AN ALGORITHM FOR THE BASIS
OF THE FINITE FOURIER TRANSFORM

T. S. Santhanam
Parks College of Saint Louis University
Cahokia, Illinois, 62206, U.S.A.

Abstract

The Finite Fourier Transformation matrix (F.F.T.) plays a central role in the formulation of quantum mechanics in a finite dimensional space studied by the author over the past couple of decades. An outstanding problem which still remains open is to find a complete basis for F.F.T. In this paper we suggest a simple algorithm to find the eigenvectors of F.F.T.

I. INTRODUCTION

The finite Fourier transform matrix (F.F.T.) plays a fundamental role in many contexts and has been studied extensively [1-3]. It is central in the discussions on finite dimensional quantum mechanics based on Weyl's commutation relations [4] studied by the author in a series of publications [5]. The eigenvalues of this matrix were determined by Schur [1] and a simple argument to recover this result has been given earlier [6]. The calculation of the eigenvectors is not straightforward and many methods have been given in particular, by Mehta [7]. In Section IV, we present a new algorithm to find the eigenvectors.

II. EIGENVALUES OF S

The F.F.T. matrix $S$, which is unitary, is defined by

$$S_{\alpha \beta} = \frac{1}{\sqrt{n}} \exp \left( \frac{2\pi i}{n} \alpha \beta \right),$$

\[\alpha, \beta = 0,1,2,...n-1\] \hspace{2cm} (2.1)

and has many interesting properties

1) \( (S^2)_{\alpha \beta} = I'_{\alpha \beta} = \delta_{\alpha + \beta, 0} \mod n \) \hspace{2cm} (2.2)

Since \( S^2 f_{\alpha} = f_{-\alpha \mod n} \) for a vector \( f_{\alpha} \) with \( n \) components, \( S^2 \) is called the parity operator

2) \( (S^4)_{\alpha \beta} = \delta_{\alpha \beta} \) \hspace{2cm} (2.3)

like the usual Fourier transform.

3) The matrix \( S \), which is by definition a symmetric matrix will diagonalize any circulant matrix.
From Equation (2-3), it is clear that the eigenvalues of $S$ are simply $\pm 1$ and $\pm i$. There is then a degeneracy of the eigenvalues. The first problem will be to determine this. Luckily, Equations (2.1)-(2.3) can be repeatedly used to fix this [6]. If $k_1$, $k_2$, $k_3$ and $k_4$ denote the multiplicity of the eigenvalues taken in the order $(1, -1, i, -i)$, Equation (2.1) implies that

$$\text{Tr } S = \frac{1}{n-1} \sum_{\ell=0}^{n-1} \exp \left( \frac{2\pi i}{n} \ell \right)^2$$

and hence

$$\text{Tr } S = \frac{1}{2} \left( 1 + i \right) \left[ 1 + \exp \left( \frac{-in}{2} \right) \right], \quad (2.4)$$

and

$$\text{Tr } S = (k_1 - k_2) + i(k_3 - k_4)$$

$$= 1 \text{ for } n = 4k + 1,$$

$$= 0 \text{ for } n = 4k + 2,$$

$$= i \text{ for } n = 4k + 3,$$

$$= (1 + i) \text{ for } n = 4k,$$

$$k = 0, 1, 2, \ldots \quad (2.5)$$

From Equation (2) we infer that

$$\text{Tr } S^2 = (k_1 + k_2) - (k_3 + k_4)$$

$$= 1 \text{ for } n \text{ odd},$$

$$= 2 \text{ for } n \text{ even}.$$
\[
\begin{array}{ccccc}
  n = 4k + 1 & n = 4k + 2 & n = 4k + 3 & n = 4k \\
  k_1 & k + 1 & k + 1 & k + 1 & k + 1 \\
  k_2 & k & k + 1 & k + 1 & k \\
  k_3 & k & k & k + 1 & k \\
  k_4 & k & k & k & k - 1 \\
\end{array}
\]

III. EIGENVECTORS OF S

Let us decompose $S$ into its primitive idempotents as

\[
S = \sum_{j=1}^{4} i^j B(j),
\]

where

\[
\begin{align*}
B(1) &= \frac{1}{2} s + \frac{1}{4} (I - I') \\
B(2) &= -\frac{1}{2} c + \frac{1}{4} (I + I'), \\
B(3) &= -\frac{1}{2} s + \frac{1}{4} (I - I'), \\
B(4) &= \frac{1}{2} c + \frac{1}{4} (I + I'),
\end{align*}
\tag{3.2}
\]

\[
\begin{align*}
C_{\alpha \beta} &= \frac{1}{\sqrt{n}} \cos \left( \frac{2\pi}{n} \alpha \beta \right) \\
s_{\alpha \beta} &= \frac{1}{\sqrt{n}} \sin \left( \frac{2\pi}{n} \alpha \beta \right),
\end{align*}
\tag{3.3}
\]

