COVARIANT DEFORMED OSCILLATOR ALGEBRAS

C. Quesne

Physique Nucléaire Théorique et Physique Mathématique, Université Libre de Bruxelles,
Campus de la Plaine CP229, Bd. du Triomphe, B1050 Brussels, Belgium

Abstract

The general form and associativity conditions of deformed oscillator algebras are reviewed. It is shown how the latter can be fulfilled in terms of a solution of the Yang-Baxter equation when this solution has three distinct eigenvalues and satisfies a Birman-Wenzl-Murakami condition. As an example, an SU_q(n) \times SU_q(m)-covariant q-bosonic algebra is discussed in some details.

1 Introduction

Since the advent of quantum groups and q-algebras (see e.g. [1] and references quoted therein), much attention has been paid to deformations of the algebras of bosonic and fermionic creation and annihilation operators [2]–[6]. Different deformations of the latter arise depending on which property of the undeformed operators is preserved.

In the simple case of the su(2) Lie algebra, two pairs of bosonic creation and annihilation operators \( a_i^\dagger, a_i, i = 1, 2 \), give rise to the Jordan-Schwinger realization

\[
J_+ = a_1^\dagger a_2, \quad J_- = a_2^\dagger a_1, \quad J_0 = \frac{1}{2}(N_1 - N_2),
\]

where \( N_i = a_i^\dagger a_i, i = 1, 2 \), are number operators. In addition, the creation operators \( a_i^\dagger, a_i^\dagger \) (as well as the modified annihilation operators \( \hat{a}_1 = a_1, \hat{a}_2 = -a_1 \)) are the components +1/2 and -1/2 of an \( su(2) \) spinor, respectively. When extending these two properties to the corresponding \( q \)-algebra \( su_q(2) \) (where \( q \) is real and positive), one gets two different sets of \( q \)-bosonic operators.

On the one hand, those first considered by Biedenharn [2], Macfarlane [3], Sun and Fu [4], give rise to a Jordan-Schwinger realization of \( su_q(2) \) of the same type as (1), where \( a_i^\dagger, a_i, i = 1, 2 \), now satisfy the relations

\[
a_i a_i = q^{N_i},
\]

while operators with different indices do still commute, and \( a_i^\dagger a_i = [N_i]_q \equiv (q^{N_i} - q^{-N_i})/(q - q^{-1}) \). However, the operators \( a_1^\dagger, a_2^\dagger \) do not transform any more under a definite representation of the algebra.

1Directeur de recherches FNRS
On the other hand, the operators $A_i^\dagger$, $A_i$, $i = 1, 2$, introduced by Pusz and Woronowicz [5], satisfy different relations

\begin{align}
A_i^\dagger A_j^\dagger - q^{-1} A_j^\dagger A_i^\dagger &= A_i A_j - q A_j A_i = 0, & i < j, \\
A_i A_j - q A_j A_i &= 0, & i \neq j, \\
A_i A_i^\dagger - q^2 A_i^\dagger A_i &= I + (q^2 - 1) \sum_{j=1}^{i-1} A_j^\dagger A_j,
\end{align}

where the two modes are not independent any more. As a result of this coupling, the operators $A_i^\dagger$, $A_i$ (as well as $\tilde{A}_1 = q^{1/2} A_2$, $\tilde{A}_2 = -q^{-1/2} A_1$) are the components $+1/2$ and $-1/2$ of an $su_q(2)$ spinor respectively, but yield an $su_q(2)$ realization that is substantially more complicated than (1). The algebra (3) has also important covariance properties under the quantum group $SU_q(2)$, dual to $su_q(2)$.

The present communication is concerned with the construction of covariant deformed oscillator algebras that generalize the Pusz-Woronowicz algebra for other quantum groups than $SU_q(2)$ (or more generally $SU_q(n)$). The method used will be based on an $R$-matrix approach similar to that applied in noncommutative differential geometry [7,8]. In Sec. 2, after reviewing the general form and associativity conditions of deformed oscillator algebras, we will show how to fulfill the latter in terms of a solution of the Yang-Baxter equation with three distinct eigenvalues. The example of an $SU_q(n) \times SU_q(m)$-covariant $q$-bosonic algebra $A_q(n,m)$ will be treated in some details in Sec. 3. Finally, in Sec. 4, an alternative derivation of the same algebra, based upon the $q$-algebra $u_q(n) + u_q(m)$ will be presented.

