COVARIANT DEFORMED OSCILLATOR ALGEBRAS

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Abstract

The general form and associativity conditions of deformed oscillator algebras are reviewed. It is shown how the latter can be fulfilled in terms of a solution of the Yang-Baxter equation when this solution has three distinct eigenvalues and satisfies a Birman-Wenzl-Murakami condition. As an example, an $SU_q(n) \times SU_q(m)$-covariant $q$-bosonic algebra is discussed in some details.

1 Introduction

Since the advent of quantum groups and $q$-algebras (see e.g. [1] and references quoted therein), much attention has been paid to deformations of the algebras of bosonic and fermionic creation and annihilation operators [2]-[6]. Different deformations of the latter arise depending on which property of the undeformed operators is preserved.

In the simple case of the $su(2)$ Lie algebra, two pairs of bosonic creation and annihilation operators $a_i^+, a_i$, $i = 1, 2$, give rise to the Jordan-Schwinger realization

$$
J_+ = a_1^+ a_2, \quad J_- = a_2^+ a_1, \quad J_0 = \frac{1}{2}(N_1 - N_2),
$$

where $N_i = a_i^+ a_i$, $i = 1, 2$, are number operators. In addition, the creation operators $a_i^+$, $a_i^+$ (as well as the modified annihilation operators $\tilde{a}_1 = a_2$, $\tilde{a}_2 = -a_1$) are the components $+1/2$ and $-1/2$ of an $su(2)$ spinor, respectively. When extending these two properties to the corresponding $q$-algebra $su_q(2)$ (where $q$ is real and positive), one gets two different sets of $q$-bosonic operators.

On the one hand, those first considered by Biedenharn [2], Macfarlane [3], Sun and Fu [4], give rise to a Jordan-Schwinger realization of $su_q(2)$ of the same type as (1), where $a_i^+$, $a_i$, $i = 1, 2$, now satisfy the relations

$$
a_i a_i^+ - q^{+1} a_i^+ a_i = q^{N_i},
$$

while operators with different indices do still commute, and $a_i^+ a_i = [N_i]_q \equiv (q^{N_i} - q^{-N_i})/(q - q^{-1})$. However, the operators $a_1^+$, $a_2^+$ do not transform any more under a definite representation of the algebra.

1 Directeur de recherches FNRS
On the other hand, the operators \( A_i^\dagger, A_i, i = 1, 2 \), introduced by Pusz and Woronowicz [5], satisfy different relations

\[
\begin{align*}
A_i^\dagger A_j^\dagger - q^{-1} A_j^\dagger A_i^\dagger &= A_i A_j - q A_j A_i = 0, & i < j, \\
A_i A_j^\dagger - q A_j A_i^\dagger &= 0, & i \neq j,
\end{align*}
\]

where the two modes are not independent any more. As a result of this coupling, the operators \( A_i^\dagger, A_2^\dagger \) (as well as \( \tilde{A}_1 = q^{1/2} A_2, \tilde{A}_2 = -q^{-1/2} A_1 \)) are the components \(+1/2\) and \(-1/2\) of an \( su_q(2) \) spinor respectively, but yield an \( su_q(2) \) realization that is substantially more complicated than (1). The algebra (3) has also important covariance properties under the quantum group \( SU_q(2) \), dual to \( su_q(2) \).

The present communication is concerned with the construction of covariant deformed oscillator algebras that generalize the Pusz-Woronowicz algebra for other quantum groups than \( SU_q(2) \) (or more generally \( SU_q(n) \)). The method used will be based on an \( R \)-matrix approach similar to that applied in noncommutative differential geometry [7,8]. In Sec. 2, after reviewing the general form and associativity conditions of deformed oscillator algebras, we will show how to fulfill the latter in terms of a solution of the Yang-Baxter equation with three distinct eigenvalues. The example of an \( SU_q(n) \times SU_q(m) \)-covariant \( q \)-bosonic algebra \( \mathcal{A}_q(n,m) \) will be treated in some details in Sec. 3. Finally, in Sec. 4, an alternative derivation of the same algebra, based upon the \( q \)-algebra \( u_q(n) + u_q(m) \) will be presented.

