Abstract

The general form and associativity conditions of deformed oscillator algebras are reviewed. It is shown how the latter can be fulfilled in terms of a solution of the Yang-Baxter equation when this solution has three distinct eigenvalues and satisfies a Birman-Wenzl-Murakami condition. As an example, an $SU_q(n) \times SU_q(m)$-covariant $q$-bosonic algebra is discussed in some details.

1 Introduction

Since the advent of quantum groups and $q$-algebras (see e.g. [1] and references quoted therein), much attention has been paid to deformations of the algebras of bosonic and fermionic creation and annihilation operators [2]–[6]. Different deformations of the latter arise depending on which property of the undeformed operators is preserved.

In the simple case of the $su(2)$ Lie algebra, two pairs of bosonic creation and annihilation operators $a_1^\dagger, a_i, i = 1, 2$, give rise to the Jordan-Schwinger realization

$$J_+ = a_1^\dagger a_2, \quad J_- = a_2^\dagger a_1, \quad J_0 = \frac{1}{2}(N_1 - N_2),$$

where $N_i = a_i^\dagger a_i, i = 1, 2$, are number operators. In addition, the creation operators $a_i^\dagger, a_i^\dagger$ (as well as the modified annihilation operators $\hat{a}_1 = a_2, \hat{a}_2 = -a_1$) are the components $+1/2$ and $-1/2$ of an $su(2)$ spinor, respectively. When extending these two properties to the corresponding $q$-algebra $su_q(2)$ (where $q$ is real and positive), one gets two different sets of $q$-bosonic operators.

On the one hand, those first considered by Biedenharn [2], Macfarlane [3], Sun and Fu [4], give rise to a Jordan-Schwinger realization of $su_q(2)$ of the same type as (1), where $a_1^\dagger, a_i, i = 1, 2$, now satisfy the relations

$$a_i^\dagger a_i^\dagger q^{N_i} = q^{N_i}, \quad \text{while operators with different indices do still commute, and } a_i^\dagger a_i = [N_i]_q = (q^{N_i} - q^{-N_i})/(q - q^{-1}).$$

However, the operators $a_1^\dagger, a_2^\dagger$ do not transform any more under a definite representation of the algebra.
On the other hand, the operators $A^+_i$, $A_i$, $i = 1, 2$, introduced by Pusz and Woronowicz [5], satisfy different relations

$$A^+_i A^+_j - q^{-1} A^+_j A^+_i = A_i A_j - q A_j A_i = 0, \quad i < j,$$
$$A_i A_j - q A_j A_i = 0, \quad i \neq j,$$
$$A_i A_j - q^2 A_j A_i = I + (q^2 - 1) \sum_{j=1}^{i-1} A^+_j A_j,$$

where the two modes are not independent any more. As a result of this coupling, the operators $A^+_1, A^+_2$ (as well as $A_1 = q^{1/2} A_2$, $A_2 = -q^{-1/2} A_1$) are the components +1/2 and −1/2 of an $su_q(2)$ spinor respectively, but yield an $su_q(2)$ realization that is substantially more complicated than (1). The algebra (3) has also important covariance properties under the quantum group $SU_q(2)$, dual to $su_q(2)$.

The present communication is concerned with the construction of covariant deformed oscillator algebras that generalize the Pusz-Woronowicz algebra for other quantum groups than $SU_q(2)$ (or more generally $SU_q(n)$). The method used will be based on an $R$-matrix approach similar to that applied in noncommutative differential geometry [7,8]. In Sec. 2, after reviewing the general form and associativity conditions of deformed oscillator algebras, we will show how to fulfill the latter in terms of a solution of the Yang-Baxter equation with three distinct eigenvalues. The example of an $SU_q(n) \times SU_q(m)$-covariant q-bosonic algebra $\mathcal{A}_q(n,m)$ will be treated in some details in Sec. 3. Finally, in Sec. 4, an alternative derivation of the same algebra, based upon the $q$-algebra $u_q(n) + u_q(m)$ will be presented.