## 2 Deformed Oscillator Algebras

Let us consider the complex algebras generated by $I$, $A_i^\dagger A_i = (A_i^\dagger)^\dagger$, $i = 1, \ldots, N$, subject to the relations [9,10]

\begin{align}
A_i^\dagger A_j^\dagger &= X_{ij,kl} A_k^\dagger A_l^\dagger, \\
A_i A_j &= X_{ji,lk} A_k A_l, \\
A_i A_j^\dagger &= \delta_{ij} + Z_{ji,ik} A_k A_l,
\end{align}

where $X$ and $Z$ are some complex $N^2 \times N^2$ matrices, and there are summations over dummy indices. As a consequence of the Hermiticity properties of the generators, $X^\ast$ is the complex conjugate of $X$, and $Z$ is a Hermitian matrix.

For these algebras to be associative, it is sufficient to require the braid transposition schemes for triples of generators. The braid scheme for $A_i^\dagger A_j^\dagger A_k^\dagger$ yields the condition

\begin{align}
X_{ij,ab} X_{bk,cz} X_{ac,xy} &= X_{jk,ab} X_{ia,xc} X_{eb,yz},
\end{align}

i.e., in compact tensor notation, the Yang-Baxter equation for $X$ (in the “braid” version)

\begin{align}
X_{12} X_{23} X_{12} &= X_{23} X_{12} X_{23}.
\end{align}
Similarly, for $A_iA_j^k A_k^l$, one gets the two conditions

$$
\delta_{ji} \delta_{kx} - X_{jk,ix} + Z_{jk,ix} - X_{jk,ab} Z_{ab,ix} = 0,
$$
and

$$
Z_{kz,ac} Z_{ja,ib} X_{bc,xy} = X_{jk,ab} Z_{bz,cy} Z_{ac,ix},
$$
which may be written in compact form as

$$
(I_{12} - X_{12})(I_{12} + Z_{12}) = 0,
$$
and

$$
Z_{23} Z_{12} X_{23} = X_{12} Z_{23} Z_{12}.
$$

From the Hermiticity properties of the generators, it follows that the remaining two triple products $A_i A_j A_k$ and $A_i A_j A_k^\dagger$ will be associative if $A_i A_j^k A_k^\dagger$ and $A_i A_j A_k$ are so. Hence, eqs. (6), (9), and (10) are the only associativity conditions of algebra (4).

Let now $R$ be any $N^2 \times N^2$ solution of the Yang-Baxter equation

$$
R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}.
$$

Then the corresponding braid matrix $\hat{R} = \tau R$, where $\tau$ is the twist operator (i.e., $\tau_{ij,kl} = \delta_{il} \delta_{jk}$), satisfies an equation similar to (6).

If $\hat{R}$ has three distinct eigenvalues $\lambda_\alpha$, $\alpha = 1, 2, 3$, and satisfies a Birman-Wenzl-Murakami (BWM) condition

$$
(\hat{R} - \lambda_1 I)(\hat{R} - \lambda_2 I)(\hat{R} - \lambda_3 I) = 0,
$$
then with each eigenspace of $\hat{R}$, one can associate two solutions of the set of associativity conditions (6), (9), and (10). In terms of the projector

$$
\mathcal{P}_\alpha = \prod_{\beta \neq \alpha} \frac{(\hat{R} - \lambda_\beta I)}{(\lambda_\alpha - \lambda_\beta)}
$$
on to the eigenspace corresponding to the eigenvalue $\lambda_\alpha$, these two solutions can be written as

$$
I - X \simeq \mathcal{P}_\alpha \quad \text{and} \quad Z = -\lambda_\alpha^{-1} \hat{R} \quad \text{or} \quad Z = -\lambda_\alpha \hat{R}^{-1}.
$$

