COVARIANT DEFORMED OSCILLATOR ALGEBRAS

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Abstract

The general form and associativity conditions of deformed oscillator algebras are reviewed. It is shown how the latter can be fulfilled in terms of a solution of the Yang-Baxter equation when this solution has three distinct eigenvalues and satisfies a Birman-Wenzl-Murakami condition. As an example, an $SU_q(n) \times SU_q(m)$-covariant $q$-bosonic algebra is discussed in some details.

1 Introduction

Since the advent of quantum groups and $q$-algebras (see e.g. [1] and references quoted therein), much attention has been paid to deformations of the algebras of bosonic and fermionic creation and annihilation operators [2]–[6]. Different deformations of the latter arise depending on which property of the undeformed operators is preserved.

In the simple case of the $su(2)$ Lie algebra, two pairs of bosonic creation and annihilation operators $a_i^+, a_i, i = 1, 2,$ give rise to the Jordan-Schwinger realization

$$J_+ = a_1^+ a_2, \quad J_- = a_2^+ a_1, \quad J_0 = \frac{1}{2}(N_1 - N_2), \quad (1)$$

where $N_i = a_i^+ a_i, i = 1, 2,$ are number operators. In addition, the creation operators $a_i^+, a_i$ (as well as the modified annihilation operators $\tilde{a}_1 = a_2, \tilde{a}_2 = -a_1$) are the components $+1/2$ and $-1/2$ of an $su(2)$ spinor, respectively. When extending these two properties to the corresponding $q$-algebra $su_q(2)$ (where $q$ is real and positive), one gets two different sets of $q$-bosonic operators.

On the one hand, those first considered by Biedenharn [2], Macfarlane [3], Sun and Fu [4], give rise to a Jordan-Schwinger realization of $su_q(2)$ of the same type as (1), where $a_i^+, a_i, i = 1, 2,$ now satisfy the relations

$$a_i a_i^+ - q^{-1} a_i^+ a_i = q^{N_i}, \quad (2)$$

while operators with different indices do still commute, and $a_i^+ a_i = [N_i]_q \equiv (q^{N_i} - q^{-N_i})/(q - q^{-1}).$ However, the operators $a_1^+, a_2^+$ do not transform any more under a definite representation of the algebra.

1 Directeur de recherches FNRS
On the other hand, the operators $A_i^\dagger$, $A_i$, $i = 1, 2$, introduced by Pusz and Woronowicz [5], satisfy different relations

$$
A_i^\dagger A_j^\dagger - q^{-1} A_j^\dagger A_i^\dagger = A_i A_j - q A_j A_i = 0, \quad i < j,
$$

$$
A_i A_j^\dagger - q A_j A_i^\dagger = 0, \quad i \neq j,
$$

$$
A_i A_i^\dagger - q^2 A_i A_i^\dagger = I + (q^2 - 1) \sum_{j=1}^{i-1} A_j A_j^\dagger,
$$

where the two modes are not independent any more. As a result of this coupling, the operators $A_i^\dagger$, $A_2^\dagger$ (as well as $\tilde{A}_1 = q^{1/2} A_2$, $\tilde{A}_2 = -q^{-1/2} A_1$) are the components $+1/2$ and $-1/2$ of an $su_q(2)$ spinor respectively, but yield an $su_q(2)$ realization that is substantially more complicated than (1). The algebra (3) has also important covariance properties under the quantum group $SU_q(2)$, dual to $su_q(2)$.

The present communication is concerned with the construction of covariant deformed oscillator algebras that generalize the Pusz-Woronowicz algebra for other quantum groups than $SU_q(2)$ (or more generally $SU_q(n)$). The method used will be based on an $R$-matrix approach similar to that applied in noncommutative differential geometry [7,8]. In Sec. 2, after reviewing the general form and associativity conditions of deformed oscillator algebras, we will show how to fulfill the latter in terms of a solution of the Yang-Baxter equation with three distinct eigenvalues. The example of an $SU_q(n) \times SU_q(m)$-covariant $q$-bosonic algebra $A_q(n,m)$ will be treated in some details in Sec. 3. Finally, in Sec. 4, an alternative derivation of the same algebra, based upon the $q$-algebra $u_q(n) + u_q(m)$ will be presented.

