \frac{L^2}{2mr^2} - \frac{\gamma}{r^\nu}. \tag{4}$$

2 Classical Solution

Let us now obtain the classical solution. By substituting the angular-momentum conservation condition

$$\dot{\varphi} = L/(mr^2) \tag{5}$$

into the energy conservation condition

$$E - V = \frac{m}{2}\dot{\varphi}^2\left[\left(\frac{dr}{d\varphi}\right)^2 + r^2\right], \tag{6}$$

one obtains

$$\left(\frac{dr}{d\varphi}\right)^2 + r^2 = \frac{2m(E-V)r^4}{L^2} \ . \tag{7}$$

This is essentially a first-order differential equation, which could be formally integrated. However, for $E = 0$, it is much more efficient to solve Eq. (7) directly. Converting to the dimensionless variable $\rho = r/a$ and substituting V into Eq. (7), we obtain

$$\left(\frac{d\rho}{d\varphi}\right)^2 + \rho^2 = \rho^{(4-\nu)} = \rho^{(2-2\mu)}. \tag{8}$$

For $\nu = 4$ the right-hand side of this equation is unity, so the solution is a cosine. This is the circular orbit $\rho = \cos\varphi$ which we will discuss in the next section. Guided by this we multiply Eq. (8) by $\rho^{2\mu-2}$ to yield

$$\left(\rho^{\mu-1}\frac{d\rho}{d\varphi}\right)^2 + \rho^{2\mu} = \left(\frac{d\rho^\mu}{\mu d\varphi}\right)^2 + (\rho^\mu)^2 = 1 \ . \tag{9}$$

Now, ρ^μ satisfies the differential equation for the trigonometric functions. Therefore, the *general* solution of Eq. (9) is given by

$$\rho^\mu = \cos\ \mu(\varphi - \varphi_0) = \cos\left[\frac{\nu-2}{2}(\varphi-\varphi_0)\right] \ , \tag{10}$$

or

$$\rho = [\cos\ \mu\,(\varphi-\varphi_0)]^{1/\mu} = \left[\cos\left(\frac{\nu-2}{2}(\varphi-\varphi_0)\right)\right]^{\frac{2}{\nu-2}} \ . \tag{11}$$

The phase, φ_0, is the integration constant.

Actually, for bound trajectories, which are the case for $\nu > 2$, the angle φ and the phase φ_0 both change value at the origin. There a particle is both at the end of a particular orbit (which starts and ends at the origin) and also at the beginning of the next orbit. φ changes value because of the use of polar coordinates and φ_0 does because of the singular nature of the potentials. One has to be careful in matching solutions for bound orbits, and I refer you to Ref. [2] for the details. For now just note that this problem can be taken care of, and we set $\varphi_0 = 0$ for the first orbit.

3 Classical Trajectories

3.1 Bound trajectories: $2 < \nu$ or $1 < \mu$

For $2 < \nu$ or $1 < \mu$, the trajectories go out of and back in to the origin in a finite amount of time. The reason for this is that the dynamic potential dominates at the origin, but the centripetal barrier dominates at a finite distance. The effective potential then asymptotes to zero from above as $r \to \infty$. This is shown in Figure 1.

