COVARIANT DEFORMED OSCILLATOR ALGEBRAS

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Abstract

The general form and associativity conditions of deformed oscillator algebras are reviewed. It is shown how the latter can be fulfilled in terms of a solution of the Yang-Baxter equation when this solution has three distinct eigenvalues and satisfies a Birman-Wenzl-Murakami condition. As an example, an $SU_q(n) \times SU_q(m)$-covariant $q$-bosonic algebra is discussed in some details.

1 Introduction

Since the advent of quantum groups and $q$-algebras (see e.g. [1] and references quoted therein), much attention has been paid to deformations of the algebras of bosonic and fermionic creation and annihilation operators [2]–[6]. Different deformations of the latter arise depending on which property of the undeformed operators is preserved.

In the simple case of the $su(2)$ Lie algebra, two pairs of bosonic creation and annihilation operators $a^+_i, a_i, i = 1, 2$, give rise to the Jordan-Schwinger realization

$$J_+ = a^+_1 a_2, \quad J_- = a^+_2 a_1, \quad J_0 = \frac{1}{2}(N_1 - N_2), \quad (1)$$

where $N_i = a^+_i a_i, i = 1, 2$, are number operators. In addition, the creation operators $a^+_1, a^+_2$ (as well as the modified annihilation operators $\bar{a}_1 = a_2, \bar{a}_2 = -a_1$) are the components $+1/2$ and $-1/2$ of an $su(2)$ spinor, respectively. When extending these two properties to the corresponding $q$-algebra $su_q(2)$ (where $q$ is real and positive), one gets two different sets of $q$-bosonic operators.

On the one hand, those first considered by Biedenharn [2], Macfarlane [3], Sun and Fu [4], give rise to a Jordan-Schwinger realization of $su_q(2)$ of the same type as (1), where $a^+_i, a_i, i = 1, 2$, now satisfy the relations

$$a_i a^+_i - q^{-1} a^+_i a_i = q^{\pm N_i}, \quad (2)$$

while operators with different indices do still commute, and $a^+_i a_i = [N_i]_q \equiv (q^{N_i} - q^{-N_i})/(q - q^{-1})$. However, the operators $a^+_1, a^+_2$ do not transform any more under a definite representation of the algebra.

1 Directeur de recherches FNRS
On the other hand, the operators $A_i^\dagger$, $A_i$, $i = 1, 2$, introduced by Pusz and Woronowicz [5], satisfy different relations

$$A_i^\dagger A_j^\dagger - q^{-1} A_j^\dagger A_i^\dagger = A_i A_j - q A_j A_i = 0, \quad i < j,$$
$$A_i^\dagger A_j^\dagger - q A_j A_i = 0, \quad i \neq j,$$
$$A_i^\dagger A_j^\dagger - q^2 A_j A_i = I + (q^2 - 1) \sum_{j=1}^{i-1} A_j^\dagger A_j,$$  

where the two modes are not independent any more. As a result of this coupling, the operators $A_i^\dagger$, $A_i^\dagger$ (as well as $A_1^\dagger = q^{1/2} A_2$, $A_2^\dagger = -q^{-1/2} A_1$) are the components $+1/2$ and $-1/2$ of an $su_q(2)$ spinor respectively, but yield an $su_q(2)$ realization that is substantially more complicated than (1). The algebra (3) has also important covariance properties under the quantum group $SU_q(2)$, dual to $su_q(2)$.

The present communication is concerned with the construction of covariant deformed oscillator algebras that generalize the Pusz-Woronowicz algebra for other quantum groups than $SU_q(2)$ (or more generally $SU_q(n)$). The method used will be based on an $R$-matrix approach similar to that applied in noncommutative differential geometry [7,8]. In Sec. 2, after reviewing the general form and associativity conditions of deformed oscillator algebras, we will show how to fulfil the latter in terms of a solution of the Yang-Baxter equation with three distinct eigenvalues. The example of an $SU_q(n) \times SU_q(m)$-covariant q-bosonic algebra $A_q(n,m)$ will be treated in some details in Sec. 3. Finally, in Sec. 4, an alternative derivation of the same algebra, based upon the $q$-algebra $u_q(n) + u_q(m)$ will be presented.

