COVARIANT DEFORMED OSCILLATOR ALGEBRAS

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Abstract

The general form and associativity conditions of deformed oscillator algebras are reviewed.
It is shown how the latter can be fulfilled in terms of a solution of the Yang-Baxter equation
when this solution has three distinct eigenvalues and satisfies a Birman-Wenzl-Murakami
condition. As an example, an $SU_q(n) \times SU_q(m)$-covariant $q$-bosonic
algebra is discussed in some details.

1 Introduction

Since the advent of quantum groups and $q$-algebras (see e.g. [1] and references quoted therein),
much attention has been paid to deformations of the algebras of bosonic and fermionic creation
and annihilation operators [2]–[6]. Different deformations of the latter arise depending on which
property of the undeformed operators is preserved.

In the simple case of the $su(2)$ Lie algebra, two pairs of bosonic creation and annihilation
operators $a_i^\dagger, a_i, i = 1, 2$, give rise to the Jordan-Schwinger realization

\[ J_+ = a_1^\dagger a_2, \quad J_- = a_2^\dagger a_1, \quad J_0 = \frac{1}{2}(N_1 - N_2), \]

where $N_i = a_i^\dagger a_i, i = 1, 2$, are number operators. In addition, the creation operators $a_1^\dagger, a_2^\dagger$ (as well
as the modified annihilation operators $\hat{a}_1 = a_2, \hat{a}_2 = -a_1$) are the components $+1/2$ and $-1/2$ of
an $su(2)$ spinor, respectively. When extending these two properties to the corresponding $q$-algebra
$su_q(2)$ (where $q$ is real and positive), one gets two different sets of $q$-bosonic operators.

On the one hand, those first considered by Biedenharn [2], Macfarlane [3], Sun and Fu [4], give
rise to a Jordan-Schwinger realization of $su_q(2)$ of the same type as (1), where $a_i^\dagger, a_i, i = 1, 2,$
now satisfy the relations

\[ a_i a_i^\dagger - q^{-1} a_i^\dagger a_i = q^{N_i}, \]

while operators with different indices do still commute, and $a_i^\dagger a_i = [N_i]_q \equiv (q^{N_i} - q^{-N_i})/(q - q^{-1}).$
However, the operators $a_1^\dagger, a_2^\dagger$ do not transform any more under a definite representation of the
algebra.

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On the other hand, the operators $A_i^\dagger, A_i, i = 1, 2,$ introduced by Pusz and Woronowicz [5], satisfy different relations

\begin{equation}
\begin{align*}
A_i^\dagger A_j - q^{-1} A_j^\dagger A_i^\dagger &= A_i A_j - q A_j A_i = 0, \quad i < j, \\
A_i A_j - q A_j A_i &= 0, \quad i \neq j, \\
A_i A_i^\dagger - q^2 A_i^\dagger A_i &= I + (q^2 - 1) \sum_{j=1}^{i-1} A_j^\dagger A_j,
\end{align*}
\end{equation}

where the two modes are not independent any more. As a result of this coupling, the operators $A_i^\dagger, A_j^\dagger$ (as well as $\tilde{A}_1 = q^{1/2} A_2, \tilde{A}_2 = -q^{-1/2} A_1$) are the components $+1/2$ and $-1/2$ of an $su_q(2)$ spinor respectively, but yield an $su_q(2)$ realization that is substantially more complicated than (1). The algebra (3) has also important covariance properties under the quantum group $SU_q(2)$, dual to $su_q(2)$.

The present communication is concerned with the construction of covariant deformed oscillator algebras that generalize the Pusz-Woronowicz algebra for other quantum groups than $SU_q(2)$ (or more generally $SU_q(n)$). The method used will be based on an $R$-matrix approach similar to that applied in noncommutative differential geometry [7,8]. In Sec. 2, after reviewing the general form and associativity conditions of deformed oscillator algebras, we will show how to fulfill the latter in terms of a solution of the Yang-Baxter equation with three distinct eigenvalues. The example of an $SU_q(n) \times SU_q(m)$-covariant $q$-bosonic algebra $A_q(n,m)$ will be treated in some details in Sec. 3. Finally, in Sec. 4, an alternative derivation of the same algebra, based upon the $q$-algebra $u_q(n) + u_q(m)$ will be presented.