These bound orbits have an opening angle at the origin of

$$\Phi_\nu = \frac{2\pi}{\nu-2} = \frac{\pi}{\mu} \ . \tag{12}$$

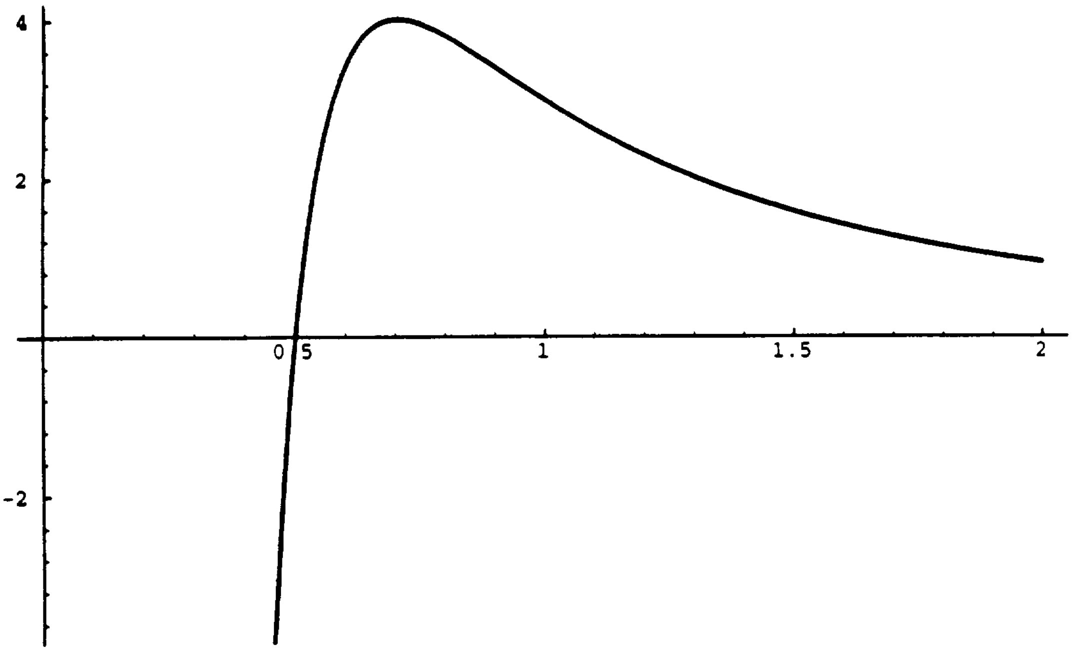

FIG. 1. The effective potential obtained from Eq. (4) for $\nu = 4$ in units of $\mathcal{E}_0/2$, as a function of $\rho = r/a$. The form is $U(\rho) = 4/\rho^2 - 1/\rho^4$.

The precession per orbit is

$$P_\nu = \left(\frac{1}{\mu} - 1\right)\pi = \left(\frac{4-\nu}{\nu-2}\right)\pi \ , \tag{13}$$

which means that if ν is a rational fraction, the trajectory will close after a finite number of orbits. The classical period of an orbit is

$$\tau_\nu = \left[\frac{ma^2}{L}\right]\frac{\sqrt{\pi}}{|\mu|}\frac{\Gamma(b)}{\Gamma(b+1/2)} \ , \qquad b \equiv \frac{1}{\mu} + \frac{1}{2} = \frac{\nu+2}{2\nu-4} > 0 \ . \tag{14}$$

(Once again, see Ref. [2] for details.)

Starting with very large ν, the first orbit describes a very thin petal. The second orbit precesses by almost $-\pi$, being a thin petal almost on the opposite side of the first orbit. As ν gets smaller, the petals become larger and the precession per orbit becomes smaller.

For example, the $\nu = 8$ case, has three petals. Here a petal is $\pi/3$ wide and the precession per orbit is $-2\pi/3$. Thus, there are three orbits before the trajectory closes. Note that here the three petals in a closed trajectory cover only half of the opening angle from the origin. We show this in Figure 2.

The case $\nu = 6$ is very interesting. The width of a petal is $\pi/2$ and the precession is $-\pi/2$ per orbit. Here, the width of a petal and the precession are exactly such that there is no overlap and also no "empty angles." It takes four orbits to close a trajectory. This is shown in Figure 3. We see that the physical solution consists of two perpendicular lemniscates (figure-eight curves composed of two opposite petals).