## 2 Deformed Oscillator Algebras

Let us consider the complex algebras generated by $I$, $A_i^\dagger A_i = (A_i^\dagger)^\dagger$, $i = 1, \ldots, N$, subject to the relations [9,10]

$$
A_i^\dagger A_j^\dagger = X_{ij,kl} A_k^\dagger A_l^\dagger,
$$

$$
A_i A_j = X_{ji,kl}^* A_k A_l,
$$

$$
A_i A_j^\dagger = \delta_{ij} + Z_{ji,kl} A_k A_l^\dagger,
$$

where $X$ and $Z$ are some complex $N^2 \times N^2$ matrices, and there are summations over dummy indices. As a consequence of the Hermiticity properties of the generators, $X^*$ is the complex conjugate of $X$, and $Z$ is a Hermitian matrix.

For these algebras to be associative, it is sufficient to require the braid transposition schemes for triples of generators. The braid scheme for $A_i^\dagger A_j^\dagger A_k^\dagger$ yields the condition

$$
X_{ij,ab} X_{bk,cz} X_{ac,xy} X_{ji,ab} = X_{jk,ab} X_{ia,zc} X_{eb,yz},
$$

i.e., in compact tensor notation, the Yang-Baxter equation for $X$ (in the "braid" version)

$$
X_{12} X_{23} X_{12} = X_{23} X_{12} X_{23}.
$$

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Similarly, for \( A_i A_j A_k \), one gets the two conditions

\[
\delta_{ji} \delta_{kj} - X_{jk,ix} + Z_{jk,ix} - X_{jk,ab} Z_{ab,ix} = 0,
\]

and

\[
Z_{kj,ac} Z_{ja,ib} X_{bc,xy} = X_{jk,ab} Z_{bc,xy} Z_{ac,ix},
\]

which may be written in compact form as

\[
(I_{12} - X_{12})(I_{12} + Z_{12}) = 0,
\]

and

\[
Z_{23} Z_{12} X_{23} = X_{12} Z_{23} Z_{12}.
\]

From the Hermiticity properties of the generators, it follows that the remaining two triple products \( A_i A_j A_k \) and \( A_i A_j A_k^\dagger \) will be associative if \( A_i A_j A_k^\dagger \) and \( A_i A_j A_k \) are so. Hence, eqs. (6), (9), and (10) are the only associativity conditions of algebra (4).

Let now \( R \) be any \( N^2 \times N^2 \) solution of the Yang-Baxter equation

\[
R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}.
\]

Then the corresponding braid matrix \( \hat{R} = \tau R \), where \( \tau \) is the twist operator (i.e., \( \tau_{ij,kl} = \delta_{ii} \delta_{jk} \)), satisfies an equation similar to (6).

If \( \hat{R} \) has three distinct eigenvalues \( \lambda_\alpha, \alpha = 1, 2, 3 \), and satisfies a Birman-Wenzl-Murakami (BWM) condition \(^2\)

\[
(\hat{R} - \lambda_1 I)(\hat{R} - \lambda_2 I)(\hat{R} - \lambda_3 I) = 0,
\]

then with each eigenspace of \( \hat{R} \), one can associate two solutions of the set of associativity conditions (6), (9), and (10). In terms of the projector

\[
P_\alpha = \prod_{\beta \neq \alpha} \frac{\hat{R} - \lambda_\beta I}{(\lambda_\alpha - \lambda_\beta)}
\]

onto the eigenspace corresponding to the eigenvalue \( \lambda_\alpha \), these two solutions can be written as

\[
I - X \simeq P_\alpha \quad \text{and} \quad Z = -\lambda_\alpha^{-1} \hat{R} \quad \text{or} \quad Z = -\lambda_\alpha \hat{R}^{-1}.
\]

Considering for instance \( Z = -\lambda_\alpha^{-1} \hat{R} \) leads to the following deformed oscillator algebra (written in compact tensor form)

\[
A_2^\dagger A_1^\dagger = SA_1^\dagger A_2^\dagger, \quad A_1 A_2 = S^* A_2 A_1, \quad A_1 A_2^\dagger = I_{12} - \lambda_\alpha^{-1} R^t \ A_2^\dagger A_2,
\]

where \( S = \tau X \) is found from (13) and (14), and \( t_1 \) means transposition with respect to the first space in the tensor product.