## 2 Deformed Oscillator Algebras

Let us consider the complex algebras generated by \( I, A_i^\dagger A_i = (A_i^\dagger)^\dagger, i = 1, \ldots, N \), subject to the relations [9,10]

\[
\begin{align*}
A_i^\dagger A_j^\dagger &= X_{ij,k} A_k^\dagger A_j^\dagger, \\
A_i A_j &= X_{ji,k}^* A_k A_i, \\
A_i A_j^\dagger &= \delta_{ij} + Z_{ji,i} A_i^\dagger A_i,
\end{align*}
\]

where \( X \) and \( Z \) are some complex \( N^2 \times N^2 \) matrices, and there are summations over dummy indices. As a consequence of the Hermiticity properties of the generators, \( X^* \) is the complex conjugate of \( X \), and \( Z \) is a Hermitian matrix.

For these algebras to be associative, it is sufficient to require the braid transposition schemes for triples of generators. The braid scheme for \( A_i^\dagger A_j^\dagger A_k^\dagger \) yields the condition

\[
X_{ij,ab} X_{bk,cz} X_{ac,xy} = X_{jk,ab} X_{ia,zc} X_{eb,yz},
\]

i.e., in compact tensor notation, the Yang-Baxter equation for \( X \) (in the “braid” version)

\[
X_{12} X_{23} X_{12} = X_{23} X_{12} X_{23}.
\]
Similarly, for $A_i A^+_j A^+_k$, one gets the two conditions
\begin{equation}
\delta_{ji} \delta_{kx} - X_{jk,ix} + Z_{jk,ix} - X_{jk,ab} Z_{ab,ix} = 0,
\end{equation}
and
\begin{equation}
Z_{kj,ac} Z_{ja,ib} X_{bc,xy} = X_{jk,ab} Z_{bz,ey} Z_{ac,ix},
\end{equation}
which may be written in compact form as
\begin{equation}
(I_{12} - X_{12})(I_{12} + Z_{12}) = 0,
\end{equation}
and
\begin{equation}
Z_{23} Z_{12} X_{23} = X_{12} Z_{23} Z_{12}.
\end{equation}
From the Hermiticity properties of the generators, it follows that the remaining two triple products $A_i A_j A_k$ and $A_i A^+_j A^+_k$ will be associative if $A_i A^+_j A^+_k$ and $A_i A^+_j A^+_k$ are so. Hence, eqs. (6), (9), and (10) are the only associativity conditions of algebra (4).

Let now $R$ be any $N^2 \times N^2$ solution of the Yang-Baxter equation
\begin{equation}
R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}.
\end{equation}
Then the corresponding braid matrix $\hat{R} = \tau R$, where $\tau$ is the twist operator (i.e., $\tau_{ij,kl} = \delta_{il} \delta_{jk}$), satisfies an equation similar to (6).

If $\hat{R}$ has three distinct eigenvalues $\lambda_\alpha$, $\alpha = 1, 2, 3$, and satisfies a Birman-Wenzl-Murakami (BWM) condition\(^2\)
\begin{equation}
(\hat{R} - \lambda_1 I)(\hat{R} - \lambda_2 I)(\hat{R} - \lambda_3 I) = 0,
\end{equation}
then with each eigenspace of $\hat{R}$, one can associate two solutions of the set of associativity conditions (6), (9), and (10). In terms of the projector
\begin{equation}
P_\alpha = \prod_{\beta \neq \alpha} \frac{(\hat{R} - \lambda_\beta I)}{(\lambda_\alpha - \lambda_\beta)}
\end{equation}
onto the eigenspace corresponding to the eigenvalue $\lambda_\alpha$, these two solutions can be written as
\begin{equation}
I - X \simeq P_\alpha \quad \text{and} \quad Z = -\lambda_\alpha^{-1} \hat{R} \quad \text{or} \quad Z = -\lambda_\alpha \hat{R}^{-1}.
\end{equation}
Considering for instance $Z = -\lambda_\alpha^{-1} \hat{R}$ leads to the following deformed oscillator algebra (written in compact tensor form)
\begin{equation}
A^+_2 A^+_1 = S A^+_1 A^+_2, \quad A_1 A_2 = S^* A_2 A_1, \quad A_1 A^+_2 = I_{12} - \lambda_\alpha^{-1} R^{t_1} A^+_2 A_1,
\end{equation}
where $S = \tau X$ is found from (13) and (14), and $t_1$ means transposition with respect to the first space in the tensor product.