## 2 Deformed Oscillator Algebras

Let us consider the complex algebras generated by $I$, $A_i^\dagger$, $A_i$, $i = 1, \ldots, N$, subject to the relations [9,10]

$$A_i^\dagger A_j^\dagger = X_{ij,kl} A_k^\dagger A_l^\dagger,$$
$$A_i A_j = X_{ji,kl}^* A_k A_l,$$
$$A_i^\dagger A_j^\dagger = \delta_{ij} + Z_{ji,kl} A_k^\dagger A_l,$$  

where $X$ and $Z$ are some complex $N^2 \times N^2$ matrices, and there are summations over dummy indices. As a consequence of the Hermiticity properties of the generators, $X^*$ is the complex conjugate of $X$, and $Z$ is a Hermitian matrix.

For these algebras to be associative, it is sufficient to require the braid transposition schemes for triples of generators. The braid scheme for $A_i^\dagger A_j^\dagger A_k^\dagger$ yields the condition

$$X_{ij,ab} X_{bk,cz} X_{ac,xy} = X_{jk,ab} X_{ia,xc} X_{eb,yz},$$  

i.e., in compact tensor notation, the Yang-Baxter equation for $X$ (in the "braid" version)

$$X_{12} X_{23} X_{12} = X_{23} X_{12} X_{23},$$  

182
Similarly, for $A_i A_j^\dagger A_k^\dagger$, one gets the two conditions
\begin{equation}
\delta_{ji} \delta_{kx} - X_{jk,ix} + Z_{jk,ix} - X_{jk,ab} Z_{ab,ix} = 0,
\end{equation}
and
\begin{equation}
Z_{kz,ac} Z_{ja,ab} X_{bc,xy} = X_{jk,ab} Z_{bz,cy} Z_{ac,ix},
\end{equation}
which may be written in compact form as
\begin{equation}
(I_{12} - X_{12})(I_{12} + Z_{12}) = 0,
\end{equation}
and
\begin{equation}
Z_{23} Z_{12} X_{23} = X_{12} Z_{23} Z_{12}.
\end{equation}
From the Hermiticity properties of the generators, it follows that the remaining two triple products $A_i A_j A_k$ and $A_i A_j A_k^\dagger$ will be associative if $A_i A_j^\dagger A_k^\dagger$ and $A_i A_j A_k^\dagger$ are so. Hence, eqs. (6), (9), and (10) are the only associativity conditions of algebra (4).

Let now $R$ be any $N^2 \times N^2$ solution of the Yang-Baxter equation
\begin{equation}
R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}.
\end{equation}
Then the corresponding braid matrix $\hat{R} = \tau R$, where $\tau$ is the twist operator (i.e., $\tau_{ij,kl} = \delta_{il} \delta_{jk}$), satisfies an equation similar to (6).

If $\hat{R}$ has three distinct eigenvalues $\lambda_{\alpha}$, $\alpha = 1, 2, 3$, and satisfies a Birman-Wenzl-Murakami (BWM) condition
\begin{equation}
(\hat{R} - \lambda_1 I)(\hat{R} - \lambda_2 I)(\hat{R} - \lambda_3 I) = 0,
\end{equation}
then with each eigenspace of $\hat{R}$, one can associate two solutions of the set of associativity conditions (6), (9), and (10). In terms of the projector
\begin{equation}
P_{\alpha} = \prod_{\beta \neq \alpha} \frac{(\hat{R} - \lambda_\beta I)}{(\lambda_{\alpha} - \lambda_\beta)},
\end{equation}
on onto the eigenspace corresponding to the eigenvalue $\lambda_{\alpha}$, these two solutions can be written as
\begin{equation}
I - X \simeq P_{\alpha} \quad \text{and} \quad Z = -\lambda_{\alpha}^{-1} \hat{R} \quad \text{or} \quad Z = -\lambda_{\alpha} \hat{R}^{-1}.
\end{equation}
Considering for instance $Z = -\lambda_{\alpha}^{-1} \hat{R}$ leads to the following deformed oscillator algebra (written in compact tensor form)
\begin{equation}
A_2^\dagger A_1^\dagger = S A_1^\dagger A_2^\dagger, \quad A_1 A_2 = S^* A_2 A_1, \quad A_1 A_2^\dagger = I_{12} - \lambda_{\alpha}^{-1} R^{t_1} A_2^\dagger A_1,
\end{equation}
where $S = \tau X$ is found from (13) and (14), and $t_1$ means transposition with respect to the first space in the tensor product.