## 2 Deformed Oscillator Algebras

Let us consider the complex algebras generated by $I, A_i^\dagger A_i = (A_i^\dagger)^\dagger, i = 1, \ldots, N,$ subject to the relations [9,10]

\begin{equation}
\begin{align*}
A_i^\dagger A_j^\dagger &= X_{ij,kl} A_k^\dagger A_l^\dagger, \\
A_i A_j &= X^*_{ji,kl} A_k A_l, \\
A_i A_j^\dagger &= \delta_{ij} + Z_{ji,kl} A_k A_l,
\end{align*}
\end{equation}

where $X$ and $Z$ are some complex $N^2 \times N^2$ matrices, and there are summations over dummy indices. As a consequence of the Hermiticity properties of the generators, $X^*$ is the complex conjugate of $X$, and $Z$ is a Hermitian matrix.

For these algebras to be associative, it is sufficient to require the braid transposition schemes for triples of generators. The braid scheme for $A_i^\dagger A_j^\dagger A_k^\dagger$ yields the condition

\begin{equation}
X_{ij,ab} X_{bk,cz} X_{ac,xy} = X_{jk,ab} X_{ia,xc} X_{eb,yz},
\end{equation}

i.e., in compact tensor notation, the Yang-Baxter equation for $X$ (in the "braid" version)

\begin{equation}
X_{12} X_{23} X_{12} = X_{23} X_{12} X_{23}.
\end{equation}
Similarly, for $A_i A_j A_k$, one gets the two conditions

$$\delta_{ji} \delta_{kx} - X_{jk,ix} + Z_{jk,ix} = 0,$$  \tag{7}

and

$$Z_{kz,ac} Z_{ja,ab} X_{bc,zy} = X_{jk,ab} Z_{bz,cy} Z_{ac,ix},$$  \tag{8}

which may be written in compact form as

$$(I_{12} - X_{12})(I_{12} + Z_{12}) = 0,$$  \tag{9}

and

$$Z_{23} Z_{12} X_{23} = X_{12} Z_{23} Z_{12}.$$  \tag{10}

From the Hermiticity properties of the generators, it follows that the remaining two triple products $A_i A_j A_k$ and $A_i A_j A_k^\dagger$ will be associative if $A_i A_j A_k^\dagger$ and $A_i A_j A_k^\dagger$ are so. Hence, eqs. (6), (9), and (10) are the only associativity conditions of algebra (4).

Let now $R$ be any $N^2 \times N^2$ solution of the Yang-Baxter equation

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}.$$  \tag{11}

Then the corresponding braid matrix $\hat{R} = \tau R$, where $\tau$ is the twist operator (i.e., $\tau_{ij,kl} = \delta_{ii} \delta_{jk}$), satisfies an equation similar to (6).

If $\hat{R}$ has three distinct eigenvalues $\lambda_{\alpha}$, $\alpha = 1, 2, 3$, and satisfies a Birman-Wenzl-Murakami (BWM) condition

$$(\hat{R} - \lambda_1 I)(\hat{R} - \lambda_2 I)(\hat{R} - \lambda_3 I) = 0,$$  \tag{12}

then with each eigenspace of $\hat{R}$, one can associate two solutions of the set of associativity conditions (6), (9), and (10). In terms of the projector

$$P_{\alpha} = \prod_{\beta \neq \alpha} \frac{(\hat{R} - \lambda_\beta I)}{(\lambda_{\alpha} - \lambda_\beta)}$$  \tag{13}

onto the eigenspace corresponding to the eigenvalue $\lambda_{\alpha}$, these two solutions can be written as

$$I - X \simeq P_{\alpha} \quad \text{and} \quad Z = -\lambda^{-1}_{\alpha} \hat{R} \quad \text{or} \quad Z = -\lambda_{\alpha} \hat{R}^{-1}.$$  \tag{14}