Several examples of such deformed oscillator algebras have been worked out so far [9]–[11]. In all cases, the solution of the Yang-Baxter equation that has been considered is the fundamental $R$-matrix of some classical quantum group. In such circumstances, the deformed oscillator algebras

\(^2\)The $SU_q(n)$-covariant algebra constructed by Pusz and Woronowicz [5] corresponds to the simpler case where $\hat{R}$ has only two distinct eigenvalues, and satisfies a Hecke condition (see Sec. 3).
are left invariant under the transformations induced by the quantum group. The construction presented here is not restricted however to such a choice, and any solution of (11) and (12) might actually be used. In a similar way, deformed oscillator algebras differing from that of Pusz-Woronowicz have been built by considering non-standard solutions of the Yang-Baxter equation and the Hecke condition [12].

The algebras constructed in refs. [9]-[11] include both standard and non-standard ones. The former [9,10] are either of $q$-bosonic or $q$-fermionic type, meaning that whenever $q \to 1$, they go over smoothly into ordinary bosonic or fermionic algebras, respectively. The latter [11], on the contrary, do not have such a smooth behaviour, but share instead some features with the quon algebra [13]. In the next section, we shall consider in more details a covariant $q$-bosonic algebra generalizing that of Pusz-Woronowicz.

3 An $SU_q(n) \times SU_q(m)$-Covariant $q$-Bosonic Algebra

The $SU_q(n)$ quantum group [1] is a complex associative algebra generated by $I$ and the noncommutative elements $T_{ij}$, $i, j = 1, \ldots, n$ of an $n \times n$ matrix $T$, subject to the relations

$$RT_1T_2 = T_2T_1R, \quad \det_q T = 1,$$

(16)

and the *-involution condition

$$T^* = (T^{-1})^t,$$

(17)

with $q$ real. In (16), $\det_q$ denotes the quantum determinant, and $R$ is the fundamental $R$-matrix associated with the $A_{n-1}$ series of Lie algebras,

$$R = q \sum_{i=1}^n e_{ii} \otimes e_{ii} + \sum_{i,j=1 \atop i \neq j}^n e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i,j=1 \atop i < j}^n e_{ij} \otimes e_{ji},$$

(18)

where $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$. The coproduct, counit and antipode are defined by

$$\Delta(T) = T_1 \otimes T_2, \quad \epsilon(T) = 1, \quad S(T) = T^{-1},$$

(19)

where $\Delta(T_{ij}) = T_{ik} \otimes T_{kj}$.

The braid matrix $\hat{R}$, corresponding to (18), is a real symmetric matrix with two distinct eigenvalues, $q$ and $-q^{-1}$. Their respective multiplicities are $\frac{1}{2}n(n+1)$ and $\frac{1}{2}n(n-1)$, i.e., the dimensions of the symmetric and antisymmetric irreps $[2\,0]_n$ and $[1^2\,0]_n$ of $SU_q(n)$. The $\hat{R}$-matrix satisfies the Hecke condition

$$(\hat{R} - qI)(\hat{R} + q^{-1}I) = 0.$$

(20)

Similar relations are valid for $SU_q(m)$. Its generators and fundamental $R$-matrix will be denoted by $T_{st}$, $s, t = 1, \ldots, m$, and $R$, respectively, to distinguish them from the corresponding quantities for $SU_q(n)$. Note that $T_{ij}$ and $T_{st}$ are assumed to commute with one another.

