A RAMANUJAN-TYPE MEASURE FOR 
THE ASKEY-WILSON POLYNOMIALS

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Abstract

A Ramanujan-type representation for the Askey-Wilson \( q \)-beta integral, admitting the transformation \( q \rightarrow q^{-1} \), is obtained. Orthogonality of the Askey-Wilson polynomials with respect to a measure, entering into this representation, is proved. A simple way of evaluating the Askey-Wilson \( q \)-beta integral is also given.

1 Introduction.

The Askey-Wilson polynomials \( p_n(x; a, b, c, d|q) \) [1], which have already become classical, represent a five-parameter system of polynomials. They satisfy the orthogonality relation

\[
\int_{-1}^{1} p_m(x; a, b, c, d|q) p_n(x; a, b, c, d|q) w(x; a, b, c, d|q) \, dx = \delta_{mn} I_n(a, b, c, d|q) \tag{1.1}
\]

with respect to the absolutely continuous measure \( d\alpha(x) = w(x)dx \), with the weight function

\[
w(x; a, b, c, d|q) = \frac{1}{\sin^2 \theta} \frac{h(\cos 2\theta, 1; q)}{\prod_{i=0}^{\infty} (1 - 2abq^i + b^2q^{2i})}, \quad x = \cos \theta, \tag{1.2}
\]

\[
h(a, b; q) = \prod_{j=0}^{\infty} (1 - 2abq^i + b^2q^{2i}).
\]

As special and limiting cases, the Askey-Wilson polynomials contain many known systems of polynomials (see, for example, [2]). In particular, the choice of the parameters \( a = -b = \sqrt{\beta}, c = -d = \sqrt{q\beta} \), leads to the continuous \( q \)-ultraspherical polynomials \( C_n(x; \beta|q) \) [3], i.e.,

\[
p_n(x; \sqrt{\beta}, -\sqrt{\beta}, \sqrt{q\beta}, -\sqrt{q\beta}|q) = \frac{(\beta^2; q)^2n(q; q)_n}{(\beta, \beta^2; q)_n} C_n(x; \beta|q), \tag{1.3}
\]

where we have used the standard notation of the theory of $q$-special functions

\[
(a; q)_n = \prod_{j=0}^{n-1} (1 - aq^j), \quad (a_1, \ldots, a_k; q)_n = \prod_{j=1}^{k} (a_j; q)_n.
\] (1.4)

In turn, from $C_n(x; \beta | q)$ one can obtain the continuous $q$-Hermite polynomials $H_n(x|q) = (q; q)_n C_n(x; 0|q)$, the Gegenbauer (ultraspherical) polynomials $C_n^\lambda (x) = \lim_{\Delta \to 1} C_n(x; (q^\lambda)^q)$, and also the Chebyshev polynomials of the first and second kinds, $T_n(x)$ and $U_n(x)$, by taking the limit $\beta \to 1$ or by putting $\beta = q$ in $C_n(x; \beta | q)$, respectively.

The key ingredient of the original proof of the orthogonality (1.1), which led to the discovery of the Askey-Wilson system of polynomials (see the discussion of this point in [4]), was the evaluation of the Askey-Wilson $q$-beta integral:

\[
I_0(a, b, c, d|q) = \int_{-1}^{1} w(x; a, b, c, d|q) dx = \frac{2\pi(abcd; q)_{\infty}}{(q, ab, ac, ad, bc, bd, cd; q)_{\infty}},
\] (1.5)

\[
\max_{\max_{v=\pm a, \pm b, \pm c, \pm d}} |v| < 1, \quad |q| < 1.
\]

The integral (1.5) has acquired its name because in a special case, when the parameters $a = q^{\alpha + 1/2}$, $b = -q^{\beta + 1/2}$, and $c = -d = q^{1/2}$; the $q \to 1^-$ limit of $I_0(a, b, c, d|q)$ is the beta function (or Euler’s integral of the first kind)

\[
\int_{-1}^{1} (1 - x)^\alpha (1 + x)^\beta dx = 2^{\alpha + \beta + 1} B(\alpha + 1, \beta + 1) = 2^{\alpha + \beta + 1} \frac{\Gamma(\alpha + 1)\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 2)}. \tag{1.6}
\]

A nonstandard form of the orthogonality on the full real line for the continuous $q$-Hermite polynomials $H_n(\sin k\pi|q)$, $q = \exp(-2\pi^2)$, was considered in [5]. It turned out that if one uses the modular transformation and the periodicity property of the $\vartheta$-function appearing in the weight function for these polynomials, the finite interval of orthogonality can be transformed into an infinite one. This technique is of interest both from a mathematical point of view and from the point of view of possible applications in theoretical physics, beginning with a number of problems, related with $q$-oscillators (see the review [6]).