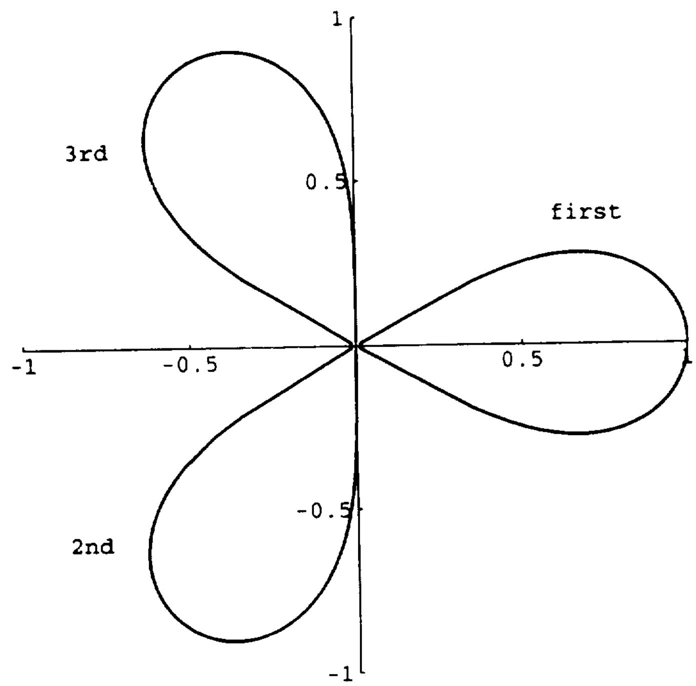

FIG. 2. The first three orbits for $\nu = 8$. Each orbit is precessed $-2\pi/3$ from the previous one, so that by the end of the 3rd orbit, the trajectory closes. In this, and later orbits, we show cartesian coordinates for orientation.

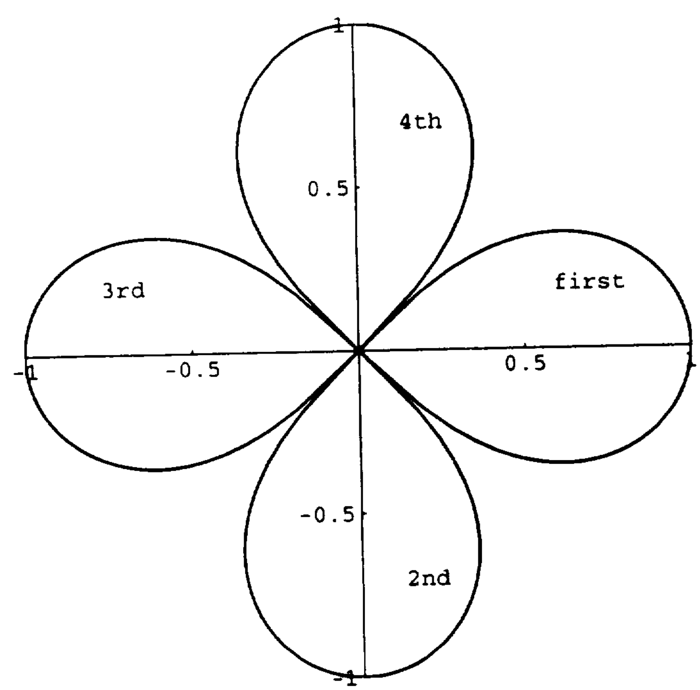

FIG. 3. The first four orbits for $\nu = 6$. Each orbit is precessed $-\pi/2$ from the previous one, so that by the end of the 4th orbit, the trajectory closes.

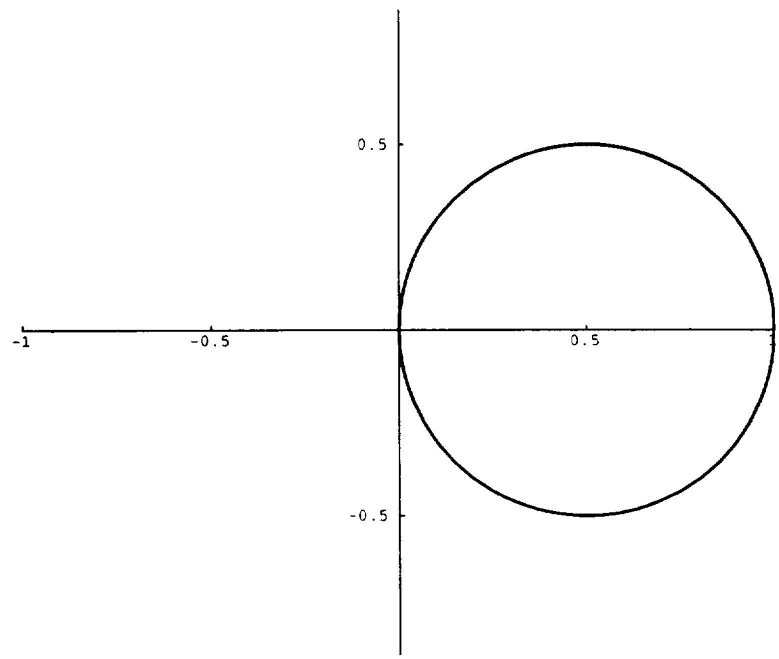

FIG. 4. The orbit for $\nu = 4$. It is a circle, and repeats itself continually.