## 2 Deformed Oscillator Algebras

Let us consider the complex algebras generated by $I$, $A^+_i A_i = (A^+_i)^\dagger$, $i = 1, \ldots, N$, subject to the relations [9,10]

$$A^+_i A^+_j = X_{ij,kl} A^+_k A^+_l,$$
$$A_i A_j = X^*_{ji,kl} A_k A_l,$$
$$A_i A^+_j = \delta_{ij} + Z_{ji,kl} A^+_k A_l,$$

where $X$ and $Z$ are some complex $N^2 \times N^2$ matrices, and there are summations over dummy indices. As a consequence of the Hermiticity properties of the generators, $X^*$ is the complex conjugate of $X$, and $Z$ is a Hermitian matrix.

For these algebras to be associative, it is sufficient to require the braid transposition schemes for triples of generators. The braid scheme for $A^+_i A^+_j A^+_k$ yields the condition

$$X_{ij,ab} X_{bk,cz} X_{ac,xy} = X_{jk,ab} X_{ia,zc} X_{eb,yz},$$

i.e., in compact tensor notation, the Yang-Baxter equation for $X$ (in the “braid” version)

$$X_{12} X_{23} X_{12} = X_{23} X_{12} X_{23}.$$
Similarly, for \( A_i A_j A_k \), one gets the two conditions
\[
\delta_{ij} \delta_{kk} - X_{jk,ix} + Z_{jk,ix} - X_{jk,ab} Z_{ab,ix} = 0, \tag{7}
\]
and
\[
Z_{kj,ac} Z_{ja,ib} X_{bc,xy} = X_{jk,ab} Z_{bc,cy} Z_{ac,ix}, \tag{8}
\]
which may be written in compact form as
\[
(I_{12} - X_{12})(I_{12} + Z_{12}) = 0, \tag{9}
\]
and
\[
Z_{23} Z_{12} X_{23} = X_{12} Z_{23} Z_{12}. \tag{10}
\]
From the Hermiticity properties of the generators, it follows that the remaining two triple products \( A_i A_j A_k \) and \( A_i A_j A_k^\dagger \) will be associative if \( A_i A_j A_k^\dagger \) and \( A_i A_j A_k^\dagger \) are so. Hence, eqs. (6), (9), and (10) are the only associativity conditions of algebra (4).

Let now \( R \) be any \( N^2 \times N^2 \) solution of the Yang-Baxter equation
\[
R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}. \tag{11}
\]
Then the corresponding braid matrix \( \hat{R} = \tau R \), where \( \tau \) is the twist operator (i.e., \( \tau_{ij,kl} = \delta_{il} \delta_{jk} \)), satisfies an equation similar to (6).

If \( \hat{R} \) has three distinct eigenvalues \( \lambda_\alpha, \alpha = 1, 2, 3 \), and satisfies a Birman-Wenzl-Murakami (BWM) condition\(^2\)
\[
(\hat{R} - \lambda_1 I)(\hat{R} - \lambda_2 I)(\hat{R} - \lambda_3 I) = 0, \tag{12}
\]
then with each eigenspace of \( \hat{R} \), one can associate two solutions of the set of associativity conditions (6), (9), and (10). In terms of the projector
\[
P_\alpha = \frac{1}{\prod_{\beta \neq \alpha} (\lambda_\alpha - \lambda_\beta)} \tag{13}
\]
on to the eigenspace corresponding to the eigenvalue \( \lambda_\alpha \), these two solutions can be written as
\[
I - X \simeq P_\alpha \quad \text{and} \quad Z = -\lambda_\alpha^{-1} \hat{R} \quad \text{or} \quad Z = -\lambda_\alpha \hat{R}^{-1}. \tag{14}
\]
Considering for instance \( Z = -\lambda_\alpha^{-1} \hat{R} \) leads to the following deformed oscillator algebra (written in compact tensor form)
\[
A_i A_j^\dagger A_k^\dagger = S A_i A_j^\dagger A_k^\dagger, \quad A_i A_j = S^* A_j A_i, \quad A_i A_j^\dagger = I_{12} - \lambda_\alpha^{-1} R_{ij} A_i A_j^\dagger A_i, \tag{15}
\]
where \( S = \tau X \) is found from (13) and (14), and \( t_1 \) means transposition with respect to the first space in the tensor product.