Several examples of such deformed oscillator algebras have been worked out so far [9]-[11]. In all cases, the solution of the Yang-Baxter equation that has been considered is the fundamental \( R \)-matrix of some classical quantum group. In such circumstances, the deformed oscillator algebras

\(^2\)The \( SU_q(n) \)-covariant algebra constructed by Pusz and Woronowicz [5] corresponds to the simpler case where \( \hat{R} \) has only two distinct eigenvalues, and satisfies a Hecke condition (see Sec. 3).
are left invariant under the transformations induced by the quantum group. The construction presented here is not restricted however to such a choice, and any solution of (11) and (12) might actually be used. In a similar way, deformed oscillator algebras differing from that of Pusz-Woronowicz have been built by considering non-standard solutions of the Yang-Baxter equation and the Hecke condition [12].

The algebras constructed in refs. [9]-[11] include both standard and non-standard ones. The former [9,10] are either of $q$-bosonic or $q$-fermionic type, meaning that whenever $q \to 1$, they go over smoothly into ordinary bosonic or fermionic algebras, respectively. The latter [11], on the contrary, do not have such a smooth behaviour, but share instead some features with the quon algebra [13]. In the next section, we shall consider in more details a covariant $q$-bosonic algebra generalizing that of Pusz-Woronowicz.

# 3 An $SU_q(n) \times SU_q(m)$-Covariant $q$-Bosonic Algebra

The $SU_q(n)$ quantum group [1] is a complex associative algebra generated by $I$ and the noncommutative elements $T_{ij}$, $i, j = 1, \ldots, n$ of an $n \times n$ matrix $T$, subject to the relations

$$RT_1T_2 = T_2T_1R, \quad \text{det}_q T = 1,$$

and the $\ast$-involution condition

$$T^\ast = (T^{-1})^t,$$

with $q$ real. In (16), $\text{det}_q$ denotes the quantum determinant, and $R$ is the fundamental $R$-matrix associated with the $A_{n-1}$ series of Lie algebras,

$$R = q \sum_{i=1}^{n} e_{ii} \otimes e_{ii} + \sum_{i,j=1 \atop i \neq j}^{n} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i,j=1 \atop i < j}^{n} e_{ij} \otimes e_{ji},$$

where $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$. The coproduct, counit and antipode are defined by

$$\Delta(T) = T_1 \otimes T_2, \quad \epsilon(T) = 1, \quad S(T) = T^{-1},$$

where $\Delta(T_{ij}) = T_{ik} \otimes T_{kj}$.

The braid matrix $\hat{R}$, corresponding to (18), is a real symmetric matrix with two distinct eigenvalues, $q$ and $-q^{-1}$. Their respective multiplicities are $\frac{1}{2}n(n+1)$ and $\frac{1}{2}n(n-1)$, i.e., the dimensions of the symmetric and antisymmetric irreps $[20]_n$ and $[1^20]_n$ of $SU_q(n)$. The $\hat{R}$-matrix satisfies the Hecke condition

$$(\hat{R} - qI)(\hat{R} + q^{-1}I) = 0.$$

Similar relations are valid for $SU_q(m)$. Its generators and fundamental $R$-matrix will be denoted by $T_{st}$, $s, t = 1, \ldots, m$, and $R$, respectively, to distinguish them from the corresponding quantities for $SU_q(n)$. Note that $T_{ij}$ and $T_{st}$ are assumed to commute with one another.

For the product $SU_q(n) \times SU_q(m)$, one can introduce a "large" $R$-matrix, $R = q^{-1}R \mathcal{R}$, of dimension $(nm)^2 \times (nm)^2$ [10]. Its matrix elements are defined by

$$R_{(is)(jt),(ku)(lv)} = q^{-1}R_{ij,kl}\mathcal{R}_{st,uv}.$$
From the properties of the two "small" braid matrices \( \hat{R} \) and \( \hat{\mathcal{R}} \), it follows that \( \hat{R} = q^{-1} \hat{R} \hat{\mathcal{R}} \) has three distinct eigenvalues \( q \), \( -q^{-1} \), and \( q^{-3} \), with respective multiplicities corresponding to the dimensions of the representations \([20]_n [20]_m \), \([20]_n [120]_m + [120]_n [20]_m \), and \([120]_n [120]_m \) of \( SU_q(n) \times SU_q(m) \), and satisfies the BWM condition (12).