Several examples of such deformed oscillator algebras have been worked out so far [9]–[11]. In all cases, the solution of the Yang-Baxter equation that has been considered is the fundamental $R$-matrix of some classical quantum group. In such circumstances, the deformed oscillator algebras

\footnote{The $SU_q(n)$-covariant algebra constructed by Pusz and Woronowicz [5] corresponds to the simpler case where $\hat{R}$ has only two distinct eigenvalues, and satisfies a Hecke condition (see Sec. 3).}
are left invariant under the transformations induced by the quantum group. The construction presented here is not restricted however to such a choice, and any solution of (11) and (12) might actually be used. In a similar way, deformed oscillator algebras differing from that of Pusz-Woronowicz have been built by considering non-standard solutions of the Yang-Baxter equation and the Hecke condition [12].

The algebras constructed in refs. [9]-[11] include both standard and non-standard ones. The former [9,10] are either of q-bosonic or q-fermionic type, meaning that whenever \( q \to 1 \), they go over smoothly into ordinary bosonic or fermionic algebras, respectively. The latter [11], on the contrary, do not have such a smooth behaviour, but share instead some features with the quon algebra [13]. In the next section, we shall consider in more details a covariant q-bosonic algebra generalizing that of Pusz-Woronowicz.

3 An \( SU_q(n) \times SU_q(m) \)-Covariant q-Bosonic Algebra

The \( SU_q(n) \) quantum group [1] is a complex associative algebra generated by \( I \) and the noncommutative elements \( T_{ij}, i, j = 1, \ldots, n \) of an \( n \times n \) matrix \( T \), subject to the relations

\[
RT_{1}T_{2} = T_{2}T_{1}R, \quad \text{det}_q T = 1, \tag{16}
\]

and the \(*\)-involution condition

\[
T^* = (T^{-1})^t, \tag{17}
\]

with \( q \) real. In (16), \( \text{det}_q \) denotes the quantum determinant, and \( R \) is the fundamental \( R \)-matrix associated with the \( A_{n-1} \) series of Lie algebras,

\[
R = q \sum_{i=1}^{n} e_{ii} \otimes e_{ii} + \sum_{i,j=1}^{n} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i,j=1}^{n} e_{ij} \otimes e_{ji}, \tag{18}
\]

where \( (e_{ij})_{kl} = \delta_{ik}\delta_{jl} \). The coproduct, counit and antipode are defined by

\[
\Delta(T) = T_1 \hat{\otimes} T_2, \quad \epsilon(T) = 1, \quad S(T) = T^{-1}, \tag{19}
\]

where \( \Delta(T_{ij}) = T_{ik} \otimes T_{kj} \).

The braid matrix \( \hat{R} \), corresponding to (18), is a real symmetric matrix with two distinct eigenvalues, \( q \) and \(-q^{-1}\). Their respective multiplicities are \( \frac{1}{2}n(n + 1) \) and \( \frac{1}{2}n(n - 1) \), i.e., the dimensions of the symmetric and antisymmetric irreps \([2^0]_n\) and \([1^2]_n\) of \( SU_q(n) \). The \( \hat{R} \)-matrix satisfies the Hecke condition

\[
(\hat{R} - qI)(\hat{R} + q^{-1}I) = 0. \tag{20}
\]

Similar relations are valid for \( SU_q(m) \). Its generators and fundamental \( R \)-matrix will be denoted by \( T_{st}, s, t = 1, \ldots, m \), and \( R \), respectively, to distinguish them from the corresponding quantities for \( SU_q(n) \). Note that \( T_{ij} \) and \( T_{st} \) are assumed to commute with one another.

For the product \( SU_q(n) \times SU_q(m) \), one can introduce a "large" \( R \)-matrix, \( R = q^{-1}R \), of dimension \((nm)^2 \times (nm)^2\) [10]. Its matrix elements are defined by

\[
R_{(is)(jt),(ku)(lv)} = q^{-1}R_{ij,kl}R_{st,uv}. \tag{21}
\]
From the properties of the two "small" braid matrices $\hat{R}$ and $\hat{R}'$, it follows that $\hat{R} = q^{-1} \hat{R} \hat{R}'$ has three distinct eigenvalues $q$, $-q^{-1}$, and $q^{-3}$, with respective multiplicities corresponding to the dimensions of the representations $[2^0]_n [2^0]_m$, $[2^0]_n [1^2]_m + [1^2]_n [2^0]_m$, and $[1^2]_n [1^2]_m$ of $SU_q(n) \times SU_q(m)$, and satisfies the BWM condition (12).