Several examples of such deformed oscillator algebras have been worked out so far [9]–[11]. In all cases, the solution of the Yang-Baxter equation that has been considered is the fundamental \( R \)-matrix of some classical quantum group. In such circumstances, the deformed oscillator algebras

\(^2\)The \( SU_q(n) \)-covariant algebra constructed by Pusz and Woronowicz [5] corresponds to the simpler case where \( \hat{R} \) has only two distinct eigenvalues, and satisfies a Hecke condition (see Sec. 3).
are left invariant under the transformations induced by the quantum group. The construction presented here is not restricted however to such a choice, and any solution of (11) and (12) might actually be used. In a similar way, deformed oscillator algebras differing from that of Pusz-Woronowicz have been built by considering non-standard solutions of the Yang-Baxter equation and the Hecke condition [12].

The algebras constructed in refs. [9]-[11] include both standard and non-standard ones. The former [9,10] are either of $q$-bosonic or $q$-fermionic type, meaning that whenever $q \to 1$, they go over smoothly into ordinary bosonic or fermionic algebras, respectively. The latter [11], on the contrary, do not have such a smooth behaviour, but share instead some features with the quon algebra [13]. In the next section, we shall consider in more details a covariant $q$-bosonic algebra generalizing that of Pusz-Woronowicz.

3 An $SU_q(n) \times SU_q(m)$-Covariant $q$-Bosonic Algebra

The $SU_q(n)$ quantum group [1] is a complex associative algebra generated by $I$ and the noncommutative elements $T_{ij}$, $i, j = 1, \ldots, n$ of an $n \times n$ matrix $T$, subject to the relations

$$RT_1T_2 = T_2T_1R, \quad \text{det}_q T = 1,$$

and the $^*$-involution condition

$$T^* = (T^{-1})^t,$$

with $q$ real. In (16), $\text{det}_q$ denotes the quantum determinant, and $R$ is the fundamental $R$-matrix associated with the $A_{n-1}$ series of Lie algebras,

$$R = q \sum_{i=1}^n e_{ii} \otimes e_{ii} + \sum_{i,j=1}^n e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i,j=1}^n e_{ij} \otimes e_{ji},$$

where $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$. The coproduct, counit and antipode are defined by

$$\Delta(T) = T_1 \hat{\otimes} T_2, \quad \epsilon(T) = 1, \quad S(T) = T^{-1},$$

where $\Delta(T_{ij}) = T_{ik} \otimes T_{kj}$.

The braid matrix $\hat{R}$, corresponding to (18), is a real symmetric matrix with two distinct eigenvalues, $q$ and $-q^{-1}$. Their respective multiplicities are $\frac{1}{2}n(n + 1)$ and $\frac{1}{2}n(n - 1)$, i.e., the dimensions of the symmetric and antisymmetric irreps $[\hat{2}]_n$ and $[1^2\hat{2}]_n$ of $SU_q(n)$. The $\hat{R}$-matrix satisfies the Hecke condition

$$(\hat{R} - qI)(\hat{R} + q^{-1}I) = 0.$$
From the properties of the two "small" braid matrices $\hat{R}$ and $\hat{R}$, it follows that $\hat{R} = q^{-1}\hat{R}\hat{R}$ has three distinct eigenvalues $q$, $-q^{-1}$, and $q^{-3}$, with respective multiplicities corresponding to the dimensions of the representations $[2^0]_n, [2^0]_m$, $[2^0]_n [1^{2^0}]_m + [1^2]_n [2^0]_m$, and $[1^2]_n [1^{2^0}]_m$ of $SU_q(n) \times SU_q(m)$, and satisfies the BWM condition (12).