The purpose of this article is to discuss the applicability of this idea to the more general case, i.e. to the Askey-Wilson $q$-beta integral (1.5) [7, 8]. To simplify consideration it will be assumed in Sections 2-4 that $|v| < 1$, $v = a, b, c, d$, and that the parameter $q = \exp(-2\pi^2)$ satisfies the requirement $0 < q < 1$. The possibility of extending these results to other values of the parameters is discussed in Section 5.

## 2 A Ramanujan-type representation for the $q$-beta integral.

From the point of view of symmetry the parametrization $x = \sin \varphi$ is most convenient; it corresponds to the change of variable $\theta = \frac{\pi}{2} - \varphi$, $-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$ in formula (1.2). Consequently, the left
side of (1.5) is equal to

\[ I_0(a, b, c, d|q) = \int_{-\pi/2}^{\pi/2} \frac{h(-\cos 2\varphi, 1; q)}{\prod_{\nu=a,d,c,d} h(\sin \varphi, \nu; q)} \, d\varphi. \]  

(2.1)

Comparison of the numerator

\[ h(-\cos 2\varphi, 1; q) = \prod_{j=0}^{\infty} (1 + 2q^j \cos 2\varphi + q^{2j}) \]

of the integral (2.1) with Jacobi's expression for the theta-function \( \vartheta_2(z, q) \equiv \vartheta_2(z|\tau) \), \( q = \exp(\pi i \tau) \) as an infinite product [9]

\[ \vartheta_2(z, q) = 2q^{1/4}(q^2; q^2)_\infty \cos z \prod_{j=1}^{\infty} (1 + 2q^{2j} \cos 2z + q^{4j}), \]  

(2.2)

shows that

\[ h(-\cos 2\varphi, 1; q) = \frac{2 \cos \varphi}{q^{1/8}(q; q)_\infty} \vartheta_2(\varphi, q^{1/2}) \]  

(2.3)

and therefore

\[ I_0(a, b, c, d|q) = \frac{2}{q^{1/8}(q; q)_\infty} \int_{-\pi/2}^{\pi/2} \frac{\vartheta_2(\varphi, q^{1/2}) \cos \varphi}{\prod_{\nu=a,b,c,d} h(\sin \varphi, \nu; q)} \, d\varphi. \]  

(2.4)

With the aid of the modular transformation [9]

\[ \vartheta_2(z|\tau) = \frac{\exp\left(-\frac{i\tau^2}{\pi^2}\right)}{(-i\tau)^{1/2}} \vartheta_4(z\tau^{-1}|-\tau^{-1}), \quad \tau = \frac{i\kappa^2}{\pi}, \]  

(2.5)

and the change of variable \( \varphi = \kappa x \), the integral (2.4) can be written as

\[ I_0(a, b, c, d|q) = \frac{2\sqrt{\pi}}{q^{1/8}(q; q)_\infty} \int_{-\pi/2\kappa}^{\pi/2\kappa} \frac{\vartheta_4\left(\frac{\pi x}{\kappa}, e^{-\pi^2/\kappa^2}\right) e^{-x^2 \cos \kappa x}}{\prod_{\nu=a,b,c,d} h(\sin \kappa x, \nu; q)} \, dx. \]  

(2.6)

Using the expansion

\[ \vartheta_4(z, q) = \sum_{k=-\infty}^{\infty} (-1)^k q^{k^2} e^{2ikz} \]  