Considering for instance $Z = -\lambda_\alpha^{-1} \hat{R}$ leads to the following deformed oscillator algebra (written in compact tensor form)

$$
A_2^\dagger A_1^l = S A_1^d A_2^l, \quad A_1 A_2 = S^* A_2 A_1, \quad A_1 A_2^l = I_{12} - \lambda_\alpha^{-1} R_{ti} A_2^l A_1,
$$
where $S = \tau X$ is found from (13) and (14), and $t_1$ means transposition with respect to the first space in the tensor product.

Several examples of such deformed oscillator algebras have been worked out so far [9]–[11]. In all cases, the solution of the Yang-Baxter equation that has been considered is the fundamental $R$-matrix of some classical quantum group. In such circumstances, the deformed oscillator algebras

\footnote{The SU$_q$(n)-covariant algebra constructed by Pusz and Woronowicz [5] corresponds to the simpler case where $\hat{R}$ has only two distinct eigenvalues, and satisfies a Hecke condition (see Sec. 3).}
are left invariant under the transformations induced by the quantum group. The construction presented here is not restricted however to such a choice, and any solution of (11) and (12) might actually be used. In a similar way, deformed oscillator algebras differing from that of Pusz-Woronowicz have been built by considering non-standard solutions of the Yang-Baxter equation and the Hecke condition [12].

The algebras constructed in refs. [9]-[11] include both standard and non-standard ones. The former [9,10] are either of $q$-bosonic or $q$-fermionic type, meaning that whenever $q \to 1$, they go over smoothly into ordinary bosonic or fermionic algebras, respectively. The latter [11], on the contrary, do not have such a smooth behaviour, but share instead some features with the quon algebra [13]. In the next section, we shall consider in more details a covariant $q$-bosonic algebra generalizing that of Pusz-Woronowicz.

3 An $SU_q(n) \times SU_q(m)$-Covariant $q$-Bosonic Algebra

The $SU_q(n)$ quantum group [1] is a complex associative algebra generated by $I$ and the noncommutative elements $T_{ij}$, $i, j = 1, \ldots, n$ of an $n \times n$ matrix $T$, subject to the relations

$$RT_1T_2 = T_2T_1R, \quad \det_q T = 1,$$

and the $*$-involution condition

$$T^* = (T^{-1})^t,$$

with $q$ real. In (16), $\det_q$ denotes the quantum determinant, and $R$ is the fundamental $R$-matrix associated with the $A_{n-1}$ series of Lie algebras,

$$R = q \sum_{i=1}^{n} e_{ii} \otimes e_{ii} + \sum_{i,j=1, i \neq j}^{n} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i,j=1}^{n} e_{ij} \otimes e_{ji},$$

where $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$. The coproduct, counit and antipode are defined by

$$\Delta(T) = T_1 \otimes T_2, \quad \epsilon(T) = 1, \quad S(T) = T^{-1},$$

where $\Delta(T_{ij}) = T_{ik} \otimes T_{kj}$.

The braid matrix $\hat{R}$, corresponding to (18), is a real symmetric matrix with two distinct eigenvalues, $q$ and $-q^{-1}$. Their respective multiplicities are $\frac{1}{2}n(n + 1)$ and $\frac{1}{2}n(n - 1)$, i.e., the dimensions of the symmetric and antisymmetric irreps $[2\mathbf{0}]_n$ and $[1^2\mathbf{0}]_n$ of $SU_q(n)$. The $\hat{R}$-matrix satisfies the Hecke condition

$$(\hat{R} - qI)(\hat{R} + q^{-1}I) = 0.$$
From the properties of the two "small" braid matrices $\hat{R}$ and $\hat{R}$, it follows that $\hat{R} = q^{-1}\hat{R}\hat{R}$ has three distinct eigenvalues $q, -q^{-1}$, and $q^{-3}$, with respective multiplicities corresponding to the dimensions of the representations $[20]_n[20]_m, [20]_n[120]_m + [120]_n[20]_m$, and $[120]_n[120]_m$ of $SU_q(n) \times SU_q(m)$, and satisfies the BWM condition (12).