By applying the results of the previous section to the antisymmetric (reducible) eigenspace of $\hat{R}$ associated with the eigenvalue $-q^{-1}$, one gets a deformed oscillator algebra of type (15), which will be denoted by $A_q(n, m)$, and whose defining relations are [10]

$$A_1^+ A_2^- = S A_1^+ A_2^-, \quad A_2 A_1 = A_1 A_2 S, \quad A_2 A_1^+ = I_{21} + q R^{1-} A_1^+ A_2^-,$$

where

$$S = \tau(I - (q + q^{-1})\mathcal{P}_A), \quad \mathcal{P}_A = \frac{(\hat{R} - q I)(\hat{R} - q^{-3} I)}{(q + q^{-1})(q^{-1} + q^{-3})},$$

and the creation and annihilation operators $A_1^+, A_{is}$ now have two indices, $i = 1, 2, \ldots, n$, and $s = 1, 2, \ldots, m$. Whenever $q \to 1$, $R$ and $S$ go over into $I$, so that (22) becomes an ordinary bosonic algebra.

The defining relations (22) of the $q$-bosonic algebra $A_q(n, m)$ may be rewritten in terms of the two "small" $R$-matrices as

$$RA_2 A_1 = A_1 A_2 R, \quad A_2 A_1^+ = I_{21} + R^{1-} R^{1-} A_1^+ A_2^-,$$

or, in a more explicit form, as

$$R_{ij, kl} A_{kl}^+ A_{jl}^+ = A_{ju}^+ A_{iu}^+ R_{uv, st}, \quad R_{ij, kl} A_{lu} A_{kl}^+ = A_{iu} A_{ju} R_{uv, st}, \quad A_{is} A_{jt}^+ = \delta_{ij} \delta_{st} + R_{ki, jl} R_{us, tv} A_{ku} A_{lv}.$$

Let us consider the map $\varphi : A_q(n, m) \to A_q(n, m) \otimes (SU_q(n) \times SU_q(m))$, defined by

$$A_{is}^+ = \varphi(A_{is}^+) = A_{is}^+ T_{ij} T_{st}, \quad \left| A_{is}^+ \right| = \varphi(A_{is}) = A_{is} T_{ij} T_{st}^\ast = T_{ij}^{-1} T_{st}^{-1} A_{is},$$

where the elements $T_{ij}$ and $T_{st}$ of $SU_q(n) \times SU_q(m)$ are assumed to commute with $A_{is}^+$ and $A_{is}$. As a consequence of (16) and its counterpart for $SU_q(m)$, this map leaves the defining relations (25) of $A_q(n, m)$ invariant. Hence, the latter is an $SU_q(n) \times SU_q(m)$-covariant algebra.

In the next section, an important part will be played by the modified annihilation operators

$$\hat{A}_{is} = A_{jt} C_{ji} C_{ts}, \quad C_{ji} = (-1)^{n-j} q^{-(n-2j+1)/2} \delta_{ji}, \quad C_{ts} = (-1)^{m-t} q^{-(m-2t+1)/2} \delta_{ts},$$

where $i' \equiv n + 1 - i$, $s' \equiv m + 1 - s$. Eq. (24) can be rewritten in terms of $A_{is}^+$ as

$$RA_2 A_1 = A_1 A_2 R, \quad R\hat{A}_1 \hat{A}_2 = \hat{A}_2 \hat{A}_1 R, \quad \hat{A}_2 A_1^+ = C_{12} C_{12} + q^2 A_1^+ \hat{A}_2 \hat{R}^{-1} \hat{R}$$

where $\hat{R}$ is defined by

$$\hat{R} = \sum_{i=1}^n e_{ii} \otimes e_{ii'} + q \sum_{i=j=1}^n e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i,j=1}^n (-q)^{i-j} e_{ij} \otimes e_{ij},$$