It is easily verified that

\[
S B(j) = i^j B(j),
\tag{3.4}
\]

thus the nonzero columns of $B(j)$ yield the eigenvectors of $S$ with eigenvalue $i^j$. Also, in analogy with the standard case, Mehta [7] has been able to express these eigenvectors in terms of Hermite functions with
IV. EIGENVECTORS OF S; AN ALTERNATE METHOD

Since the F.F.T. matrix S satisfies Equation (2.1) we construct the matrix

\[ T = S^3 + S^2 S_d + S S_d^2 + S_d^3 \]

\[ = I' (S + S_d) + (S + S_d) S_d^2, \quad (4.1) \]

where

\[ S_d = \text{diagonal } S. \quad (4.2) \]

We find that

\[ S T = S (S^3 + S^2 S_d + S S_d^2 + S_d^3) \]

\[ = (I + S^3 S_d + S^2 S_d^2 + S S_d^3) \]

\[ = (S_d^2 + S^3 + S^2 S_d + S S_d^2) S_d \]

\[ = T S_d. \quad (4.3) \]

If T is nonsingular,

\[ T^+ S T = S_d \quad (4.4) \]

Therefore, the columns of T automatically provide the eigenvectors of S. The degenerate eigenvectors of S corresponding to the repeated eigenvalues can be made orthonormal by using Gram-Schmidt process. This will render T unitary. While the process is quite general, we shall illustrate this for some special cases.

**Case of n = 2**

\[ S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad (4.5) \]
and
\[ S_d = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]  

(4.6)

Since \( S^2 = S_d^2 = I \),

(4.7)

We get from Equation (4.1)
\[ T = 2 (S + S_d), \]

\[ = 2 \begin{pmatrix} 1 + \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -1 - \frac{1}{\sqrt{2}} \end{pmatrix} \]  

(4.8)

We unitarized matrix of the eigenvectors of \( S \) is therefore
\[ U_2 = \frac{1}{\sqrt{2 \sqrt{2} (\sqrt{2} + 1)}} \begin{pmatrix} \sqrt{2} + 1 & 1 \\ 1 & - (\sqrt{2} + 1) \end{pmatrix} \]  

(4.9)

\textbf{case of } n = 3

\[ S = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \epsilon & \epsilon^2 \\ 1 & \epsilon^2 & \epsilon \end{pmatrix}, \]

\[ \epsilon = \exp \frac{2\pi i}{3}. \]  

(4.10)

From Equation (2.8) we see that
\[ S_d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & i \end{pmatrix} \]  

(4.11)

one finds from Equation (4.1) that the unitarized matrix of the eigenvectors of \( S \) is
\[ u_3 = \frac{1}{\sqrt{2} \sqrt{3} (\sqrt{3} + 1)} \begin{pmatrix} \sqrt{3} + 1 & \sqrt{2} & 0 \\ 1 & \frac{1 + \sqrt{3}}{\sqrt{2}} & i \sqrt{3} + \sqrt{3} \\ 1 & \frac{1 + \sqrt{3}}{\sqrt{2}} & -i \sqrt{3} + \sqrt{3} \end{pmatrix}. \] (4.12)

**case of n = 4**

In this case we have

\[ S = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -i \\ 1 & -i & -1 & i \end{pmatrix} \] (4.13)

and

\[ S_d = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \] (4.14)

It is easily calculated that

\[ T = \begin{pmatrix} 3 & 1 & 1 & 0 \\ 1 & 1 & -1 & 2i \\ 1 & -1 & -1 & 0 \\ 1 & 1 & -1 & -2i \end{pmatrix} \] (4.15)

The first two column vectors correspond to the eigenvalue = +1, the third one to -1 and the last to -i.
By a simple use of Gram-Schmidt orthogonalization procedure one can find the unitarized matrix corresponding to the eigenvectors of $S$ as

$$U_4 = \frac{1}{\sqrt{2\sqrt{4} (\sqrt{4} + 1)}} \begin{pmatrix} 3 & 0 & \sqrt{3} & 0 \\ 1 & \sqrt{2} & -\sqrt{3} & i\sqrt{6} \\ 1 & -2\sqrt{2} & -\sqrt{3} & 0 \\ 1 & \sqrt{2} & -\sqrt{3} & -i\sqrt{6} \end{pmatrix}$$  \hspace{1cm} (4.16)
ACKNOWLEDGMENTS

It is my pleasure to thank Professor Bernardo Wolf for inviting me to the harmonic oscillator conference and the organizers of the conference for their hospitality.
REFERENCES


[10] This can be easily generalized to $A^{n-1} + A^{n-2}A_d + A^{n-3}A_d^2 + \ldots + A_{d}^{n-2} + A_{d}^{n-1}$ for the case of a general involution matrix satisfying the relation $A^N = I$. 248