(2.7)

and taking into account the uniform convergence of the series (2.7) in any bounded domain of values of \( z \) [9], we substitute (2.7) into (2.6) and integrate this series termwise, i.e.,

\[ I_0(a, b, c, d|q) = \frac{2\sqrt{\pi}}{q^{1/8}(q; q)_\infty} \sum_{k=-\infty}^{\infty} (-1)^k \int_{-\pi/2\kappa}^{\pi/2\kappa} e^{-(x^2+\pi^2/\kappa^2) \cos \kappa x} \frac{\cos \kappa x \, dx}{\prod_{\nu=a,b,c,d} h(\sin \kappa x, \nu; q)}. \]  

(2.8)
The change of variable \( x_k = x + \frac{\pi}{\kappa} k \), \( x_k^{\text{min}} = \frac{\pi}{\kappa}(k - \frac{1}{2}) \leq x_k \leq \frac{\pi}{\kappa}(k + \frac{1}{2}) = x_k^{\text{max}} \) and an account for the relation \( x_k^{\text{max}} - x_{k-1}^{\text{min}} \) allows to sum the right-hand side of (2.8) with respect to \( k \) and represent (2.8) in the form

\[
I_0(a, b, c, d|q) = \frac{2\sqrt{\pi}}{q^{1/8}(q; q)_{\infty}} I_0(a, b, c, d|q) = \frac{2\sqrt{\pi}}{q^{1/8}(q; q)_{\infty}} \int_{-\infty}^{\infty} \frac{e^{-x^2} \cos \kappa x dx}{\prod_{\nu=a,b,c,d} \ell(h(\sin \kappa x; v; q))}.
\]

(2.9)

Thus, combining formulas (1.5) and (2.9) yields the following representation for the Askey-Wilson q-beta integral [7]

\[
\tilde{I}_0(a, b, c, d|q) \equiv \int_{-\infty}^{\infty} \tilde{\rho}(\kappa x; a, b, c, d|q) e^{-x^2} \cos \kappa x dx = \frac{\sqrt{\pi} q^{1/4}(abcd; q)_{\infty}}{(ab, ac, ad, bc, bd, cd; q)_{\infty}},
\]

(2.10)

where, in accordance with the definition (1.2),

\[
\tilde{\rho}(x; a, b, c, d|q) = \prod_{\nu=a,b,c,d} h^{-1}(\sin x, v; q) = \prod_{\nu=a,b,c,d} e_q(i v e^{-i \theta}) e_q(-i v e^{i \theta}),
\]

(2.11)

and \( e_q(z) = (z; q)_{\infty}^{-1} \) is the \( q \)-exponential function [2].

We note that each factor \( h^{-1}(\sin \kappa x, v; q), v = a, b, c, d, \) in the integrand (2.10) is represented as

\[
h^{-1}(\sin \kappa x, v; q) = \sum_{n=0}^{\infty} (iv)^n \sum_{k=0}^{n} (-1)^k \frac{\exp[-i(n - 2k)\kappa x]}{(q; q)_k(q; q)_{n-k}},
\]

(2.12)

if one uses the generating function for the continuous \( q \)-Hermite polynomials \( H_n(x|q) \)

\[
(te^{i \theta}, te^{-i \theta}; q)_{\infty}^{-1} = \sum_{n=0}^{\infty} \frac{H_n(\cos \theta|q)}{(q; q)_n} t^n, \quad |t| < 1,
\]

(2.13)

and their explicit representation [2]

\[
H_n(\cos \theta|q) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q e^{i(n-2k)\theta},
\]

(2.14)

where the symbol \( \left[ \begin{array}{c} n \\ k \end{array} \right]_q \) denotes the \( q \)-binomial coefficient [2]. Therefore the integration over \( x \) in (2.10) is reduced to the Fourier transformation formula for the ground state of the linear harmonic oscillator

\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-x^2/2 + i xy) dx = \exp(-y^2/2).
\]

(2.15)

An explicit evaluation of the nonstandard form of the Askey-Wilson q-beta integral (2.10) will be discussed in greater detail in Section 4.