For the product $SU_q(n) \times SU_q(m)$, one can introduce a "large" $R$-matrix, $R = q^{-1}R\,R$, of dimension $(nm)^2 \times (nm)^2$ [10]. Its matrix elements are defined by

$$R_{(is)(jt),(ku)(lv)} = q^{-1}R_{ij,kl}\,R_{st,uv}.$$  

(21)
From the properties of the two "small" braid matrices $\hat{R}$ and $\hat{R}$, it follows that $\hat{R} = q^{-1} \hat{R} \hat{R}$ has three distinct eigenvalues $q$, $-q^{-1}$, and $q^{-3}$, with respective multiplicities corresponding to the dimensions of the representations $[20]_n[20]_m$, $[20]_n[120]_m + [120]_n[20]_m$, and $[120]_n[120]_m$ of $SU_q(n) \times SU_q(m)$, and satisfies the BWM condition (12).

By applying the results of the previous section to the antisymmetric (reducible) eigenspace of $\hat{R}$ associated with the eigenvalue $-q^{-1}$, one gets a deformed oscillator algebra of type (15), which will be denoted by $A_q(n,m)$, and whose defining relations are [10]

$$A_i^+ A_j^+ = S A_i^+ A_j^+ A_1 A_2 S, \quad A_2 A_i^+ = A_1 A_2 S, \quad A_2 A_1^+ = I_{21} + q R I A_1^+ A_2,$$

where

$$S = \tau(I - (q + q^{-1})P_A), \quad P_A = \frac{(\hat{R} - q I)(\hat{R} - q^{-2} I)}{(q + q^{-1})(q^{-1} + q^{-3})},$$

and the creation and annihilation operators $A_i^+$, $A_i$ now have two indices, $i = 1, 2, \ldots, n$, and $s = 1, 2, \ldots, m$. Whenever $q \to 1$, $R$ and $S$ go over into $I$, so that (22) becomes an ordinary bosonic algebra.

The defining relations (22) of the $q$-bosonic algebra $A_q(n,m)$ may be rewritten in terms of the two "small" $R$-matrices as

$$RA_i^+ A_j^+ = A_i^+ A_j^+ R, \quad RA_2 A_1 = A_1 A_2 R, \quad A_2 A_1^+ = I_{21} I_{21} + R^{st} R^{lt} A_1^+ A_2,$$

or, in a more explicit form, as

$$R_{ij,kl} A_{lk}^+ A_{it} = A_{ju}^+ A_{iu} R_{uv, st}, \quad R_{ij,kl} A_{it} A_{kt} = A_{iu}^+ A_{ju} R_{uv, st},$$

$$A_{is}^+ A_{jt} = \delta_{ij} \delta_{st} + R_{ki, jl} R_{us, tv} A_{ku}^+ A_{lv}.$$

Let us consider the map $\varphi: A_q(n,m) \to A_q(n,m) \otimes (SU_q(n) \times SU_q(m))$, defined by

$$A_i^+ = \varphi(A_i^+) = A_i^+ T_{ij} T_{is}, \quad A_i = \varphi(A_i) = A_i T_{ij}^* T_{is} = T_{ij}^* T_{is}^{-1} A_i,$$

where the elements $T_{ij}$ and $T_{is}$ of $SU_q(n) \times SU_q(m)$ are assumed to commute with $A_i^+$ and $A_i$. As a consequence of (16) and its counterpart for $SU_q(m)$, this map leaves the defining relations (25) of $A_q(n,m)$ invariant. Hence, the latter is an $SU_q(n) \times SU_q(m)$-covariant algebra.