By applying the results of the previous section to the antisymmetric (reducible) eigenspace of $\hat{R}$ associated with the eigenvalue $-q^{-1}$, one gets a deformed oscillator algebra of type (15), which will be denoted by $A_q(n, m)$, and whose defining relations are [10]

$$A_1^\dagger A_1^\dagger = SA_1^\dagger A_2^\dagger, \quad A_2A_1 = A_1A_2S, \quad A_2A_1^\dagger = I_{21} + qR^{ij}A_1^\dagger A_2,$$  \hspace{1cm} (22)

where

$$S = \tau(I - (q + q^{-1})P_A), \quad P_A = \frac{(\hat{R} - q I)(\hat{R} - q^{-3} I)}{(q + q^{-1})(q^{-1} + q^{-3})},$$  \hspace{1cm} (23)

and the creation and annihilation operators $A_1^\dagger, A_2$ now have two indices, $i = 1, 2, \ldots, n$, and $s = 1, 2, \ldots, m$. Whenever $q \to 1$, $\hat{R}$ and $S$ go over into $I$, so that (22) becomes an ordinary bosonic algebra.

The defining relations (22) of the $q$-bosonic algebra $A_q(n, m)$ may be rewritten in terms of the two "small" $\hat{R}$-matrices as

$$RA_1^\dagger A_2^\dagger = A_1^\dagger A_2^\dagger R, \quad RA_2A_1 = A_1A_2R, \quad A_2A_1^\dagger = I_{21}T_{21} + R^{ij}\tau^iA_1^\dagger A_2,$$  \hspace{1cm} (24)

or, in a more explicit form, as

$$R_{ijkl}A_{ks}^\dagger A_{lt} = A_{ju}^\dagger A_{iu}^\dagger R_{uv, st}, \quad R_{ijkl}A_{lt}A_{ks} = A_{iu}^\dagger A_{ju}^\dagger R_{uv, st}, \quad A_{is}^\dagger A_{jt} = \delta_{ij}\delta_{st} + R_{ki,jl}R_{us, tv}A_{ku}^\dagger A_{lv}. \hspace{1cm} (25)$$

Let us consider the map $\varphi : A_q(n, m) \to A_q(n, m) \otimes (SU_q(n) \times SU_q(m))$, defined by

$$A_1^i_s = \varphi(A_1^i_s) = A_1^i_s T_{ij} T_{st}, \quad A_2^i_s = \varphi(A_2^i_s) = A_2^i_s T_{ij} T_{st}^* = T_{ij}^{-1}T_{st}^{-1}A_2^i_s,$$  \hspace{1cm} (26)

where the elements $T_{ij}$ and $T_{st}$ of $SU_q(n) \times SU_q(m)$ are assumed to commute with $A_1^i_s$ and $A_2^i_s$. As a consequence of (16) and its counterpart for $SU_q(m)$, this map leaves the defining relations (25) of $A_q(n, m)$ invariant. Hence, the latter is an $SU_q(n) \times SU_q(m)$-covariant algebra.

In the next section, an important part will be played by the modified annihilation operators

$$\hat{A}_s = A_{jt}C_{ji}C_{ts}, \quad C_{ji} = (-1)^{n-j}q^{-(n-2j+1)/2}\delta_{ji}, \quad C_{ts} = (-1)^{m-i}q^{-(m-2s+1)/2}\delta_{ts}, \hspace{1cm} (27)$$

where $i' \equiv n + 1 - i, s' \equiv m + 1 - s$. Eq. (24) can be rewritten in terms of $A_1^i_s, \hat{A}_s$ as

$$RA_1^i_1A_2^i_2 = A_1^i_1A_2^i_2R, \quad R\hat{A}_1\hat{A}_2 = \hat{A}_2\hat{A}_1R, \quad \hat{A}_2A_1^i = C_{12}C_{12} + q^2A_1^i\hat{A}_2\hat{R}^{-1}\hat{R}^{-1},$$  \hspace{1cm} (28)