As mentioned above, the weight function (1.2) with the parameters \( a = -b = \beta^{-1/2}, c = -d = a q^{1/2}, \) corresponds to the continuous \( q \)-ultraspherical polynomials \( C_n(x|\beta|q) \). The relations [2]

\[
(a; q)_{\infty} = (a, aq; q^2)_{\infty}, \quad (a, -a; q)_{\infty} = (a^2; q^2)_{\infty},
\]
enable the representation (2.10) for this particular case to be simplified to

\[
\int_{-\infty}^{\infty} \frac{\exp(-x^2 + i\kappa x)dx}{(-\beta \exp(2i\kappa x), -\beta \exp(-2i\kappa x); q)_\infty} = \frac{\sqrt{\pi q^{1/8}}(\beta, q\beta; q)_\infty}{(\beta^2; q)_\infty}.
\]

(2.16)

If one compares (2.16) with the Ramanujan integral \( q = \exp(-2k^2), |q| < 1 \) [10, 11]

\[
\int_{-\infty}^{\infty} e^{-x^2 + 2mz} e_q(aa^{1/2} e^{2ikx}) e_q(bq^{1/2} e^{-2ikx}) dx = \frac{\sqrt{\pi e^{m^2}}}{(qab; q)_\infty} E_q(aq e^{2imk}) E_q(bq e^{-2imk}),
\]

(2.17)

it is easy to verify that (2.16) agrees with (2.17) if one sets \( 2m = ik = i\kappa \) and \( a = b = -\beta q^{1/2} \).

### 3 Orthogonality of the Askey-Wilson polynomials with respect to the measure \( \rho(\kappa x; a, b, c, d|q) \).

A direct proof of the orthogonality for the Askey-Wilson polynomials

\[
\int_{-\infty}^{\infty} p_m(\sin \kappa x; a, b, c, d|q)p_n(\sin \kappa x; a, b, c, d|q)\rho(\kappa x; a, b, c, d|q) \exp(-x^2) \cos \kappa x dx =
\]

\[
= \delta_{mn} \tilde{I}_n(a, b, c, d|q)
\]

(3.1)

with respect to the weight function appearing in the nonstandard integral representation (2.10), is analogous to the proof of eigenfunctions orthogonality for the Sturm-Liouville differential equation [12]. Indeed, the difference differentiation formula for the Askey-Wilson polynomials [1]

\[
\sin \kappa \partial_x p_n(\sin \kappa x; a, b, c, d|q) =
\]

\[
= q^{-n/2}(1 - q^n)(1 - abcdq^{n-1}) \cos \kappa x p_{n-1}(\sin \kappa x; aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2}|q)
\]

(3.2)

provides a lowering operator for these polynomials. To find a raising operator one can use the relation

\[
w(\sin \varphi; a, b, c, d|q) = \frac{2q_2(\varphi, q^{1/2})}{q^{1/8}(q; q)_\infty} \rho(\varphi; a, b, c, d|q),
\]

(3.3)

which follows from (1.2), (2.3) and (2.11), and write the difference equation for the Askey-Wilson polynomials [1] in the form

\[
\sin \kappa \partial_x \left[ \frac{\vartheta_2(\kappa x, q^{1/2})}{\cos \kappa x} \rho(\kappa x; aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2}|q) \sin \kappa \partial_x p_n(\sin \kappa x; a, b, c, d|q) \right] =
\]

\[
= (1 - q^{-n})(1 - abcdq^{n-1}) \cos \kappa x \vartheta_2(\kappa x, q^{1/2}) \rho(\kappa x; a, b, c, d|q) p_n(\sin \kappa x; a, b, c, d|q).
\]

(3.4)
Now, using the difference differentiation formula (3.2) in the left-hand side of (3.4) and the periodicity property of the \( \vartheta_2 \)-function [9],

\[
\vartheta_2(z \pm \pi \tau, q) = q^{-1} \exp(\mp 2iz) \vartheta_2(z, q), \quad q = \exp(\pi i \tau),
\]

we arrive at the raising operator

\[
(p_0 \cos 2\kappa x - \cos 2\kappa x p_0) p_n(\sin \kappa x; a, b, c, d|q) = q^{-1} \exp(T_2iz) p_n(\sin \kappa x; a, b, c, d|q),
\]

\[
(\sin 2\kappa x \cos \kappa \partial_x - \cos 2\kappa x \sin \kappa \partial_x) \rho(\kappa x; aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2}|q)
\]

\[
p_{n-1}(\sin \kappa x; aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2}|q) = q^{1-n} \cos \kappa x \rho(\kappa x; a, b, c, d|q) p_n(\sin \kappa x; a, b, c, d|q).
\]