In the next section, an important part will be played by the modified annihilation operators

$$\hat{A}_i = A_j C_{ji} C_{ts}, \quad C_{ji} = (-1)^{n-j} q^{-(n-2j+1)/2} \delta_{ji}, \quad C_{ts} = (-1)^{m-i} q^{-(m-2t+1)/2} \delta_{ts},$$

where $i' \equiv n + 1 - i$, $s' \equiv m + 1 - s$. Eq. (24) can be rewritten in terms of $A_i^+$, $\hat{A}_i$ as

$$R A_i^+ A_j^+ = A_i^+ A_j^+ R, \quad R \hat{A}_1 \hat{A}_2 = \hat{A}_2 \hat{A}_1 R, \quad \hat{A}_2 A_1^+ = C_{12} C_{12} + q^2 A_i^+ \hat{A}_2 \hat{R}^{-1} \hat{R}^{-1},$$

where $\hat{R}$ is defined by

$$\hat{R} = \sum_{i=1}^n e_{ii} \otimes e_{i'i'} + q \sum_{i,j=1 \atop i \neq j}^n e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i,j=1 \atop i < j}^n (-q)^{i-j+1} e_{ij} \otimes e_{i'j'},$$
and a similar definition holds for $\hat{R}$. Under map $\varphi$ of eq. (26), $\hat{A}_{is}$ is transformed into

\[ \hat{A}_{is}' = \varphi(\hat{A}_{is}) = \hat{A}_{jt} \hat{T}_{js}, \quad \hat{T} \equiv C^{-1}(T^{-1})^{t}C, \quad \hat{T} \equiv C^{-1}(T^{-1})^{t}C. \] (30)

Finally, combining eqs. (18) and (25) yields the detailed form of the $\mathcal{A}_q(n,m)$ defining relations

\[ A_{is}^\dagger A_{jt}^\dagger - q^{-1} A_{jt}^\dagger A_{is}^\dagger = 0, \quad s < t, \]
\[ A_{is}^\dagger A_{js}^\dagger - q^{-1} A_{js}^\dagger A_{is}^\dagger = 0, \quad i < j, \]
\[ A_{is}^\dagger A_{jt} - A_{jt}^\dagger A_{is} = 0, \quad i > j, \quad s < t, \]
\[ A_{is}^\dagger A_{jt} - A_{jt}^\dagger A_{is} = -(q - q^{-1})A_{js}^\dagger A_{it}^\dagger, \quad i < j, \quad s < t, \] (31)

and

\[ A_{is} A_{jt}^\dagger - A_{jt}^\dagger A_{is} = 0, \quad i \neq j, \quad s \neq t, \]
\[ A_{is} A_{js}^\dagger - q A_{js}^\dagger A_{is} = (q - q^{-1}) \sum_{t=1}^{s-1} A_{jt}^\dagger A_{it}, \quad i \neq j, \]
\[ A_{is} A_{it}^\dagger - q A_{it}^\dagger A_{is} = (q - q^{-1}) \sum_{j=1}^{s-1} A_{jt}^\dagger A_{js}, \quad s \neq t, \] (32)

\[ A_{is} A_{is}^\dagger - q^2 A_{is}^\dagger A_{is} = I + (q^2 - 1) \left( \sum_{j=1}^{s-1} A_{js}^\dagger A_{js} + \sum_{t=1}^{s-1} A_{it}^\dagger A_{it} \right) - (q^{-2} - 1) \sum_{j=1}^{s-1} \sum_{t=1}^{s-1} A_{jt}^\dagger A_{jt} \right), \]

together with the Hermitian conjugates of (31). Whenever $m = 1$, substituting $A_{i1}^\dagger, A_i$ for $A_{i1}^\dagger, A_{i1}$ in (31) and (32) yields the Pusz-Woronowicz relations (3) for arbitrary $n$ values. Hence, the covariant $q$-bosonic algebra $\mathcal{A}_q(n,m)$ is a generalization of that of Pusz-Woronowicz for values of $m$ greater than 1.