By applying the results of the previous section to the antisymmetric (reducible) eigenspace of \( \hat{R} \) associated with the eigenvalue \( -q^{-1} \), one gets a deformed oscillator algebra of type (15), which will be denoted by \( A_q(n, m) \), and whose defining relations are \([10]\)

\[
A_i^\dagger A_j^\dagger = S A_i^\dagger A_j^\dagger, \quad A_2 A_1 = A_1 A_2 S, \quad A_2 A_1^\dagger = I_{21} + q R^t A_1^\dagger A_2, \tag{22}
\]

where

\[
S = \tau(I - (q + q^{-1}) \mathcal{P}_A), \quad \mathcal{P}_A = \frac{(\hat{R} - qI)(\hat{R} - q^{-3}I)}{(q + q^{-1})(q^{-1} + q^{-3})}, \tag{23}
\]

and the creation and annihilation operators \( A_i^\dagger, A_i \) now have two indices, \( i = 1, 2, \ldots, n \), and \( s = 1, 2, \ldots, m \). Whenever \( q \to 1 \), \( R \) and \( S \) go over into \( I \), so that (22) becomes an ordinary bosonic algebra.

The defining relations (22) of the \( q \)-bosonic algebra \( A_q(n, m) \) may be rewritten in terms of the two "small" \( \hat{R} \)-matrices as

\[
R A_1^\dagger A_2^\dagger = A_2^\dagger A_1^\dagger R, \quad R A_2 A_1 = A_1 A_2 R, \quad A_2 A_1^\dagger = I_{21} I_{21} + R^t R^t A_1^\dagger A_2, \tag{24}
\]

or, in a more explicit form, as

\[
R_{ij,kl} A_{ks}^\dagger A_{lt}^\dagger = A_{ju}^\dagger R_{uv, st}, \quad R_{ij,kl} A_{lt} A_{ks} = A_{iu} R_{uv, st}, \quad A_{is} A_j^\dagger = \delta_{ij} \delta_{st} + R_{ki,kl} R_{us, tv} A_{ku}^\dagger A_{lu}. \tag{25}
\]

Let us consider the map \( \varphi : A_q(n, m) \to A_q(n, m) \otimes (SU_q(n) \times SU_q(m)) \), defined by

\[
A_{is}^\dagger = \varphi(A_{is}^\dagger) = A_{jt}^\dagger T_{is} T_{jt}, \quad |A_{is}^\dagger = \varphi(A_{is}) = A_{jt} T_{is} T_{jt}^* = T_{ij}^{-1} T_{st}^{-1} A_{jt}, \tag{26}
\]

where the elements \( T_{ij} \) and \( T_{st} \) of \( SU_q(n) \times SU_q(m) \) are assumed to commute with \( A_i^\dagger \) and \( A_i \). As a consequence of (16) and its counterpart for \( SU_q(m) \), this map leaves the defining relations (25) of \( A_q(n, m) \) invariant. Hence, the latter is an \( SU_q(n) \times SU_q(m) \)-covariant algebra.

In the next section, an important part will be played by the modified annihilation operators

\[
A_{is} = A_{jt} C_{jt} C_{is}, \quad C_{ji} = (-1)^{n-j} q^{-(n-2j+1)/2} \delta_{ji}, \quad C_{is} = (-1)^{m-i} q^{-(m-2i+1)/2} \delta_{is}, \tag{27}
\]

where \( i' = n + 1 - i, s' = m + 1 - s \). Eq. (24) can be rewritten in terms of \( A_{is}^\dagger, \hat{A}_{is} \) as

\[
R A_1^\dagger A_2^\dagger = A_2^\dagger A_1^\dagger R, \quad \hat{R} A_1 A_2 = \hat{A}_1 \hat{A}_2 R, \quad \hat{A}_2 A_1^\dagger = C_{12} C_{12} + q R A_1^\dagger A_2 \hat{R}^{-1} \hat{R}^{-1}, \tag{28}
\]