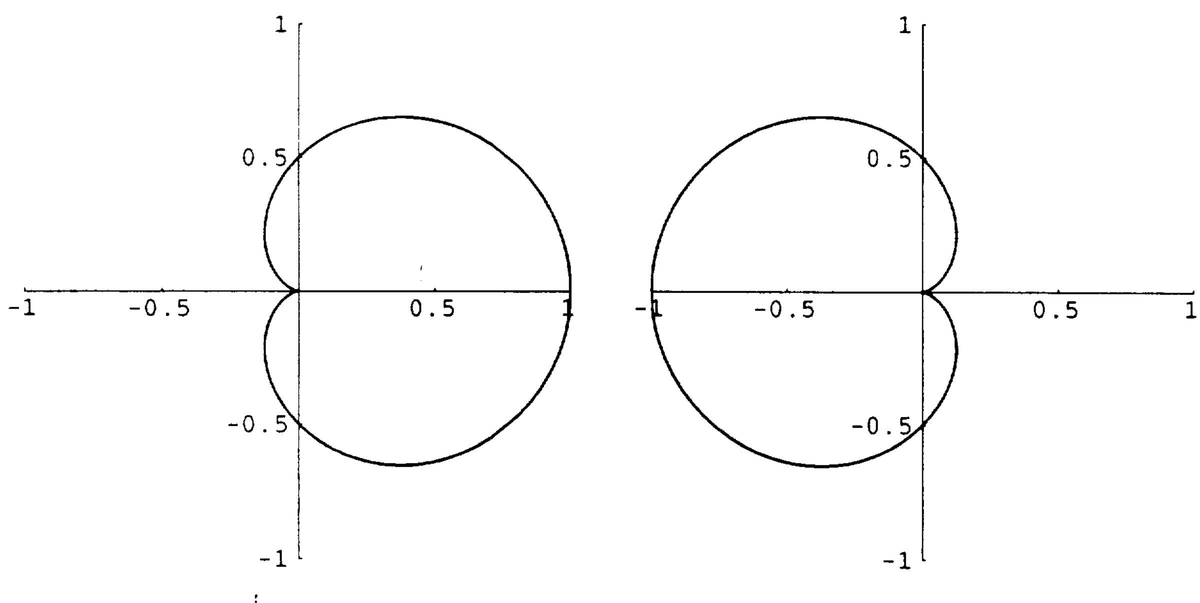

FIG 5. The first two orbits for $\nu = 3$. Each orbit is precessed π from the previous one, so that by the end of the 2nd orbit, the trajectory closes.

When we reach $\nu = 4$, the petals have widened so much that they form a circle. The circle starts at the origin, travels symmetrically about the positive x-axis, and returns to the origin. The precession is zero, so the orbit continually repeats itself. In Figure 4 we show this orbit.

As ν becomes less than 4, we can think of a petal obtaining a width greater than π, i.e., an orbit consists of two spirals, one out and one in, at opposite ends of the orbit.

Consider the special case $\nu = 3$. The width of the double-spiral orbit is still given by the formula for Φ_ν, and is 2π. Therefore, the first orbit begins and ends towards the negative x-axis. The precession is π, so the trajectory closes after two orbits. We show this case in Figure 5.

As ν approaches 2. the spirals become tighter and tighter and the precession (now clockwise) becomes larger. In fact, the spirals' angular variation as well as the orbit's precession both become infinite in magnitude as ν approaches 2.

3.2 Unbound trajectories: $\nu \leq 2$ or $\mu \leq 0$

When ν reaches 2, there is a singular change. First, the double spiral becomes infinite in angular width. But also, the joining of the two sides of the double spiral at $\rho = 1$ and $\varphi = 0$ breaks down. It is as if a tightly-wound double spring broke. The ends spiral out to infinity. This special case is a Cotes' (infinite) spiral. It takes an infinite time to reach infinity from the origin.

When the potential parameter ν just leaves that of the infinite spiral, that is, when one barely has $\nu < 2$ or $\mu < 0$, there is another change. Although the two ends of the entire trajectory still reach to infinity and the spirals in and out almost have infinite angular widths, the distance of closest approach jumps from $\rho = 0$ to $\rho = 1$.