We are now in a position to give a direct proof of the orthogonality relation (3.1). We multiply both sides of the equality (3.6) by \( p_m(\sin \kappa x; a, b, c, d|q) \exp(-x^2) \) and integrate in \( x \) over the full real line. As a result we obtain in the right-hand side,

\[
q^{1-n} \int_{-\infty}^{\infty} p_m(\sin \kappa x; a, b, c, d|q) p_n(\sin \kappa x; a, b, c, d|q) \rho(\kappa x; a, b, c, d|q) e^{-x^2} \cos \kappa x \, dx \equiv q^{1-n} I_{mn}(a, b, c, d|q).
\]

The left-hand side

\[
\int_{-\infty}^{\infty} dx \rho(\kappa x; aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2}|q) p_{n-1}(\sin \kappa x; aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2}|q),
\]

can be integrated by parts with the aid of (3.2) and the evident relations

\[
\int_{-\infty}^{\infty} dx f(x) \cos \kappa \partial_x \varphi(x) = \int_{-\infty}^{\infty} dx \varphi(x) \cos \kappa \partial_x f(x),
\]

\[
\int_{-\infty}^{\infty} dx \sin \kappa \partial_x \varphi(x) = - \int_{-\infty}^{\infty} dx \varphi(x) \sin \kappa \partial_x f(x),
\]

which apply to (3.8) because the function \( \rho(\kappa x; aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2}|q) \) has no singularities inside of the strip \(-\kappa \leq y \leq \kappa, \ -\infty < x < \infty \) in the complex plane \( z = x + iy \). This leads to

\[
q^{1-n} (1 - q^m) (1 - abcdq^{m-1}) I_{m-1n-1}(aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2}|q).
\]

Equating the right-hand (3.7) and left-hand (3.10) sides thus yields

\[
q^{m-n} I_{mn}(a, b, c, d|q) = (1 - q^m) (1 - abcdq^{m-1}) I_{m-1n-1}(aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2}|q).
\]
We now interchange \( m \) and \( n \) in (3.11) and take into account that the integral \( I_{mn}(a, b, c, d|q) \) is symmetric in \( m \) and \( n \) due to the definition (3.7), i.e.,

\[
q^{\frac{n-m}{2}} I_{mn}(a, b, c, d|q) = (1 - q^n)(1 - abcdq^{n-1}) I_{m-n-1}(aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2}|q).
\]  

(3.11')

Finally, multiplying both sides of (3.11) by \((1 - q^n)(1 - abcdq^{n-1})\) and of (3.11') by \((1 - q^m)(1 - abcdq^{m-1})\) and subtracting the second expression from the first, we get

\[
(q^{\frac{n-m}{2}} - q^{\frac{m-n}{2}})(1 - abcdq^{m+n-1}) I_{mn}(a, b, c, d|q) = 0.
\]  

(3.12)

From (3.12) it follows that \( I_{mn}(a, b, c, d|q) = \delta_{mn} I_n(a, b, c, d|q) \), confirming the orthogonality (3.1) of the Askey-Wilson polynomials for \( m \neq n \) [8].

We note that as special and limiting cases, (3.1) contains the orthogonality relations for other known sets of polynomials, such as the continuous \( q \)-ultraspherical polynomials \( C_n(x; \beta|q) \), the continuous \( q \)-Hermite polynomials \( H_n(x; q) = (q; q)_n C_n(x; 0|q) \) (the corresponding special case of (3.1), when the all parameters \( a, b, c, d \) are equal to zero, is considered in [5]), the Chebyshev polynomials of the first and second kinds, \( T_n(x) \) and \( U_n(x) \), and so on.