4 Alternative Derivation in Terms of $u_q(n) + u_q(m)$

An alternative approach to the construction of covariant deformed oscillator algebras, based upon $q$-algebras, has been developed elsewhere [14,15]. In the case of the algebra $\mathcal{A}_q(n,m)$ introduced in the previous section, one considers the $q$-algebra $u_q(n) + u_q(m)$. The Cartan-Chevalley generators of $u_q(n)$ are denoted by $E_{ii} = (E_{ii})^\dagger$, $i = 1, 2, \ldots, n$, $E_{i,i+1}$, $E_{i+1,i} = (E_{i,i+1})^\dagger$, $i = 1, 2, \ldots, n - 1$, and satisfy the commutation relations

\[ [E_{ii}, E_{jj}] = 0, \quad [E_{ii}, E_{j,j+1}] = (\delta_{ij} - \delta_{i,j+1})E_{j,j+1}, \]
\[ [E_{ii}, E_{j+1,j}] = (\delta_{i,j+1} - \delta_{ij})E_{j+1,j}, \quad [E_{i,i+1}, E_{j+1,j}] = \delta_{ij} [H_i]_q, \] (33)

together with the quadratic and cubic $q$-Serre relations. In (33), $H_i \equiv E_{ii} - E_{i+1,i+1}$. The algebra is endowed with a Hopf algebra structure with coproduct $\Delta$, counit $\epsilon$, and antipode $S$, defined by

\[ \Delta(E_{ii}) = E_{ii} \otimes I + I \otimes E_{ii}, \quad \Delta(E_{i,i+1}) = E_{i,i+1} \otimes q^{H_i/2} + q^{-H_i/2} \otimes E_{i,i+1}, \]

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\[
\Delta(E_{i+1,i}) = E_{i+1,i} \otimes q^{H_i/2} + q^{-H_i/2} \otimes E_{i+1,i},
\]
\[
\epsilon(E_{ii}) = \epsilon(E_{i+1,i}) = \epsilon(E_{i+1,i}) = 0,
\]
\[
S(E_{ii}) = -E_{ii}, \quad S(E_{i+1,i}) = -qE_{i+1,i}, \quad S(E_{i+1,i}) = -q^{-1}E_{i+1,i}.
\]

The Cartan-Chevalley generators of \(u_q(m)\) are denoted by \(E_i, s = 1, 2, \ldots, m, \ E_{s+1,i}, \ E_{s,i+1}, s = 1, 2, \ldots, m-1,\) and satisfy relations similar to (33)-(36), while commuting with the generators of \(u_q(n)\).

In the approach based upon \(u_q(n) + u_q(m)\), the \(q\)-bosonic creation operators \(A_{is}^+, i = 1, 2, \ldots, n, s = 1, 2, \ldots, m,\) belonging to \(A_q(n,m)\), are defined as the components of a double irreducible tensor \(T^{[0][0]}_{[n][m]}\) with respect to this \(q\)-algebra. This means that they fulfill the relations

\[
\Delta(A_{i,s}^+) = \delta_{i,s}A_{i,s}^+, \quad \Delta(A_{s+1,i}^+) = \delta_{s-1,i}A_{s-1,i+1}^+, \quad \Delta(A_{s,i+1}^+) = \delta_{s,i}A_{s+1,i}^+,
\]

where, for any \(u_q(n) + u_q(m)\) generator \(X, X(A_{i,s}^+) = \sum X_i S(X_q),\) with \(\Delta(X) = \sum X_i \otimes X_q.\) The modified annihilation operators \(\hat{A}_{i,s}, i = 1, 2, \ldots, n, s = 1, 2, \ldots, m,\) of eq. (27), are similarly defined as the components of a double irreducible tensor \(T^{[0][0]}_{[n][m]}\) with respect to \(u_q(n) + u_q(m),\) and satisfy the relations

\[
\Delta(\hat{A}_{i,s}) = -\delta_{i,s+1}\hat{A}_{i,s}, \quad \Delta(\hat{A}_{s+1,i}) = \delta_{s,i-1}\hat{A}_{s-1,i+1}, \quad \Delta(\hat{A}_{s,i+1}) = \delta_{s,i}\hat{A}_{s+1,i+1},
\]