As the value of ν decreases, the value of the angular width of the trajectory, now given by $\Phi_\nu = \pi/|\mu|$, also decreases accordingly. By the time $\nu = 1$, the angular width has decreased to 2π. Eventually it becomes less than π, meaning the orbit comes in and out in the same half plane. This happens for $\nu < 0$, i.e., when the force becomes repulsive.

When $0 < \nu < 2$, the repulsive centripetal barrier dominates at small r whereas the attractive potential $V = -\gamma/r^\nu$ dominates at large r. A typical shape is familiar from the Kepler problem. Therefore, for $0 < \nu < 2$, the $E = 0$ classical orbits are all unbounded. The distance, a, now has a completely different interpretation. It is now the distance of closest approach. Even so, the formal solution (10) remains valid for negative values of μ.

As a first example consider the case $\nu = 3/2$ or $\mu = -1/4$. This orbit has a total angular width of 4π. It is shown in the two drawings of Figure 6. The large-scale first drawing shows the trajectory coming in from the top, performing some gyration, and going out at the bottom. The small-scale second drawing shows the trajectory winding around twice near the origin, with the distance of closest approach being one.

A second example is the exact Kepler potential, $\nu = 1$ or $\mu = -1/2$. Eq. (10) gives

$$\rho^{-1/2} = \cos\varphi/2 \, , \tag{15}$$

so that

$$\frac{1}{\rho} = (\cos\varphi/2)^2 = \frac{1+\cos\varphi}{2} \, . \tag{16}$$

This is the famous parabolic orbit for the Kepler problem with $E = 0$. This orbit is shown in the first drawing of Figure 7. The parabola yields an angular width of 2π, as it should.

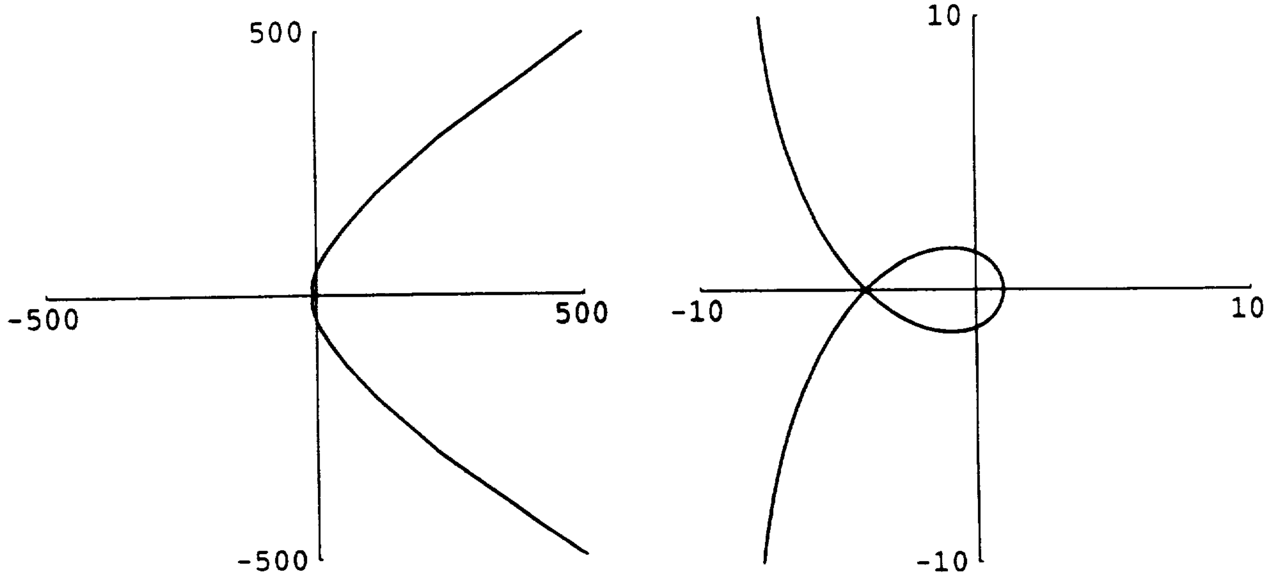

FIG. 6. A large-scale view, and a small-scale view near the origin, of the trajectory for $\nu = 3/2$.