where $\hat{R}$ is defined by

$$\hat{R} = \sum_{i=1}^{n} e_{ii} \otimes e_{i'i'} + q \sum_{i,j=1}^{n} \sum_{i 
eq j} \sum_{i < j} ((q - q^{-1}) \sum_{i,j=1}^{n} (-q)^{i-j+1}e_{ij} \otimes e_{i'j'}) \hspace{1cm} (29)$$

185
and a similar definition holds for \( \tilde{\mathcal{R}} \). Under map \( \varphi \) of eq. (26), \( \hat{A}_{is} \) is transformed into

\[
\hat{A}_{is}' = \varphi(\hat{A}_{is}) = \hat{A}_{jt}\tilde{T}_{ij}\tilde{T}_{is}, \quad \tilde{T} \equiv C^{-1}(T^{-1})^t C, \quad \tilde{T} \equiv C^{-1}(T^{-1})^t C. \tag{30}
\]

Finally, combining eqs. (18) and (25) yields the detailed form of the \( A_q(n, m) \) defining relations

\[
\begin{align*}
A_{is}^\dagger A_{jt}^\dagger - q^{-1} A_{jt}^\dagger A_{is}^\dagger &= 0, & s < t, \\
A_{is}^\dagger A_{js}^\dagger - q^{-1} A_{js}^\dagger A_{is}^\dagger &= 0, & i < j, \\
A_{is}^\dagger A_{jt} - A_{jt}^\dagger A_{is} &= 0, & i > j, \quad s < t, \\
A_{is}^\dagger A_{jt} - A_{jt}^\dagger A_{is} &= -(q - q^{-1}) A_{js}^\dagger A_{it}^\dagger, & i < j, \quad s < t,
\end{align*}
\]

and

\[
\begin{align*}
A_{is}A_{jt}^\dagger - A_{jt}^\dagger A_{is} &= 0, & i \neq j, \quad s \neq t, \\
A_{is}A_{js}^\dagger - q A_{js}^\dagger A_{is} &= (q - q^{-1}) \sum_{t=1}^{s-1} A_{jt}^\dagger A_{it}, & i \neq j, \\
A_{is}A_{it}^\dagger - q A_{it}^\dagger A_{is} &= (q - q^{-1}) \sum_{j=1}^{s-1} A_{jt}^\dagger A_{js}, & s \neq t, \\
A_{is}A_{is}^\dagger - q^2 A_{is}^\dagger A_{is} &= I + (q^2 - 1) \left( \sum_{j=1}^{s-1} A_{js}^\dagger A_{js} + \sum_{t=1}^{s-1} A_{it}^\dagger A_{it} \right) - (q^{-2} - 1) \left( \sum_{j=1}^{s-1} A_{jt}^\dagger A_{jt} \right),
\end{align*}
\]

together with the Hermitian conjugates of (31). Whenever \( m = 1 \), substituting \( A_{i1}^\dagger, A_i \) for \( A_{i1}^\dagger, A_{i1} \) in (31) and (32) yields the Pusz-Woronowicz relations (3) for arbitrary \( n \) values. Hence, the covariant \( q \)-bosonic algebra \( A_q(n, m) \) is a generalization of that of Pusz-Woronowicz for values of \( m \) greater than 1.