where \( \hat{R} \) is defined by

\[
\hat{R} = \sum_{i=1}^n e_{ii} \otimes e_{ii'} + q \sum_{i,j=1 \atop i \neq j}^n e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i,j=1 \atop i < j}^n (-q)^{i-j+1} e_{ij} \otimes e_{i'j'}. \tag{29}
\]
and a similar definition holds for $\tilde{\mathcal{R}}$. Under map $\varphi$ of eq. (26), $\hat{A}_{is}$ is transformed into

$$\hat{A}_{is}' = \varphi(\hat{A}_{is}) = \hat{A}_{jt}\hat{T}_{j}^{T}\hat{T}_{is}, \quad \hat{T} \equiv C^{-1}(T^{-1})^{T}C, \quad \hat{T} \equiv C^{-1}(T^{-1})^{T}C. \quad (30)$$

Finally, combining eqs. (18) and (25) yields the detailed form of the $\mathcal{A}_{q}(n, m)$ defining relations

$$\begin{align*}
A_{is}^{\dagger}A_{is}^{\dagger} - q^{-1}A_{is}^{\dagger}A_{is} &= 0, \quad s < t, \\
A_{is}^{\dagger}A_{js}^{\dagger} - q^{-1}A_{js}^{\dagger}A_{is} &= 0, \quad i < j, \\
A_{is}^{\dagger}A_{jt}^{\dagger} - A_{jt}^{\dagger}A_{is} &= 0, \quad i > j, \quad s < t, \\
A_{is}^{\dagger}A_{jt}^{\dagger} - A_{jt}^{\dagger}A_{is} &= -(q - q^{-1})A_{js}^{\dagger}A_{it}^{\dagger}, \quad i < j, \quad s < t, \quad (31)
\end{align*}$$

and

$$\begin{align*}
A_{is}A_{jt}^{\dagger} - A_{jt}^{\dagger}A_{is} &= 0, \quad i \neq j, \quad s \neq t, \\
A_{is}A_{js}^{\dagger} - qA_{js}^{\dagger}A_{is} &= (q - q^{-1})\sum_{t=1}^{s-1}A_{jt}^{\dagger}A_{it}, \quad i \neq j, \\
A_{is}A_{it}^{\dagger} - qA_{it}^{\dagger}A_{is} &= (q - q^{-1})\sum_{j=1}^{s-1}A_{jt}^{\dagger}A_{js}, \quad s \neq t, \\
A_{is}A_{is}^{\dagger} - q^{2}A_{is}^{\dagger}A_{is} &= I + (q^{2} - 1)\left(\sum_{j=1}^{s-1}A_{js}^{\dagger}A_{js} + \sum_{t=1}^{s-1}A_{it}^{\dagger}A_{it}\right)
\end{align*} \quad (32)$$

together with the Hermitian conjugates of (31). Whenever $m = 1$, substituting $A_{i1}^{\dagger}, A_{i}$ for $A_{i1}^{\dagger}, A_{i1}$ in (31) and (32) yields the Pusz-Woronowicz relations (3) for arbitrary $n$ values. Hence, the covariant $q$-bosonic algebra $\mathcal{A}_{q}(n, m)$ is a generalization of that of Pusz-Woronowicz for values of $m$ greater than 1.

### 4 Alternative Derivation in Terms of $u_{q}(n) + u_{q}(m)$

An alternative approach to the construction of covariant deformed oscillator algebras, based upon $q$-algebras, has been developed elsewhere [14,15]. In the case of the algebra $\mathcal{A}_{q}(n, m)$ introduced in the previous section, one considers the $q$-algebra $u_{q}(n) + u_{q}(m)$. The Cartan-Chevalley generators of $u_{q}(n)$ are denoted by $E_{ii} = (E_{ii})^{\dagger}$, $i = 1, 2, \ldots, n$, $E_{i,i+1}$, $E_{i+1,i} = (E_{i+1,i})^{\dagger}$, $i = 1, 2, \ldots, n - 1$, and satisfy the commutation relations