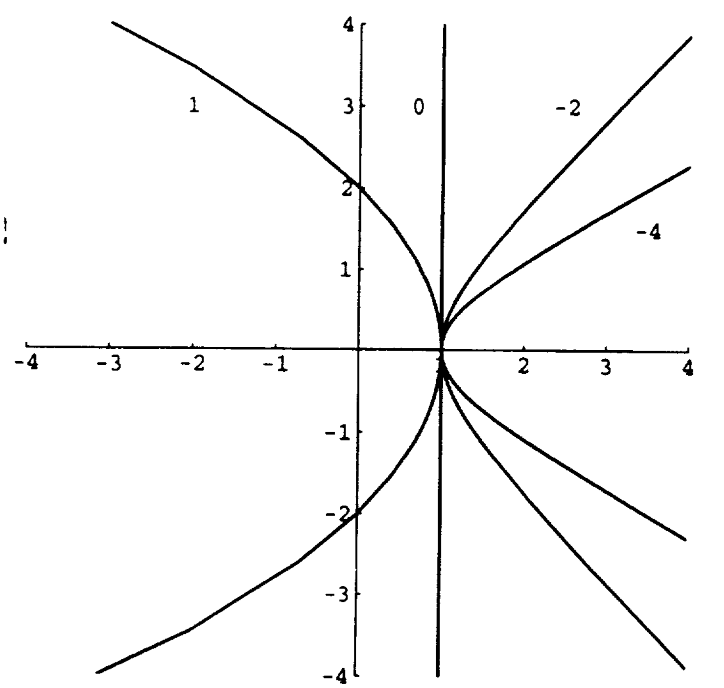

FIG 7. From left to right, he trajectories for the cases i) $\nu = 1$, ii) $\nu = 0$, iii) $\nu = -2$, and iv) $\nu = -4$. The curves are labeled by the numbers ν.

If we formally set $\nu = 0$ in the expression (3), we get a negative constant potential $V(r) = -\gamma$. Therefore, in this case the force vanishes and we have a free particle. Its orbit must be a straight line. However, Eq. (8) shows that one still has the same type of solution, Eq. (11). Here it is

$$\rho = [\cos\varphi]^{-1} , \qquad x = r\cos\varphi = a . \tag{17}$$

This is the equation for a vertical straight line that crosses the x-axis at $x = a$, as required by the initial conditions. This orbit is shown in the second drawing Figure 7, it subtending an angular width of π from the origin.

For $\nu < 0$ or $\mu < -1$ the potentials $V(r)$ in Eq. (3) are repulsive and negative-valued for all $r > 0$, with $V(r)$ going to $-\infty$ at large distances. Since both the potential, $V(r)$, and the centripetal potential decrease monotonically, the effective potential has no minima or maxima. Even so, for $E = 0$ these unbounded orbits behave qualitatively like those for $0 \leq \nu < 2$. The quantity a now labels the distance of closest approach and the solutions are given by the same expression (10), which is also valid for all $\mu < 0$:

$$\rho = [\cos\mu\varphi]^{1/\mu} = [\cos|\mu|\varphi]^{-1/|\mu|} , \qquad \mu < 0 . \tag{18}$$

The most famous special case of these potentials is the "inverted" harmonic-oscillator potential, with $\nu = \mu = -2$. The orbit is given by $\rho = [\cos 2\varphi]^{-1/2}$, so that

$$1 = \frac{r^2}{a^2}\cos 2\varphi = \frac{r^2}{a^2}(\cos^2\varphi - \sin^2\varphi) = \frac{x^2}{a^2} - \frac{y^2}{a^2} . \tag{19}$$

Thus, the trajectory is a special hyperbolic orbit, whose minor and major axes are equal, $b^2 = a^2$. We show this orbit as the third drawing in Figure 7. Now the angular width has decreased to $\pi/2$.

As the last case, we consider the orbit for $\nu = -4$ or $\mu = -3$. This orbit is shown in the last drawing of Figure 7. The orbit subtends an angle of $\pi/3$, again as it should. One sees that as ν becomes more and more negative, the orbits will become narrower and narrower. This is just as in the bound case, where the petals became narrower and narrower as ν became more and more positive.