### 4 Alternative Derivation in Terms of \( u_q(n) + u_q(m) \)

An alternative approach to the construction of covariant deformed oscillator algebras, based upon \( q \)-algebras, has been developed elsewhere [14,15]. In the case of the algebra \( A_q(n, m) \) introduced in the previous section, one considers the \( q \)-algebra \( u_q(n) + u_q(m) \). The Cartan-Chevalley generators of \( u_q(n) \) are denoted by \( E_{ii} = (E_{ii})^\dagger, i = 1, 2, \ldots, n, E_{i,i+1}, E_{i+1,i} = (E_{i,i+1})^\dagger, i = 1, 2, \ldots, n - 1 \), and satisfy the commutation relations

\[
\begin{align*}
[E_{ii}, E_{jj}] &= 0, \quad [E_{ii}, E_{jj+1}] = (\delta_{ij} - \delta_{i,j+1}) E_{jj+1}, \\
[E_{ii}, E_{i+1,j}] &= (\delta_{i,j+1} - \delta_{ij}) E_{i+1,j}, \quad [E_{i+1,i}, E_{j+1,j}] = \delta_{ij} [H_i]_q, \tag{33}
\end{align*}
\]

together with the quadratic and cubic \( q \)-Serre relations. In (33), \( H_i \equiv E_{ii} - E_{i+1,i+1} \). The algebra is endowed with a Hopf algebra structure with coproduct \( \Delta \), counit \( \epsilon \), and antipode \( S \), defined by

\[
\Delta(E_{ii}) = E_{ii} \otimes I + I \otimes E_{ii}, \quad \Delta(E_{i,i+1}) = E_{i,i+1} \otimes q^{H_i/2} + q^{-H_i/2} \otimes E_{i,i+1},
\]

186
\[ \Delta(E_{i+1,i}) = E_{i+1,i} \otimes q^{H_i/2} + q^{-H_i/2} \otimes E_{i+1,i}, \]  
\[ \epsilon(E_{ii}) = \epsilon(E_{i+1,i}) = \epsilon(E_{i+1,i}) = 0, \]  
\[ S(E_{ii}) = -E_{ii}, \quad S(E_{i+1,i}) = -q E_{i+1,i}, \quad S(E_{i+1,i}) = -q^{-1} E_{i+1,i}. \]  

The Cartan-Chevalley generators of \( u_q(m) \) are denoted by \( E_{ss}, s = 1, 2, \ldots, m, \) \( E_{s,s+1}, E_{s+1,s}, \) \( s = 1, 2, \ldots, m-1, \) and satisfy relations similar to (33)-(36), while commuting with the generators of \( u_q(n) \).

In the approach based upon \( u_q(n) + u_q(m) \), the \( q \)-bosonic creation operators \( A^+_i, i = 1, 2, \ldots, n, \) \( s = 1, 2, \ldots, m, \) belonging to \( A_q(n,m), \) are defined as the components of a double irreducible tensor \( T^{[0][1]}_{n,[0]} \) with respect to this \( q \)-algebra. This means that they fulfil the relations

\[ E_{jj}(A^+_i) = \delta_{jj} A^+_i, \quad E_{j,j+1}(A^+_i) = \delta_{j,j-1} A^+_i, \quad E_{j+1,j}(A^+_i) = \delta_{j,j+1} A^+_i, \]  
\[ S_{jj}(A^+_i) = \delta_{jj} A^+_i, \quad S_{j,j+1}(A^+_i) = \delta_{j,j-1} A^+_i, \quad S_{j+1,j}(A^+_i) = \delta_{j,j+1} A^+_i, \]  

where, for any \( u_q(n) + u_q(m) \) generator \( X \), \( X(A^+_i) \) denotes the quantum adjoint action \( X(A^+_i) = \sum_r X^+_r S(X^+_r) \), with \( \Delta(X) = \sum_r X^+_r \otimes X^+_r \). The modified annihilation operators \( \tilde{A}_{i,s}, i = 1, 2, \ldots, n, \) \( s = 1, 2, \ldots, m \), of eq. (27), are similarly defined as the components of a double irreducible tensor \( T^{[0][0]}_{n,[0]} \) with respect to \( u_q(n) + u_q(m) \), and satisfy the relations

\[ E_{jj}(\tilde{A}_{i,s}) = -\delta_{jj} \tilde{A}_{i,s}, \quad E_{j,j+1}(\tilde{A}_{i,s}) = \delta_{j,j-1} \tilde{A}_{i,s}, \quad E_{j+1,j}(\tilde{A}_{i,s}) = \delta_{j,j+1} \tilde{A}_{i,s}, \]  
\[ S_{jj}(\tilde{A}_{i,s}) = -\delta_{jj} \tilde{A}_{i,s}, \quad S_{j,j+1}(\tilde{A}_{i,s}) = \delta_{j,j-1} \tilde{A}_{i,s}, \quad S_{j+1,j}(\tilde{A}_{i,s}) = \delta_{j,j+1} \tilde{A}_{i,s}. \]  