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and a similar definition holds for $\mathcal{R}$. Under map $\varphi$ of eq. (26), $\hat{A}_{is}$ is transformed into

$$
\hat{A}'_{is} = \varphi(\hat{A}_{is}) = \hat{A}_{jt} \hat{T}_j \hat{T}_i,
\hat{T} \equiv C^{-1}(T^{-1})^t C, \quad \hat{T} \equiv C^{-1}(T^{-1})^t C.
$$

(30)

Finally, combining eqs. (18) and (25) yields the detailed form of the $\mathcal{A}_q(n, m)$ defining relations

$$
\begin{align*}
A_{is}^t A_{jt}^t - q^{-1} A_{it}^t A_{jt}^t & = 0, \quad s < t, \\
A_{is}^t A_{js}^t - q^{-1} A_{js}^t A_{is}^t & = 0, \quad i < j, \\
A_{is}^t A_{jt}^t - A_{jt}^t A_{is}^t & = 0, \quad i > j, \quad s < t, \\
A_{is}^t A_{jt}^t - A_{jt}^t A_{is}^t & = -(q - q^{-1}) A_{js}^t A_{it}^t, \quad i < j, \quad s < t,
\end{align*}
$$

(31)

and

$$
\begin{align*}
A_{is} A_{jt}^t - A_{jt}^t A_{is} & = 0, \quad i \neq j, \quad s \neq t, \\
A_{is} A_{js}^t - q A_{js}^t A_{is} & = (q - q^{-1}) \sum_{t=1}^{s-1} A_{jt}^t A_{it}, \quad i \neq j, \\
A_{is} A_{it}^t - q A_{it}^t A_{is} & = (q - q^{-1}) \sum_{j=1}^{s-1} A_{jt}^t A_{js}, \quad s \neq t, \\
A_{is} A_{is}^t - q^2 A_{is}^t A_{is} & = I + (q^2 - 1) \left( \sum_{j=1}^{i-1} A_{js}^t A_{js} + \sum_{t=1}^{s-1} A_{it}^t A_{it} \right) \\
& \quad - (q^{-2} - 1) \sum_{j=1}^{i-1} \sum_{t=1}^{s-1} A_{jt}^t A_{jt},
\end{align*}
$$

(32)

together with the Hermitian conjugates of (31). Whenever $m = 1$, substituting $A_{i1}^t, A_i$ for $A_{i1}, A_i$ in (31) and (32) yields the Pusz-Woronowicz relations (3) for arbitrary $n$ values. Hence, the covariant $q$-bosonic algebra $\mathcal{A}_q(n, m)$ is a generalization of that of Pusz-Woronowicz for values of $m$ greater than 1.

4 Alternative Derivation in Terms of $u_q(n) + u_q(m)$

An alternative approach to the construction of covariant deformed oscillator algebras, based upon $q$-algebras, has been developed elsewhere [14,15]. In the case of the algebra $\mathcal{A}_q(n, m)$ introduced in the previous section, one considers the $q$-algebra $u_q(n) + u_q(m)$. The Cartan-Chevalley generators of $u_q(n)$ are denoted by $E_{ii} = (E_{ii})^t, i = 1, 2, \ldots, n, E_{i,i+1}, E_{i+1,i} = (E_{i+1,i})^t, i = 1, 2, \ldots, n - 1$, and satisfy the commutation relations

$$
\begin{align*}
[E_{ii}, E_{jj}] & = 0, \quad [E_{ii}, E_{j,j+1}] = (\delta_{ij} - \delta_{i,j+1}) E_{j,j+1}, \\
[E_{ii}, E_{j+1,j}] & = (\delta_{i,j+1} - \delta_{ij}) E_{j+1,j}, \quad [E_{i,i+1}, E_{j+1,j}] = \delta_{ij} [H_i]_q,
\end{align*}
$$