Considering for instance $Z = -\lambda^{-1}_{\alpha} \hat{R}$ leads to the following deformed oscillator algebra (written in compact tensor form)

$$A_i^\dagger A_j^\dagger = S A_i^\dagger A_j^\dagger, \quad A_1 A_2 = S^* A_2 A_1, \quad A_i A_j^\dagger = I_{12} - \lambda^{-1}_{\alpha} R^{t_1} A_i^\dagger A_j^\dagger,$$  \tag{15}

where $S = \tau X$ is found from (13) and (14), and $t_1$ means transposition with respect to the first space in the tensor product.

Several examples of such deformed oscillator algebras have been worked out so far [9]-[11]. In all cases, the solution of the Yang-Baxter equation that has been considered is the fundamental $R$-matrix of some classical quantum group. In such circumstances, the deformed oscillator algebras

\footnote{The $SU_q(n)$-covariant algebra constructed by Pusz and Woronowicz [5] corresponds to the simpler case where $\hat{R}$ has only two distinct eigenvalues, and satisfies a Hecke condition (see Sec. 3).}
are left invariant under the transformations induced by the quantum group. The construction presented here is not restricted however to such a choice, and any solution of (11) and (12) might actually be used. In a similar way, deformed oscillator algebras differing from that of Pusz-Woronowicz have been built by considering non-standard solutions of the Yang-Baxter equation and the Hecke condition [12].

The algebras constructed in refs. [9]–[11] include both standard and non-standard ones. The former [9,10] are either of $q$-bosonic or $q$-fermionic type, meaning that whenever $q \to 1$, they go over smoothly into ordinary bosonic or fermionic algebras, respectively. The latter [11], on the contrary, do not have such a smooth behaviour, but share instead some features with the quon algebra [13]. In the next section, we shall consider in more details a covariant $q$-bosonic algebra generalizing that of Pusz-Woronowicz.

### 3 An $SU_q(n) \times SU_q(m)$-Covariant $q$-Bosonic Algebra

The $SU_q(n)$ quantum group [1] is a complex associative algebra generated by $I$ and the noncommutative elements $T_{ij}$, $i, j = 1, \ldots, n$ of an $n \times n$ matrix $T$, subject to the relations

$$RT_1T_2 = T_2T_1R, \quad \det_q T = 1,$$

and the $*$-involution condition

$$T^* = (T^{-1})^t,$$

with $q$ real. In (16), $\det_q$ denotes the quantum determinant, and $R$ is the fundamental $R$-matrix associated with the $A_{n-1}$ series of Lie algebras,

$$R = q \sum_{i=1}^{n} e_{ii} \otimes e_{ii} + \sum_{i,j=1, i \neq j}^{n} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i,j=1}^{n} e_{ij} \otimes e_{ji},$$

where $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$. The coproduct, counit and antipode are defined by

$$\Delta(T) = T_1 \otimes T_2, \quad \epsilon(T) = 1, \quad S(T) = T^{-1},$$

where $\Delta(T_{ij}) = T_{ik} \otimes T_{kj}$.

The braid matrix $\hat{R}$, corresponding to (18), is a real symmetric matrix with two distinct eigenvalues, $q$ and $-q^{-1}$. Their respective multiplicities are $\frac{1}{2}n(n + 1)$ and $\frac{1}{2}n(n - 1)$, i.e., the dimensions of the symmetric and antisymmetric irreps $[2\hat{0}]_n$ and $[1^2\hat{0}]_n$ of $SU_q(n)$. The $\hat{R}$-matrix satisfies the Hecke condition

$$(\hat{R} - qI)(\hat{R} + q^{-1}I) = 0.$$  

Similar relations are valid for $SU_q(m)$. Its generators and fundamental $R$-matrix will be denoted by $T_{st}$, $s, t = 1, \ldots, m$, and $R$, respectively, to distinguish them from the corresponding quantities for $SU_q(n)$. Note that $T_{ij}$ and $T_{st}$ are assumed to commute with one another.