The operators \( A^+_i \) and \( \tilde{A}_{i,s} \) can be explicitly written down in terms of \( m \) independent copies of the Pusz-Woronowicz operators [14]. By using such expressions and exploiting the tensorial character of the operators, it is straightforward to prove that their \( q \)-commutation relations are given in coupled form by

\[ [A^+_0, A^+_0]_{[0][0]} = [A^+_1, A^+_1]_{[1][2]} = [\tilde{A}, \tilde{A}]_{[0][0]} = [\tilde{A}, \tilde{A}]_{[0][0]} = 0, \]  
\[ [\tilde{A}, A^+_0]_{[1][0]} = [\tilde{A}, A^+_1]_{[0][0]} = [\tilde{A}, A^+_1]_{[0][0]} = 0, \]  
\[ [\tilde{A}, A^+_1]_{[0][0]} = \sqrt{n} q^{[0][0]/4}. \]  

where, for simplicity's sake, the row labels of the coupled \( u_q(n) + u_q(m) \) irreps have been dropped. In (41), the coupled \( q \)-commutator of two double irreducible tensors \( T^{[1][1]}_{n,[1]} \) and \( U^{[1][1]}_{[1][1]} \) is defined by [14]

\[ [T^{[1][1]}_{n,[1]} m, U^{[1][1]}_{[1][1]} m]_{(M_1)(n)} = \sum_{[1][1]} m = 0, \]  
\[ [T^{[1][1]}_{n,[1]} m, U^{[1][1]}_{[1][1]} m]_{(M_1)(n)} = \sum_{[1][1]} m = 0, \]  

\[ [T^{[1][1]}_{n,[1]} m, U^{[1][1]}_{[1][1]} m]_{(M_1)(n)} = \sum_{[1][1]} m = 0. \]  

Here

\[ \epsilon = \phi([\lambda_1]_n) + \phi([\lambda_1]_m) - \phi([\lambda_2]_n) - \phi([\lambda_2]_m) + \phi([\lambda_2]_m) - \phi([\lambda_2]_m), \]  
\[ \phi([\lambda_1]_n) = \frac{1}{2} \sum_{i=1}^n (n + 1 - 2s) \lambda_{1i}, \quad \phi([\lambda_2]_m) = \frac{1}{2} \sum_{s=1}^m (m + 1 - 2s) \lambda_{2s}, \]
and
\[
\left[T^{(\lambda_1)_n(\lambda_2)_m} \times U^{(\lambda'_1)_n(\lambda'_2)_m}\right]_{(M_1)_n(M_2)_m}^{(\lambda_1)_n(\lambda_2)_m} = \sum_{(\mu_1)_n(\mu_2)_m(\nu_1)_n(\nu_2)_m} \langle [\lambda_1]_n(\mu_1)_n, [\lambda'_1]_n(\mu'_1)_n [\lambda_2]_m(\mu_2)_m, [\lambda'_2]_m(\mu'_2)_m \rangle_q 
\times T^{(\lambda_1)_n(\lambda_2)_m} U^{(\lambda'_1)_n(\lambda'_2)_m},
\]

(44)

where \(\langle , | \rangle_q\) denotes a \(u_q(n)\) or \(u_q(m)\) Wigner coefficient.

By using the values of the latter, eq. (41) can be written in an explicit form [14]. The resulting relations coincide with eqs. (31) and (32), thereby proving the equivalence of the two constructions of \(A_q(n,m)\) based upon \(SU_q(n) \times SU_q(m)\) and \(u_q(n) + u_q(m)\), respectively.

References