4 Quantum Solution

Consider the radial Schrödinger equation with angular-momentum quantum number l:

$$ER_l = \left[-\frac{\hbar^2}{2m}\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2}\right) + V(r)\right] R_l . \tag{20}$$

This Schrödinger equation is exactly solvable for the potential of Eq. (3) for all $E = 0$ and all $-\infty < \nu < \infty$. To see this, set $E = 0$ in Eq. (20), change variables to ρ, and then multiply by $-\rho^2$. One finds

$$0 = \left[\rho^2\frac{d^2}{d\rho^2} + 2\rho\frac{d}{d\rho} - l(l+1) + \frac{g^2}{\rho^{2\mu}}\right] R_l(r) . \tag{21}$$

This is a well-known differential equation of mathematical physics. For $\nu \neq 2$ or $\mu \neq 0$, the solution can be directly given as

$$R_l(r) = \frac{1}{\rho^{1/2}} \mathbf{J}_{\left(\frac{2l+1}{|\nu-2|}\right)} \left(\frac{2g}{|\nu - 2| \rho^{\left(\frac{\nu-2}{2}\right)}} \right) = \frac{1}{\rho^{1/2}} \mathbf{J}_{\left(\frac{l+1/2}{|\mu|}\right)} \left(\frac{g}{|\mu| \rho^{\mu}} \right) , \quad \mu \neq 0 . \tag{22}$$

One actually has to be careful about when an absolute value of μ is called for in the labels of the solution, and whether the J Bessel functions are called for *vs.* the Y function. These details are given in Ref. [3].

5 Properties of the Quantum Solution

5.1 Normalizable bound states: $2 < \nu$ or $1 < \mu$

The normalization constants for the wave functions would have to be of the form

$$N_l^{-2} = \int_0^\infty \frac{r^2 dr}{\rho} \mathbf{J}^2_{\left(\frac{l+1/2}{|\mu|}\right)} \left(\frac{g}{|\mu| \rho^{\mu}} \right) . \tag{23}$$

Changing variables first from r to ρ and then from ρ to $z = g/(|\mu|\rho^{\mu})$, and being careful about the limits of integration for all μ, one obtains

$$N_l^{-2} = \frac{a^3}{|\mu|} \left(\frac{g}{|\mu|} \right)^{2/\mu} I_l , \tag{24}$$

where

$$I_l = \int_0^\infty \frac{dz}{z^{(1+2/\mu)}} \mathbf{J}^2_{\left(\frac{l+1/2}{|\mu|}\right)}(z) . \tag{25}$$

This integral is convergent and given by

$$I_l = \frac{1}{2\pi^{1/2}} \frac{\Gamma\left(\frac{1}{2} + \frac{1}{\mu}\right)}{\Gamma\left(1 + \frac{1}{\mu}\right)} \frac{\Gamma\left(\frac{l+1/2}{|\mu|} - \frac{1}{\mu}\right)}{\Gamma\left(1 + \frac{l+1/2}{|\mu|} + \frac{1}{\mu}\right)} , \tag{26}$$

if the following two conditions are satisfied:

$$\frac{2l+1}{|\mu|} + 1 > \frac{2}{\mu} + 1 > 0 . \tag{27}$$

Eqs. (26) and (27) lead to two sets of normalizable states. The first is when

$$\mu > 0 \quad \text{or} \quad \nu > 2 , \qquad l > 1/2 . \tag{28}$$

These are ordinary bound states and result because the effective potential asymptotes to zero from above, as in Figure 1. In this case, for $E = 0$, the wave function can reach infinity only by tunneling through an infinite forbidden region. That takes forever, and so the state is bound.

Note that the condition on l in Eq. (28) is the minimum nonzero angular momentum allowed in quantum mechanics, $l_{min} = 1$. This agrees with the classical orbit solution which is bound for any nonzero angular momentum. Also, the above $E = 0$ solutions exist for all $g^2 > 0$, and not just for discrete values of the coupling constant.

5.2 Free states: $-2 \leq \nu \leq 2$ or $-2 \leq \mu \leq 0$

For $-2 \leq \nu \leq 2$ or $-2 \leq \mu \leq 0$ and $l \geq 1$ (as well as the solutions with $l = 0$ and $0 < \mu$ or $2 < \nu$) the solutions are free, continuum solutions. This is in analogy to the classical case, where the trajectories are normal and free.