$$\begin{align*}
[E_{ii}, E_{jj}] &= 0, \quad [E_{ii}, E_{j,j+1}] = (\delta_{ij} - \delta_{i,j+1})E_{j,j+1}, \\
[E_{ii}, E_{j+1,i}] &= (\delta_{i,j+1} - \delta_{ij})E_{j+1,j}, \quad [E_{i,i+1}, E_{j+1,i}] = \delta_{ij}[H_{i}]_{q}, \quad (33)
\end{align*}$$

together with the quadratic and cubic $q$-Serre relations. In (33), $H_{i} \equiv E_{ii} - E_{i+1,i+1}$. The algebra is endowed with a Hopf algebra structure with coproduct $\Delta$, counit $\epsilon$, and antipode $S$, defined by

$$\begin{align*}
\Delta(E_{ii}) &= E_{ii} \otimes I + I \otimes E_{ii}, \quad \Delta(E_{i,i+1}) = E_{i,i+1} \otimes q^{H_{i}/2} + q^{-H_{i}/2} \otimes E_{i,i+1},
\end{align*}$$
\[ \Delta(E_{i+1,i}) = E_{i+1,i} \otimes q^{H_i/2} + q^{-H_i/2} \otimes E_{i+1,i}, \]  
\[ \epsilon(E_{ii}) = \epsilon(E_{i+1,i}) = 0, \]  
\[ S(E_{ii}) = qE_{i+1,i}, \quad S(E_{i+1,i}) = -q^{-1}E_{i+1,i}. \]  
(34)  
(35)  
(36)

The Cartan-Chevalley generators of \( u_q(m) \) are denoted by \( E_{ss}, s = 1, 2, \ldots, m, \) \( E_{s,s+1}, E_{s+1,s}, \) \( s = 1, 2, \ldots, m-1, \) and satisfy relations similar to (33)-(36), while commuting with the generators of \( u_q(n) \).

In the approach based upon \( u_q(n) + u_q(m) \), the \( q \)-bosonic creation operators \( A_{ii}^+, i = 1, 2, \ldots, n, \) \( s = 1, 2, \ldots, m, \) belonging to \( u_q(n,m) \), are defined as the components of a double irreducible tensor \( T^{[0]_n[0]_m} \) with respect to this \( q \)-algebra. This means that they fulfill the relations

\[ E_{jj}(A_{is}^+) = \delta_{ji}A_{is}^+, \quad E_{j,j+1}(A_{is}^+) = \delta_{ji-1}A_{i-1,s}^+, \quad E_{j+1,j}(A_{is}^+) = \delta_{ji}A_{i+1,s}^+, \]  
(37)

\[ E_{ii}(A_{is}^+) = \delta_{is}A_{is}^+, \quad E_{i,i+1}(A_{is}^+) = \delta_{i,s-1}A_{i,s-1}, \quad E_{i+1,i}(A_{is}^+) = \delta_{is}A_{i+1,s+1}, \]  
(38)

where, for any \( u_q(n) + u_q(m) \) generator \( X, X(A_{is}^+) \) denotes the quantum adjoint action \( X(A_{is}^+) = \sum_r X_r^iA_{is}^*S_r^{ii}X_r^j \), with \( \Delta(X) = \sum_r X_r^i \otimes X_r^j \). The modified annihilation operators \( \hat{A}_{is}, i = 1, 2, \ldots, n, \) \( s = 1, 2, \ldots, m, \) of eq. (27), are similarly defined as the components of a double irreducible tensor \( T^{[0]_n[0]_m} \) with respect to \( u_q(n) + u_q(m) \), and satisfy the relations

\[ E_{jj}(\hat{A}_{is}) = -\delta_{ji}\hat{A}_{is}, \quad E_{j,j+1}(\hat{A}_{is}) = \delta_{ji-1}\hat{A}_{i-1,s}, \quad E_{j+1,j}(\hat{A}_{is}) = \delta_{ji}\hat{A}_{i+1,s}, \]  
(39)

\[ E_{ii}(\hat{A}_{is}) = -\delta_{is}\hat{A}_{is}, \quad E_{i,i+1}(\hat{A}_{is}) = \delta_{i,s-1}\hat{A}_{i,s-1}, \quad E_{i+1,i}(\hat{A}_{is}) = \delta_{is}\hat{A}_{i+1,s+1}, \]  
(40)