For the product $SU_q(n) \times SU_q(m)$, one can introduce a “large” $R$-matrix, $R = q^{-1}R \mathcal{R}$, of dimension $(nm)^2 \times (nm)^2$ [10]. Its matrix elements are defined by

$$R_{(is)(jt),(ku)(lv)} = q^{-1}R_{ij,kl}\mathcal{R}_{st,uv}.$$
From the properties of the two "small" braid matrices $\hat{R}$ and $\hat{R}$, it follows that $\hat{R} = q^{-1}\hat{R}\hat{R}$ has three distinct eigenvalues $q$, $-q^{-1}$, and $q^{-3}$, with respective multiplicities corresponding to the dimensions of the representations $[2\,0]_n[2\,0]_m$, $[2\,0]_n[1^2\,0]_m + [1^2\,0]_n[2\,0]_m$, and $[1^2\,0][1^2\,0]_m$ of $SU_q(n) \times SU_q(m)$, and satisfies the BWM condition (12).

By applying the results of the previous section to the antisymmetric (reducible) eigenspace of $\hat{R}$ associated with the eigenvalue $-q^{-1}$, one gets a deformed oscillator algebra of type (15), which will be denoted by $A_q(n, m)$, and whose defining relations are [10]

$$A_i^\dagger A_j^\dagger = S A_i^\dagger A_j^\dagger, \quad A_i A_j = A_j A_i S, \quad A_i A_j^\dagger = I_{21} + q R_{ij} A_i^\dagger A_j^\dagger,$$

where

$$S = \tau(I - (q + q^{-1})P_A), \quad P_A = \frac{(\hat{R} - qI)(\hat{R} - q^{-3}I)}{(q + q^{-1})(q^{-1} + q^{-3})},$$

and the creation and annihilation operators $A_i^\dagger$, $A_i$ now have two indices, $i = 1, 2, \ldots, n$, and $s = 1, 2, \ldots, m$. Whenever $q \to 1$, $R$ and $S$ go over into $I$, so that (22) becomes an ordinary bosonic algebra.

The defining relations (22) of the $q$-bosonic algebra $A_q(n, m)$ may be rewritten in terms of the two "small" $\hat{R}$-matrices as

$$RA_i^\dagger A_j^\dagger = A_i^\dagger A_j^\dagger R, \quad RA_i A_j = A_j A_i R, \quad A_i A_j^\dagger = I_{21} T_{21} + R_{ij} T_{ij} A_i^\dagger A_j^\dagger,$$

or, in a more explicit form, as

$$R_{ij,kl} A_{ik}^\dagger A_{jl}^\dagger = A_j^\dagger A_i^\dagger R_{uv, st},$$

$$R_{ij,kl} A_{ik} A_{jl} = A_i A_j R_{uv, st},$$

$$A_{is} A_j^\dagger = \delta_{ij} \delta_{st} + R_{ki, jl} R_{us, tv} A_{ku} A_{lv}.$$

Let us consider the map $\varphi : A_q(n, m) \to A_q(n, m) \otimes (SU_q(n) \times SU_q(m))$, defined by

$$A_{is}^\dagger = \varphi(A_{is}^\dagger) = A_{is}^\dagger T_{is}^\dagger T_{is},$$

$$A_{is} = \varphi(A_{is}) = A_{is} T_{is}^\dagger T_{is}^\dagger = T_{ij}^{-1} T_{st}^{-1} A_{js},$$

where the elements $T_{ij}$ and $T_{st}$ of $SU_q(n) \times SU_q(m)$ are assumed to commute with $A_{is}^\dagger$ and $A_{is}$. As a consequence of (16) and its counterpart for $SU_q(m)$, this map leaves the defining relations (25) of $A_q(n, m)$ invariant. Hence, the latter is an $SU_q(n) \times SU_q(m)$-covariant algebra.