(33)

together with the quadratic and cubic $q$-Serre relations. In (33), $H_i \equiv E_{ii} - E_{i+1,i+1}$. The algebra is endowed with a Hopf algebra structure with coproduct $\Delta$, counit $\epsilon$, and antipode $S$, defined by

$$
\Delta(E_{ii}) = E_{ii} \otimes I + I \otimes E_{ii}, \quad \Delta(E_{i,i+1}) = E_{i,i+1} \otimes q^{H_{i+1}/2} + q^{-H_{i+1}/2} \otimes E_{i,i+1},
$$
\[ \Delta(E_{i+1,i}) = E_{i+1,i} \otimes q^{H_1/2} + q^{-H_1/2} \otimes E_{i+1,i}, \]  \[ \epsilon(E_{i,i}) = \epsilon(E_{i+1,i}) = 0, \]  \[ S(E_{i,i}) = -E_{i,i}, \quad S(E_{i+1,i}) = -qE_{i+1,i}, \quad S(E_{i+1,1}) = -q^{-1}E_{i+1,1}. \]

The Cartan-Chevalley generators of \( u_q(m) \) are denoted by \( \mathcal{E}_{s,s}, s = 1, 2, \ldots, m, \) \( \mathcal{E}_{s,s+1}, \mathcal{E}_{s+1,s}, s = 1, 2, \ldots, m-1, \) and satisfy relations similar to (33)-(36), while commuting with the generators of \( u_q(n). \)

In the approach based upon \( u_q(n) \oplus u_q(m), \) the \( q \)-bosonic creation operators \( A_{i,s}^\dagger, i = 1, 2, \ldots, n, \) \( s = 1, 2, \ldots, m, \) belonging to \( \mathcal{A}_q(n,m), \) are defined as the components of a double irreducible tensor \( T^{[6]_m[10]_n} \) with respect to this \( q \)-algebra. This means that they fulfil the relations

\[ E_{jj}(A_{i,s}^\dagger) = \delta_{j,i} A_{i,s}^\dagger, \quad E_{jj+1}(A_{i,s}^\dagger) = \delta_{j,i} A_{i+1,s}^\dagger, \quad E_{j+1,j}(A_{i,s}^\dagger) = \delta_{j,i} A_{i+1,s}^\dagger, \]  \[ E_{tt}(A_{i,s}^\dagger) = \delta_{t,i} A_{i,s}^\dagger, \quad E_{t,t+1}(A_{i,s}^\dagger) = \delta_{t,i} A_{i+1,s}^\dagger, \]  \[ E_{t+1,t}(A_{i,s}^\dagger) = \delta_{t,i} A_{i+1,s}^\dagger. \]

where, for any \( u_q(n) + u_q(m) \) generator \( X, X(A_{i,s}^\dagger) \) denotes the quantum adjoint action \( X(A_{i,s}^\dagger) = \sum_r X_r^1 \otimes X_r^2, \) with \( \Delta(X) = \sum_r X_r^1 \otimes X_r^2. \) The modified annihilation operators \( \hat{A}_{i,s}, i = 1, 2, \ldots, n, \) \( s = 1, 2, \ldots, m, \) of eq. (27), are similarly defined as the components of a double irreducible tensor \( T^{[6-1]_n[6-1]_m} \) with respect to \( u_q(n) + u_q(m), \) and satisfy the relations

\[ E_{jj}(\hat{A}_{i,s}) = -\delta_{j,i} \hat{A}_{i,s}, \quad E_{jj+1}(\hat{A}_{i,s}) = \delta_{j,i} \hat{A}_{i-1,s}, \quad E_{j+1,j}(\hat{A}_{i,s}) = \delta_{j,i} \hat{A}_{i+1,s}, \]  \[ E_{tt}(\hat{A}_{i,s}) = -\delta_{t,i} \hat{A}_{i,s}, \quad E_{t,t+1}(\hat{A}_{i,s}) = \delta_{t,i} \hat{A}_{i-1,s}, \quad E_{t+1,t}(\hat{A}_{i,s}) = \delta_{t,i} \hat{A}_{i+1,s}. \]