4 Evaluation of the integrals \( \tilde{I}_n(a, b, c, d|q) \).

Iterating the recurrence relation

\[
\tilde{I}_n(a, b, c, d|q) = (1 - q^n)(1 - abcdq^{n-1}) \tilde{I}_{n-1}(aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2}|q),
\]

(4.1)

which follows from (3.11) or (3.11') when \( m = n \), allows to express the normalization integrals \( \tilde{I}_n(a, b, c, d|q) \), \( n = 1, 2, ... \), through a known value of the Askey-Wilson \( q \)-beta integral \( I_0(a, b, c, d|q) \), i.e.

\[
\tilde{I}_n(a, b, c, d|q) = \frac{(q, ab, ac, ad, bc, bd, cd; q)_n}{(1 - abcdq^{2n-1})(abcd; q)_{n-1}} \tilde{I}_0(a, b, c, d|q).
\]  

(4.2)

It only remains to evaluate the integral \( \tilde{I}_0(a, b, c, d|q) \) itself. To this end, having defined the symmetrical \( \rho_+(x) \) and antisymmetrical \( \rho_-(x) \) combinations with respect to the inversion \( x \rightarrow -x \),

\[
\rho_{\pm}(x; a, b, c, d|q) = \frac{1}{2}[\rho(x; a, b, c, d|q) \pm \rho(-x; a, b, c, d|q)],
\]

(4.3)

it is convenient to rewrite (2.10) as

\[
\tilde{I}_0(a, b, c, d|q) = \int_{-\infty}^{\infty} dx \exp(-x^2 + i\kappa x)\rho_+(\kappa x; a, b, c, d|q).
\]  

(4.4)

Let us carry out the replacements \( v \rightarrow v\sqrt{q} \), \( v = a, b, c, d \), and the subsequent shift of the variable of integration \( x \rightarrow x + i\kappa \) in (4.4). (We remind that the function \( \rho(\kappa z; aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2}|q) \) does not have singularities in the strip \(-\kappa \leq y \leq \kappa, -\infty < x < \infty\) of the complex plane \( z = x + iy \).) Then, taking into account that in accordance with the definitions (1.2) and (2.11)

\[
\rho(\kappa(x + i\kappa); aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2}|q) = \rho(\kappa x; a, b, c, d|q) \prod_{v=a,b,c,d} (1 + iv \exp(i\kappa x)),
\]

(4.5)
we obtain

\[ \tilde{I}_0(aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2}|q) = (1 - s_2)\tilde{I}_0(a, b, c, d|q) + \]

\[ + s_4 \int_{-\infty}^{\infty} dx \exp(-x^2 + 3i\kappa x)\rho_+(\kappa x; a, b, c, d|q) - is_3 \int_{-\infty}^{\infty} dx \exp(-x^2 + 2i\kappa x)\rho_-(\kappa x; a, b, c, d|q), \]

where

\[ s_2 = ab + ac + ad + bc + bd + cd, \]

\[ s_3 = abc + abd + acd + bcd, \quad s_4 = abcd. \]

It remains only to express the second and third integrals in the right-hand side of (4.6) in terms of \( \tilde{I}_0(a, b, c, d|q) \). To that end one can use the \( n = 1 \) case of (3.6)

\[ (\sin 2\kappa x \cos \kappa \partial_x - \cos 2\kappa x \sin \kappa \partial_x)\rho(\kappa x; aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2}|q) = \]

\[ \left[ (1 - s_4) \sin 2\kappa x + (s_3 - s_1) \cos \kappa x \right] \rho(\kappa x; a, b, c, d|q), \]

taking into account that \( p_0(x; a, b, c, d|q) = 1, \quad p_1(x; a, b, c, d|q) = 2(1 - s_4)x + s_3 - s_1 \) and \( s_1 = a + b + c + d \). The symmetrization of (4.8) leads to the relations

\[ (\sin 2\kappa x \cos \kappa \partial_x - \cos 2\kappa x \sin \kappa \partial_x)\rho_\pm(\kappa x; aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2}|q) = \]

\[ = (1 - s_4) \sin 2\kappa x \rho_\pm(\kappa x; a, b, c, d|q) + (s_3 - s_1) \cos \kappa x \rho_\mp(\kappa x; a, b, c, d|q). \]

Multiplying both sides of the equality (4.9) for the antisymmetrical combination \( \rho_-(\kappa x) \) by \( \exp(-x^2) \) and integrating over the variable \( x \) yields

\[ (1 - s_4) \int_{-\infty}^{\infty} dx \exp(-x^2 + 2i\kappa x)\rho_-(\kappa x; a, b, c, d|q) = i(s_1 - s_3)\tilde{I}_0(a, b, c, d|q). \]