By applying the results of the previous section to the antisymmetric (reducible) eigenspace of $\hat{R}$ associated with the eigenvalue $-q^{-1}$, one gets a deformed oscillator algebra of type (15), which will be denoted by $A_q(n,m)$, and whose defining relations are [10]

$$A_2 A_1 = A_1 A_2 S, \quad A_2 A_1^\dagger = I_{21} + q R^{(l')} A_1^\dagger A_2,$$  \hspace{1cm} (22)

where

$$S = \tau(I - (q + q^{-1}) P_A), \quad P_A = \frac{(\hat{R} - q I)(\hat{R} - q^{-3} I)}{(q + q^{-1})(q^{-1} + q^{-3})},$$  \hspace{1cm} (23)

and the creation and annihilation operators $A_1^\dagger$, $A_s$ now have two indices, $i = 1, 2, \ldots, n$, and $s = 1, 2, \ldots, m$. Whenever $q \rightarrow 1$, $R$ and $S$ go over into $I$, so that (22) becomes an ordinary bosonic algebra.

The defining relations (22) of the $q$-bosonic algebra $A_q(n,m)$ may be rewritten in terms of the two "small" $R$-matrices as

$$RA_1^\dagger A_2 = A_1^\dagger A_2^\dagger R, \quad RA_1 A_2 = A_1 A_2 R, \quad A_2 A_1^\dagger = I_{21} + R^{(l')} R^{(l')} A_1^\dagger A_2,$$  \hspace{1cm} (24)

or, in a more explicit form, as

$$R_{ij,kl} A_{ks}^\dagger A_{lt}^\dagger = A_{ju}^\dagger A_{iu}^\dagger R_{uv, st}, \quad R_{ij,kl} A_{lt} A_{ks} = A_{iu}^\dagger A_{ju}^\dagger R_{uv, st}, \quad A_{is} A_{jt}^\dagger = \delta_{ij} \delta_{st} + R_{ki, jl} R_{us, tv} A_{ku}^\dagger A_{lv}.$$  \hspace{1cm} (25)

Let us consider the map $\varphi : A_q(n,m) \rightarrow A_q(n,m) \otimes (SU_q(n) \times SU_q(m))$, defined by

$$A_{is}^\dagger = \varphi(A_{is}) = A_{jt}^\dagger T_{jt} T_{st}, \quad A_{is} = \varphi(A_{is}) = A_{jt}^\dagger T_{jt} T_{st} = T_{ij}^{-1} T_{st}^{-1} A_{jt},$$  \hspace{1cm} (26)

where the elements $T_{ij}$ and $T_{st}$ of $SU_q(n) \times SU_q(m)$ are assumed to commute with $A_{is}^\dagger$ and $A_{is}$. As a consequence of (16) and its counterpart for $SU_q(m)$, this map leaves the defining relations (25) of $A_q(n,m)$ invariant. Hence, the latter is an $SU_q(n) \times SU_q(m)$-covariant algebra.

In the next section, an important part will be played by the modified annihilation operators

$$\tilde{A}_{is} = A_{jt} C_{ji} C_{ts}, \quad C_{ji} = (-1)^{n-j} q^{-(n-2j+1)/2} \delta_{j,i'}, \quad C_{ts} = (-1)^{m-i} q^{-(m-2t+1)/2} \delta_{t,s'},$$  \hspace{1cm} (27)

where $i' \equiv n + 1 - i$, $s' \equiv m + 1 - s$. Eq. (24) can be rewritten in terms of $A_{is}^\dagger$, $\tilde{A}_{is}$ as

$$RA_1^\dagger A_2 = A_1^\dagger A_2^\dagger \tilde{R}, \quad R \tilde{A}_1 \tilde{A}_2 = \tilde{A}_2 \tilde{A}_1 \tilde{R}, \quad \tilde{A}_2 A_1^\dagger = C_{12} C_{12} + q^2 A_1^\dagger \tilde{A}_2 \tilde{R}^{-1} \tilde{R}^{-1},$$  \hspace{1cm} (28)