The operators \(A_{i,s}^+\) and \(\hat{A}_{i,s}\) can be explicitly written down in terms of \(m\) independent copies of the Pusz-Woronowicz operators [14]. By using such expressions and exploiting the tensorial character of the operators, it is straightforward to prove that their \(q\)-commutation relations are given in coupled form by

\[
[A_{i,s}^+, A_{i',s'}^+]^{[0][0]} = [A_{i,s}^+, A_{i',s'}^+]^{[0][0]} = [\hat{A}_{i,s}^+, \hat{A}_{i',s'}^+]^{[0][0]} = [\hat{A}_{i,s}^+, \hat{A}_{i',s'}^+]^{[0][0]} = [\hat{A}_{i,s}^+, \hat{A}_{i',s'}^+]^{[0][0]} = [\hat{A}_{i,s}^+, \hat{A}_{i',s'}^+]^{[0][0]} = 0,
\]

where, for simplicity's sake, the row labels of the coupled \(u_q(n) + u_q(m)\) irreps have been dropped.

In (41), the coupled \(q\)-commutator of two double irreducible tensors \(T^{[1][2]}_{[n][m]}\) and \(U^{[1][2]}_{[n][m]}\) is defined by [14]

\[
[T^{[1][2]}_{[n][m]}, U^{[1][2]}_{[n][m]}]^{[1][2]}_{[n][m]} = [T^{[1][2]}_{[n][m]}, U^{[1][2]}_{[n][m]}]^{[1][2]}_{[n][m]} = [\hat{A}_{i,s}^+, \hat{A}_{i',s'}^+]^{[0][0]} = [\hat{A}_{i,s}^+, \hat{A}_{i',s'}^+]^{[0][0]} = [\hat{A}_{i,s}^+, \hat{A}_{i',s'}^+]^{[0][0]} = 0,
\]

Here

\[
\epsilon = \phi([\lambda_1][n]) + \phi([\lambda_2][m]) - \phi([\lambda_1][n]) + \phi([\lambda_2][m]) + \phi([\lambda_2][m]) - \phi([\lambda_2][m]),
\]

\[
\phi([\lambda_1][n]) = \frac{1}{2} \sum_{i=1}^{n} (n + 1 - 2i) \lambda_{1i}, \quad \phi([\lambda_2][m]) = \frac{1}{2} \sum_{s=1}^{m} (m + 1 - 2s) \lambda_{2s},
\]

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and
\[
[T^{[\lambda_1]_n[\lambda_2]_m} \times U^{[\lambda'_1]_n[\lambda'_2]_m}]^{[\Lambda_1]_n[\Lambda_2]_m}_{(M_1)_n(M_2)_m}
= \sum_{(\mu_1)_n(\mu_2)_m(\mu'_1)_n(\mu'_2)_m} \langle [\lambda_1]_n(\mu_1)_n, [\Lambda_1]_n(M_1)_n \rangle_q \langle [\lambda'_2]_m(\mu'_2)_m, [\Lambda_2]_m(M_2)_m \rangle_q
\]
\[
\times T^{[\lambda'_1]_n[\lambda'_2]_m}_{(\mu'_1)_n(\mu'_2)_m} U^{[\lambda_1]_n[\lambda_2]_m}_{(\mu_1)_n(\mu_2)_m},
\]
where \(\langle , | \rangle_q\) denotes a \(u_q(n)\) or \(u_q(m)\) Wigner coefficient.

By using the values of the latter, eq. (41) can be written in an explicit form [14]. The resulting relations coincide with eqs. (31) and (32), thereby proving the equivalence of the two constructions of \(\mathcal{A}_q(n, m)\) based upon \(SU_q(n) \times SU_q(m)\) and \(u_q(n) + u_q(m)\), respectively.

References