Now we multiply both sides of (4.9) for \( \rho_+(\kappa x; aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2}|q) \) by \( \exp(-x^2 + i\kappa x) \) and integrate over \( x \). Using (4.10), the result can be written as

\[ \int_{-\infty}^{\infty} dx \exp(-x^2 + 3i\kappa x)\rho_+(\kappa x; a, b, c, d|q) = \]

\[ = \left[ 1 - \frac{(s_3 - s_1)^2}{(1 - s_4)^2} \right] \tilde{I}_0(a, b, c, d|q) - \frac{1 - q}{1 - s_4} \tilde{I}_0(aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2}|q). \]
Substituting (4.10) and (4.11) into (4.6), we find

\[(1 - abcd)(1 - qabcd) \tilde{I}_0(aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2}q) = \]

\[= (1 - ab)(1 - ac)(1 - ad)(1 - bc)(1 - bd)(1 - cd) \tilde{I}_0(a, b, c, d|q).\]

Since \(0 < q < 1\), by iterating equation (4.12) one can express the Askey-Wilson \(q\)-beta integral (2.10) with arbitrary parameters in terms of its value for vanishing parameters \(a, b, c, d\), i.e.,

\[\tilde{I}_0(a, b, c, d|q) = \frac{(abcd; q)_\infty}{(ab, ac, ad, bc, bd, cd; q)_\infty} \tilde{I}_0(0, 0, 0, 0|q).\]  \hspace{1cm} (4.13)

The value of \(\tilde{I}_0(0, 0, 0, 0|q)\) is easily found from (2.10) and (3.1) with the aid of the Fourier transformation formula (2.15) for the quadratically decreasing exponential function, i.e.,

\[\tilde{I}_0(0, 0, 0, 0|q) = \int_{-\infty}^{\infty} dx \exp(-x^2 + i\kappa x) = \sqrt{\pi} q^{1/8}.\]  \hspace{1cm} (4.14)

Combining formulas (4.13) and (4.14) leads to

\[\tilde{I}_0(a, b, c, d|q) = \frac{\sqrt{\pi} q^{1/8}(abcd; q)_\infty}{(ab, ac, ad, bc, bd, cd; q)_\infty},\]  \hspace{1cm} (4.15)

which is the known value of the Askey-Wilson \(q\)-beta integral [1]

\[I_0(a, b, c, d|q) = \frac{2\sqrt{\pi}}{q^{1/8}(q; q)\_\infty} \tilde{I}_0(a, b, c, d|q) = \frac{2\pi (abcd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty}.\]  \hspace{1cm} (4.15')

Substituting (4.15) into (4.2), we finally obtain the explicit form for the normalization integral

\[\tilde{I}_n(a, d, c, d|q) = \frac{\sqrt{\pi} q^{1/8}(q; q)_n(abcdq^{n-1}; q)_\infty}{(1 - abcdq^{n-1})(abq^n, acq^n, adq^n, bcq^n, bdq^n, cdq^n; q)_\infty}.\]  \hspace{1cm} (4.16)

The complications arising in the evaluation of the standard form of the Askey-Wilson \(q\)-beta integral (1.5) can be illustrated by the following short quotation from reference [4]: "This was surprisingly hard, and it has taken over five years before relatively simple ways of evaluating this integral were found".