where $\tilde{R}$ is defined by

$$\tilde{R} = \sum_{i=1}^n e_{ii} \otimes e_{i' i'} - q \sum_{i,j=1, i \neq j'}^n e_{ij} \otimes e_{jj'} + (q - q^{-1}) \sum_{i,j=1}^n (-q)_{i-j+1} e_{ij} \otimes e_{i' j'},$$  \hspace{1cm} (29)
and a similar definition holds for \( \hat{R} \). Under map \( \varphi \) of eq. (26), \( \hat{A}_i \) is transformed into

\[
\hat{A}_i' = \varphi(\hat{A}_i) = \hat{A}_j \hat{T}_{ji} \hat{T}_{ts}, \quad \hat{T} \equiv C^{-1}(T^{-1})^t C, \quad \hat{T} \equiv C^{-1}(T^{-1})^t C. \tag{30}
\]

Finally, combining eqs. (18) and (25) yields the detailed form of the \( \mathcal{A}_q(n, m) \) defining relations

\[
A_{is}^\dagger A_{it}^\dagger - q^{-1} A_{it}^\dagger A_{is}^\dagger = 0, \quad s < t, \\
A_{is}^\dagger A_{js}^\dagger - q^{-1} A_{js}^\dagger A_{is}^\dagger = 0, \quad i < j, \\
A_{is}^\dagger A_{jt}^\dagger - A_{jt}^\dagger A_{is}^\dagger = 0, \quad i > j, \quad s < t, \\
A_{is}^\dagger A_{jt}^\dagger - A_{jt}^\dagger A_{is}^\dagger = -(q - q^{-1}) A_{js}^\dagger A_{it}^\dagger, \quad i < j, \quad s < t, \tag{31}
\]

and

\[
A_{is} A_{jt}^\dagger - A_{jt}^\dagger A_{is} = 0, \quad i \neq j, \quad s \neq t, \\
A_{is} A_{js}^\dagger - q A_{js}^\dagger A_{is} = (q - q^{-1}) \sum_{t=1}^{s-1} A_{jt}^\dagger A_{it}, \quad i \neq j, \\
A_{is} A_{it}^\dagger - q A_{it}^\dagger A_{is} = (q - q^{-1}) \sum_{j=1}^{s-1} A_{jt}^\dagger A_{js}, \quad s \neq t, \\
A_{is} A_{is}^\dagger - q^2 A_{is}^\dagger A_{is} = I + (q^2 - 1) \left( \sum_{j=1}^{t-1} A_{js}^\dagger A_{jt} + \sum_{t=i}^{t-1} A_{it}^\dagger A_{tt} \right) - (q^{-2} - 1) \sum_{j=1}^{t-1} \sum_{t=1}^{t-1} A_{jt}^\dagger A_{jt}, \tag{32}
\]

together with the Hermitian conjugates of (31). Whenever \( m = 1 \), substituting \( A_{i+1}^\dagger, A_i \) for \( A_{i+1}^\dagger, A_i \) in (31) and (32) yields the Pusz-Woronowicz relations (3) for arbitrary \( n \) values. Hence, the covariant \( q \)-bosonic algebra \( \mathcal{A}_q(n, m) \) is a generalization of that of Pusz-Woronowicz for values of \( m \) greater than 1.

### 4 Alternative Derivation in Terms of \( u_q(n) + u_q(m) \)

An alternative approach to the construction of covariant deformed oscillator algebras, based upon \( q \)-algebras, has been developed elsewhere [14,15]. In the case of the algebra \( \mathcal{A}_q(n, m) \) introduced in the previous section, one considers the \( q \)-algebra \( u_q(n) + u_q(m) \). The Cartan-Chevalley generators of \( u_q(n) \) are denoted by \( E_{ii} = (E_{ii})^\dagger \), \( i = 1, 2, \ldots, n \), \( E_{i,i+1}, E_{i+1,i} = (E_{i,i+1})^\dagger \), \( i = 1, 2, \ldots, n-1 \), and satisfy the commutation relations

\[
[E_{ii}, E_{jj}] = 0, \quad [E_{ii}, E_{j,j+1}] = (\delta_{ij} - \delta_{i,j+1}) E_{j,j+1}, \\
[E_{ii}, E_{j+1,i}] = (\delta_{i,j+1} - \delta_{ij}) E_{j+1,j}, \quad [E_{i,i+1}, E_{j+1,i}] = \delta_{ij} [H_i]_q. \tag{33}
\]