The operators \( A_{is}^+ \) and \( \hat{A}_{is} \) can be explicitly written down in terms of \( m \) independent copies of the Pusz-Woronowicz operators [14]. By using such expressions and exploiting the tensorial character of the operators, it is straightforward to prove that their \( q \)-commutation relations are given in coupled form by

\[
\left[ A_{1}^+, A_{1}^+ \right]^{[20]_n[20]_m} = \left[ A_{1}^+, A_{1}^+ \right]^{[20]_n[20]_m} = \left[ A_{1}^+, A_{1}^+ \right]^{[0]-2}n[0]-2m = \left[ A_{1}^+, A_{1}^+ \right]^{[0]-2}n[0]-2m = 0, \\
\left[ \hat{A}_{1}, A_{1}^+ \right]^{[10]-1}n[10]-1m = \left[ \hat{A}_{1}, A_{1}^+ \right]^{[10]-1}n[10]-1m = \left[ \hat{A}_{1}, A_{1}^+ \right]^{[0]}n[0]-1m = 0, \\
\left[ \hat{A}_{1}, A_{1}^+ \right]^{[0]}n[0]-1m = \sqrt{[n]_q}q^{|m|}_q, 
\]

(41)

where, for simplicity's sake, the row labels of the coupled \( u_q(n) + u_q(m) \) irreps have been dropped. In (41), the coupled \( q \)-commutator of two double irreducible tensors \( T^{[A_1]_n[A_2]_m} \) and \( U^{[A_1]_n[A_2]_m} \) is defined by [14]

\[
\left[ T^{[A_1]_n[A_2]_m}, U^{[A_1]_n[A_2]_m} \right]^{[A_1]_n[A_2]_m} = \left[ T^{[A_1]_n[A_2]_m}, U^{[A_1]_n[A_2]_m} \right]^{[A_1]_n[A_2]_m} = \left[ T^{[A_1]_n[A_2]_m}, U^{[A_1]_n[A_2]_m} \right]^{[A_1]_n[A_2]_m} = (-1)^{q^{|a|}} U^{[A_1]_n[A_2]_m} T^{[A_1]_n[A_2]_m} + (-1)^{q^{|a|}} U^{[A_1]_n[A_2]_m} T^{[A_1]_n[A_2]_m}. 
\]

(42)

Here

\[
\epsilon = \phi([A_1]_n) + \phi([A_1]_n) - \phi([A_1]_n) + \phi([A_1]_n) + \phi([A_2]_m) + \phi([A_2]_m) - \phi([A_2]_m), \\
\phi([A_1]_n) = \frac{1}{2} \sum_{i=1}^{n} (n + 1 - 2i) \lambda_{1i}, \quad \phi([A_2]_m) = \frac{1}{2} \sum_{s=1}^{m} (m + 1 - 2s) \lambda_{2s}, 
\]

(43)
and

\[ T^{(\lambda_1)_{n_1}(\lambda_2)_{m_1}} \times U^{(\lambda_1')_{n_1}(\lambda_2')_{m_1}} \] 

\[ = \sum_{(\mu_1)_{n_1}(\mu_2)_{n_2}} \langle \lambda_1|_{n_1}(\mu_1)|_{n_1}\rangle \langle \lambda_1'|_{n_1}(\mu_2')|_{n_2}\rangle \langle \lambda_2|_{m_2}(\mu_2)|_{m_2}\rangle \langle \lambda_2'|_{m_2}(\mu_2')|_{m_2}\rangle \]

\[ \times T^{(\lambda_1)_{n_1}(\lambda_2)_{m_1}} U^{(\lambda_1')_{n_1}(\lambda_2')_{m_1}} \]  

(44)

where \( \langle \ , \ | \rangle_q \) denotes a \( u_q(n) \) or \( u_q(m) \) Wigner coefficient.

By using the values of the latter, eq. (41) can be written in an explicit form [14]. The resulting relations coincide with eqs. (31) and (32), thereby proving the equivalence of the two constructions of \( \mathcal{A}_q(n,m) \) based upon \( SU_q(n) \times SU_q(m) \) and \( u_q(n) + u_q(m) \), respectively.

References