The operators \( A_{i,s}^\dagger \) and \( \hat{A}_{i,s} \) can be explicitly written down in terms of \( m \) independent copies of the Pusz-Woronowicz operators [14]. By using such expressions and exploiting the tensorial character of the operators, it is straightforward to prove that their \( q \)-commutation relations are given in coupled form by

\[ [A_{i,s}^\dagger, A_{i,s}^\dagger]^{[20]_n[120]_m} = [A_{i,s}^\dagger, A_{i,s}^\dagger]^{[20]_m[20]_n} = [\hat{A}_{i,s}, \hat{A}_{i,s}]^{[6-2]_n[6]_m} = [\hat{A}_{i,s}, \hat{A}_{i,s}]^{[6-2]_m[6]_n} = 0, \]  \[ [\hat{A}_{i,s}, A_{i,s}^\dagger]^{[6-1]_n[6-1]_m} = [\hat{A}_{i,s}, A_{i,s}^\dagger]^{[6-2]_n[6]_m} = [\hat{A}_{i,s}, A_{i+s-1,s}^\dagger]^{[6]_m[6-1]_n} = 0, \]  \[ [\hat{A}_{i,s}, A_{i,s}^\dagger]^{[6]_m[6-1]_n} = \sqrt{[n]_q[2m]_q}. \]

where, for simplicity's sake, the row labels of the coupled \( u_q(n) + u_q(m) \) irreps have been dropped. In (41), the coupled \( q \)-commutator of two double irreducible tensors \( T^{[\lambda_1]_m[\lambda_2]_n} \) and \( U^{[\lambda_1]_m[\lambda_2]_n} \) is defined by [14]

\[ [T^{[\lambda_1]_m[\lambda_2]_n}, U^{[\lambda_1]_m[\lambda_2]_n}]^{[\lambda_1]_m[\lambda_2]_n} = [T^{[\lambda_1]_m[\lambda_2]_n}, U^{[\lambda_1]_m[\lambda_2]_n}]^{[\lambda_1]_m[\lambda_2]_n} = \left(-1\right)^{q_\alpha} [U^{[\lambda_1]_m[\lambda_2]_n} \times T^{[\lambda_1]_m[\lambda_2]_n}]^{[\lambda_1]_m[\lambda_2]_n}. \]  \[ \epsilon = \phi([\lambda_1]_n) + \phi([\lambda_2]_m) - \phi([\lambda_1]_n) + \phi([\lambda_2]_m) - \phi([\lambda_2]_m), \]  \[ \phi([\lambda_1]_n) = \frac{1}{2} \sum_{i=1}^{n}(n + 1 - 2i)\lambda_{1i}, \quad \phi([\lambda_2]_m) = \frac{1}{2} \sum_{s=1}^{m}(m + 1 - 2s)\lambda_{2s}, \]  \[ \phi([\lambda_2]_m) = \frac{1}{2} \sum_{s=1}^{m}(m + 1 - 2s)\lambda_{2s}, \]
and

\[
[T(\lambda_1)_n^{(\lambda_2)}_m \times U(\lambda_1')_n^{(\lambda_2')_m}](\Lambda_1)_n^{(\Lambda_2)_m} = \sum_{(\mu_1)_n^{(\mu_2)_m}(\mu_1')_n^{(\mu_2')_m}} \langle [\lambda_1]_n^{(\mu_1)_n}[\lambda_1']_n^{(\mu_1')_n}][\Lambda_1]_n^{(M_1)_n}\rangle_q \langle [\lambda_2]_m^{(\mu_2)_m}[\lambda_2']_m^{(\mu_2')_m}][\Lambda_2]_m^{(M_2)_m}\rangle_q
\]

where \(\langle , | \rangle_q\) denotes a \(u_q(n)\) or \(u_q(m)\) Wigner coefficient.

By using the values of the latter, eq. (41) can be written in an explicit form [14]. The resulting relations coincide with eqs. (31) and (32), thereby proving the equivalence of the two constructions of \(A_q(n,m)\) based upon \(SU_q(n) \times SU_q(m)\) and \(u_q(n) + u_q(m)\), respectively.

References