Together with the quadratic and cubic \( q \)-Serre relations. In (33), \( H_i \equiv E_{ii} - E_{i+1,i+1}. \) The algebra is endowed with a Hopf algebra structure with coproduct \( \Delta \), counit \( \epsilon \), and antipode \( S \), defined by

\[
\Delta(E_{ii}) = E_{ii} \otimes I + I \otimes E_{ii}, \quad \Delta(E_{i,i+1}) = E_{i,i+1} \otimes q^{H_{i}/2} + q^{-H_{i}/2} \otimes E_{i,i+1},
\]

186
\[ \Delta(E_{i+1,i}) = E_{i+1,i} \otimes q^{H_{i}/2} + q^{-H_{i}/2} \otimes E_{i+1,i}, \]  
\[ \epsilon(E_{ii}) = \epsilon(E_{i+1,i}) = \epsilon(E_{i+1,i+1}) = 0, \]  
\[ S(E_{ii}) = -E_{ii}, \quad S(E_{i+1,i}) = -qE_{i+1,i}, \quad S(E_{i+1,i+1}) = -q^{-1}E_{i+1,i}. \] 

The Cartan-Chevalley generators of \( u_q(m) \) are denoted by \( \mathcal{E}_s \), \( s = 1, 2, \ldots, m \), \( \mathcal{E}_{s+1} \), \( \mathcal{E}_{s+1} \), \( s = 1, 2, \ldots, m-1 \), and satisfy relations similar to (33)-(36), while commuting with the generators of \( u_q(n) \).

In the approach based upon \( u_q(n) + u_q(m) \), the \( q \)-bosonic creation operators \( A_{it}^\dagger \), \( i = 1, 2, \ldots, n \), \( s = 1, 2, \ldots, m \), belonging to \( \mathcal{A}_{it} \), \( n \), \( m \), are defined as the components of a double irreducible tensor \( T^{[0][0]}_{[0][0]} \) with respect to this \( q \)-algebra. This means that they fulfill the relations

\[ E_{jj}(A_{it}^\dagger) = \delta_{jt}A_{it}^\dagger, \quad E_{jj+1}(A_{it}^\dagger) = \delta_{jt}A_{i,t+1}\]  
\[ \mathcal{E}_{it}(A_{it}^\dagger) = \delta_{it}A_{it}^\dagger, \quad \mathcal{E}_{it+1}(A_{it}^\dagger) = \delta_{it}A_{it,s}, \quad \mathcal{E}_{it+1,s}(A_{it}^\dagger) = \delta_{it}A_{it+1,s}, \]  

where, for any \( u_q(n) + u_q(m) \) generator \( X \), \( X(A_{it}^\dagger) \) denotes the quantum adjoint action \( X(A_{it}^\dagger) = \sum_r X_{it}^r A_{it}^\dagger S(X_r^r) \), with \( \Delta(X) = \sum_r X_{it}^r \otimes X_{it}^r \). The modified annihilation operators \( \tilde{A}_{it} \), \( i = 1, 2, \ldots, n \), \( s = 1, 2, \ldots, m \), of eq. (27), are similarly defined as the components of a double irreducible tensor \( T^{[0][0]}_{[0][0]} \) with respect to \( u_q(n) + u_q(m) \), and satisfy the relations

\[ E_{jj}(\tilde{A}_{it}) = -\delta_{jt}\tilde{A}_{it}, \quad E_{jj+1}(\tilde{A}_{it}) = \delta_{jt}\tilde{A}_{i,t+1}, \quad E_{j+1,j}(\tilde{A}_{it}) = \delta_{jt}\tilde{A}_{i,t}, \]  
\[ \mathcal{E}_{it}(\tilde{A}_{it}) = -\delta_{it}\tilde{A}_{it}, \quad \mathcal{E}_{it+1}(\tilde{A}_{it}) = \delta_{it}\tilde{A}_{it,s}, \quad \mathcal{E}_{it+1,s}(\tilde{A}_{it}) = \delta_{it}\tilde{A}_{it+1,s}. \]  