In the next section, an important part will be played by the modified annihilation operators $A_{is}', \tilde{A}_{is}'$ as

$$A_{is}' = A_{is} C_{is} C_{is}, \quad C_{is} = (-1)^{n-i} q^{-(n-2+i)/2} \delta_{ij}, \quad C_{is}' = (-1)^{m-i} q^{-(m-2+i)/2} \delta_{is},$$

where $i' = n + 1 - i$, $s' = m + 1 - s$. Eq. (24) can be rewritten in terms of $A_{is}^\dagger$, $\tilde{A}_{is}'$ as

$$RA_i^\dagger A_j^\dagger = A_i^\dagger A_j^\dagger R, \quad R\tilde{A}_i \tilde{A}_j = \tilde{A}_j \tilde{A}_i R, \quad \tilde{A}_i A_j^\dagger = C_{12} C_{12} + q^2 A_i^\dagger \tilde{A}_j R^{-1} \tilde{R}^{-1},$$

where $R$ is defined by

$$R = \sum_{i=1}^{n} e_{ii} \otimes e_{ii'} + q \sum_{i,j=1}^{n} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i,j=1}^{n} (-q)^{i-j+1} e_{ij} \otimes e_{ii'},$$

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and a similar definition holds for $\hat{R}$. Under map $\varphi$ of eq. (26), $\hat{A}_{is}$ is transformed into

$$\hat{A}_{is}' = \varphi(\hat{A}_{is}) = \hat{A}_{jt} \hat{T}_{j} \hat{T}_{is}, \quad \hat{T} = C^{-1}(T^{-1})^t C, \quad \hat{T} = C^{-1}(T^{-1})^t C. \tag{30}$$

Finally, combining eqs. (18) and (25) yields the detailed form of the $A_q(n, m)$ defining relations

$$A_{is}^\dagger A_{jt}^\dagger - q^{-1} A_{is}^\dagger A_{jt}^\dagger = 0, \quad s < t, \tag{31}$$

$$A_{is}^\dagger A_{js}^\dagger - q^{-1} A_{is}^\dagger A_{js}^\dagger = 0, \quad i < j,$n

$$A_{is}^\dagger A_{jt} - A_{jt}^\dagger A_{is} = 0, \quad i > j, \quad s < t, \tag{32}$$

$$A_{is}^\dagger A_{jt} - A_{jt}^\dagger A_{is} = -(q - q^{-1}) A_{js}^\dagger A_{it}^\dagger, \quad i < j, \quad s < t,$n

and together with the Hermitian conjugates of (31). Whenever $m = 1$, substituting $A_1^\dagger, A_i$ for $A_{i1}^\dagger, A_{ii}$ in (31) and (32) yields the Pusz-Woronowicz relations (3) for arbitrary $n$ values. Hence, the covariant $q$-bosonic algebra $A_q(n, m)$ is a generalization of that of Pusz-Woronowicz for values of $m$ greater than 1.

4 Alternative Derivation in Terms of $u_q(n) + u_q(m)$

An alternative approach to the construction of covariant deformed oscillator algebras, based upon $q$-algebras, has been developed elsewhere [14,15]. In the case of the algebra $A_q(n, m)$ introduced in the previous section, one considers the $q$-algebra $u_q(n) + u_q(m)$. The Cartan-Chevalley generators of $u_q(n)$ are denoted by $E_{ii} = (E_{ii})^\dagger, i = 1, 2, \ldots, n, E_{ii+1}, E_{i+1,i} = (E_{i+1,i})^\dagger, i = 1, 2, \ldots, n - 1,$ and satisfy the commutation relations

$$[E_{ii}, E_{jj}] = 0, \quad [E_{ii}, E_{j,j+1}] = (\delta_{ij} - \delta_{i,j+1}) E_{j,j+1}, \tag{33}$$