5.3 Unbound yet normalizable states: $\nu < -2$ or $\mu < -2$

There is another class of normalizable solutions which is quite surprising. For any l and all $\nu < -2$ or $\mu < -2$, one can verify that the conditions of Eq. (27) are also satisfied. Thus, even though one here has a repulsive potential that falls off faster than the inverse-harmonic oscillator and the states are *not* bound, the solutions are *normalizable*!

The corresponding classical solutions yield infinite orbits, for which the particle needs only a finite time to reach infinity [2]. But it is known that a classical potential which yields trajectories with a finite travel time to infinity also yields a discrete spectrum in the quantum case. This discrete spectrum is obtained by imposing particular boundary conditions on the solutions, which defines a self-adjoint extension of the Hamiltonian. (See Ref. [1].)

5.4 Bound states in arbitrary dimensions

One can easily generalize the problem of the last section to arbitrary D space dimensions. Doing so yields another surprising physical result.

To obtain the D-dimensional analogue of Eq. (21), one simply has to replace 2ρ by $(D-1)\rho$ and $l(l+1)$ by $l(l+D-2)$. The solutions follow similarly as

$$
\begin{aligned}
R_{l,D} &= \frac{1}{\rho^{D/2-1}} \mathbf{J}_{\left(\frac{2l+D-2}{|\nu-2|}\right)} \left(\frac{2g}{|\nu-2|\rho^{\left(\frac{\nu-2}{2}\right)}} \right) \\
&= \frac{1}{\rho^{D/2-1}} \mathbf{J}_{\left(\frac{l+D/2-1}{|\mu|}\right)} \left(\frac{g}{|\mu|\rho^{\mu}} \right) .
\end{aligned}
\tag{29}
$$

To find out which states are now normalizable one first has to change the integration measure from $r^2 dr$ to $r^{D-1} dr$ and then continue as before. The end result is that if the wave functions are normalizable, the normalization constant is given by

$$
N_{l,D}^{-2} = \frac{a^D}{|\mu|} \left(\frac{g}{|\mu|} \right)^{2/\mu} I_{l,D} \ , \tag{30}
$$

where

$$
I_{l,D} = \frac{1}{2\pi^{1/2}} \frac{\Gamma\left(\frac{1}{2}+\frac{1}{\mu}\right)}{\Gamma\left(1+\frac{1}{\mu}\right)} \frac{\Gamma\left(\frac{l+D/2-1}{|\mu|}-\frac{1}{\mu}\right)}{\Gamma\left(1+\frac{l+D/2-1}{|\mu|}+\frac{1}{\mu}\right)} \ , \tag{31}
$$

which is defined and convergent for

$$
\frac{2l+D-2}{|\mu|} + 1 > \frac{2}{\mu} + 1 > 0 \ . \tag{32}
$$

This yields the surprising result that there are bound states for all $\nu > 2$ or $\mu > 0$ when $l > 2 - D/2$. Explicitly this means that the minimum allowed l for there to be zero-energy bound states are:

$$
\begin{aligned}
D = 2 \ , \quad l_{min} &= 2 \ , \\
D = 3 \ , \quad l_{min} &= 1 \ , \\
D = 4 \ , \quad l_{min} &= 1 \ , \\
D > 4 \ , \quad l_{min} &= 0 \ .
\end{aligned}
\tag{33}
$$

This effect of dimensions is purely quantum mechanical, and exists for all central potentials. Classically, the number of dimensions involved in a central potential problem has no intrinsic effect on the dynamics. The orbit remains in two dimensions, and the problem is decided by the form of the effective potential, U, which contains only the angular momentum barrier and the dynamical potential.

In quantum mechanics there are two places where an effect of dimension appears. The first is in the factor $l(l + D + 2)$ of the angular-momentum barrier. The second is more fundamental. It is due to the operator

$$
U_{qm} = -\frac{(D-1)}{\rho}\frac{d}{d\rho} \ . \tag{34}
$$

This is a new contribution to the "effective potential," and can be calculated [1]. The end result is that given in Eq. (33).

The dimensional effect produces what amounts to an additional centrifugal barrier which can bind the wave function at the threshold, even though the expectation value of the angular momentum vanishes.

6 Closing Comment

I hope you have found this discussion of anharmonic power potentials entertaining and enlightening. Jamil and I certainly have. The intuition obtained into the workings and relationships between classical and quantum physics has been delightful to us, to say the least.

Thank you very much.

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