5 The transformation \(q \to q^{-1}\).

It is necessary to emphasize that the nonstandard orthogonality relation (3.1) admits the transformation \(q \to q^{-1}\) [7, 8]. The standard form of the Askey-Willson integral (1.5) does not in general have this property. Even in the simplest case of vanishing parameters \(a, b, c\) and \(d\), which corresponds to the continuous \(q\)-Hermite polynomials \(H_n(x|q)\), the definition of a weight function for the system of polynomials \(h_n(x; q) = i^{-n}H_n(ix|q^{-1})\) requires a special analysis [13, 14].
Since
\[(z; q^{-1})_\infty = (qz; q)_\infty^{-1},\] (5.1)
the change \(q \to q^{-1}\) (i.e. \(\kappa \to i\kappa\)) in the function \(\rho(\kappa x; a, b, c, d|q)\) appearing in (2.10) and (3.1), transforms it into
\[
\rho(i\kappa x; a, b, c, d|q^{-1}) = \prod_{v=a,b,c,d} (ivq^{\kappa x}, -ivq^{-\kappa x}, q)_\infty = \prod_{v=a,b,c,d} E_q(ivq^{\kappa x})E_q(-ivq^{-\kappa x}), \tag{5.2}
\]
where \(E_q(z) = e_q^{-1}(-z) = (-z; q)_\infty\) [2]. Therefore, under the transformation \(q \to q^{-1}\), the orthogonality relation (3.1) for the Askey-Wilson polynomials with the parameter \(q < 1\) converts into the following orthogonality relation for the Askey-Wilson polynomials with \(q > 1\):
\[
\int_{-\infty}^{\infty} p_m(i\sinh \kappa x; a, b, c, d|q^{-1}) p_n(i\sinh \kappa x; a, b, c, d|q^{-1}) e^{-x^2} \cosh \kappa x dx = \\
\delta_{mn} \tilde{I}_n(a, b, c, d|q^{-1}) \tag{5.3}
\]
The explicit form of \(\tilde{I}_n(a, b, c, d|q^{-1})\) is readily obtained from (4.16), upon making use of the formulas (5.1) and \((a; q^{-1})_n = (a^{-1}; q)_n(a^{-n}q^{-n(n-1)/2})\) [2].

On the other hand, with the aid of the explicit representation for the Askey-Wilson polynomials [1, 2]
\[
p_n(\sin \varphi; a, b, c, d|q) = (ab, ac, ad; q)_n a^{-n} \phi_3 \left[ q^{-n}, abcdq^{n-1}, ia\varphi - iae^{-i\varphi} \right]_{ab, ac, ad; q, q} \tag{5.4}
\]
and the inversion formula (with respect to the transformation \(q \to q^{-1}\)) for the basic hypergeometric series \(\phi_3\) (see [2], p.21, exercise 1.4(i)), it is easy to show that
\[
p_n(x; a, b, c, d|q^{-1}) = (-1)^n (abcd)_n q^{-\frac{3}{2}n(n-1)} p_n(x; a^{-1}, b^{-1}, c^{-1}, d^{-1}|q). \tag{5.5}
\]
Consequently, from (5.3) and (5.5) it follows the orthogonality relation
\[
\int_{-\infty}^{\infty} p_m(i\sinh \kappa x; a^{-1}, b^{-1}, c^{-1}, d^{-1}|q) p_n(i\sinh \kappa x; a^{-1}, b^{-1}, c^{-1}, d^{-1}|q) \rho(i\kappa x; a, b, c, d|q^{-1})^* e^{-x^2} \cosh \kappa x dx = \\
(1 - q^{2n-1}/abcd)(1/abcd; q)_{n-1} \tilde{I}_o(a, b, c, d|q^{-1}) \delta_{mn} \tag{5.6}
\]
for the Askey-Wilson polynomials with the parameters \(|v| > 1, v = a, b, c, d\) and \(0 < q < 1\). The value of the integral \(\tilde{I}_o(a, b, c, d|q^{-1})\) is simple to obtain from (4.15) by means of the formula (5.1).
6 Concluding remarks.

The orthogonality relations (3.1) and (5.6) are bound to be related by the Fourier transformation for the Askey-Wilson functions, analogous to the well-known transformation for the harmonic oscillator wave functions $H_n(x) \exp(-x^2/2)$ (or Hermite functions in the terminology of mathematicians [15, 16]) connecting the coordinate and momentum realizations in quantum mechanics. It should be interesting to compare this Fourier transformation with the $q$-transformations, that reproduce the Askey-Wilson polynomials [17, 18]. For the $q$-Hermite functions $H_n(\sin \kappa x | q) \exp(-x^2/2)$, $q = \exp(-2\kappa^2)$, which are the simplest case of the Askey-Wilson functions with vanishing parameters $a, b, c$, and $d$, such Fourier transformation has the form [5]

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(ixy - x^2/2)H_n(\sin \kappa x | q)dx = i^n q^{n^2/4}h_n(\sinh \kappa y | q) \exp(-y^2/2).$$

The general case needs to be analyzed in greater detail.

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