$$[E_{ii}, E_{j+1,j}] = (\delta_{i,j+1} - \delta_{ij}) E_{j+1,j}, \quad [E_{i,i+1}, E_{j+1,j}] = \delta_{ij}[H_i]_q,$n

together with the quadratic and cubic $q$-Serre relations. In (33), $H_i \equiv E_{ii} - E_{i+1,i+1}$. The algebra is endowed with a Hopf algebra structure with coproduct $\Delta$, counit $\epsilon$, and antipode $S$, defined by

$$\Delta(E_{ii}) = E_{ii} \otimes I + I \otimes E_{ii}, \quad \Delta(E_{i,i+1}) = E_{i,i+1} \otimes q^{H_{i+1}/2} + q^{-H_{i+1}/2} \otimes E_{i,i+1},$$

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\[ \Delta(E_{i+1,i}) = E_{i+1,i} \otimes q^{H_i/2} + q^{-H_i/2} \otimes E_{i+1,i}, \quad (34) \]

\[ \epsilon(E_{ii}) = \epsilon(E_{i+1,i}) = \epsilon(E_{i+1,i+1}) = 0, \quad (35) \]

\[ S(E_{ii}) = -E_{ii}, \quad S(E_{i+1,i}) = -qE_{i+1,i}, \quad S(E_{i+1,i+1}) = -q^{-1}E_{i+1,i+1}. \quad (36) \]

The Cartan-Chevalley generators of \( u_q(m) \) are denoted by \( E_{ss}, s = 1, 2, \ldots, m, E_{s,s+1}, E_{s+1,s}, \) \( s = 1, 2, \ldots, m-1, \) and satisfy relations similar to (33)-(36), while commuting with the generators of \( u_q(n). \)

In the approach based upon \( u_q(n) + u_q(m) \), the \( q \)-bosonic creation operators \( A_{is}, i = 1, 2, \ldots, n, \) \( s = 1, 2, \ldots, m, \) belonging to \( u_q(n,m), \) are defined as the components of a double irreducible tensor \( T^{[10]_n[10]^m} \) with respect to this \( q \)-algebra. This means that they fulfill the relations

\[ E_{jj}(A_{is}) = \delta_{ji} A_{is}, \quad E_{j+1,j}(A_{is}) = \delta_{ji} A_{i-1,s}, \quad E_{j+1,j}(A_{is}) = \delta_{ji} A_{i+1,s}, \quad (37) \]

\[ E_{tt}(A_{is}) = \delta_{ts} A_{is}, \quad E_{t+1,t}(A_{is}) = \delta_{ts} A_{i,s-1}, \quad E_{t+1,t}(A_{is}) = \delta_{ts} A_{i,s+1}. \quad (38) \]

where, for any \( u_q(n) + u_q(m) \) generator \( X, X(A_{is}) \) denotes the quantum adjoint action \( X(A_{is}) = \sum X_i \otimes X_s \), with \( \Delta(X) = \sum X_i \otimes X_s^2 \). The modified annihilation operators \( \hat{A}_{is}, i = 1, 2, \ldots, n, \) \( s = 1, 2, \ldots, m, \) of eq. (27), are similarly defined as the components of a double irreducible tensor \( T^{[0]_n[0]^m} \) with respect to \( u_q(n) + u_q(m) \), and satisfy the relations

\[ E_{jj}(\hat{A}_{is}) = -\delta_{ji} \hat{A}_{is}, \quad E_{j+1,j}(\hat{A}_{is}) = \delta_{ji} \hat{A}_{i-1,s}, \quad E_{j+1,j}(\hat{A}_{is}) = \delta_{ji} \hat{A}_{i+1,s}, \quad (39) \]

\[ E_{tt}(\hat{A}_{is}) = -\delta_{ts} \hat{A}_{is}, \quad E_{t+1,t}(\hat{A}_{is}) = \delta_{ts} \hat{A}_{i,s-1}, \quad E_{t+1,t}(\hat{A}_{is}) = \delta_{ts} \hat{A}_{i,s+1}. \quad (40) \]