The operators \( A_{it}^\dagger \) and \( \tilde{A}_{it} \) can be explicitly written down in terms of \( m \) independent copies of the Pusz-Woronowicz operators [14]. By using such expressions and exploiting the tensorial character of the operators, it is straightforward to prove that their \( q \)-commutation relations are given in coupled form by

\[ [A_{it}^\dagger, A_{i't'}^\dagger]^{[0][0]}_{[0][0]} = [A_{i't'}^\dagger, A_{it}^\dagger]^{[1][2]}_{[0][0]} = [A_{i't'}^\dagger, A_{it}^\dagger]^{[0][0]}_{[1][2]} = [A_{i't'}^\dagger, A_{it}^\dagger]^{[0][0]}_{[0][1]} = 0, \]
\[ [\tilde{A}_{it}, A_{i't'}^\dagger]^{[0][0]}_{[0][0]} = [\tilde{A}_{i't'}, A_{it}^\dagger]^{[1][2]}_{[0][0]} = [\tilde{A}_{i't'}, A_{it}^\dagger]^{[0][0]}_{[1][2]} = [\tilde{A}_{i't'}, A_{it}^\dagger]^{[0][0]}_{[0][1]} = 0, \]
\[ [\tilde{A}_{it}, A_{i't'}^\dagger]^{[0][0]}_{[0][0]} = \sqrt{|n|_q|m|_q}, \]

where, for simplicity's sake, the row labels of the coupled \( u_q(n) + u_q(m) \) irreps have been dropped. In (41), the coupled \( q \)-commutator of two double irreducible tensors \( T^{[0][0]}_{[0][0]} \) and \( U^{[0][0]}_{[0][0]} \) is defined by [14]

\[ [T^{[0][0]}_{[0][0]}, U^{[0][0]}_{[0][0]}]^{[0][0]}_{[0][0]} = -[T^{[0][0]}_{[0][0]}, U^{[0][0]}_{[0][0]}]^{[0][0]}_{[0][0]} = -[T^{[0][0]}_{[0][0]}, U^{[0][0]}_{[0][0]}]^{[0][0]}_{[0][0]} = 0, \]

\[ [T^{[0][0]}_{[0][0]}, U^{[0][0]}_{[0][0]}]^{[0][0]}_{[0][0]} = \sqrt{|n|_q|m|_q}. \]  

Here

\[ \epsilon = \phi([\lambda_1]_n) + \phi([\lambda'_1]_n) - \phi([\lambda_2]_m) + \phi([\lambda_2]_m) + \phi([\lambda'_2]_m) - \phi([\lambda'_2]_m), \]

\[ \phi([\lambda_1]_n) = \frac{1}{2} \sum_{i=1}^{n} (n + 1 - 2i) \lambda_1, \quad \phi([\lambda_2]_m) = \frac{1}{2} \sum_{s=1}^{m} (m + 1 - 2s) \lambda_2, \]  

187
and

\[ T[^{\lambda_1}_{n}[\lambda_2]_m \times U[^{\lambda'_1}_{n}[\lambda'_2]_m]^{\Lambda_1}_{m}(M_1)_{n}(M_2)_{m}]^{\Lambda_2}_{m} = \sum_{(\mu_1)_{n}(\mu_2)_{m}(\mu'_1)_{n}(\mu'_2)_{m}} \langle [\lambda_1]_{n}(\mu_1)_{n}, [\lambda'_1]_{n}(\mu'_1)_{n} | [\lambda_2]_{m}(\mu_2)_{m}, [\lambda'_2]_{m}(\mu'_2)_{m} \rangle_q \times T[^{\lambda_1}_{n}[\lambda_2]_m]^{\Lambda_1}_{m} U[^{\lambda'_1}_{n}[\lambda'_2]_m]^{\Lambda_2}_{m}, \]

where \( \langle \cdot, | \cdot \rangle_q \) denotes a \( u_q(n) \) or \( u_q(m) \) Wigner coefficient.

By using the values of the latter, eq. (41) can be written in an explicit form [14]. The resulting relations coincide with eqs. (31) and (32), thereby proving the equivalence of the two constructions of \( A_q(n, m) \) based upon \( SU_q(n) \times SU_q(m) \) and \( u_q(n) + u_q(m) \), respectively.

References