The operators \( A_{is} \) and \( \hat{A}_{is} \) can be explicitly written down in terms of \( m \) independent copies of the Pusz-Woronowicz operators [14]. By using such expressions and exploiting the tensorial character of the operators, it is straightforward to prove that their \( q \)-commutation relations are given in coupled form by

\[ [A^\dagger, A^\dagger]^{[20]^n[12]^m} = [A^\dagger, A^\dagger]^{[12]^n[20]^m} = [\hat{A}, \hat{A}]^{[0]^n[2]^m} = [\hat{A}, \hat{A}]^{[2]^n[0]^m} = 0, \]

\[ [\hat{A}, A^\dagger]^{[10]_n[10]_m} = [\hat{A}, A^\dagger]^{[10]_m[10]_n} = 0, \]

\[ [\hat{A}, A^\dagger]^{[0]_n[0]_m} = \sqrt{n} q^{-m}, \quad (41) \]

where, for simplicity’s sake, the row labels of the coupled \( u_q(n) + u_q(m) \) irreps have been dropped. In (41), the coupled \( q \)-commutator of two double irreducible tensors \( T^{[A_1]_n[A_2]_m} \) and \( U^{[A'_1]_n[A'_2]_m} \) is defined by [14]

\[ [T^{[A_1]_n[A_2]_m}, U^{[A'_1]_n[A'_2]_m}]^{[A_1]_n[A_2]_m}_{(M_1)_n(M_2)_m} = (-1)^{\epsilon} q^a [U^{[A'_1]_n[A'_2]_m} T^{[A_1]_n[A_2]_m}]^{[A_1]_n[A_2]_m}_{(M_1)_n(M_2)_m}. \quad (42) \]

Here

\[ \epsilon = \phi([A_1]_n) + \phi([A'_1]_n) - \phi([A_1]_n) + \phi([A_2]_m) + \phi([A'_2]_m) - \phi([A_2]_m), \]

\[ \phi([A_1]_n) = \frac{1}{2} \sum_{i=1}^n (n + 1 - 2i) \lambda_{1i}, \quad \phi([A_2]_m) = \frac{1}{2} \sum_{s=1}^m (m + 1 - 2s) \lambda_{2s}, \quad (43) \]
and

\[ T^{(\lambda_1)_{n}[(\lambda_2)_{m} \times U^{[\lambda'_1]_{n}[(\lambda'_2)_{m}]}_{(M_1)_{n}(M_2)_{m}}}_{(\mu_1)_{n}(\mu_2)_{m}} \]

\[ = \sum_{(\mu_1)_{n}(\mu_2)_{m}(\mu'_1)_{n}(\mu'_2)_{m}} \langle [\lambda_1]_n (\mu_1)_n, [\lambda'_1]_n (\mu'_1)_n | [\lambda_1]_n (M_1)_n \rangle_q \langle [\lambda_2]_m (\mu_2)_m | [\lambda'_2]_m (M_2)_m \rangle_q 
\times T^{(\lambda_1)_{n}[(\lambda_2)_{m}]}_{(\mu_1)_{n}(\mu_2)_m} U^{[\lambda'_1]_{n}[(\lambda'_2)_{m}]}_{(\mu'_1)_{n}(\mu'_2)_{m}}, \]

(44)

where \( \langle , , \rangle_q \) denotes a \( u_q(n) \) or \( u_q(m) \) Wigner coefficient.

By using the values of the latter, eq. (41) can be written in an explicit form [14]. The resulting relations coincide with eqs. (31) and (32), thereby proving the equivalence of the two constructions of \( A_q(n, m) \) based upon \( SU_q(n) \times SU_q(m) \) and \( u_q(n) + u_q(m) \), respectively.

References